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# A Novel Approach for Multiplicative Linguistic Group Decision Making Based on Symmetrical Linguistic Chi-Square Deviation and VIKOR Method

**Zhiwei Gong [1,2], Jian Lin [2,*] and Ling Weng [2]**

1 Center for Discrete Mathematics and Theoretical Computer Science, Fuzhou University, Fuzhou 350116, China; zwgong@fafu.edu.cn
2 College of Computer and Information Sciences, Fujian Agriculture and Forestry University, Fuzhou 350002, China; wenglingmoo@163.com
* Correspondence: linjian36@163.com or linjian@fafu.edu.cn

**Abstract:** Most linguistic-based approaches to multi-attribute group decision making (MAGDM) use symmetric, uniformly distributed sets of additive linguistic terms to express the opinions of decision makers. However, in reality, there are also some problems that require the use of asymmetric, uneven, i.e., non-equilibrium, multiplicative linguistic term sets to express the evaluation. The purpose of this paper is to propose a new approach to MAGDM under multiplicative linguistic information. The aggregation of linguistic data is an important component in MAGDM. To solve this problem, we define a chi-square for measuring the difference between multiplicative linguistic term sets. Furthermore, the linguistic generalized weighted logarithm multiple averaging ($LGWLMA$) operator and linguistic generalized ordered weighted logarithm multiple averaging ($LGOWLMA$) operator are proposed based on chi-square deviation. On the basis of the proposed two operators, we develop a novel approach to GDM with multiplicative linguistic term sets. Finally, the evaluation of transport logistics enterprises is developed to illustrate the validity and practicality of the proposed approach.

**Keywords:** multiplicative symmetrical linguistic information; aggregation operator; linguistic chi-square deviation; group decision making



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## 1. Introduction

Group decision making (GDM) is an important and interesting research topic at present. It is done by inviting a large number of decision makers (DMs) with experience and knowledge to share their opinions on the evaluation of each alternative [1]. According to the evaluation information provided by DMs, the alternatives are comprehensively evaluated and the best alternative is finally obtained. With the increasingly complex social and economic environment, the probability that a single DM can consider the problem comprehensively is extremely low [2]. Therefore, a considerable amount of research has applied the GDM process to various types of practical problems [3–6], such as emergency preparedness selection [7,8], the evaluation of pro-environmental behavior [9], waste incineration plant and wind farm siting [10,11], healthcare facility selection [12], and so on. As an important form of GDM, MAGAM develops some methods for selecting the most desirable option from a pre-provided set of options based on the opinion information given by all decision makers on multiple influential attributes.

In a MAGDM environment, the evaluation of experts cannot be expressed precisely in quantitative form, but rather in qualitative form, due to human characteristics that cause judgments to carry inherent subjectivity and ambiguity. In such cases, experts or DMs prefer to use linguistic variables instead of numbers to express their judgments [13]. For example, when experts try to assess the "excellence" of a project, they use linguistic terms such as "good", "average", "poor", etc. In order to enhance the robustness and the diversity of usage scenarios of classical decision models [14], linguistic variables have been intensively investigated [15–20] and applied in many fields [10,21–23].

In the existing studies, many linguistic variables are assessed with uniformly and symmetrically distributed sets. However, there are also some issues that require unevenly distributed sets of linguistic terms to assess their variables. For example, negative reviews usually have a greater impact on the final choice of other users than positive reviews when shopping online since DMs tend to be risk-averse in the realistic situations [24]. Therefore, Xu [25] proposed an unbalanced linguistic label set called the multiplicative linguistic term set. Xu [25] demonstrated many practical advantages over additive linguistic term sets in specific contexts. Xu [26] proposed the definition of incomplete multiplicative linguistic preference relations. An effective method to solve the group decision problem with multiplicative linguistic information is developed. Meng et al. [27] devised a new type of linguistic variables, called dual multiplicative linguistic variables ($DMLVs$), and introduced a novel group decision-making method based on the consistency and consensus analysis of dual multiplicative linguistic preference relations ($DMLPRs$). Xie et al. [28] analyzed the dual probabilistic multiplicative linguistic preference relations ($DPMLPRs$) based upon the dual probabilistic multiplicative linguistic term sets ($DPMLTSs$). Then, the comparable degree between the $DPMLPRs$ and the consensus of the group $DPMLPR$ are studied. To compare two uncertain multiplicative linguistic variables, Xia et al. [29] defined a possibility degree formula and investigated its properties. Then, a group decision-making method was devised to deal with the case in which the preference is expressed by an uncertain multiplicative linguistic variable ($UMLV$).

To deal with MAGDM in such settings, several new aggregation operators have been developed for aggregating multiplicative linguistic information. Xu [25] proposed the extended $OWA$ ($EOWA$) operator and extended $OWG$ ($EOWG$) operator to aggregate multiplicative linguistic information. Tang [30] further defined an aggregation operator for fusing multiplicative linguistic variables, namely the extended linguistic geometric mean ($ELGM$) operator. The above studies all focus on the decision making of the identified linguistic variables. However, in some cases, it may still be difficult for decision makers to provide accurate linguistic values due to a lack information, time constraints, or limited expertise of the decision maker, etc. Considering this problem, Xu [31] proposed the uncertain linguistic weighted geometric mean ($ULWGM$) operator, the $ULOWG$ operator, and the induced $ULOWG$ operator to aggregate uncertain multiplicative linguistic information. Lin [32] proposed an integrated algorithm for linguistic group decision making based on the deviation measure and $ULHWG$ operator. Zhang [33] promoted the continuous $OWG$ ($C-OWG$) operator under uncertain linguistic environments. However, multiplicative linguistic aggregation operators are seldom studied in comparison to additive linguistic information. The aggregation operators studied above focus on original data. In the decision-making process, the decision makers are more interested in analyzing the difference between the original data and aggregation result. Therefore, the purpose of this paper is to develop a new multiplicative linguistic aggregation operator considering deviation. The ranking of alternatives is one of the most crucial steps of MAGDM. At present, there are many alternative ranking methods, such as TOPSIS [34,35], VIKOR [36], PROMETHEE [37], MULTIMOORA [38], and ELECTRE III [39], etc. Wojciech Sałabun et al. [40] performed simulation experiments on the TOPSIS, VIKOR, COPRAS, and PROMETHEE II methods to calculate the similarity of their obtained final rankings. Saroj Kumar Patel [41] compared the application of multi-criteria decision-making methods such as the TOPSIS and VIKOR methods for alternative industrial robot selection. Bartłomiej Kizielewicz et al. [42] proposed an MCDA-based method that considers fuzzy versions of TOPSIS, VIKOR, MMOORA, and WASPAS for evaluating the decision problem of choosing the best housing solution. Opricovic et al. [43,44] compared the VIKOR method with the TOPSIS and PROMETHEE methods and concluded that VIKOR can be compared with other methods to obtain the best results. The VIKOR approach finds a compromise solution to a problem and assists decision makers in the final decision. A compromise is a suitable solution that comes closest to an ideal solution. In addition, the VIKOR method can better retain complete decision information in the decision-making process. Therefore, scholars have become increasingly interested in this approach in recent years. However, the VIKOR method has been applied less in the study of multiplicative linguistic group decision making. Based on this, considering the superiority of VIKOR compared with other traditional multi-attribute decision methods, a multiplicative linguistic multi-attribute decision model based on symmetrical linguistic chi-square deviation and VIKOR is proposed.

The rest of this paper is organized as follows. Section 2 presents some preliminary knowledge. Section 3 proposes the deviation-based linguistic geometric aggregation operators and proves some of its properties. In Section 4, we propose a MAGDM approach that is based on this aggregation operator and give a numerical example to illustrate the feasibility of the proposed approach.

## 2. Preliminaries

### 2.1. Multiplicative Linguistic Scale and Its Operational Laws

The linguistic assessment scale is the basis of linguistic decision making. In [25], Xu defined a multiplicative symmetrical linguistic scale.

**Definition 1.** *Let*

$$S = \{s_\alpha | \alpha = \frac{1}{u}, \cdots, \frac{1}{2}, 1, 2, \cdots, u\}$$

*be a totally ordered discrete linguistic term set, and let S satisfy the following conditions:*

(1) $s_\alpha > s_\beta$ *if* $\alpha > \beta$.
(2) *There is the reciprocal operator,* $rec(s_\alpha) = s_\beta$*, such that* $\alpha\beta = 1$*. In particular,* $rec(s_1) = s_1$,

*where u is a positive integer and* $s_\alpha$ *refers to linguistic terms.*

In particular, $s_{\frac{1}{u}}$ and $s_u$ represent, respectively, the lower and upper limits of the linguistic terms actually used by decision makers. In order to preserve all the given information, Xu [25] extended the discrete linguistic term set $S$ to a continuous linguistic term set $\bar{S} = \{s_\alpha | \alpha \in [1/t, t]\}$, where $t > u$. If $s_\alpha \in S$, then we call $s_\alpha$ the original linguistic term; otherwise, we call $s_\alpha$ the virtual linguistic term. To facilitate the calculation of linguistic information, Xu [25] proposed the following linguistic operational laws:

**Definition 2.** *Let* $s_\alpha, s_\beta \in \bar{S}$*, and* $\lambda \in [0, 1]$*, then*

(1) $s_\alpha \bigotimes s_\beta = s_{\alpha\beta}$;
(2) $(s_\alpha)^\lambda = s_{\alpha^\lambda}$.

**Lemma 1.** *Let* $s_\alpha, s_{\alpha_1}, s_{\alpha_2} \in \bar{S}$*, and* $\lambda, \lambda_1, \lambda_2 \in [0, 1]$*, then*

(1) $s_{\alpha_1} \bigotimes s_{\alpha_2} = s_{\alpha_2} \bigotimes s_{\alpha_1}$;
(2) $(s_{\alpha_1} \bigotimes s_{\alpha_2})^\lambda = (s_{\alpha_1})^\lambda \bigotimes (s_{\alpha_2})^\lambda$;
(3) $(s_\alpha)^{\lambda_1} \bigotimes (s_\alpha)^{\lambda_2} = (s_\alpha)^{\lambda_1 + \lambda_2}$.

### 2.2. The Linguistic Geometric Aggregation Operators

The question of how to aggregate linguistic information is an important topic; in [45], Xu defined the $LG$ operator and $LWG$ operator as follows:

**Definition 3.** *Let* $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n} \in \bar{S}$ *be n linguistic variables. If* $LG : \bar{S}^n \to \bar{S}$ *satisfies*

$$LG(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = (s_{\alpha_1} \bigotimes s_{\alpha_2} \bigotimes \cdots \bigotimes s_{\alpha_n})^{\frac{1}{n}} = s_{\left(\prod_{j=1}^{n} \alpha_j\right)^{\frac{1}{n}}},$$

*then LG is called the linguistic geometric operator.*

**Definition 4.** *Let $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n} \in \bar{S}$ be n linguistic variables. If $LWG : \bar{S}^n \to \bar{S}$ satisfies*

$$LWG(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = (s_{\alpha_1})^{w_1} \bigotimes (s_{\alpha_2})^{w_2} \bigotimes \cdots \bigotimes (s_{\alpha_n})^{w_n} = s_{\prod_{j=1}^{n} \alpha_j^{w_j}},$$

*where $w = (w_1, w_2, \cdots, w_n)^T$ is the weighting vector of linguistic variables and $w_j \geq 0$, $\sum_{j=1}^{n} w_j = 1$, then LWG is called the linguistic geometric weighted averaging operator.*

In particular, if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the $LWG$ operator is reduced to a linguistic geometric ($LG$) operator.

To investigate the extension of the $OWG$ operator in a linguistic environment, Xu [25] defined the following $LOWG$ operator.

**Definition 5.** *A LOWG operator of dimension n is a mapping $LOWG : \bar{S}^n \to \bar{S}$ that is associated with a weighting vector $w = (w_1, w_2, \cdots, w_n)^T$, such that $w_j \geq 0$, $\sum_{j=1}^{n} w_j = 1$, and aggregates a collection of linguistic variables $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}$ according to the following expression:*

$$LOWG(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = (s_{\beta_1})^{w_1} \bigotimes (s_{\beta_2})^{w_2} \bigotimes \cdots \bigotimes (s_{\beta_n})^{w_n} = s_{\prod_{j=1}^{n} \beta_j^{w_j}},$$

*where $s_{\beta_j}$ is the jth largest of $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}$, and LOWG is thus called the linguistic ordered weighted geometric averaging operator.*

In particular, if $w = (1, 0, \cdots, 0)^T$, then the $LOWG$ operator is reduced to a linguistic maximum ($LM_1$) operator; if $w = (0, 0, \cdots, 1)^T$, then the $LOWG$ operator is reduced to a linguistic minimum ($LM_2$) operator, and if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the $LOWG$ operator is reduced to a linguistic geometric ($LG$) operator.

Based on the $LWG$ operators and $LOWG$ operators, in [46], Xu defined the $LHG$ operator as follows.

**Definition 6.** *An LHG operator of dimension n is a mapping $LHG : \bar{S}^n \to \bar{S}$ that is associated with an exponential weighting vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, such that $\omega_j \geq 0$, $\sum_{j=1}^{n} \omega_j = 1$, and aggregates a collection of linguistic variables $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}$ according to the following expression:*

$$LHG(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = (s_{\beta_1})^{\omega_1} \bigotimes (s_{\beta_2})^{\omega_2} \bigotimes \cdots \bigotimes (s_{\beta_n})^{\omega_n},$$

*where $s_{\beta_i}$ is the jth largest of $\bar{s}_{\alpha_1}, \bar{s}_{\alpha_2}, \cdots, \bar{s}_{\alpha_n}$ ($\bar{s}_{\alpha_i} = (s_{\alpha_i})^{nw_1}, i = 1, 2, \cdots, n$), $w = (w_1, w_2, \cdots, w_n)^T$ is the weight vector of the $s_{\alpha_i} (i = 1, 2, \cdots, n)$ with $w_j \geq 0$, $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient, which plays the role of balance; thus, LHG is called the linguistic hybrid geometric operator.*

In particular, if $\omega = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the $LHG$ operator is degenerated to the $LWG$ operator, and if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the $LHG$ operator is reduced to the $LOWG$ operator.

*2.3. The Penalty Function and BUM Function*

In [47], Calvo et al. were the first to theoretically study the relationship between the aggregation operator and the penalty function. In [48,49], Calvo and Grabisch et al. described the conditions for constructing an aggregation function from a penalty function in a general sense.

**Definition 7.** *Let $I$ be a subset of the set of real numbers. If the function $P : I^{n+1} \to I$ satisfies the following three conditions:*

(1) $P(X, y) \geq 0$ *for all* $X \in I^n$ *and* $y \in I$*;*
(2) $P(X, y) = 0$ *if* $X = Y$ *and* $Y = (y, y, \cdots, y)$*;*
(3) *For every fixed* $X$*, the set of minimizers of* $P(X, y)$ *is either a singleton or an interval.*

*then $P$ is called a penalty function.*

For deriving the associated weight of the *OWA* operator, Yager [50] proposed the following BUM function-based formula:

$$w_i = Q(\frac{i}{n}) - Q(\frac{i-1}{n}), i = 1, 2, \cdots, n. \tag{1}$$

The BUM function $Q$ is called the fuzzy semantic quantization operator, such that $f(0) = 0$, $f(1) = 1$, and $f(x) \leq f(y)$ for all $x < y$. The BUM function reflects the decision makers' tendency towards risk. For example, $Q(x) = x^q$ can be selected according to decision makers' risk preference. If $q \in (0, 1)$, then the decision maker is optimistic. If $q = 1$, then the decision maker is neutral. If $q > 1$, then the decision maker is pessimistic. The BUM function can also be expressed as follows:

$$Q(r) = \begin{cases} 0, & r < \alpha \\ \dfrac{r - \alpha}{\beta - \alpha}, & \alpha \leq r \leq \beta, \\ 1, & r > \beta \end{cases} \tag{2}$$

where $\alpha, \beta$ and $r$ are in the range of $[0, 1]$, the fuzzy linguistic "as many as possible " , "more " and "at least half " corresponding to the pair $(\alpha, \beta) = (0.5, 1)$, $(\alpha, \beta) = (0.3, 0.8)$, and $(\alpha, \beta) = (0, 0.5)$, respectively.

## 3. Linguistic Geometric Aggregation Operators Based on Symmetrical Linguistic Chi-Square Deviation

Let $\bar{S} = \{s_\alpha | \alpha \in [1/t, t]\}$ be an extended multiplicative linguistic scale of $S = \{s_\alpha | \alpha = \frac{1}{u}, \cdots, \frac{1}{2}, 1, 2, \cdots, u\}$. In general, $t$ is a constant slightly larger than $u$. Inspired by the existing studies [46,48,51], in this section, we will introduce a novel aggregation operator by minimizing a new penalty function and obtain the expression of the proposed operator.

### *3.1. Linguistic Generalized Weighted Logarithm Multiple Averaging Operator*

**Definition 8.** *Let $s_\alpha \in \bar{S}$, $T : \bar{S} \to [1/t, t]$ be a mapping, such that $T(s_\alpha) = \alpha$.*

Obviously, $T$ is a bijection and $T^{-1}(\alpha) = s_\alpha$. For all $\alpha, \beta \in (\frac{1}{u}, u)$, we have $tT(s_\alpha) > 1$ and $tT(s_\beta) > 1$. It follows that $\log_{tT(s_\beta)}^{tT(s_\alpha)} = \frac{\log(tT(s_\alpha))}{\log(tT(s_\beta))} > 0$. Since

$$((\log_{tT(s_\beta)}^{tT(s_\alpha)})^\lambda - 1)^2 = (\log_{tT(s_\beta)}^{tT(s_\alpha)})^{2\lambda} - 2(\log_{tT(s_\beta)}^{tT(s_\alpha)})^\lambda + 1 \geq 0,$$

and $\log_{tT(s_\beta)}^{tT(s_\alpha)} = \left(\log_{tT(s_\alpha)}^{tT(s_\beta)}\right)^{-1}$, we have $\left(\log_{tT(s_\beta)}^{tT(s_\alpha)}\right)^\lambda + \left(\log_{tT(s_\alpha)}^{tT(s_\beta)}\right)^\lambda - 2 \geq 0$. The equal sign holds if and only if $\alpha = \beta$.

**Definition 9.** *Let $s_\alpha, s_\beta \in S = \{s_\alpha | \alpha = \frac{1}{u}, \cdots, \frac{1}{2}, 1, 2, \cdots, u\}$, $D(s_\alpha, s_\beta)$ be the symmetrical linguistic chi-square deviation between $s_\alpha$ and $s_\beta$, if*

$$D(s_\alpha, s_\beta) = \frac{((\log_{tT(s_\beta)}^{tT(s_\alpha)})^\lambda - 1)^2}{\left(\log_{tT(s_\beta)}^{tT(s_\alpha)}\right)^\lambda} \tag{3}$$

*that is,*

$$D(s_\alpha, s_\beta) = \left(\log_{tT(s_\beta)}^{tT(s_\alpha)}\right)^\lambda + \left(\log_{tT(s_\alpha)}^{tT(s_\beta)}\right)^\lambda - 2. \tag{4}$$

Obviously, the smaller the value of $D(s_\alpha, s_\beta)$ , the smaller the difference between two multiplicative linguistic values $s_\alpha$ and $s_\beta$. Therefore, Equations (3) and (4) can effectively describe the deviation between two multiplicative linguistic values.

Let $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}$ be a set of linguistic variables and $w = (w_1, w_2, \cdots, w_n)^T$ be a weighting vector satisfying $w_j \geq 0$, $\sum\limits_{j=1}^{n} w_j = 1$. Assume that the aggregation operator of dimension $n$ is a mapping $f$ determined by the following formula: $s_\beta = f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n})$. In the aggregation process, the deviation between the arguments $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}$ and the aggregation result $s_\beta$ should be minimized. In order to reduce the deviation between $s_{\alpha_i}(i = 1, 2, \cdots, n)$ and $s_\beta$, we can also construct the penalty function $F$ and the minimization problem as follows:

$$minF = \sum_{i=1}^{n} w_i D(s_{\alpha_i}, s_\beta) = \sum_{i=1}^{n} w_i \left( \left( \frac{\ln(tT(s_{\alpha_i}))}{\ln(tT(s_\beta))} \right)^\lambda + \left( \frac{\ln(tT(s_\beta))}{\ln(tT(s_{\alpha_i}))} \right)^\lambda - 2 \right),$$

where $\lambda$ is a parameter such that $\lambda \in (-\infty, +\infty)$ and $\lambda \neq 0$. According to the necessary conditions for the existence of extremum, we take the partial derivative of $F$ with respect to $\beta$, and then we have

$$\frac{\partial F}{\partial \beta} = \sum_{i=1}^{n} w_i \left( \frac{\ln(tT(s_{\alpha_i}))}{\ln(tT(s_\beta))} \right)^{\lambda-1} \left( -\frac{\ln(tT(s_{\alpha_i}))}{\ln^2(tT(s_\beta))} \right) \frac{\lambda}{\beta} + \sum_{i=1}^{n} w_i \left( \frac{\ln(tT(s_\beta))}{\ln(tT(s_{\alpha_i}))} \right)^{\lambda-1} \frac{1}{\ln(tT(s_{\alpha_i}))} \frac{\lambda}{\beta}.$$

Let $\frac{\partial F}{\partial \beta} = 0$; we obtain the formula as follows:

$$\beta = \frac{1}{t} \exp\left\{ \left( \frac{\sum\limits_{i=1}^{n} w_i \ln^\lambda(tT(s_{\alpha_i}))}{\sum\limits_{i=1}^{n} w_i / \ln^\lambda(tT(s_{\alpha_i}))} \right)^{\frac{1}{2\lambda}} \right\}. \tag{5}$$

From Equation (5), the linguistic generalized weighted logarithm multiple averaging $(LGWLMA)$ operator can be defined as follows.

**Definition 10.** *An $LGWLMA$ operator of dimension $n$ is a mapping $LGWLMA : \bar{S}^n \to \bar{S}$ that is associated with a weighting vector $w = (w_1, w_2, \cdots, w_n)^T$, such that $w_j \geq 0$, $\sum\limits_{j=1}^{n} w_j = 1$, according to the following expression:*

$$LGWLMA(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\frac{1}{t} \exp\left\{ \left( \frac{\sum_{i=1}^{n} w_i \ln^\lambda(tT(s_{\alpha_i}))}{\sum_{i=1}^{n} w_i / \ln^\lambda(tT(s_{\alpha_i}))} \right)^{\frac{1}{2\lambda}} \right\}}.$$

*where $\lambda$ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.*

By using different cases of the parameter $\lambda$ in the $LGWLMA$ operator, we can obtain different types of aggregation operator.

**Remark 1.** *If $\lambda = 1$ or $\lambda = -1$, then we obtain the linguistic weighted logarithm multiple averaging $(LWLMA)$ operator:*

$$LWLMA(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\frac{1}{t} \exp\left\{ \left( \frac{\sum_{i=1}^{n} w_i \ln(tT(s_{\alpha_i}))}{\sum_{i=1}^{n} w_i / \ln(tT(s_{\alpha_i}))} \right)^{\frac{1}{2}} \right\}}.$$

**Remark 2.** *If $\lambda \to 0$, then the LGWLMA operator reduces to the linguistic weighted logarithm geometric averaging $(LWLGA)$ operator. Since*

$$\lim_{\lambda\to 0}\ln\beta = \lim_{\lambda\to 0}\left(\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\alpha_i}))}{\sum_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\alpha_i}))}\right)^{\frac{1}{2\lambda}} + \ln\frac{1}{t} = \prod_{i=1}^{n}\ln^{w_i}(tT(s_{\alpha_i})) + \ln\frac{1}{t}.$$

*It follows that*

$$\lim_{\lambda\to 0} LGWLMA(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = LWLGA(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\frac{1}{t}\exp\left\{\prod_{i=1}^{n}\ln^{w_i}(tT(s_{\alpha_i}))\right\}}.$$

*3.2. The LGOWLMA Operator and Its Desirable Properties*

If we rearrange the arguments in descending order in the $LGWLMA$ operator, then we can obtain the linguistic generalized ordered weighted logarithm multiple averaging $(LGOWLMA)$ operator, which can be defined as follows.

**Definition 11.** *A LGOWLMA operator of dimension n is a mapping $LGOWLMA : \bar{S}^n \to \bar{S}$ that is associated with a weighting vector $w = (w_1, w_2, \cdots, w_n)^T$, such that $w_j \geq 0$, $\sum_{j=1}^{n} w_j = 1$, according to the following expression:*

$$LGOWLMA(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\frac{1}{t}\exp\left\{\left(\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i}))}\right)^{\frac{1}{2\lambda}}\right\}}. \tag{6}$$

*where $s_{\beta_i}$is the ith largest of arguments $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}$ and $\lambda$ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.*

Similar to the $LGWLMA$ operator, the $LGOWLMA$ operator has some special cases as follows.

**Remark 3.** *If $\lambda = 1$ or $\lambda = -1$, then we obtain the LOWLMA operator:*

$$LOWLMA(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\frac{1}{t}\exp\left\{\left(\frac{\sum_{i=1}^{n} w_i \ln(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i/\ln(tT(s_{\beta_i}))}\right)^{\frac{1}{2}}\right\}}.$$

**Remark 4.** *If $\lambda \to 0$, then the LGOWLMA operator reduces to the LOWLGA operator. That is,*

$$\lim_{\lambda\to 0} LGOWLMA(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = LOWLGA(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\frac{1}{t}\exp\left\{\prod_{i=1}^{n}\ln^{w_i}(tT(s_{\beta_i}))\right\}}.$$

Let $\varphi(\lambda)$ be the $LGOWLMA$ operator and let

$$g(\lambda) = \left(\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i}))}\right)^{\frac{1}{2\lambda}}.$$

It is clear that $g(-\lambda) = g(\lambda)$; hence, we have $\varphi(-\lambda) = \varphi(\lambda)$. Obviously, the $LGOWLMA$ operator is an even function with respect to the parameter $\lambda$. For the sake of simplicity, assume $\lambda > 0$. The $LGOWLMA$ operator is monotonic, commutative, idempotent and bounded. These properties are presented as follows.

**Theorem 1 (Monotonicity).** *Let $f$ be the LGOWLMA operator. If $s_{\alpha_i} \geq s_{\alpha'_i}$ for $i = 1, 2, \cdots, n$, then*

$$f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \geq f(s_{\alpha'_1}, s_{\alpha'_2}, \cdots, s_{\alpha'_n}).$$

**Proof.** Let

$$\gamma = \frac{1}{t} \exp\left\{ \left( \frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))} \right)^{\frac{1}{2\lambda}} \right\},$$

then

$$\ln \gamma = \ln \frac{1}{t} + \left( \frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))} \right)^{\frac{1}{2\lambda}},$$

$$\frac{\partial \ln \gamma}{\partial \beta_i} = \frac{1}{2\lambda} \left( \frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))} \right)^{\frac{1}{2\lambda}-1}$$
$$\cdot \frac{(w_i/\alpha_i)\lambda \left( \ln^{\lambda-1}(tT(s_{\beta_i})) \sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})) + \sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i})) \Big/ \ln^{\lambda+1}(tT(s_{\beta_i})) \right)}{(\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})))^2}.$$

Obviously, $\partial \ln \gamma / \partial \beta_i \geq 0$ holds, i.e., $\ln \gamma$ is a monotonically increasing function of $\beta_i$. Then, $\gamma$ increases monotonically with respect to $\beta_i$. Based on Equation (6), it is clear that function $f$ increases monotonically with respect to $\alpha_i$. Accordingly, we have $f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \geq f(s_{\alpha'_1}, s_{\alpha'_2}, \cdots, s_{\alpha'_n})$. □

**Theorem 2 (Idempotency).** *Let $f$ be the LGOWLMA operator. If $S_{\alpha_i} = S_{\alpha}$ for $i = 1, 2, \cdots, n$, then $f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\alpha}$.*

**Proof.** Let $s_{\alpha_i} = s_{\alpha}$ for $i = 1, 2, \cdots, n$, then $s_{\alpha_i} = s_{\bar{\alpha}}$, where

$$\bar{\alpha} = \frac{1}{t} \exp\left\{ \left( \frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\alpha}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\alpha}))} \right)^{\frac{1}{2\lambda}} \right\} = \frac{1}{t} \exp\{\ln(tT(s_{\alpha}))\} = \alpha.$$

Therefore, we have $f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\alpha}$. □

**Theorem 3 (Commutativity).** *Let $f$ be the LGOWLMA operator, and $(s_{\alpha'_1}, s_{\alpha'_2}, \cdots, s_{\alpha'_n})$ is any permutation of the linguistic arguments $(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n})$, then*

$$f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = f(s_{\alpha'_1}, s_{\alpha'_2}, \cdots, s_{\alpha'_n}).$$

**Proof.** Let

$$f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\gamma},$$
$$f(s_{\alpha'_1}, s_{\alpha'_2}, \cdots, s_{\alpha'_n}) = s_{\gamma'}.$$

where

$$\gamma = \frac{1}{t}\exp\left\{\left(\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))}\right)^{\frac{1}{2\lambda}}\right\},$$

and

$$\gamma' = \frac{1}{t}\exp\left\{\left(\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta'_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta'_i}))}\right)^{\frac{1}{2\lambda}}\right\}.$$

Since $(s_{\alpha'_1}, s_{\alpha'_2}, \cdots, s_{\alpha'_n})$ is any permutation of the linguistic arguments $(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n})$, we have $s_{\beta_i} = s_{\beta'_i}$ for $i = 1, 2, \cdots, n$. Thus, we obtain

$$f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = f(s_{\alpha'_1}, s_{\alpha'_2}, \cdots, s_{\alpha'_n}).$$

□

**Theorem 4 (Boundedness).** *Let $f$ be the LGOWLMA operator, $LM_1$ be the linguistic maximum operator and $LM_2$ be the linguistic minimum operator. Then,*

$$LM_2(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \leq f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \leq LM_1(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}).$$

**Proof.** It is clear by Theorems 1 and 2. □

**Theorem 5.** *Let $f$ be the LGOWLMA operator, and $w = (w_1, w_2, \cdots, w_n)^T$ be the weighting vector satisfying $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.*

(1) *If $w_1 \geq w_2 \geq \cdots \geq w_n$, then*

$$f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \geq s_{\frac{1}{t}\exp\left\{\left(\frac{\sum_{i=1}^{n} \ln^{\lambda}(tT(s_{\alpha_i}))}{\sum_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\alpha_i}))}\right)^{\frac{1}{2\lambda}}\right\}}.$$

(2) *If $w_1 \leq w_2 \leq \cdots \leq w_n$, then*

$$f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \leq s_{\frac{1}{t}\exp\left\{\left(\frac{\sum_{i=1}^{n} \ln^{\lambda}(tT(s_{\alpha_i}))}{\sum_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\alpha_i}))}\right)^{\frac{1}{2\lambda}}\right\}}.$$

**Proof.** From Equation (6),

$$f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) = s_{\frac{1}{t}\exp\left\{\left(\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i}))}\right)^{\frac{1}{2\lambda}}\right\}},$$

where $s_{\beta_i}$ is the $i$th largest of arguments $s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}$. It is clear that $s_{\beta_1} \geq s_{\beta_2} \geq \cdots \geq s_{\beta_n} \geq \frac{1}{t}$. If $w_1 \geq w_2 \geq \cdots \geq w_n$, then

$$\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))} - \frac{\sum_{i=1}^{n} \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} 1 / \ln^{\lambda}(tT(s_{\beta_i}))}$$

$$
\begin{aligned}
&= \frac{\sum\limits_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_i})) - \sum\limits_{i=1}^{n} \ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i}))}{\sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_i}))} \\
&= \frac{\sum\limits_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{j=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_j})) - \sum\limits_{j=1}^{n} \ln^{\lambda}(tT(s_{\beta_j})) \sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i}))}{\sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_i}))} \\
&= \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} w_i \big(\ln(tT(s_{\beta_i}))/\ln(tT(s_{\beta_j}))\big)^{\lambda} - \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} w_i \big(\ln(tT(s_{\beta_j}))/\ln(tT(s_{\beta_i}))\big)^{\lambda}}{\sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_i}))} \\
&= \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} w_i \Big( \big(\ln(tT(s_{\beta_i}))/\ln(tT(s_{\beta_j}))\big)^{\lambda} - \big(\ln(tT(s_{\beta_j}))/\ln(tT(s_{\beta_i}))\big)^{\lambda} \Big)}{\sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_i}))} \\
&= \frac{\sum\limits_{i<j} w_i \left( \left(\frac{\ln(tT(s_{\beta_i}))}{\ln(tT(s_{\beta_j}))}\right)^{\lambda} - \left(\frac{\ln(tT(s_{\beta_j}))}{\ln(tT(s_{\beta_i}))}\right)^{\lambda} \right) + \sum\limits_{j>i} w_i \left( \left(\frac{\ln(tT(s_{\beta_i}))}{\ln(tT(s_{\beta_j}))}\right)^{\lambda} - \left(\frac{\ln(tT(s_{\beta_j}))}{\ln(tT(s_{\beta_i}))}\right)^{\lambda} \right)}{\sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_i}))} \\
&= \frac{\sum\limits_{i<j} (w_i - w_j) \Big( \big(\ln(tT(s_{\beta_i}))/\ln(tT(s_{\beta_j}))\big)^{\lambda} - \big(\ln(tT(s_{\beta_j}))/\ln(tT(s_{\beta_i}))\big)^{\lambda} \Big)}{\sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i})) \sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_i}))} \geq 0.
\end{aligned}
$$

It follows that

$$
\frac{\sum\limits_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum\limits_{i=1}^{n} w_i/\ln^{\lambda}(tT(s_{\beta_i}))} \geq \frac{\sum\limits_{i=1}^{n} \ln^{\lambda}(tT(s_{\beta_i}))}{\sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\beta_i}))} = \frac{\sum\limits_{i=1}^{n} \ln^{\lambda}(tT(s_{\alpha_i}))}{\sum\limits_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\alpha_i}))}.
$$

which means that

$$
f(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \geq s_{\frac{1}{t}\exp\left\{ \left( \frac{\sum_{i=1}^{n} \ln^{\lambda}(tT(s_{\alpha_i}))}{\sum_{i=1}^{n} 1/\ln^{\lambda}(tT(s_{\alpha_i}))} \right)^{\frac{1}{2\lambda}} \right\}}.
$$

Obviously, case (2) of Theorem 5 can be proven in a similar way. In sum, the proof of Theorem 5 is complete. □

**Theorem 6.** *Let $f$ be the LGOWLMA operator. For weighting vector $w = (w_1, w_2, \cdots, w_n)^T$ and $w = (w'_1, w'_2, \cdots, w'_n)^T$, which satisfies $w_i \geq 0$, $w'_i \geq 0$, and $\sum\limits_{i=1}^{n} w_i = 1$, $\sum\limits_{i=1}^{n} w'_i = 1$ for $i = 1, 2, \cdots, n$. If $w_i/w_{i+1} \geq w'_i/w'_{i+1}$ for $i = 1, 2, \cdots, n-1$, then*

$$
f_w(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \geq f_{w'}(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}).
$$

**Proof.** If $w_i/w_{i+1} \geq w'_i/w'_{i+1}$ for $i = 1, 2, \cdots, n-1$, we have

$$
\frac{w_i}{w'_i} \geq \frac{w_{i+1}}{w'_{i+1}}, \quad i = 1, 2, \cdots, n-1.
$$

Assume that $w_i/w'_i = v_i$, then $w_i = v_i w'_i$ and $v_i \geq v_{i+1}$ for $i = 1, 2, \cdots, n-1$, which means that $v_i \geq v_j$ for $i < j$. Therefore, we have

$$
\begin{aligned}
&\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))} - \frac{\sum_{i=1}^{n} w'_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w'_i / \ln^{\lambda}(tT(s_{\beta_i}))} \\
&= \frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))} - \frac{\sum_{j=1}^{n} w'_j \ln^{\lambda}(tT(s_{\beta_j}))}{\sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j}))} \\
&= \frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i})) \sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j})) - \sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})) \sum_{j=1}^{n} w'_j \ln^{\lambda}(tT(s_{\beta_j}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})) \sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j}))} \\
&= \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w'_j \Big( \big( \ln(tT(s_{\beta_i})) / \ln(tT(s_{\beta_j})) \big)^{\lambda} - \big( \ln(tT(s_{\beta_j})) / \ln(tT(s_{\beta_i})) \big)^{\lambda} \Big)}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})) \sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j}))} \\
&= \frac{\sum_{i<j} w_i w'_j \left( \left( \frac{\ln(tT(s_{\beta_i}))}{\ln(tT(s_{\beta_j}))} \right)^{\lambda} - \left( \frac{\ln(tT(s_{\beta_j}))}{\ln(tT(s_{\beta_i}))} \right)^{\lambda} \right) + \sum_{i>j} w_i w'_j \left( \left( \frac{\ln(tT(s_{\beta_i}))}{\ln(tT(s_{\beta_j}))} \right)^{\lambda} - \left( \frac{\ln(tT(s_{\beta_j}))}{\ln(tT(s_{\beta_i}))} \right)^{\lambda} \right)}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})) \sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j}))} \\
&= \frac{\sum_{i<j} w_i w'_j \left( \left( \frac{\ln(tT(s_{\beta_i}))}{\ln(tT(s_{\beta_j}))} \right)^{\lambda} - \left( \frac{\ln(tT(s_{\beta_j}))}{\ln(tT(s_{\beta_i}))} \right)^{\lambda} \right) + \sum_{i<j} w_j w'_i \left( \left( \frac{\ln(tT(s_{\beta_j}))}{\ln(tT(s_{\beta_i}))} \right)^{\lambda} - \left( \frac{\ln(tT(s_{\beta_i}))}{\ln(tT(s_{\beta_j}))} \right)^{\lambda} \right)}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})) \sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j}))} \\
&= \frac{\sum_{i<j} (w_i w'_j - w_j w'_i) \Big( \big( \ln(tT(s_{\beta_i})) / \ln(tT(s_{\beta_j})) \big)^{\lambda} - \big( \ln(tT(s_{\beta_j})) / \ln(tT(s_{\beta_i})) \big)^{\lambda} \Big)}{\sum_{i=1}^{n} w_i / \ln^{\lambda}((tT(s_{\beta_i}))) \sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j}))} \\
&= \frac{\sum_{i<j} (v_i w'_i w'_j - v_j w'_j w'_i) \Big( \big( \ln(tT(s_{\beta_i})) / \ln(tT(s_{\beta_j})) \big)^{\lambda} - \big( \ln(tT(s_{\beta_j})) / \ln(tT(s_{\beta_i})) \big)^{\lambda} \Big)}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})) \sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j}))} \\
&= \frac{\sum_{i<j} w'_i w'_j (v_i - v_j) \Big( \big( \ln(tT(s_{\beta_i})) / \ln(tT(s_{\beta_j})) \big)^{\lambda} - \big( \ln(tT(s_{\beta_j})) / \ln(tT(s_{\beta_i})) \big)^{\lambda} \Big)}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i})) \sum_{j=1}^{n} w'_j / \ln^{\lambda}(tT(s_{\beta_j}))}.
\end{aligned}
$$

Since $v_i \geq v_j$ and $\big( \ln(tT(s_{\beta_i})) / \ln(tT(s_{\beta_j})) \big)^{\lambda} - \big( \ln(tT(s_{\beta_j})) / \ln(tT(s_{\beta_i})) \big)^{\lambda} \geq 0$ for $i < j$, we obtain

$$
\frac{\sum_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))} - \frac{\sum_{i=1}^{n} w'_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum_{i=1}^{n} w'_i / \ln^{\lambda}(tT(s_{\beta_i}))} \geq 0.
$$

Namely, we have

$$\exp\left\{\frac{\sum\limits_{i=1}^{n} w_i \ln^{\lambda}(tT(s_{\beta_i}))}{\sum\limits_{i=1}^{n} w_i / \ln^{\lambda}(tT(s_{\beta_i}))}\right\} \geq \exp\left\{\frac{\sum\limits_{i=1}^{n} w_i' \ln^{\lambda}(tT(s_{\beta_i}))}{\sum\limits_{i=1}^{n} w_i' / \ln^{\lambda}(tT(s_{\beta_i}))}\right\}.$$

Based on Equation (6), we can obtain

$$f_w(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}) \geq f_{w'}(s_{\alpha_1}, s_{\alpha_2}, \cdots, s_{\alpha_n}).$$

which completes the proof of Theorem 6. □

## 4. Group Decision Making Based on Multiplicative Linguistic Aggregation Operator and Linguistic VIKOR Method

### *4.1. A Chi-Square Deviation-Based Linguistic VIKOR Method for Group Decision Making under Multiplicative Linguistic Environment*

Consider a multiple attribute group decision-making problem under a linguistic setting. Let $X = \{x_1, x_2, \cdots, x_m\}$ be a set of $m$ feasible alternatives, and $C = \{c_1, c_2, \cdots, c_n\}$ be a set of attributes. $w = (w_1, w_2, \cdots, w_n)^T$ is the weighting vector of attributes satisfying $w_i \geq 0$ and $\sum\limits_{i=1}^{n} w_i = 1$. Let $D = \{d_1, d_2, \cdots, d_l\}$ be the set of decision makers. $V = (v_1, v_2, \cdots, v_l)^T$ is the completely unknown weighting vector of decision makers satisfying $v_k \geq 0$ and $\sum\limits_{k=1}^{l} v_k = 1$. Assume that each decision maker gives their own decision matrix $R_k = \left(\tilde{r}_{ij}^{(k)}\right)_{m\times n}$, where $\tilde{r}_{ij}^{(k)}$ is the evaluation result given by decision maker $d_k \in D$ under the attribute $c_j \in C$ for the alternative $x_i \in X$.

The linguistic decision-making algorithm is based on the linguistic VIKOR method and the *LGOWLMA* operator can be summarized as the following framework (see Figure 1).

The process of this linguistic group decision-making algorithm involves the following steps.

**Step 1.** Under the attribute $c_i \in C$, decision maker $e_k$ measures the alternative $x_i$ according to the multiplicative linguistic scale $S = \{s_\alpha | \alpha = \frac{1}{t}, \cdots, \frac{1}{2}, 1, 2, \cdots, t\}$ and obtains attribute value $\tilde{r}_{ij}^{(k)}$. Thus, the linguistic decision matrix $R_k = \left(\tilde{r}_{ij}^{(k)}\right)_{m\times n}$ is constructed.

**Step 2.** Utilize Equation (1) and the BUM function $Q(x) = x^{\frac{3}{2}}$ to calculate the weighting vector of decision maker $V = (v_1, v_2, \cdots, v_l)^T$, which satisfies $v_k \geq 0$ and $\sum\limits_{k=1}^{l} v_k = 1$.

**Step 3.** Based on Definition 11, use the *LGOWLM* operator

$$\tilde{r}_{ij} = s_{\frac{1}{t}\exp\left\{\left(\frac{\sum\limits_{k=1}^{l} w_i \ln^{\lambda}(tT(\tilde{r}_{ij}^{(k)}))}{\sum\limits_{k=1}^{l} w_i / \ln^{\lambda}(tT(\tilde{r}_{ij}^{(k)}))}\right)^{\frac{1}{2\lambda}}\right\}}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n.$$

to aggregate all the decision matrices $R_k$ into a collective decision matrix $R = (\tilde{r}_{ij})_{m\times n}$.

**Step 4.** Based on Definition 8, the entropy weight method [52] is adopted to obtain the weighting vector of attributes $w = (w_1, w_2, \cdots, w_n)^T$, where

$$w_j = \frac{1 + (1/\ln m)\sum\limits_{i=1}^{m}\left(\frac{T(\tilde{r}_{ij})}{\sum\limits_{i=1}^{m} T(\tilde{r}_{ij})} \ln \frac{T(\tilde{r}_{ij})}{\sum\limits_{i=1}^{m} T(\tilde{r}_{ij})}\right)}{\sum\limits_{j=1}^{n}\left(1 + (1/\ln m)\sum\limits_{i=1}^{m}\left(\frac{T(\tilde{r}_{ij})}{\sum\limits_{i=1}^{m} T(\tilde{r}_{ij})} \ln \frac{T(\tilde{r}_{ij})}{\sum\limits_{i=1}^{m} T(\tilde{r}_{ij})}\right)\right)}, \tag{7}$$

and satisfies $w_j \geq 0$ and $\sum\limits_{j=1}^{n} w_j = 1$.

**Step 5.** Find positive and negative ideal solutions as follows:

$$(\tilde{r}_j^{+})_{1\times n} = (\max_{1\leq i\leq m} \tilde{r}_{i1}, \max_{1\leq i\leq m} \tilde{r}_{i2}, \cdots, \max_{1\leq i\leq m} \tilde{r}_{in}), \tag{8}$$

$$(\tilde{r}_j^{-})_{1\times n} = (\min_{1\leq i\leq m} \tilde{r}_{i1}, \min_{1\leq i\leq m} \tilde{r}_{i2}, \cdots, \min_{1\leq i\leq m} \tilde{r}_{in}). \tag{9}$$

**Step 6.** Based on the ratio of the linguistic chi-square deviation between each alternative to the positive ideal solution and the negative ideal solution, we denote

$$\mathbb{S}^{+} = \max_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} w_j \frac{\big(\log_{tT(\tilde{r}_{ij})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_{ij})}\big)^{\lambda} - 2}{\big(\log_{tT(\tilde{r}_j^{-})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_j^{-})}\big)^{\lambda} - 2} \right\}, \tag{10}$$

$$\mathbb{S}^{-} = \min_{1\leq i\leq m} \left\{ \sum_{j=1}^{n} w_j \frac{\big(\log_{tT(\tilde{r}_{ij})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_{ij})}\big)^{\lambda} - 2}{\big(\log_{tT(\tilde{r}_j^{-})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_j^{-})}\big)^{\lambda} - 2} \right\}, \tag{11}$$

$$\mathbb{R}^{+} = \max_{1\leq i\leq m} \max_{1\leq j\leq n} \left\{ w_j \frac{\big(\log_{tT(\tilde{r}_{ij})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_{ij})}\big)^{\lambda} - 2}{\big(\log_{tT(\tilde{r}_j^{-})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_j^{-})}\big)^{\lambda} - 2} \right\}, \tag{12}$$

$$\mathbb{R}^{-} = \min_{1\leq i\leq m} \max_{1\leq j\leq n} \left\{ w_j \frac{\big(\log_{tT(\tilde{r}_{ij})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_{ij})}\big)^{\lambda} - 2}{\big(\log_{tT(\tilde{r}_j^{-})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_j^{-})}\big)^{\lambda} - 2} \right\}. \tag{13}$$

**Step 7.** Calculate interest ratio $\mathbb{Q} = (\mathbb{Q}_i)_{m\times 1}$, and arrange $\mathbb{Q}_i$ in ascending order.

$$\begin{aligned} \mathbb{Q}_i =& \frac{\delta}{\mathbb{S}^{+} - \mathbb{S}^{-}} \Big( \sum_{j=1}^{n} w_j \frac{\big(\log_{tT(\tilde{r}_{ij})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_{ij})}\big)^{\lambda} - 2}{\big(\log_{tT(\tilde{r}_j^{-})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_j^{-})}\big)^{\lambda} - 2} - \mathbb{S}^{-} \Big) \\ &+ \frac{1-\delta}{\mathbb{R}^{+} - \mathbb{R}^{-}} \Big( \max_{1\leq j\leq n} \left\{ w_j \frac{\big(\log_{tT(\tilde{r}_{ij})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_{ij})}\big)^{\lambda} - 2}{\big(\log_{tT(\tilde{r}_j^{-})}^{tT(\tilde{r}_j^{+})}\big)^{\lambda} + \big(\log_{tT(\tilde{r}_j^{+})}^{tT(\tilde{r}_j^{-})}\big)^{\lambda} - 2} \right\} - \mathbb{R}^{-} \Big), \end{aligned} \tag{14}$$

where $\delta$ is the decision mechanism coefficient. If $\delta > 0.5$, it belongs to the risk preference type; if $\delta = 0.5$, it belongs to the risk-neutral type; if $\delta < 0.5$, it belongs to the risk aversion type.

**Step 8.** Arrange the alternative $x_i$ according to the value of $\mathbb{Q}_i$, and select the optimal alternative(s).

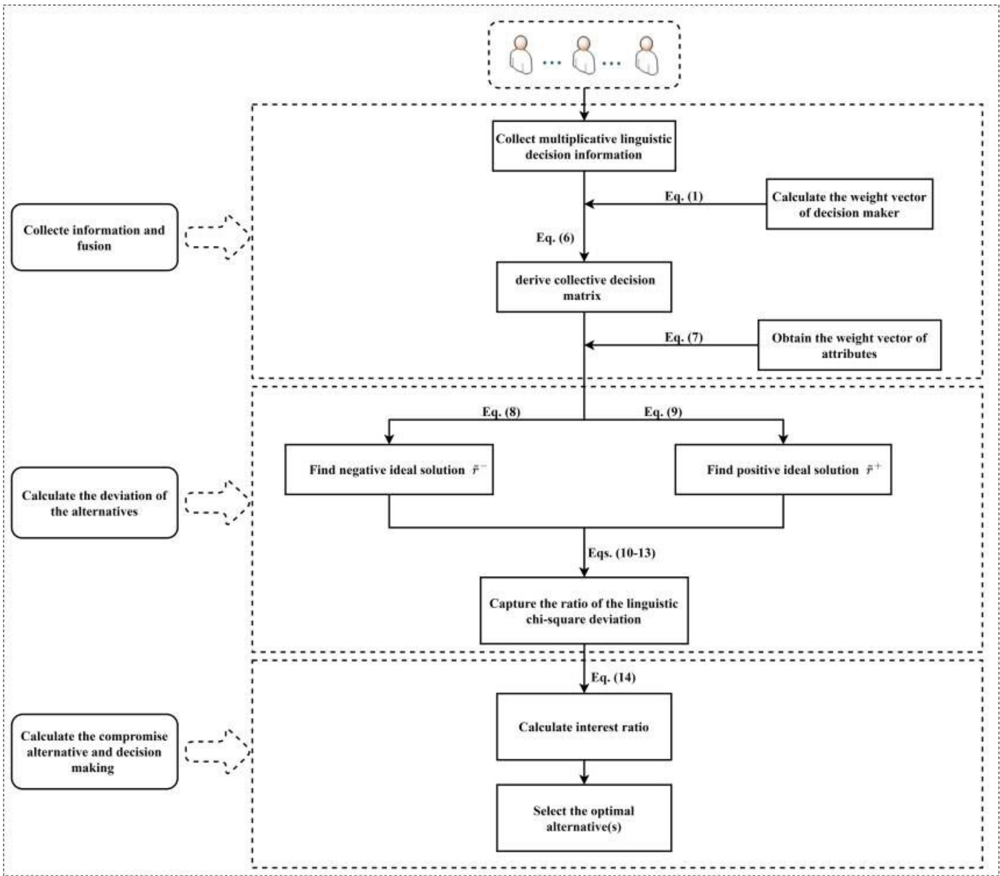


**Figure 1.** The framework of the proposed linguistic VIKOR method.

*4.2. Comparison and Analysis*

In the following, we discuss the application of the proposed group decision-making method in evaluating transport logistics enterprises. Five transport logistics enterprises $\{x_1, x_2, x_3, x_4, x_5\}$ are evaluated using four main evaluation indexes: equipment and facilities($c_1$), management and services($c_2$), personnel quality($c_3$), informatization degree($c_4$). Assume that four decision makers $\{d_1, d_2, d_3, d_4\}$ evaluated five enterprises with four attributes using the following multiplicative linguistic scale

$$S = \{s_{1/5} = extrmely\ poor, s_{1/4} = very\ poor, s_{1/3} = poor, s_{1/2} = slightly\ poor,$$
$$s_1 = fair, s_2 = slightly\ good, s_3 = good, s_4 = very\ good, s_5 = extremly\ good\}.$$

4.2.1. Using the Proposed Approach to Select the Optimal Alternative

**Step 1.** The decision matrices given by decision makers are shown in Tables 1–4.

**Table 1.** Decision matrix $R_1$ provided by decision maker $d_1$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $s_1$ | $s_{1/2}$ | $s_3$ | $s_{1/2}$ |
| $x_2$ | $s_4$ | $s_1$ | $s_3$ | $s_{1/2}$ |
| $x_3$ | $s_{1/2}$ | $s_{1/2}$ | $s_1$ | $s_{1/4}$ |
| $x_4$ | $s_2$ | $s_2$ | $s_1$ | $s_1$ |
| $x_5$ | $s_5$ | $s_3$ | $s_2$ | $s_2$ |

**Table 2.** Decision matrix $R_2$ provided by decision maker $d_2$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $s_2$ | $s_{1/2}$ | $s_4$ | $s_{1/4}$ |
| $x_2$ | $s_3$ | $s_1$ | $s_3$ | $s_{1/4}$ |
| $x_3$ | $s_{1/2}$ | $s_{1/4}$ | $s_1$ | $s_{1/2}$ |
| $x_4$ | $s_2$ | $s_1$ | $s_2$ | $s_1$ |
| $x_5$ | $s_3$ | $s_2$ | $s_1$ | $s_3$ |

**Table 3.** Decision matrix $R_3$ provided by decision maker $d_3$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $s_1$ | $s_{1/3}$ | $s_3$ | $s_{1/3}$ |
| $x_2$ | $s_3$ | $s_2$ | $s_2$ | $s_{1/4}$ |
| $x_3$ | $s_{1/3}$ | $s_{1/2}$ | $s_2$ | $s_{1/2}$ |
| $x_4$ | $s_3$ | $s_2$ | $s_3$ | $s_1$ |
| $x_5$ | $s_4$ | $s_4$ | $s_2$ | $s_1$ |

**Table 4.** Decision matrix $R_4$ provided by decision maker $d_4$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $s_2$ | $s_{1/2}$ | $s_4$ | $s_1$ |
| $x_2$ | $s_4$ | $s_1$ | $s_2$ | $s_{1/4}$ |
| $x_3$ | $s_{1/4}$ | $s_{1/2}$ | $s_1$ | $s_{1/4}$ |
| $x_4$ | $s_3$ | $s_3$ | $s_1$ | $s_{1/2}$ |
| $x_5$ | $s_5$ | $s_2$ | $s_1$ | $s_2$ |

**Step 2.** Select function $Q(x) = x^{\frac{3}{2}}$, and utilize Equation (1) to calculate the weighting vector of decision maker, and we obtain

$$
\begin{aligned}
v_1 &= Q(\frac{1}{4}) - Q(0) = (\frac{1}{4})^{\frac{3}{2}} - 0 = 0.125, \\
v_2 &= Q(\frac{2}{4}) - Q(\frac{1}{4}) = (\frac{2}{4})^{\frac{3}{2}} - (\frac{1}{4})^{\frac{3}{2}} = 0.229, \\
v_3 &= Q(\frac{3}{4}) - Q(\frac{2}{4}) = (\frac{3}{4})^{\frac{3}{2}} - (\frac{2}{4})^{\frac{3}{2}} = 0.296, \\
v_4 &= Q(1) - Q(\frac{3}{4}) = 1 - (\frac{3}{4})^{\frac{3}{2}} = 0.350.
\end{aligned}
$$

That is, $v = (0.125, 0.229, 0.296, 0.350)$.

**Step 3.** Let $\lambda = 1$. Based on Definition 11, the *LGOWLMA* operator is utilized to aggregate all the decision matrices $R_k(k = 1, 2, 3, 4)$ into a collective decision matrix $R$, as shown in Table 5.

**Table 5.** Group linguistic decision matrix $R$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $s_{1.24}$ | $s_{0.42}$ | $s_{3.31}$ | $s_{0.33}$ |
| $x_2$ | $s_{3.31}$ | $s_{1.08}$ | $s_{2.29}$ | $s_{0.26}$ |
| $x_3$ | $s_{0.32}$ | $s_{0.34}$ | $s_{1.08}$ | $s_{0.29}$ |
| $x_4$ | $s_{2.29}$ | $s_{1.59}$ | $s_{1.29}$ | $s_{0.75}$ |
| $x_5$ | $s_{3.88}$ | $s_{2.36}$ | $s_{1.24}$ | $s_{1.59}$ |

**Step 4.** By Definition 8, utilize the entropy weight method to obtain the weighting vector of attributes

$$w_1 = \frac{1 + (1/\ln 5)\sum_{i=1}^{5}\left(\frac{T(\tilde{r}_{i1})}{\sum_{i=1}^{5} T(\tilde{r}_{i1})} \ln \frac{T(\tilde{r}_{i1})}{\sum_{i=1}^{5} T(\tilde{r}_{i1})}\right)}{\sum_{j=1}^{4}\left(1 + (1/\ln 5)\sum_{i=1}^{5}\left(\frac{T(\tilde{r}_{i1})}{\sum_{i=1}^{5} T(\tilde{r}_{i1})} \ln \frac{T(\tilde{r}_{i1})}{\sum_{i=1}^{5} T(\tilde{r}_{i1})}\right)\right)} \approx 0.260,$$

$$w_2 = \frac{1 + (1/\ln 5)\sum_{i=1}^{5}\left(\frac{T(\tilde{r}_{i2})}{\sum_{i=1}^{5} T(\tilde{r}_{i2})} \ln \frac{T(\tilde{r}_{i2})}{\sum_{i=1}^{5} T(\tilde{r}_{i2})}\right)}{\sum_{j=1}^{4}\left(1 + (1/\ln 5)\sum_{i=1}^{5}\left(\frac{T(\tilde{r}_{i2})}{\sum_{i=1}^{5} T(\tilde{r}_{i2})} \ln \frac{T(\tilde{r}_{i2})}{\sum_{i=1}^{5} T(\tilde{r}_{i2})}\right)\right)} \approx 0.275,$$

$$w_3 = \frac{1 + (1/\ln 5)\sum_{i=1}^{5}\left(\frac{T(\tilde{r}_{i3})}{\sum_{i=1}^{5} T(\tilde{r}_{i3})} \ln \frac{T(\tilde{r}_{i3})}{\sum_{i=1}^{5} T(\tilde{r}_{i3})}\right)}{\sum_{j=1}^{4}\left(1 + (1/\ln 5)\sum_{i=1}^{5}\left(\frac{T(\tilde{r}_{i3})}{\sum_{i=1}^{5} T(\tilde{r}_{i3})} \ln \frac{T(\tilde{r}_{i3})}{\sum_{i=1}^{5} T(\tilde{r}_{i3})}\right)\right)} \approx 0.128,$$

$$w_4 = \frac{1 + (1/\ln 5)\sum_{i=1}^{5}\left(\frac{T(\tilde{r}_{i4})}{\sum_{i=1}^{5} T(\tilde{r}_{i4})} \ln \frac{T(\tilde{r}_{i4})}{\sum_{i=1}^{5} T(\tilde{r}_{i4})}\right)}{\sum_{j=1}^{4}\left(1 + (1/\ln 5)\sum_{i=1}^{5}\left(\frac{T(\tilde{r}_{i4})}{\sum_{i=1}^{5} T(\tilde{r}_{i4})} \ln \frac{T(\tilde{r}_{i4})}{\sum_{i=1}^{5} T(\tilde{r}_{i4})}\right)\right)} \approx 0.337.$$

That is,

$$(w_1, w_2, w_3, w_4) = (0.260, 0.275, 0.128, 0.337).$$

**Step 5.** According to Equations (8) and (9), we find positive and negative ideal solutions as follows:

$$(\tilde{r}_i^+)_{1\times 4} = (s_{3.88}, s_{2.36}, s_{3.31}, s_{1.59}),$$
$$(\tilde{r}_i^-)_{1\times 4} = (s_{0.32}, s_{0.34}, s_{1.08}, s_{0.26}).$$

**Step 6.** Based on the ratio of the linguistic chi-square deviation between each alternative to the positive ideal solution and the negative ideal solution:

$$\mathbb{S}^+ = 0.8731, \mathbb{S}^- = 0.0909, \mathbb{R}^+ = 0.3370, \mathbb{R}^- = 0.0820.$$

**Step 7.** Calculate interest ratio $\mathbb{Q} = (\mathbb{Q}_i)_{m\times 1}$, and arrange $\mathbb{Q}_i$ in ascending order.

$$(\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_4, \mathbb{Q}_5)^T = (0.2811, 0.6725, 0.8784, 0.0049, 0.0175)^T.$$

Therefore, interest ratios $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_4, \mathbb{Q}_5$ are arranged in ascending order

$$\mathbb{Q}_4 \prec \mathbb{Q}_5 \prec \mathbb{Q}_1 \prec \mathbb{Q}_2 \prec \mathbb{Q}_3.$$

**Step 8.** Arrange the alternative $x_i$ according to the value of $\mathbb{Q}_i$, and we obtain a descending order of $x_i$:

$$x_4 \succ x_5 \succ x_1 \succ x_2 \succ x_3.$$

Therefore, we obtain $x_4$ as the best alternative.

#### 4.2.2. Using Xu's Approach to Select the Optimal Alternative

In order to better understand the difference between our proposed approach and the existing approach, we adopted Xu's approach [25] to solve the above problems.

**Step 1.** The decision matrices given by decision makers are shown in Tables 1–4.

**Step 2.** Select function $Q(x) = x^{\frac{3}{2}}$, and utilize Equations (1) to calculate the weighting vector of decision maker, and we obtain $v = (0.125, 0.229, 0.296, 0.350)$.

**Step 3.** Based on Definition 5, the $LOWG$ operator is utilized to aggregate all the decision matrices $R_k(k = 1, 2, 3, 4)$ into a collective decision matrix $\hat{R} = (\hat{r}_{ij})_{5\times4}$, as shown in Table 6.

**Table 6.** Group linguistic decision matrix $\hat{R}$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $x_1$ | $s_{1.28}$ | $s_{0.43}$ | $s_{3.32}$ | $s_{0.38}$ |
| $x_2$ | $s_{3.32}$ | $s_{1.09}$ | $s_{2.31}$ | $s_{0.27}$ |
| $x_3$ | $s_{0.35}$ | $s_{0.39}$ | $s_{1.09}$ | $s_{0.32}$ |
| $x_4$ | $s_{2.31}$ | $s_{1.65}$ | $s_{1.34}$ | $s_{0.78}$ |
| $x_5$ | $s_{3.91}$ | $s_{2.39}$ | $s_{1.28}$ | $s_{1.65}$ |

**Step 4.** Based on Definition 3, the $LG$ operator is utilized to aggregate the $i$th row of the matrix $\hat{R}$. That is,

$$l_i = LG(\hat{r}_{i1}, \hat{r}_{i2}, \hat{r}_{i3}, \hat{r}_{i4}),$$

and then we obtain the global preference values as the following:

$$l_1 = s_{0.9129}, l_2 = s_{1.2257}, l_3 = s_{0.4671}, l_4 = s_{1.4128}, l_5 = s_{2.1077}.$$

Thus, we have $l_5 > l_4 > l_2 > l_1 > l_3$.

**Step 5.** Arrange the alternative $x_i$ according to the value of $l_i$, and we obtain a descending order of $x_i$:

$$x_5 \succ x_4 \succ x_2 \succ x_1 \succ x_3.$$

Therefore, we obtain $x_5$ as the best alternative.

The comparison results of our proposed approach and Xu's approach are shown in Figure 2.

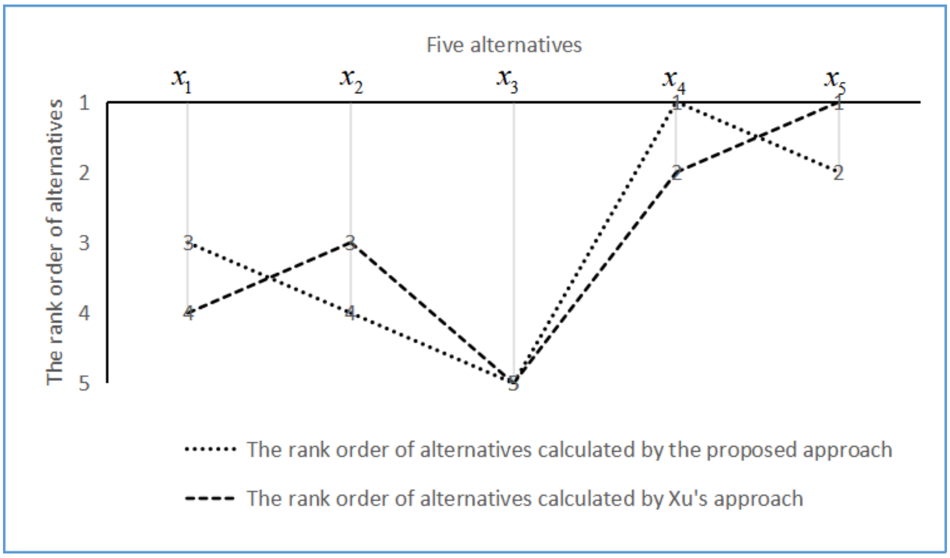


**Figure 2.** The ranking of alternatives obtained by the two approaches.

From the final ranking of alternatives, it can be seen that there are obvious differences between the two methods. In this paper, the VIKOR method can fully consider maximizing group benefits and minimizing individual losses. The final result is a compromise solution in which all attributes of the alternative give in to each other. Xu's approach does not take into account the deviation of each attribute, so it cannot give the order of each alternative more reasonably. In addition, this paper uses the entropy weight method to determine the weight of attributes, which can fully consider the importance of different attributes. However, the characteristics of attributes are not considered by the linguistic geometric average operator. Therefore, the ranking result derived by the proposed approach can more effectively and flexibly reflect the real level of each alternative.

## 5. Conclusions

In this paper, a new multi-attribute group decision-making method is developed under a multiplicative linguistic environment, since, in some cases, it is more reasonable to use a multiplicative linguistic term set to provide its evaluation value. The main innovations and advantages of this study are as follows.

(1) A novel linguistic chi-square deviation formula is proposed to better express the deviation between multiplicative linguistic values.

(2) Two new linguistic deviation-based tools for multiplicative information aggregation are proposed: the linguistic generalized weighted logarithmic multiple averaging ($LGWLMA$) operator and the linguistic generalized ordered weighted logarithmic multiple averaging ($LGOWLMA$) operator.

(3) The proposed method considers the non-uniform linguistic scale and is more suitable for representing the decision information of multiplicative linguistic structure.

The above theoretical analysis and numerical calculation results show that the proposed method is simple, intuitive and has no information loss. The multiplicative linguistic VIKOR method can be applied to economic forecasting, asset evaluation, environmental governance, intelligent decision making and other fields. We will continue working on the extension of the proposed method to a complex decision-making information environment. In particular, we will focus on the group decision-making problems with heterogeneous linguistic information.

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