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A New Numerical Approach for Variable-Order Time-Fractional Modified Subdiffusion Equation via Riemann–Liouville Fractional Derivative

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Abstract: Fractional differential equations describe nature adequately because of the symmetry properties that describe physical and biological processes. In this paper, a new approximation is found for the variable-order (VO) Riemann–Liouville fractional derivative (RLFD) operator; on that basis, an efficient numerical approach is formulated for VO time-fractional modified subdiffusion equations (TFMSDE). Complete theoretical analysis is performed, such as stability by the Fourier series, consistency, and convergence, and the feasibility of the proposed approach is also discussed. A numerical example illustrates that the proposed scheme demonstrates high accuracy, and that the obtained results are more feasible and accurate.

Keywords: implicit difference scheme; variable-order fractional modified subdiffusion equation; stability; consistency; convergence

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1. Introduction

VO-FDEs are generalizations of constant-order fractional differential equations (FDEs). Sometime, constant-order FDEs cannot describe the complex processes in porous media and medium structures because of changes with time [1]. Samko et al. first discussed the concept of VO integral and differential operators of the RLFD-type formula and investigated their mathematical properties [2]. Lorenzo and Hartley [3] discussed the basic concept of variable- and distributed-order fractional operators. They introduced definitions based on the Riemann–Liouville definition and discussed the behavior of new operators. Many researchers have worked on VO-FDEs with various numerical methods. For example, Younes [4] considered the moving least-squares method and finite difference method to find the solution of 2D VO fractional diffusion-wave equations. They solved some nonlinear numerical examples with complex geometries and the obtained results confirmed the accuracy and efficiency of the proposed method. A numerical scheme was proposed for 1D VO anomalous subdiffusion equations by Chen et al. [5]. They used fourth-order approximation for spatial and first-order temporal approximation and also proposed an improved numerical scheme for better accuracy. Xu et al. [6] developed a numerical scheme for multiterm FDE of VO, and reported a stability and convergence investigation through mathematical induction. The tested examples showed a strong level of accuracy. Wang et al. [7] formulated the numerical inversion of inverse problems with the homotopy regularization method by utilizing Legendre polynomials. Bhrawy and Zaky [8] formulated the Jacobi spectral collocation method on the basis of Legendre polynomials, and the Cahebyshev method for VO fractional 1D and 2D Cable equations. The numerical solution demonstrated a powerful method with a high level of accuracy for VO-FDE. The two new approximations were derived for the VO time derivative from the first to the second order by Zhao et al. [9]. They utilized superdiffusion and subdiffusion problems to demonstrate the effectiveness of the proposed approximation. Shen et al. [10] used the Caputo definition for the diffusion equation of VO and developed a numerical scheme, discussing the theoretical analysis via the von Neumann method. Sun et al. [11] presented a recent survey on VO-FDEs. They discussed initial existing definitions, numerical methods, and a summary on physical models and their applications. The Chebbyshev cardinal functional was constructed for VO delay factional models by Avazzaddeh et al. [12]. They obtained an algebraic system of equations with an operational matrix that reduced the cost of computation. Zayernouri and Karniadakis [13] proposed a spectral collocation method for linear and nonlinear VO space and time FDEs. They obtained differentiation matrices by collocating the fractional VO, and successfully solved many problems, such as time and space fractional advection, advection–diffusion, and Burger's equations to confirm the efficiency of the proposed method. Ali [14] investigated the new approximation for fractional Riemann–Liouville derivatives and also successfully applied the same approximation to VO-FDEs. They analyzed the theoretical analysis of the Fourier series method and showed that the proposed schemes were unconditionally stable and convergent. Ma et al. [15] considered the system of equations of VO and solved by Adams–Bashforth–Moulton algorithm. The numerical results reported that the suggested method was powerful and efficient. VOs and with variable coefficient FDEs were solved by Katsikadelis [16]. They contributed both implicit and explicit numerical approaches for linear and nonlinear VO-FDEs. Akgül et al. [17] successfully discussed the reproducing kernel method for VO-FDEs, which is more efficient and workable. In another study, Jia et al. [18] studied the simplified reproducing kernel technique for VO-FDE. They proved the theoretical analysis for the proposed scheme. Dehghan [19] worked on a numerical method on the basis of shifted Legendre polynomials for VO-FDEs, and reduced the equation into a system of algebraic equations by using Legendre polynomials. They compared the numerical method with two existing methods, and the results showed a great level of performance. Chen [20] solved a two-dimensional VO-TFMSDE for the first time. They used the Grünwald-Letnikov formula for VO time-fractional derivatives and central difference approximation for second-order space derivatives, and developed an implicit scheme. They found stability, convergence, and solvability via the Fourier series method. A numerical method for improving temporal accuracy was also developed for the proposed scheme. Ali et al. [21] derived an approximation for a VO fractional integral operator and solved the diffusion equation of VO. They investigated the theoretical analysis and compared the numerical results with an existing method that had shown better accuracy. Hijaz et al. [22] formulated a novel numerical approach for noninteger-order nonlinear differential equations. The noninteger order derivative was in a Caputo sense, and the numerical results were powerful and suitable for differential equations of noninteger order. The explicit difference scheme was considered for VO fractional heat equations with a linear forcing term by Sweilam and Mrawm [23]. They discretized the VO Caputo fractional derivative and discussed the stability by means of the Gerschgorin theorem. The results confirmed the effectiveness of the developed scheme via some VO heat equation models. Babaei et al. [24] developed the Chebyshev collocation technique of the sixth kind for the solution of nonlinear VO fractional integrodifferential equations. They reduced the VO fractional equation into a system of algebraic equations, and established the convergence and rate of convergence for the proposed approach. They discussed the robustness of the method and found it to be faster and more accurate. Wang and Zheng [25] proved the well-posedness and singularity of the solutions for VO FDEs. They used the FDS and discussed different cases of singularity of VO at $t = 0$ and the theoretical results. Kaur et al. [26] formulated a new approach and solved the fractional-order advection problem, the fractional derivative in the sense of a conformable time sense. The solution was obtained with the differential transform method, and the obtained results were compared with the existing literature. Abbas [27] considered the eigenvalue approach and found the exact solution in the Laplacian domain without any restriction. They discussed the effect of all parameters by increasing the value of τ, decreasing the values of all studied fields. Alzahrani et al. [28] used Laplacian Fourier transformation with the eigenvalue method for 2D porous material and the obtained outcomes for different types of conductivity. Further related studies can be found in [29–35].

The above cited literature shows that more efficient and better numerical schemes are needed to investigate VO-FDS. In this paper, our aim is to formulate a new numerical approach for VO-RLFD operators that is approximated through Jumrie properties, and the partial derivative with respect to time is replaced by backward difference approximation. The obtained approximation of the fractional derivative is used, and the space derivatives are approximated with the finite difference method for VO-TFMSDE. Theoretical analysis from the aspects of stability, consistency, and convergence is discussed, and a numerical experiment for 2D RSP-HGSGF with a fractional derivative is presented for further confirmation.

Consider the 2D VO-TFMSDE [36]:

$$\frac{\partial w(x,y,t)}{\partial t} = \left[{}_0^R D_t^{1-\gamma(x,y,t)} + {}_0^R D_t^{1-\sigma(x,y,t)} \right] \left(\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2} \right) + G(x,y,t). \quad (1)$$

The initial and boundary conditions are:

$$\begin{array}{c} w(x,y,0) = q_1(x,y), 0 \le x,y \le L, \\ w(0,y,t) = q_2(y,t),\ w(L,y,t) = q_3(y,t),\ 0 \le y \le L, 0 \le t \le T, \\ w(x,0,t) = q_4(x,t),\ \ w(x,L,t) = q_5(x,t), 0 \le x \le L, 0 \le t \le T, \end{array} \quad (2)$$

where $q_1(x,y), q_2(y,t), q_3(y,t), q_4(x,t)$ and $q_5(x,t)$ are defined values, ${}_0^R D_t^{1-\gamma(x,y,t)}$ and ${}_0^R D_t^{1-\sigma(x,y,t)}$ are VO fractional partial derivatives with respect to time, and u represents the quantity of concentration function.

The remaining paper is organized as follows: Section 2, briefly discusses the preliminaries. Section 3 explains the implicit difference scheme and, Sections 3.1 and 3.2 provide the complete theoretical analysis of stability, consistency, and convergence. In Section 4, the numerical experiments are presented. Lastly, the conclusion is discussed in Section 5.

2. Preliminaries

In VO fractional theory, many definitions have been developed. Here, we introduce the VO-RLFD property and its approximations.

Definition 1. *The VO-RLFD of order $n-1 < \alpha(x,y,t) < n$, $n \in \mathcal{N}$ for function $w(x,y,t)$ can be written as follows:*

$${}_0^R D_t^{\gamma(x,y,t)} w(x,y,t_n) = \frac{1}{\Gamma\gamma(x,y,t)} \frac{d}{dt} \int_0^{t_n} \frac{w(x,y,\eta)}{(t_n - \eta)^{1-\gamma(x,y,t)}} d\eta. \quad (3)$$

For the VO-RLFD, we have an important property, $(0 < \gamma(x,y,t) < 1)$:

$${}_0^R D_t^{\gamma(x,y,t)} t^m = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(n-\gamma(x,y,t)+1)} t^{m-\gamma(x,y,t)}, & n-1 < \gamma(x,y,t) < n, \\ 0, & \gamma(x,y,t) = n. \end{cases} \quad (4)$$

Definition 2. *The VO Riemann–Liouville fractional integral operator can be defined as follows:*

$${}_0 I_t^{\gamma(x,y,t)} w(x,y,t) = \frac{1}{\Gamma(\gamma(x,y,t))} \int_0^{t_n} \frac{w(x,y,\eta)}{(t_n - \eta)^{1-\gamma(x,y,t)}} d\eta, \quad (5)$$

To formulate a new approximation for VO-RLFD at grid point (x_i, y_j, t_n):

$$ {}_0^RD_t^{1-\gamma(x_i,y_j,t_n)} w(x_i, y_j, t_n) = \frac{1}{\Gamma(\gamma(x_i, y_j, t_n))} \frac{\partial}{\partial t} \int_0^{t_n} \frac{w(x_i, y_j, \xi)}{(t_n - \xi)^{1-\gamma(x_i,y_j,t_n)}} d\xi, \tag{6} $$

and

$$ \begin{aligned} {}_0^RD_t^{1-\gamma(x_i,y_j,t_n)} w(x_i, y_j, t_n) &= \frac{\partial}{\partial t} \frac{1}{\Gamma(\gamma(x_i,y_j,t_n))} \int_0^{t_n} \frac{w(x_i,y_j,\xi)}{(t_n-\xi)^{1-\gamma(x_i,y_j,t_n)}} d\xi, \\ &= \frac{\partial}{\partial t} \frac{1}{\Gamma(\gamma(x_i,y_j,t_n))} \int_0^{t_n} (t_n - \xi)^{\gamma(x_i,y_j,t_n)-1} w(x_i, y_j, \xi) d\xi. \end{aligned} \tag{7} $$

Applying the Jumarie property from [37] to Equation (7) above, we obtain

$$ \begin{aligned} &= \frac{\partial}{\partial t} \frac{1}{\gamma(x_i,y_j,t_n)\Gamma(\gamma(x_i,y_j,t_n))} \int_0^{t_n} w(x_i, y_j, \xi)(d\xi)^{\gamma(x_i,y_j,t_n)}, \\ &= \frac{\partial}{\partial t} \frac{1}{\Gamma(\gamma(x_i,y_j,t_n)+1)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} w(x_i, y_j, \xi)(d\xi)^{\gamma(x_i,y_j,t_n)}, \\ &= \frac{\partial}{\partial t} \frac{1}{\Gamma(\gamma(x_i,y_j,t_n)+1)} \sum_{k=0}^{n-1} w(x_i, y_j, t_{n-k}) \int_{t_k}^{t_{k+1}} \xi^0 (d\xi)^{\gamma(x_i,y_j,t_n)}. \end{aligned} $$

Here, using the following Jumarie property in the above equation:

$$ \int_0^t \xi^{\alpha(x_i,y_j,t_n)} (d\xi)^{\beta(x_i,y_j,t_n)} = \frac{\Gamma(\alpha(x_i, y_j, t_n) + 1)\Gamma(\beta(x_i, y_j, t_n) + 1)}{\Gamma(\alpha(x_i, y_j, t_n) + \beta(x_i, y_j, t_n) + 1)} t^{\alpha(x_i,y_j,t_n)+\beta(x_i,y_j,t_n)}, \tag{8} $$

$$ = \frac{\partial}{\partial t} \frac{\tau^{\gamma(x_i,y_j,t_n)}}{\Gamma(\gamma(x_i,y_j,t_n)+1)} \sum_{k=0}^{n-1} w(x_i, y_j, t_{n-k}) \Big((k+1)^{\gamma(x_i,y_j,t_n)} - (k)^{\gamma(x_i,y_j,t_n)} \Big), $$

$$ {}_0^RD_t^{1-\gamma(x_i,y_j,t_n)} w(x_i, y_j, t_n) = \frac{\tau^{\gamma(x_i,y_j,t_n)-1}}{\Gamma(\gamma(x_i,y_j,t_n)+1)} \sum_{k=0}^{n-1} b_k^{(\gamma(x_i,y_j,t_n))} (w(x_i, y_j, t_{n-k}) - w(x_i, y_j, t_{n-k-1})), $$

and $b_k^{(\gamma(x_i,y_j,t_n))} = (k+1)^{\gamma(x_i,y_j,t_n)} - (k)^{\gamma(x_i,y_j,t_n)}, k = 0, 1, 2, \ldots, n-1.$

Lemma 1. *The* $\gamma(x, y, t)$ $(0 < \gamma(x, y, t) < 1)$*-order RLFD of function* $w(x, y, t)$ *on* $[0, T]$ *can be defined in discretized form as follows:*

$$ {}_0^RD_t^{1-\gamma(x_i,y_j,t_n)} w(x_i, y_j, t_n) = \frac{\tau^{\gamma_{i,j}^n - 1}}{\Gamma(\gamma_{i,j}^n + 1)} \sum_{k=0}^{n-1} b_k^{(\gamma_{i,j}^n)} (w_{i,j}^{n-k} - w_{i,j}^{n-k-1}). \tag{9} $$

Lemma 2. *Coefficients* $b_k^{(\gamma_{i,j}^n)}$ $(k = 0, 1, 2, 3 \ldots)$ *satisfy the following properties [21]:*

(i) $b_0^{(\gamma_{i,j}^n)} = 1, b_k^{(\gamma_{i,j}^n)} > 0, k = 0, 1, 2, \ldots,$

(ii) $b_k^{(\gamma_{i,j}^n)} > b_{k+1}^{(\gamma_{i,j}^n)}, k = 0, 1, 2, \ldots,$

(iii) *There exists a positive constant* $C > 0$*, such that* $\tau \le C b_k^{(\gamma_{i,j}^n)} \tau^{(\gamma_{i,j}^n)}, k = 1, 2, \ldots$

(vi) $\sum_{k=0}^{n} b_k^{(\gamma_{i,j}^n)} \tau^{(\gamma_{i,j}^n)} \le T^{(\gamma_{i,j}^n)}$

3. Implicit Difference Scheme

To construct the IDS for VO-TFMSDE, we used Lemma 1 for the VO fractional derivative part and the difference approximation for the space derivatives. We obtain

$$ \begin{aligned} w_{i,j}^n - w_{i,j}^{n-1} &= \frac{A\tau^{\gamma_{i,j}^n}}{\Gamma(\gamma_{i,j}^n+1)\Delta x^2} \sum_{k=0}^{n-1} b_k^{(\gamma_{i,j}^n)} \delta_x^2 \Big(w_{i,j}^{n-k} - w_{i,j}^{n-k-1} \Big) \\ &+ \frac{A\tau^{\gamma_{i,j}^n}}{\Gamma(\gamma_{i,j}^n+1)\Delta y^2} \sum_{k=0}^{n-1} b_k^{(\gamma_{i,j}^n)} \delta_y^2 \Big(w_{i,j}^{n-k} - w_{i,j}^{n-k-1} \Big) \\ &+ \frac{B\tau^{\sigma_{i,j}^n}}{\Gamma(\sigma_{i,j}^n+1)\Delta x^2} \sum_{k=0}^{n-1} b_k^{(\sigma_{i,j}^n)} \delta_x^2 \Big(w_{i,j}^{n-k} - w_{i,j}^{n-k-1} \Big) \\ &+ \frac{B\tau^{\sigma_{i,j}^n}}{\Gamma(\sigma_{i,j}^n+1)\Delta y^2} \sum_{k=0}^{n-1} b_k^{(\sigma_{i,j}^n)} \delta_y^2 \Big(w_{i,j}^{n-k} - w_{i,j}^{n-k-1} \Big) + \tau G_{i,j}^n. \end{aligned} \tag{10} $$

From Equation (10), the IDS for VO-TFMSDE (1), as

$$w_{i,j}^n - w_{i,j}^{n-1} = P_1^{i,j,n}\delta_x^2 w_{i,j}^n - P_1^{i,j,n} b_{n-1}^{(\gamma_{i,j}^n)}\delta_x^2 w_{i,j}^0 + P_2^{i,j,n}\delta_y^2 u_{i,j}^n - P_2^{i,j,n} b_{n-1}^{(\gamma_{i,j}^n)}\delta_y^2 w_{i,j}^0 - \sum_{k=1}^{n-1}(b_{k-1}^{(\gamma_{i,j}^n)} - b_k^{(\gamma_{i,j}^n)})(P_1^{i,j,n}\delta_x^2 w_{i,j}^{n-k} + P_2^{i,j,n}\delta_y^2 w_{i,j}^{n-k}) + P_3^{i,j,n}\delta_x^2 u_{i,j}^n - P_3^{i,j,n} b_{n-1}^{(\sigma_{i,j}^n)}\delta_x^2 w_{i,j}^0 + P_4^{i,j,n}\delta_y^2 w_{i,j}^n - P_4^{i,j,n} b_{n-1}^{(\sigma_{i,j}^n)}\delta_y^2 w_{i,j}^0 - \sum_{k=1}^{n-1}(b_{k-1}^{(\sigma_{i,j}^n)} - b_k^{(\sigma_{i,j}^n)})(P_3^{i,j,n}\delta_x^2 w_{i,j}^{n-k} + P_4^{i,j,n}\delta_y^2 w_{i,j}^{n-k}) + \tau G_{i,j}^n, \quad (11)$$

with

$$\begin{gathered} w_{i,j}^0 = q_1(x_i, y_j), \\ w_{0,j}^k = q_2(y_j, t_k),\ w_{M_x,j}^k = q_3(y_j, t_k), \\ w_{i,0}^k = q_4(x_i, t_k),\ w_{i,M_y}^k = q_5(x_i, t_k), \\ 0 \le x \le L_x, 0 \le y \le L_y,\ \ 0 \le t \le T. \end{gathered} \quad (12)$$

Here,

$$\begin{gathered} P_1^{i,j,n} = \frac{A\tau^{\gamma_{i,j}^n}}{\Gamma(\gamma_{i,j}^n+1)(\Delta x)^2},\ P_2^{i,j,n} = \frac{A\tau^{\gamma_{i,j}^n}}{\Gamma(\gamma_{i,j}^n+1)(\Delta y)^2}, \\ P_3^{i,j,n} = \frac{B\tau^{\sigma_{i,j}^n}}{\Gamma(\sigma_{i,j}^n+1)(\Delta x)^2},\ P_4^{i,j,n} = \frac{B\tau^{\sigma_{i,j}^n}}{\Gamma(\sigma_{i,j}^n+1)(\Delta y)^2}, \end{gathered}$$

and

$$\delta_x^2 w_{i,j}^n = w_{i+1,j}^n - 2w_{i,j}^n + w_{i-1,j}^n,\ \delta_y^2 w_{i,j}^n = w_{i,j+1}^n - 2w_{i,j}^n + w_{i,j-1}^n.$$

3.1. Stability Analysis

Using the Fourier series to find the stability of the VO-TFMSDE following similar analysis in [38], let $W_{i,j}^n$ represent the exact solution for Equation (11). We have

$$W_{i,j}^n - W_{i,j}^{n-1} = P_1^{i,j,n}\delta_x^2 W_{i,j}^n - P_1^{i,j,n} b_{n-1}^{(\gamma_{i,j}^n)}\delta_x^2 W_{i,j}^0 + P_2^{i,j,n}\delta_y^2 W_{i,j}^n - P_2^{i,j,n} b_{n-1}^{(\gamma_{i,j}^n)}\delta_y^2 W_{i,j}^0 - \sum_{k=1}^{n-1}(b_{k-1}^{(\gamma_{i,j}^n)} - b_k^{(\gamma_{i,j}^n)})(P_1^{i,j,n}\delta_x^2 W_{i,j}^{n-k} + P_2^{i,j,n}\delta_y^2 W_{i,j}^{n-k}) + P_3^{i,j,n}\delta_x^2 W_{i,j}^n - P_3^{i,j,n} b_{n-1}^{(\sigma_{i,j}^n)}\delta_x^2 W_{i,j}^0 + P_4^{i,j,n}\delta_y^2 W_{i,j}^n - P_4^{i,j,n} b_{n-1}^{(\sigma_{i,j}^n)}\delta_y^2 W_{i,j}^0 - \sum_{k=1}^{n-1}(b_{k-1}^{(\sigma_{i,j}^n)} - b_k^{(\sigma_{i,j}^n)})(P_3^{i,j,n}\delta_x^2 W_{i,j}^{n-k} + P_4^{i,j,n}\delta_y^2 W_{i,j}^{n-k}) + \tau G_{i,j}^n. \quad (13)$$

The error can be defined as follows:

$$\Phi_{i,j}^n = W_{i,j}^n - w_{i,j}^n. \quad (14)$$

Error $\Phi_{i,j}^n$ satisfies (13). We have

$$\Phi_{i,j}^n - \Phi_{i,j}^{n-1} = P_1^{i,j,n}(\Phi_{i+1,j}^n - 2\Phi_{i,j}^n + \Phi_{i-1,j}^n) - P_1^{i,j,n} b_{n-1}^{(\gamma_{i,j}^n)}(\Phi_{i+1,j}^0 - 2\Phi_{i,j}^0 + \Phi_{i-1,j}^0) + P_2^{i,j,n}(\Phi_{i,j+1}^n - 2\Phi_{i,j}^n + \Phi_{i,j-1}^n) - P_2^{i,j,n} b_{n-1}^{(\gamma_{i,j}^n)}(\Phi_{i,j+1}^0 - 2\Phi_{i,j}^0 + \Phi_{i,j-1}^0) - \sum_{k=1}^{n-1}(b_{k-1}^{(\gamma_{i,j}^n} - b_k^{(\gamma_{i,j}^n)})\Big(P_1^{i,j,n}(\Phi_{i+1,j}^{n-k} - 2\Phi_{i,j}^{n-k} + \Phi_{i-1,j}^{n-k}) + P_2^{i,j,n}(\Phi_{i,j+1}^{n-k} - 2\Phi_{i,j}^{n-k} + \Phi_{i,j-1}^{n-k})\Big) + P_3^{i,j,n}(\Phi_{i+1,j}^n - 2\Phi_{i,j}^n + \Phi_{i-1,j}^n) - P_3^{i,j,n} b_{n-1}^{(\sigma_{i,j}^n)}(\Phi_{i+1,j}^0 - 2\Phi_{i,j}^0 + \Phi_{i-1,j}^0) + P_4^{i,j,n}(\Phi_{i,j+1}^n - 2\Phi_{i,j}^n + e_{i,j-1}^n) - P_4^{i,j,n} b_{n-1}^{(\sigma_{i,j}^n)}(\Phi_{i,j+1}^0 - 2\Phi_{i,j}^0 + \Phi_{i,j-1}^0) - \sum_{k=1}^{n-1}(b_{k-1}^{(\sigma_{i,j}^n)} - b_k^{(\sigma_{i,j}^n)})\Big(P_3^{i,j,n}(\Phi_{i+1,j}^{n-k} - 2\Phi_{i,j}^{n-k} + \Phi_{i-1,j}^{n-k}) + P_4^{i,j,n}(\Phi_{i,j+1}^{n-k} - 2\Phi_{i,j}^{n-k} + \Phi_{i,j-1}^{n-k})\Big). \quad (15)$$

In addition,

$$\begin{gathered} \Phi_{0,j}^n = \Phi_{i,0}^n = 0, \Phi_{i,j}^0 = 0, \\ \Phi_{i,M_y}^n = \Phi_{M_x,j}^n = 0. \end{gathered}$$

Here, following the same approach as that in [36,38], suppose that

$$\Phi_{i,j}^{n} = \xi^{n} e^{\sqrt{-1}(\sigma_1 i\Delta x + \sigma_2 j\Delta y)}, \tag{16}$$

where $\sigma_1 = 2\pi l_1/L_x$, $\sigma_2 = 2\pi l_2/L_y$; putting Equation (16) in Equation (15) and dividing both sides by $e^{\sqrt{-1}(\sigma_1 i\Delta x)} e^{\sqrt{-1}(\sigma_2 j\Delta y)}$ and then put $(e^{\sqrt{-1}(\sigma_1 i\Delta x)} + e^{-\sqrt{-1}(\sigma_1 i\Delta x)}) = 2 - 4\sin^2(\frac{\sigma_1 \Delta x}{2})$, and $(e^{\sqrt{-1}(\sigma_2 j\Delta y)} + e^{-\sqrt{-1}(\sigma_2 j\Delta y)}) = 2 - 4\sin^2(\frac{\sigma_2 \Delta y}{2})$, we have

$$\xi^{n} = \frac{1}{(1+\mu_1^{i,j,n}+\mu_2^{i,j,n})}\Big(\xi^{n-1} + (\mu_1^{i,j,n} b_{n-1}^{(\gamma_{i,j}^{n})} + \mu_2^{i,j,n} b_{n-1}^{(\sigma_{i,j}^{n})})\xi^{0} + \mu_1^{i,j,n}\sum_{k=1}^{n-1}(b_{k-1}^{(\gamma_{i,j}^{n})} - b_{k}^{(\gamma_{i,j}^{n})})\xi^{n-k} + \mu_2^{i,j,n}\sum_{k=1}^{n-1}(b_{k-1}^{(\sigma_{i,j}^{n})} - b_{s}^{(\sigma_{i,j}^{n})})\xi^{n-k}\Big), \tag{17}$$

where

$$\mu_1^{i,j,n} = 4\Big(P_1^{i,j,n}\sin^2(\tfrac{\sigma_1\Delta x}{2}) + P_2^{i,j,n}\sin^2(\tfrac{\sigma_2\Delta y}{2})\Big),$$
$$\mu_2^{i,j,n} = 4\Big(P_3^{i,j,n}\sin^2(\tfrac{\sigma_1\Delta x}{2}) + P_4^{i,j,n}\sin^2(\tfrac{\sigma_2\Delta y}{2})\Big).$$

Proposition 1. *If* ξ^{k} $(n = 1, 2, \ldots, N)$ *Equation (17) is satisfied. Then, it must be proven that* $|\xi^{n}| \le |\xi^{0}|$.

Proof. To prove the proposition via the induction method, let us take $n = 1$ in Equation (17). We obtain

$$\xi^{1} = \frac{(1 + b_0^{\gamma_{i,j}^{1}}\mu_1^{i,j,1} + b_0^{(\sigma_{i,j}^{1})}\mu_2^{i,j,1})\xi^{0}}{(1+\mu_1^{i,j,1}+\mu_2^{i,j,1})},$$

and as $\mu_1^{i,j,1} \ge 0$, $\mu_2^{i,j,1} \ge 0$, $b_0^{(\gamma_{i,j}^{1})} = 1$, and $b_0^{(\sigma_{i,j}^{1})} = 1$, so

$$|\xi^{1}| \le |\xi^{0}|.$$

supposing that

$$|\xi^{p}| \le |\xi^{0}|; \quad p = 1, 2, \ldots, n-1.$$

Here, $0 < \gamma_{i,j}^{n}, \sigma_{i,j}^{n} < 1$, from Equation (17) and Lemma 2:

$$\begin{aligned}
|\xi^{n}| \le{}& \frac{1}{(1+\mu_1^{i,j,n}+\mu_2^{i,j,n})}\Big[|\xi^{n-1}| + (\mu_1^{i,j,n} b_{n-1}^{(\gamma_{i,j}^{n})} + \mu_2^{i,j,n} b_{n-1}^{(\sigma_{i,j}^{n})})|\xi^{0}| + \mu_1^{i,j,n}\sum_{k=1}^{n-1}(b_{k-1}^{(\gamma_{i,j}^{n})} - \\
& b_{k}^{(\gamma_{i,j}^{n})})|\xi^{n-k}| + \mu_2^{i,j,n}\sum_{k=1}^{n-1}(b_{k-1}^{(\sigma_{i,j}^{n})} - b_{k}^{(\sigma_{i,j}^{n})})|\xi^{n-k}|\Big],\\
\le{}& \frac{1}{(1+\mu_1^{i,j,n}+\mu_2^{i,j,n})}\Big[1 + (\mu_1^{i,j,n} b_{n-1}^{(\gamma_{i,j}^{n})} + \mu_2^{i,j,n} b_{n-1}^{(\sigma_{i,j}^{n})}) + \mu_1^{i,j,n}\sum_{k=1}^{n-1}(b_{s-1}^{(\gamma_{i,j}^{n})} - \\
& b_{k}^{(\gamma_{i,j}^{n})}) + \mu_2^{i,j,n}\sum_{k=1}^{n-1}(b_{k-1}^{(\sigma_{i,j}^{n})} - b_{k}^{(\sigma_{i,j}^{n})})\Big]|\xi^{0}|,\\
={}& \frac{1}{(1+\mu_1^{i,j,n}+\mu_2^{i,j,n})}\Big[1 + \mu_1^{i,j,n} b_{n-1}^{(\gamma_{i,j}^{n})} + \mu_2^{i,j,n} b_{n-1}^{(\sigma_{i,j}^{n})} + \mu_1^{i,j,n}(1 - b_{n-1}^{(\gamma_{i,j}^{n})}) + \\
& \mu_2^{i,j,n}(1 - b_{n-1}^{(\sigma_{i,j}^{n})})\Big]|\xi^{0}|\\
={}& \Big[\frac{1+\mu_1^{i,j,n}+\mu_2^{i,j,n}}{1+\mu_1^{i,j,n}+\mu_2^{i,j,n}}\Big]|\xi^{0}|
\end{aligned}$$

$$|\xi^{n}| \le |\xi^{0}|. \tag{18}$$

This completes the proof. □

Proposition 1 concludes that the solution of Equation (11) satisfies

$$\| \xi^n \|_2 \leq \| \xi^0 \|_2,$$

Hence, the scheme in Equation (11) is unconditionally stable.

3.2. Consistency

The consistency of 2D VO-TFMSDE is investigated in this section. Let the approximate and exact solutions be represented by w and W, respectively. If the approximated solution of TFMSDE is $G_{i,j}^k(w) = 0$, then the local truncation error is $G_{i,j}^k(w) = T_{i,j}^k$ at point (x_i, y_j, t_k).

Theorem 1. *The truncation error $T(x, y, t)$ of the finite difference scheme is:*

$$T(x, y, t) = O(\tau + \Delta x^2 + \Delta y^2).$$

Proof.

$$\begin{aligned}
T_{i,j}^n &= w_{i,j}^n - w_{i,j}^{n-1} \\
&\quad - P_1^{i,j,n} \sum_{k=0}^{n-1} b_k^{(\gamma_{i,j}^n)} \Big[(w_{i+1,j}^{n-k} - 2w_{i,j}^{n-k} + w_{i-1,j}^{n-k}) - (w_{i+1,j}^{n-k-1} - 2w_{i,j}^{n-k-1} + w_{i-1,j}^{n-k-1} \Big] \\
&\quad - P_2^{i,j,n} \sum_{k=0}^{n-1} b_k^{(\gamma_{i,j}^n)} \Big[(w_{i,j+1}^{n-k} - 2w_{i,j}^{n-k} + w_{i,j-1}^{n-k}) - (w_{i,j+1}^{n-k-1} - 2w_{i,j}^{n-k-1} + w_{i,j-1}^{n-k-1}) \Big] \\
&\quad - P_3^{i,j,n} \sum_{k=0}^{n-1} b_k^{(\sigma_{i,j}^n)} \Big[(w_{i+1,j}^{n-k} - 2w_{i,j}^{n-k} + w_{i-1,j}^{n-k}) - (w_{i+1,j}^{n-k-1} - 2w_{i,j}^{n-k-1} + w_{i-1,j}^{n-k-1}) \Big] \\
&\quad - P_4^{i,j,n} \sum_{k=0}^{n-1} b_k^{(\sigma_{i,j}^n)} \Big[(w_{i,j+1}^{n-k} - 2w_{i,j}^{n-k} + w_{i,j-1}^{n-k}) - (w_{i,j+1}^{n-k-1} - 2w_{i,j}^{n-k-1} + w_{i,j-1}^{n-k-1}) \Big].
\end{aligned} \tag{19}$$

With the Taylor series expansion, we can write

$$\begin{aligned}
T_{i,j}^n &= w_{i,j}^n - \left(w_{i,j}^n - \tau \frac{\partial w}{\partial t}\Big|_{i,j}^n + \frac{\tau^2}{2} \frac{\partial^2 w}{\partial t^2}\Big|_{i,j}^n + \dots \right) \\
&\quad - P_1^{i,j,n} \sum_{k=0}^{n-1} b_k^{(\gamma_{i,j}^n)} \Bigg[\Bigg(\left(w_{i,j}^{n-k} + (\Delta x) \frac{\partial w}{\partial x}\Big|_{i,j}^{n-k} + \frac{(\Delta x)^2}{2} \frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 w}{\partial x^3}\Big|_{i,j}^{n-k} + \dots \right) \\
&\quad - 2w_{i,j}^{n-k} + \left(w_{i,j}^{n-k} - (\Delta x) \frac{\partial w}{\partial x}\Big|_{i,j}^{n-k} + \frac{(\Delta x)^2}{2} \frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 w}{\partial x^3}\Big|_{i,j}^{n-k} + \dots \right) \Bigg) \\
&\quad - \Bigg(\left(w_{i,j}^{n-k-1} + (\Delta x) \frac{\partial w}{\partial x}\Big|_{i,j}^{n-k-1} + \frac{(\Delta x)^2}{2} \frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k-1} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 w}{\partial x^3}\Big|_{i,j}^{n-k-1} + \dots \right) \\
&\quad - 2w_{i,j}^{n-k-1} + \left(w_{i,j}^{n-k-1} - (\Delta x) \frac{\partial w}{\partial x}\Big|_{i,j}^{n-k-1} + \frac{(\Delta x)^2}{2} \frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k-1} - \frac{(\Delta x)^3}{3!} \frac{\partial^3 w}{\partial x^3}\Big|_{i,j}^{n-k-1} + \dots \right) \Bigg) \Bigg] \\
&\quad - P_2^{i,j,n} \sum_{k=0}^{n-1} b_k^{(\gamma_{i,j}^n)} \Bigg[\Bigg(\left(w_{i,j}^{n-k} + (\Delta y) \frac{\partial w}{\partial y}\Big|_{i,j}^{n-k} + \frac{(\Delta y)^2}{2} \frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k} + \frac{(\Delta y)^3}{3!} \frac{\partial^3 w}{\partial y^3}\Big|_{i,j}^{n-k} + \dots \right) \\
&\quad - 2w_{i,j}^{n-k} + \left(w_{i,j}^{n-k} - (\Delta y) \frac{\partial w}{\partial y}\Big|_{i,j}^{n-k} + \frac{(\Delta y)^2}{2} \frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k} - \frac{(\Delta y)^3}{3!} \frac{\partial^3 w}{\partial y^3}\Big|_{i,j}^{n-k} + \dots \right) \Bigg) \\
&\quad - \Bigg(\left(w_{i,j}^{n-k-1} + (\Delta y) \frac{\partial w}{\partial y}\Big|_{i,j}^{n-k-1} + \frac{(\Delta y)^2}{2} \frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k-1} + \frac{(\Delta y)^3}{3!} \frac{\partial^3 w}{\partial y^3}\Big|_{i,j}^{n-k-1} + \dots \right) \\
&\quad - 2w_{i,j}^{n-k-1} + \left(w_{i,j}^{n-k-1} - (\Delta y) \frac{\partial w}{\partial y}\Big|_{i,j}^{n-k-1} + \frac{(\Delta y)^2}{2} \frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k-1} - \frac{(\Delta y)^3}{3!} \frac{\partial^3 w}{\partial y^3}\Big|_{i,j}^{n-k-1} + \dots \right) \Bigg) \Bigg] \\
&\quad - P_3^{i,j,n} \sum_{k=0}^{n-1} b_k^{(\sigma_{i,j}^n)} \Bigg[\Bigg(\left(w_{i,j}^{n-k} + (\Delta x) \frac{\partial w}{\partial x}\Big|_{i,j}^{n-k} + \frac{(\Delta x)^2}{2} \frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 w}{\partial x^3}\Big|_{i,j}^{n-k} + \dots \right)
\end{aligned}$$

$$
\begin{aligned}
&- 2w_{i,j}^{n-k} + \left(w_{i,j}^{n-k} - (\Delta x)\frac{\partial w}{\partial x}\Big|_{i,j}^{n-k} + \frac{(\Delta x)^2}{2}\frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 w}{\partial x^3}\Big|_{i,j}^{n-k} + \dots\right)\Big) \\
&- \Big(\Big(w_{i,j}^{n-k-1} + (\Delta x)\frac{\partial w}{\partial x}\Big|_{i,j}^{n-k-1} + \frac{(\Delta x)^2}{2}\frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k-1} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 w}{\partial x^3}\Big|_{i,j}^{n-k-1} + \dots\Big) \\
&- 2w_{i,j}^{n-k-1} + \left(w_{i,j}^{n-k-1} - (\Delta x)\frac{\partial w}{\partial x}\Big|_{i,j}^{n-k-1} + \frac{(\Delta x)^2}{2}\frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k-1} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 w}{\partial x^3}\Big|_{i,j}^{n-k-1} + \dots\right)\Big)\Big] \\
&- P_4^{i,j,n}\sum_{k=0}^{n-1} b_s^{(\sigma_{i,j}^n)}\Big[\Big(\Big(w_{i,j}^{n-k} + (\Delta y)\frac{\partial u}{\partial y}\Big|_{i,j}^{n-k} + \frac{(\Delta y)^2}{2}\frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k} + \frac{(\Delta y)^3}{3!}\frac{\partial^3 w}{\partial y^3}\Big|_{i,j}^{n-k} + \dots\Big) \\
&- 2w_{i,j}^{n-k} + \left(w_{i,j}^{n-k} - (\Delta y)\frac{\partial w}{\partial y}\Big|_{i,j}^{n-k} + \frac{(\Delta y)^2}{2}\frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k} - \frac{(\Delta y)^3}{3!}\frac{\partial^3 w}{\partial y^3}\Big|_{i,j}^{n-k} + \dots\right)\Big) \\
&- \Big(\Big(w_{i,j}^{n-k-1} + (\Delta y)\frac{\partial w}{\partial y}\Big|_{i,j}^{n-k-1} + \frac{(\Delta y)^2}{2}\frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k-1} + \frac{(\Delta y)^3}{3!}\frac{\partial^3 w}{\partial y^3}\Big|_{i,j}^{n-k-1} + \dots\Big) \\
&- 2w_{i,j}^{n-k-1} + \left(w_{i,j}^{n-k-1} - (\Delta y)\frac{\partial w}{\partial y}\Big|_{i,j}^{n-k-1} + \frac{(\Delta y)^2}{2}\frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k-1} - \frac{(\Delta y)^3}{3!}\frac{\partial^3 w}{\partial y^3}\Big|_{i,j}^{n-k-1} + \dots\right)\Big)\Big].
\end{aligned} \tag{20}
$$

After simplification, we obtain

$$
\begin{aligned}
T_{i,j}^n &= \tau\frac{\partial w}{\partial t}\Big|_{i,j}^{n} - P_1^{i,j,n}\textstyle\sum_{k=0}^{n-1} b_s^{(\gamma_{i,j}^n)}\left[(\Delta x)^2\frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k} - (\Delta x)^2\frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k-1}\right] \\
&- P_2^{i,j,n}\textstyle\sum_{k=0}^{n-1} b_k^{(\gamma_{i,j}^n)}\left[(\Delta y)^2\frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k} - (\Delta y)^2\frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k-1}\right] \\
&- P_3^{i,j,n}\textstyle\sum_{k=0}^{n-1} b_k^{(\sigma_{i,j}^n)}\left[(\Delta x)^2\frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k} - (\Delta x)^2\frac{\partial^2 w}{\partial x^2}\Big|_{i,j}^{n-k-1}\right] \\
&- P_4^{i,j,n}\textstyle\sum_{k=0}^{n-1} b_k^{(\sigma_{i,j}^n)}\left[(\Delta y)^2\frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k} - (\Delta y)^2\frac{\partial^2 w}{\partial y^2}\Big|_{i,j}^{n-k-1}\right].
\end{aligned} \tag{21}
$$

Hence,

$$T_{i,j}^k = O(\tau + \Delta x^2 + \Delta y^2).$$

This theorem shows that this method is consistent because $\tau \to 0$, $\Delta x \to 0$, $\Delta y \to 0$. □

Theorem 2 (**Lax equivalence theorem (see [39,40])**)**.** *If the method is consistent and stable, then it is convergent.*

4. Numerical Experiments

In this section, we solve a numerical example as a 2D VO-TFMSDE for different values of VO to check the feasibility of the proposed IDS. The error between the numerical and exact solutions follows the references in equations,
i.e., the E_∞ error formula is as follows:

$$E_\infty = \max_{0\le i\le M_x-1, 0\le j\le M_y-1, 0\le n\le N} |w(x_i, y_j, t_n) - w_{i,j}^n|. \tag{22}$$

Example 1. *Consider the following 2D VO-TFMSDE [20].*

$$\frac{\partial w}{\partial t} = \left[{}_0^R D_t^{1-\gamma(x,y,t)} + {}_0^R D_t^{1-\sigma(x,y,t)}\right]\left(\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right) + G(x,y,t), \tag{23}$$

where $G(x,y,t) = 2e^{(x+y)}\left(t - \frac{2t^{1+\gamma(x,y,t)}}{\Gamma(2+\gamma(x,y,t))} - \frac{2t^{1+\sigma(x,y,t)}}{\Gamma(2+\sigma(x,y,t))}\right)$, *and the closed form solution is* $w(x,y,t) = t^2 e^{x+y}$.

Example 2. *Consider the two-dimensional RSP-HGSGF with fractional derivative*

$$
\begin{aligned}
u_t(x,y,t) ={}_0 D_t{}^{1-\beta(x,y,t)}\Big(u_{xx}(x,y,t) + u_{yy}(x,y,t)\Big) + u_{xx}(x,y,t) + \\
u_{yy}(x,y,t) + f(x,y,t), \ \ 0 \le t \le T,
\end{aligned} \tag{24}
$$

where

$$f(x,y,t) = 2e^{x+y}\Big(t - t^2 - 2\frac{t^{1+\beta(x,y,t)}}{\Gamma(2+\beta(x,y,t))}\Big). \quad (25)$$

The exact solution is given by

$$u(x,t) = e^{x+y}t^2. \quad (26)$$

In the present study, the IDS was developed on the basis of a new approximation for RLFD operator that was derived in Lemma 1 for 2D VO-TFMSDE. The numerical results were compared with the exact solution and with the results in [20]. Chen [20] numerically solved the VO-TFMSDE and treated the VO fractional derivative by Grünwald-Letnikov. In Table 1, the proposed IDS shows better accuracy at various values of space and time steps $(\Delta x, \Delta y, \Delta t)$ and derivative of VO $(\gamma(x,y,t), \sigma(x,y,t))$, and Table 2 shows the accuracy with the exact solution. For more confirmation, Table 3 shows the even better accuracy of the suggested scheme for two-dimensional RSP-HGSGF with fractional derivatives. Figures 1 and 2 represent the exact and approximate solutions, respectively, at $\gamma(x,y,t) = e^{(xyt)-2.5}, \sigma(x,y,t) = e^{-(xyt)-1.8}, T = 1, y = 0.125, N = 64$, which were more similar, and the approximated solution converged to the exact solution. Figures 3 and 4 represent the comparison of IDS with the exact solution, which showed high accuracy at $\gamma = \frac{1-(xyt)^2}{10}, \frac{2-cos(xyt)}{20}, \sigma = e^{xyt} - cos(xyt)30, \frac{3+(xyt)^2-(xt)^3}{30}, T = 1, y = 0.091, 0.1$ and $N = 121, 160$ respectively.

Table 1. Comparison of IDS (11) at $T = 1.0,\ \Delta x = \Delta y = h$ and for different values of $\gamma(x,y,t)$, $\sigma(x,y,t)$.

		$h^2 = \tau = \frac{1}{25}$		$h^2 = \tau = \frac{1}{100}$	
$\gamma(x,y,t)$	$\sigma(x,y,t)$	[20]	Scheme	[20]	Scheme
$\frac{1-(xyt)^2}{10}$	$\frac{e^{xyt}+cos(xyt)}{30}$	8.0355×10^{-3}	5.8320×10^{-3}	2.0696×10^{-3}	1.2832×10^{-3}
$\frac{2-cos(xyt)}{20}$	$\frac{3+(xy)^2-(xt)^3}{30}$	8.0183×10^{-3}	6.2580×10^{-3}	2.0640×10^{-3}	1.3293×10^{-3}
$\frac{3-sin^2(xyt)}{30}$	$\frac{5+(xy)^3-(xt)^4}{50}$	8.1723×10^{-3}	5.0709×10^{-3}	2.1095×10^{-3}	9.6731×10^{-4}
$\frac{5-x+y^2-t^3}{50}$	$\frac{9+x-y^2-t^3}{80}$	8.2124×10^{-3}	4.8675×10^{-3}	2.1195×10^{-3}	6.3245×10^{-4}
$\frac{1-xyt+sin(xyt)}{10}$	$\frac{1+cos(xy)-(xt)^3}{30}$	8.0406×10^{-3}	3.2527×10^{-3}	2.0718×10^{-3}	4.6502×10^{-4}
$\frac{1-(xyt)^3+cos^2(xyt)}{10}$	$\frac{6^{(xyt)}-sin^3(xyt)}{60}$	8.1988×10^{-3}	5.4178×10^{-4}	2.1130×10^{-3}	2.3773×10^{-4}

Table 2. Numerical results of IDS (11) at $T = 1.0,\ \Delta x = \Delta y = h$ and for different values of $\gamma(x,y,t)$, $\sigma(x,y,t)$.

$\gamma(x,y,t)$	$\sigma(x,y,t)$	$h^2 = \tau = \frac{1}{4}$	$h^2 = \tau = \frac{1}{16}$	$h^2 = \tau = \frac{1}{64}$
$\frac{1-(xyt)^2}{10}$	$\frac{e^{xyt}+cos(xyt)}{30}$	3.7765×10^{-2}	9.2040×10^{-3}	1.9846×10^{-3}
$\frac{2-cos(xyt)}{20}$	$\frac{3+(xy)^2-(xt)^3}{30}$	3.9540×10^{-2}	9.8408×10^{-3}	2.1700×10^{-3}
$\frac{e^{(xyt)}-(xyt)}{8}$	$\frac{e^{(xyt)}-(xyt)^3}{12}$	2.3184×10^{-2}	8.2456×10^{-3}	2.1890×10^{-3}
$e^{(xyt)-2.5}$	$e^{-(xyt)-1.8}$	2.2517×10^{-2}	7.9011×10^{-3}	2.0444×10^{-3}
$\frac{3-sin^2(xyt)}{30}$	$\frac{5+(xy)^3-(xt)^4}{50}$	3.5056×10^{-2}	8.1540×10^{-3}	1.6455×10^{-3}
$\frac{5-x+y^2-t^3}{50}$	$\frac{9+x-y^2-t^3}{80}$	3.4383×10^{-2}	7.8799×10^{-3}	1.5577×10^{-3}
$\frac{1+(xyt)^5}{9}$	$\frac{\sqrt{xyt}+1}{15}$	2.7685×10^{-2}	6.1931×10^{-3}	1.1278×10^{-3}
$\frac{1-xyt+sin(xyt)}{10}$	$\frac{1+cos(xy)-(xt)^3}{30}$	2.7378×10^{-2}	5.5218×10^{-3}	8.9180×10^{-4}
$\frac{1-(xyt)^3+cos^2(xyt)}{10}$	$\frac{6^{(xyt)}-sin^3(xyt)}{60}$	3.0216×10^{-3}	7.0405×10^{-4}	4.1117×10^{-4}

Table 3. Comparison of numerical scheme for (24) with exact solution (26), maximal error at $T = 1.0$.

$\beta(x,y,t)$	$h^2 = \tau = \frac{1}{4}$	$h^2 = \tau = \frac{1}{16}$	$h^2 = \tau = \frac{1}{64}$	$h^2 = \tau = \frac{1}{100}$
$sin(xyt + \frac{2\pi}{5})$	2.7906×10^{-2}	8.0080×10^{-3}	2.0703×10^{-3}	1.3204×10^{-3}
$\frac{e^{xyt}-(xyt)}{8}$	2.6989×10^{-3}	6.1874×10^{-4}	4.8372×10^{-4}	3.5197×10^{-4}
$\frac{\sqrt{xyt}+1}{15}$	6.0945×10^{-3}	7.6964×10^{-4}	1.0488×10^{-4}	2.1621×10^{-4}
$\frac{e^{xyt}-sin(xyt)}{10}$	5.2639×10^{-3}	3.7005×10^{-4}	1.8890×10^{-4}	2.0941×10^{-4}
$e^{xyt-2.5}$	5.6529×10^{-3}	5.3719×10^{-4}	2.0541×10^{-4}	1.6042×10^{-4}
$\frac{e^{xyt}-(xyt)^3}{12}$	5.5681×10^{-3}	5.2462×10^{-4}	1.8347×10^{-4}	1.4969×10^{-4}
$\frac{1+(xyt)^5}{9}$	4.3292×10^{-3}	3.1150×10^{-5}	2.7668×10^{-4}	9.3363×10^{-5}

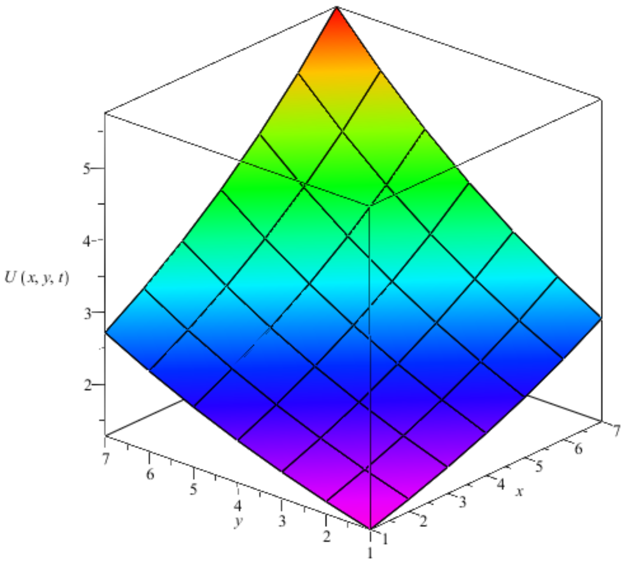

Figure 1. Exact solution.

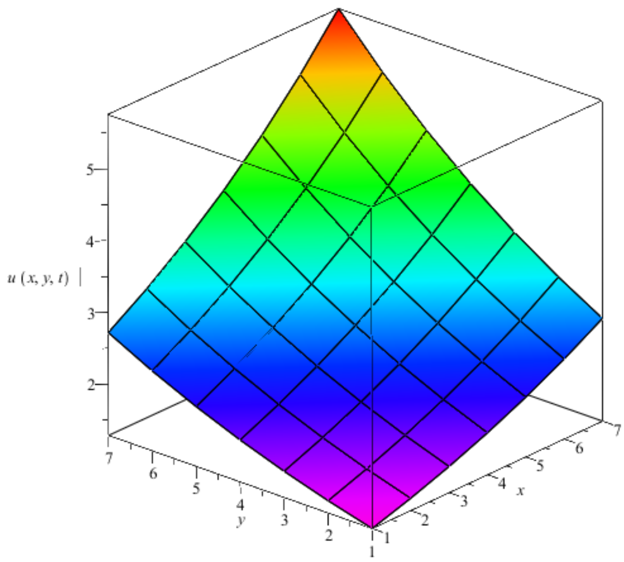

Figure 2. Approximate solution.

The comparison plot of the exact and approximate solutions of Equation (23) at $\gamma = e^{(xyt)-2.5}, \sigma = e^{-(xyt)-1.8}, T = 1, y = 0.125$ and $N = 64$.

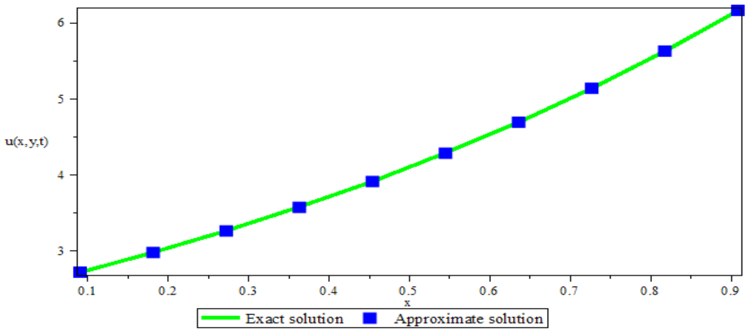

Figure 3. Comparison of the IDS and the exact solution of Equation (23) at $\gamma = \frac{1-(xyt)^2}{10}$, $\sigma = \frac{e^{xyt}-cos(xyt)}{30}$, $T = 1, y = 0.091$ and $N = 121$.

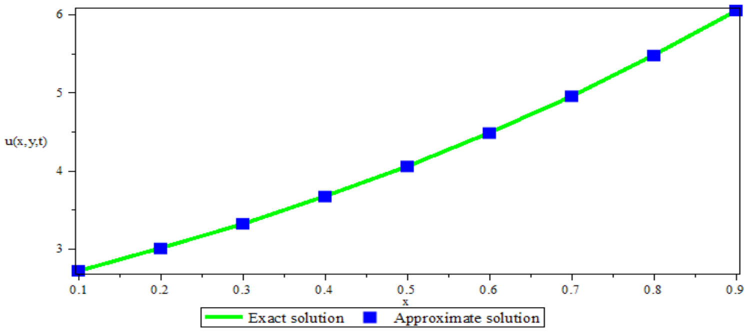

Figure 4. Comparison of the IDS and the exact solution of Equation (23) at $\gamma = \frac{2-cos(xyt)}{20}$, $\sigma = \frac{3+(xyt)^2-(xt)^3}{30}$, $T = 1, y = 0.1$ and $N = 160$.

5. Conclusions

In this study, we formulated the numerical approximation of a VO-RLFD operator and developed a new IDS. The proposed scheme was successfully applied to a 2D VO-TFMSDE and RSP-HGSGF with a fractional derivative. Theoretical analysis regarding stability on a Fourier approach, consistency, and convergence was studied for 2D VO-TFMSDE. The numerical results are reported in the form of error tables, and 3D and 2D plots. Moreover, the solution was tested with a numerical example, and the theoretical analysis was confirmed with the numerical results. The proposed approach is highly effective and efficient for VO-FDEs. This scheme can also be extended to other types of VO fractional differential equations.

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