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Complex Interval-Valued q-Rung Orthopair Fuzzy Hamy Mean Operators and Their Application in Decision-Making Strategy

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Abstract: This paper deals with uncertainty, asymmetric information, and risk modelling in a complex power system. The uncertainty is managed by using probability and decision theory methods. Multi-attribute decision-making (MADM) technique is a very effective and well-known tool to investigate fuzzy information more effectively. However, the selection of houses cannot be carried out by utilizing symmetry information, because enterprises does not have complete information, so asymmetric information should be used when selecting enterprises. Hamy mean (HM) operator is a feasible tool to handle strategic decision-making problems because it can capture the order between the finite input terms. Additionally, the complex interval-valued q-rung orthopair fuzzy (CIVq-ROF) setting is a broadly flexible and massively dominant technique to operate problematic and awkward data in actual life problems. The major contribution of this analysis is how to aggregate the collection of alternatives into a singleton set, for this we analyzed the technique of CIVq-ROF Hamy mean (CIVq-ROFHM) operator and CIVq-ROF weighted Hamy mean (Cq-ROFWHM) operator and some well-known results are deliberated. Keeping the advantages of the parameters in HM operators, we discussed the specific cases of the invented operators. To investigate the decision-making problems based on CIVq-ROF information, we suggested the following multi-attribute decision-making (MADM) technique to determine the beneficial term from the finite group of alternatives with the help of evaluating several examples. This manuscript showed how to make decisions when there is asymmetric information about enterprises. Finally, based on the evaluating examples, we try to discover the sensitive analysis and supremacy of the invented operators to find the flexibility and dominancy of the diagnosed approaches.

Keywords: complex interval-valued q-rung orthopair fuzzy sets; weighted Hamy mean operators; decision-making strategy

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1. Introduction

One of the hardest problems is given in the form of ambiguity and inconsistency, which is involved in every field of life and affects the resultant values during the decision-making process. To reduce the affected ratio of data from ambiguity, Atanassov [1] initiated the tool of the intuitionistic fuzzy set (IFS) by extending the tool of fuzzy set, which was familiarized by Zadeh [2]. As a theory, the experts must consider fuzzy variables to show their reasons. Various attempts have been made by the distinct intellectuals in managing the data values based on distinct aggregation operators [3–5]. On the other hand, the fundamental theory of interval-valued IFS (IVIFS) was exposed by Atanassov [6]. IVIFS indicates two terms in the shape of interval values whose sum of upper terms lies in [0, 1]. Among all these diverse ideas, one is to determine the beneficial optimal, several well-known implementations are discussed based on IFS and IVIFS [7–10]. After that, Garg [11] reflected the theory of interval-valued Pythagorean fuzzy set (IVPFS) by adding the new tool, called the sum of the square of the duplet lying within [0, 1]. IVPFS has broadly generalized the prevailing Pythagorean fuzzy set (IFS) [12]. Under the circumstances, many professors have diagnosed their theories and techniques [13–16]. Serious complications have occurred when an expert suggested $([0.7, 0.9], [0.6, 0.8])$ for truth and falsity grade, then $0.9^2 + 0.8^2 = 0.81 + 0.64 = 1.45 > 1$. For this, Joshi et al. [17] diagnosed the interval-valued q-rung orthopair fuzzy set (IVq-ROFS), with a tool: $0 \leq \mathcal{M}_E^{+\mathfrak{D}} + \mathcal{N}_E^{+\mathfrak{D}} \leq 1$. IVq-ROFS has broadly generalized the prevailing q-rung orthopair fuzzy set (q-ROFS) [18]. Under the circumstances, many professors have diagnosed their theories and techniques [19–22].

The above-used analysis discovers that several decision-making tools have been diagnosed in the availability of IVIFS, IVPFS, and IVq-ROFSs. One thing is clear, the above tools are handled only with one-dimensional data. However, several genuine awkward life dilemmas contain two-dimensional information, that is, data associated with the attributes and periodicity of the parameters regarding the dilemma. To portray the above kind of data, the expert will have to choose two or more q-ROFS/IVq-ROFS which may grow performance time and the number of calculations needed while resolving the dilemma. Continuously, to employ the periodic data in the truth grade (TG), Ramot et al. [23] proposed the mathematical work of complex FS (CFS). As a theory, the experts must consider complex fuzzy variables to show their reasons. Various attempts have been made by distinct intellectuals in managing the data values based on distinct aggregation operators [24,25]. Moreover, to include the hesitation term along with the periodic data in the conception of CFS, Alkouri and Salleh [26] utilized the novel theme of complex IFS (CIFS) and the theory of complex IVIFS (CIVIFS), diagnosed by Garg and Rani [27]. After that, Ali et al. [28] reflected the theory of complex IVPFS (CIVPFS) by adding the new tool, called the sum of the square of the real part (also for imaginary parts) of the duplet lying within [0, 1]. CIVIFS has broadly generalized the prevailing complex PFS (CPFS) [29]. Under the circumstances, many professors have diagnosed their theories and techniques [30–32]. Furthermore, Garg et al. [33] diagnosed the complex IVq-ROFS (CIVq-ROFS), with a tool: $0 \leq \mathcal{M}_E^{+\mathfrak{D}} + \mathcal{N}_E^{+\mathfrak{D}} \leq 1$ and $0 \leq \varphi_{\mathcal{M}_E}^{+\mathfrak{D}} + \varphi_{\mathcal{N}_E}^{+\mathfrak{D}} \leq 1$. CIVq-ROFS has broadly generalized the prevailing complex q-ROFS (Cq-ROFS) [34]. Under the circumstances, many professors have diagnosed their theories and techniques [35,36].

All scholars have acknowledged the existing concepts of IVIFS, IVPFS, IVq-ROFS, CIVIFS, CIVPFS, and CIVq-ROFS for implementation in the region of similarity measures, aggregation operators, and especially hybrid operators to enhance and improve the superiority of the invented approaches. In the presence of the above scenario, the theory of Muirhead means operators for CIVq-ROFS were invented by Garg et al. [37] which contained the truth and falsity grade in the terms of polar coordinates whose real and imaginary parts are the shape of interval values. Further, the theory of Dombi HM operators for IFS was exposed by Li et al. [38] which can easily depict awkward and unreliable information mentioned in genuine life dilemmas. The theory of Dombi HM operators based on interval-valued IFS was initiated by Wu et al. [39]. The main theory of HM operators based on IFS was also developed by Liang [40]. Similarly, the HM operators under the consideration of PFS were developed by Li et al. [41] with a rule the sum of the square of the duplet will be contained in the unit interval and HM operators based on q-ROFS were exposed by Wang et al. [42].

CPFS and Cq-ROFS have a lot of application in the scenario of different fields, but due to ambiguity of the strategic decision-making dilemmas, in optional cases, the piratical decision-makers are not suitable to give their suggestion in the shape of one-dimension supporting and supporting against grades. Continuously, generalization of various prevailing ideas might be massively dominant and valuable to describe the awkwardness due to his/her disinclined decision in awkward decision-making troubles. Therefore, to give massive space to the intellectuals, it is noticeable to ask the individual to explain their decisions through intervals. Thus, motivated by this concept, in this analysis, we diagnose the well-known theory operators based on the CIVq-ROF information. Yet, the theory of HM operator has not developed in the consideration of CIVq-ROF settings. In the presence of the CIVq-ROF setting, both grades are in the form of complex values such as polar coordinates whose amplitude and phase terms have represented the extent and additional data in duplet sets, usually concerned with periodicity. In the prevailing q-ROFS/IVq-ROFS information, we observed that there is only one parameter to express the data which results in data loss in various cases. For instance, assume a various initiative decides to organize a biometric system (BMS) in all hostels over the university. For this, the administration appoints some experts who provide two-dimensional data related to each BMS, (i) classical BMS and (ii) manufacture date of the BMS. The task is very challenging for experts to select the best BMS and the prevailing theories are unsuitable to perfectly address it. Therefore, for managing such data, the theory of CIVq-ROF information is more reliable. Because it contains two grades, and each grade includes two-dimensional information such as amplitude and phase term which represented the model and production date of the BMS. The proposed idea is also valuable for medical research, as well as the establishment of databases for biometric and facial recognition, audio and image segmentations. The notion of CIVq-ROF sets and Hamy mean operators are very closely related to the notion of symmetry. Based on symmetry, we can talk about the mixture of both theories. The main advantages of this scenario are implemented here:

1. For $0 \leq \mathcal{M}_E^{+2} + \mathcal{N}_E^{+2} \leq 1$ and $0 \leq {\varphi^+_{\mathcal{M}_E}}^2 + {\varphi^+_{\mathcal{N}_E}}^2 \leq 1$ in CIVq-ROFS, then we get CIVPFSs.
2. For $0 \leq \mathcal{M}_E^{+1} + \mathcal{N}_E^{+1} \leq 1$ and $0 \leq {\varphi^+_{\mathcal{M}_E}}^1 + {\varphi^+_{\mathcal{N}_E}}^1 \leq 1$ in CIVq-ROFS, then we get CIVIFSs.
3. For $\mathcal{M}^- = \mathcal{M}^+, \varphi^-_{\mathcal{M}} = \varphi^+_{\mathcal{M}}, \mathcal{N}^- = \mathcal{N}^+$ and $\varphi^-_{\mathcal{N}} = \varphi^+_{\mathcal{N}}$ with $0 \leq \mathcal{M}_E^{+\mathfrak{D}} + \mathcal{N}_E^{+\mathfrak{D}} \leq 1$ and $0 \leq {\varphi^+_{\mathcal{M}_E}}^{\mathfrak{D}} + {\varphi^+_{\mathcal{N}_E}}^{\mathfrak{D}} \leq 1$ in CIVq-ROFS, then we get Cq-ROFSs.
4. For $\mathcal{M}^- = \mathcal{M}^+, \varphi^-_{\mathcal{M}} = \varphi^+_{\mathcal{M}}, \mathcal{N}^- = \mathcal{N}^+$ and $\varphi^-_{\mathcal{N}} = \varphi^+_{\mathcal{N}}$ with $0 \leq \mathcal{M}_E^{+2} + \mathcal{N}_E^{+2} \leq 1$ and $0 \leq {\varphi^+_{\mathcal{M}_E}}^2 + {\varphi^+_{\mathcal{N}_E}}^2 \leq 1$ in CIVq-ROFS, then we get CPFSs.
5. For $\mathcal{M}^- = \mathcal{M}^+, \varphi^-_{\mathcal{M}} = \varphi^+_{\mathcal{M}}, \mathcal{N}^- = \mathcal{N}^+$ and $\varphi^-_{\mathcal{N}} = \varphi^+_{\mathcal{N}}$ with $0 \leq \mathcal{M}_E^{+1} + \mathcal{N}_E^{+1} \leq 1$ and $0 \leq {\varphi^+_{\mathcal{M}_E}}^1 + {\varphi^+_{\mathcal{N}_E}}^1 \leq 1$ in CIVq-ROFS, then we get CIFSs.

Using the beneficial and important worth of CIVq-ROFS, the major aspects of the presented approaches are diagnosed here:

1. To invent the CIVq-ROFHM operator and Cq-ROFWHM operator with some well-known results are deliberated.
2. To discuss the specific cases of the invented operators.
3. To expose a MADM technique under the invented operators is to determine the beneficial term from the finite group of alternatives with the help of evaluating several examples.
4. To discover the sensitive analysis and supremacy of the invented operators is to find the flexibility and dominancy of the diagnosed approaches.

A major analysis of this construction is followed: In Section 2, to continuously extend the prevailing works, some existing CIVq-ROFSs, HM operator, and several ideas such as algebraic laws, score value (SV), accuracy value (AV), and how we investigate their relation, are recalled. In Section 3, we invented the CIVq-ROFHM operator and Cq-ROFWHM operator and some well-known results are deliberated. To discuss the specific cases of the invented operators. In Section 4, we exposed a MADM technique under the invented operators to determine the beneficial term from the finite group of alternatives with the help of evaluating several examples. We discovered the sensitive analysis and supremacy of the invented operators to find the flexibility and dominancy of the diagnosed approaches. Section 5 discusses the conclusion.

2. Preliminaries

In this study, to continuously extend the prevailing works, we recall several existing theories, called CIVq-ROFSs, HM operator, and several ideas such as algebraic laws, SV, AV, and how we investigate their relation. These theories are massively informative for new proposed work.

Definition 1 ([33]). *A CIVq-ROFS is particularized by:*

$$\mathfrak{k} = \left\{ \left(\sigma, \mathcal{M}'_E(\sigma), \mathcal{N}'_E(\sigma) \right) : \sigma \in X \right\} \tag{1}$$

where $\mathcal{M}'_E = \left[\mathcal{M}^-_E, \mathcal{M}^+_E\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}_E}, \varphi^+_{\mathcal{M}_E}]}$ *and* $\mathcal{N}'_E = \left[\mathcal{N}^-_E, \mathcal{N}^+_E\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}_E}, \varphi^+_{\mathcal{N}_E}]}$ *with* $0 \le \mathcal{M}^{+\mathfrak{D}}_E + \mathcal{N}^{+\mathfrak{D}}_E \le 1$ *and* $0 \le \varphi^{+\ \mathfrak{D}}_{\mathcal{M}_E} + \varphi^{+\ \mathfrak{D}}_{\mathcal{N}_E} \le 1$. *The CIVq-ROF number is stated by:* $\mathfrak{k} = (\mathcal{M}'_E, \mathcal{N}'_E) = \left(\left[\mathcal{M}^-_E, \mathcal{M}^+_E\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}_E}, \varphi^+_{\mathcal{M}_E}]}, \left[\mathcal{N}^-_E, \mathcal{N}^+_E\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}_E}, \varphi^+_{\mathcal{N}_E}]} \right)$. *Suppose* $\mathfrak{k} = \left([\mathcal{M}^-, \mathcal{M}^+].e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}}, \varphi^+_{\mathcal{M}}]}, [\mathcal{N}^-, \mathcal{N}^+].e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}}, \varphi^+_{\mathcal{N}}]} \right), \mathfrak{k}_1 = \left(\left[\mathcal{M}^-_1, \mathcal{M}^+_1\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}_1}, \varphi^+_{\mathcal{M}_1}]}, \left[\mathcal{N}^-_1, \mathcal{N}^+_1\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}_1}, \varphi^+_{\mathcal{N}_1}]} \right)$ *and* $\mathfrak{k}_2 = \left(\left[\mathcal{M}^-_2, \mathcal{M}^+_2\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}_2}, \varphi^+_{\mathcal{M}_2}]}, \left[\mathcal{N}^-_2, \mathcal{N}^+_2\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}_2}, \varphi^+_{\mathcal{N}_2}]} \right)$, *stated the CIVq-ROFNs, we defined several algebraic laws, such that*

$$\mathfrak{k}^c = \left(\left[\mathcal{N}^-, \mathcal{N}^+\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}}, \varphi^+_{\mathcal{N}}]}, \left[\mathcal{M}^-, \mathcal{M}^+\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}}, \varphi^+_{\mathcal{M}}]} \right) \tag{2}$$

$$\mathfrak{k}_1 \vee \mathfrak{k}_2 = \begin{pmatrix} \left[max\left(\mathcal{M}^-_1, \mathcal{M}^-_2\right), max\left(\mathcal{M}^+_1, \mathcal{M}^+_2\right)\right].e^{\varsigma 2\pi.[max(\varphi^-_{\mathcal{M}_1}(\eta), \varphi^-_{\mathcal{M}_2}(\eta)), max(\varphi^+_{\mathcal{M}_1}(\eta), \varphi^+_{\mathcal{M}_2}(\eta))]}, \\ \left[min\left(\mathcal{N}^-_1, \mathcal{N}^-_2\right), min\left(\mathcal{N}^+_1, \mathcal{N}^+_2\right)\right].e^{\varsigma 2\pi.[min(\varphi^-_{\mathcal{N}_1}(\eta), \varphi^-_{\mathcal{N}_2}(\eta)), min(\varphi^+_{\mathcal{N}_1}(\eta), \varphi^+_{\mathcal{N}_2}(\eta))]} \end{pmatrix} \tag{3}$$

$$\mathfrak{k}_1 \wedge \mathfrak{k}_2 = \begin{pmatrix} \left[min\left(\mathcal{M}^-_1, \mathcal{M}^-_2\right), min\left(\mathcal{M}^+_1, \mathcal{M}^+_2\right)\right].e^{\varsigma 2\pi.[min(\varphi^-_{\mathcal{M}_1}(\eta), \varphi^-_{\mathcal{M}_2}(\eta)), min(\varphi^+_{\mathcal{M}_1}(\eta), \varphi^+_{\mathcal{M}_2}(\eta))]}, \\ \left[max\left(\mathcal{N}^-_1, \mathcal{N}^-_2\right), max\left(\mathcal{N}^+_1, \mathcal{N}^+_2\right)\right].e^{\varsigma 2\pi.[max(\varphi^-_{\mathcal{N}_1}(\eta), \varphi^-_{\mathcal{N}_2}(\eta)), max(\varphi^+_{\mathcal{N}_1}(\eta), \varphi^+_{\mathcal{N}_2}(\eta))]} \end{pmatrix} \tag{4}$$

$$\mathfrak{k}_1 \oplus \mathfrak{k}_2 = \begin{pmatrix} \left[\left(\mathcal{M}^{-\mathfrak{D}}_1 + \mathcal{M}^{-\mathfrak{D}}_2 - \mathcal{M}^{-\mathfrak{D}}_1 \mathcal{M}^{-\mathfrak{D}}_2\right)^{\frac{1}{\mathfrak{D}}}, \left(\mathcal{M}^{+\mathfrak{D}}_1 + \mathcal{M}^{+\mathfrak{D}}_2 - \mathcal{M}^{+\mathfrak{D}}_1 \mathcal{M}^{+\mathfrak{D}}_2\right)^{\frac{1}{\mathfrak{D}}} \right]. \\ e^{\varsigma 2\pi.[(\varphi^{-\mathfrak{D}}_{\mathcal{M}_1} + \varphi^{-\mathfrak{D}}_{\mathcal{M}_2} - \varphi^{-\mathfrak{D}}_{\mathcal{M}_1} \varphi^{-\mathfrak{D}}_{\mathcal{M}_2})^{\frac{1}{\mathfrak{D}}}, (\varphi^{+\mathfrak{D}}_{\mathcal{M}_1} + \varphi^{+\mathfrak{D}}_{\mathcal{M}_2} - \varphi^{+\mathfrak{D}}_{\mathcal{M}_1} \varphi^{+\mathfrak{D}}_{\mathcal{M}_2})^{\frac{1}{\mathfrak{D}}}]}, \\ \left[\left(\mathcal{N}^-_1 \mathcal{N}^-_2\right), \left(\mathcal{N}^+_1 \mathcal{N}^+_2\right) \right].e^{\varsigma 2\pi.[\varphi^-_{\mathcal{N}_1} \varphi^-_{\mathcal{N}_2}, \varphi^+_{\mathcal{N}_1} \varphi^+_{\mathcal{N}_2}]} \end{pmatrix} \tag{5}$$

$$\mathfrak{k}_1 \otimes \mathfrak{k}_2 = \begin{pmatrix} \left[\left(\mathcal{M}^-_1 \mathcal{M}^-_2\right), \left(\mathcal{M}^+_1 \mathcal{M}^+_2\right) \right].e^{\varsigma 2\pi.[\varphi^-_{\mathcal{M}_1} \varphi^-_{\mathcal{M}_2}, \varphi^+_{\mathcal{M}_1} \varphi^+_{\mathcal{M}_2}]}, \\ \left[\left(\mathcal{N}^{-\mathfrak{D}}_1 + \mathcal{N}^{-\mathfrak{D}}_2 - \mathcal{N}^{-\mathfrak{D}}_1 \mathcal{N}^{-\mathfrak{D}}_2\right)^{\frac{1}{\mathfrak{D}}}, \left(\mathcal{N}^{+\mathfrak{D}}_1 + \mathcal{N}^{+\mathfrak{D}}_2 - \mathcal{N}^{+\mathfrak{D}}_1 \mathcal{N}^{+\mathfrak{D}}_2\right)^{\frac{1}{\mathfrak{D}}} \right]. \\ e^{\varsigma 2\pi.[(\varphi^{-\mathfrak{D}}_{\mathcal{N}_1} + \varphi^{-\mathfrak{D}}_{\mathcal{N}_2} - \varphi^{-\mathfrak{D}}_{\mathcal{N}_1} \varphi^{-\mathfrak{D}}_{\mathcal{N}_2})^{\frac{1}{\mathfrak{D}}}, (\varphi^{+\mathfrak{D}}_{\mathcal{N}_1} + \varphi^{+\mathfrak{D}}_{\mathcal{N}_2} - \varphi^{+\mathfrak{D}}_{\mathcal{N}_1} \varphi^{+\mathfrak{D}}_{\mathcal{N}_2})^{\frac{1}{\mathfrak{D}}}]} \end{pmatrix} \tag{6}$$

$$\gamma \mathfrak{k} = \begin{pmatrix} \left[\left(1 - \left(1 - \mathcal{M}^{-\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(1 - \mathcal{M}^{+\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}} \right].e^{\varsigma 2\pi[(1-(1-\varphi^{-\ \mathfrak{D}}_{\mathcal{M}})^{\gamma})^{\frac{1}{\mathfrak{D}}}, (1-(1-\varphi^{+\ \mathfrak{D}}_{\mathcal{M}})^{\gamma})^{\frac{1}{\mathfrak{D}}}]}, \\ [\mathcal{N}^{-\gamma}, \mathcal{N}^{+\gamma}].e^{\varsigma 2\pi[\varphi^{-\gamma}_{\ \mathcal{N}}, \varphi^{+\gamma}_{\ \mathcal{N}}]} \end{pmatrix} \tag{7}$$

$$\mathfrak{k}^{\gamma} = \begin{pmatrix} [\mathcal{M}^{-\gamma}, \mathcal{M}^{+\gamma}].e^{\varsigma 2\pi[\varphi^{-\gamma}_{\ \mathcal{M}}, \varphi^{+\gamma}_{\ \mathcal{M}}]}, \\ \left[\left(1 - \left(1 - \mathcal{N}^{-\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(1 - \mathcal{N}^{+\mathfrak{D}}\right)^{\gamma}\right)^{\frac{1}{\mathfrak{D}}} \right].e^{\varsigma 2\pi[(1-(1-\varphi^{-\ \mathfrak{D}}_{\mathcal{N}})^{\gamma})^{\frac{1}{\mathfrak{D}}}, (1-(1-\varphi^{+\ \mathfrak{D}}_{\mathcal{N}})^{\gamma})^{\frac{1}{\mathfrak{D}}}]} \end{pmatrix} \tag{8}$$

It is still very awkward, how we find the relation among any two CIVq-ROF numbers, for this, we recall two new definitions.

Definition 2 ([33]). *Suppose* $\mathfrak{k} = \left(\left[\mathcal{M}^-_E, \mathcal{M}^+_E\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}_E}, \varphi^+_{\mathcal{M}_E}]}, \left[\mathcal{N}^-_E, \mathcal{N}^+_E\right].e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}_E}, \varphi^+_{\mathcal{N}_E}]} \right)$, *stated the CIVq-ROFN. The SV and AV are particularized by:*

$$\dot{\mathrm{S}}(\mathfrak{k}) = \frac{1}{4} \left| \mathcal{M}^{-\mathfrak{D}}_E - \mathcal{N}^{-\mathfrak{D}}_E + \varphi^{-\mathfrak{D}}_{\mathcal{M}_E} - \varphi^{-\mathfrak{D}}_{\mathcal{N}_E} + \mathcal{M}^{+\mathfrak{D}}_E - \mathcal{N}^{+\mathfrak{D}}_E + \varphi^{+\mathfrak{D}}_{\mathcal{M}_E} - \varphi^{+\mathfrak{D}}_{\mathcal{N}_E} \right|, \dot{\mathrm{S}}(\mathfrak{k}) \in [0, 1] \tag{9}$$

$$H(\mathfrak{k}) = \frac{1}{4} \left| \mathcal{M}^{-\mathfrak{D}}_E + \mathcal{N}^{-\mathfrak{D}}_E + \varphi^{-\mathfrak{D}}_{\mathcal{M}_E} + \varphi^{-\mathfrak{D}}_{\mathcal{N}_E} + \mathcal{M}^{+\mathfrak{D}}_E + \mathcal{N}^{+\mathfrak{D}}_E + \varphi^{+\mathfrak{D}}_{\mathcal{M}_E} + \varphi^{+\mathfrak{D}}_{\mathcal{N}_E} \right|, H(\mathfrak{k}) \in [0, 1] \tag{10}$$

Then, by using Equations (9) and (10), we obtained:

1. *When* $\dot{\check{S}}(\mathfrak{k}_1) \prec \dot{\check{S}}(\mathfrak{k}_2)$, *then* $\mathfrak{k}_1 \prec \mathfrak{k}_2$;
2. *When* $\dot{\check{S}}(\mathfrak{k}_1) = \dot{\check{S}}(\mathfrak{k}_2)$, *then*
   (1) *When* $H(\mathfrak{k}_1) \prec H(\mathfrak{k}_2)$, *then* $\mathfrak{k}_1 \prec \mathfrak{k}_2$;
   (2) *When* $H(\mathfrak{k}_1) = H(\mathfrak{k}_2)$, *then* $\mathfrak{k}_1 = \mathfrak{k}_2$.

Definition 3 ([41]). *A HM operator is particularized by:*

$$\mathrm{HM}^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu) = \frac{\sum_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(\prod_{\varrho=1}^{\eta} \mathfrak{k}_{\varsigma_\varrho} \right)^{\frac{1}{\eta}}}{C_\mu^\eta} \tag{11}$$

where $\eta = 1, 2, \ldots, \mu, \varsigma_1, \varsigma_2, \ldots, \varsigma_\eta$, *stated positive integers, in the form of the parameter of* $\{1, 2, \ldots, \mu\}$ *of* μ *positive integers with* $C_\mu^\eta = \frac{\mu!}{\eta!(\mu-\eta)!}$, *stated the binomial coefficient (BC).*

3. Hamy Mean Operators under CIVq-ROFNs

HM operator is a feasible tool to handle strategic decision-making dilemmas because it can capture the order between the finite input terms. Additionally, the CIVq-ROF setting is a broadly flexible and massively dominant technique to operate problematic and awkward data in actual life dilemmas. Motivated by these tools, the technique of CIVq-ROFHM operator and Cq-ROFWHM operator with some well-known results are deliberated. Keeping the advantages of the parameters in HM operators, we discussed the specific cases of the invented operators. The group of every CIVq-ROFNs, stated by $\mathfrak{k}_\varsigma = \left(\left[\mathcal{M}_\varsigma^-, \mathcal{M}_\varsigma^+ \right] e^{\varsigma 2\pi [\varphi_{\mathcal{M}_\varsigma}^-, \varphi_{\mathcal{M}_\varsigma}^+]}, \left[\mathcal{N}_\varsigma^-, \mathcal{N}_\varsigma^+ \right] e^{\varsigma 2\pi [\varphi_{\mathcal{N}_\varsigma}^-, \varphi_{\mathcal{N}_\varsigma}^+]} \right)$, $(\varsigma = 1, 2, \ldots, \mu)$, with weight vector $\tilde{\omega} = \left(\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n \right)^T$, $\tilde{\omega}_\varsigma \in [0, 1]$, $\sum_{\varsigma=1}^{n} \tilde{\omega}_\varsigma = 1$.

Definition 4. *The CIVq-ROFHM operator is particularized by:*

$$CIVq - ROFHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu) = \frac{\oplus_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(\otimes_{\varrho=1}^{\eta} \left(\mathfrak{k}_{\varsigma_\varrho} \right) \right)^{\frac{1}{\eta}}}{C_\mu^\eta} \tag{12}$$

where $\eta = 1, 2, \ldots, \mu, \varsigma_1, \varsigma_2, \ldots, \varsigma_\eta$, *stated positive integers, in the form of the parameter of* $\{1, 2, \ldots, \mu\}$ *of* μ *positive integers with* $C_\mu^\eta = \frac{\mu!}{\eta!(\mu-\eta)!}$ *, stated the BC.*

Theorem 1. *To obtain the mathematical form of Equation (12), we use the Def. (1), such that:*

$$\begin{aligned} & CIVq - ROF\mathrm{HM}^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu) = \frac{\oplus_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(\otimes_{\varrho=1}^{\eta} \left(\mathfrak{k}_{\varsigma_\varrho} \right) \right)^{\frac{1}{\eta}}}{C_\mu^\eta} \\ & = \left(\begin{array}{c} \left[\left(1 - \left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\left(\prod_{\varrho=1}^{\eta} \mathcal{M}_{\varsigma_\varrho}^- \right)^{\frac{1}{\eta}} \right)^{\mathfrak{D}} \right) \right)^{\frac{1}{C_\mu^\eta}} \right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\left(\prod_{\varrho=1}^{\eta} \mathcal{M}_{\varsigma_\varrho}^+ \right)^{\frac{1}{\eta}} \right)^{\mathfrak{D}} \right) \right)^{\frac{1}{C_\mu^\eta}} \right)^{\frac{1}{\mathfrak{D}}} \right] \\ e^{\varsigma 2\pi \left[\left(1 - \left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\left(\prod_{\varrho=1}^{\eta} \varphi_{\mathcal{M}_{\varsigma_\varrho}}^- \right)^{\frac{1}{\eta}} \right)^{\mathfrak{D}} \right) \right)^{\frac{1}{C_\mu^\eta}} \right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\left(\prod_{\varrho=1}^{\eta} \varphi_{\mathcal{M}_{\varsigma_\varrho}}^+ \right)^{\frac{1}{\eta}} \right)^{\mathfrak{D}} \right) \right)^{\frac{1}{C_\mu^\eta}} \right)^{\frac{1}{\mathfrak{D}}} \right]}, \\ \left[\left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\prod_{\varrho=1}^{\eta} \left(1 - \left(\mathcal{N}_{\varsigma_\varrho}^- \right)^{\mathfrak{D}} \right) \right)^{\frac{1}{\eta}} \right)^{\frac{1}{\mathfrak{D}}} \right)^{\frac{1}{C_\mu^\eta}}, \left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\prod_{\varrho=1}^{\eta} \left(1 - \left(\mathcal{N}_{\varsigma_\varrho}^+ \right)^{\mathfrak{D}} \right) \right)^{\frac{1}{\eta}} \right)^{\frac{1}{\mathfrak{D}}} \right)^{\frac{1}{C_\mu^\eta}} \right] \\ e^{\varsigma 2\pi \left[\left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\prod_{\varrho=1}^{\eta} \left(1 - \left(\varphi_{\mathcal{N}_{\varsigma_\varrho}}^- \right)^{\mathfrak{D}} \right) \right)^{\frac{1}{\eta}} \right)^{\frac{1}{\mathfrak{D}}} \right)^{\frac{1}{C_\mu^\eta}}, \left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\prod_{\varrho=1}^{\eta} \left(1 - \left(\varphi_{\mathcal{N}_{\varsigma_\varrho}}^+ \right)^{\mathfrak{D}} \right) \right)^{\frac{1}{\eta}} \right)^{\frac{1}{\mathfrak{D}}} \right)^{\frac{1}{C_\mu^\eta}} \right]} \end{array} \right) \end{aligned} \tag{13}$$

Proof of this Theorem 1 is given in Appendix A. Using invented works, we expose idempotency, monotonicity, and boundedness.

Proposition 1 (Idempotency). *When* $\mathfrak{k} = \mathfrak{k}_\varsigma$, *then:*

$$CIVq - ROFHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu) = \mathfrak{k} \tag{14}$$

Proof of this Property 1 is given in Appendix B.

Proposition 2 (Monotonicity). *When* $\mathcal{M}^-_{\varsigma\varrho} \succcurlyeq \mathcal{M}''^-_{\varsigma\varrho}, \varphi^-_{\mathcal{M}_{\varsigma\varrho}} \succcurlyeq \varphi^-_{\mathcal{M}''_{\varsigma\varrho}}, \mathcal{M}^+_{\varsigma\varrho} \succcurlyeq \mathcal{M}''^+_{\varsigma\varrho}$, $\varphi^+_{\mathcal{M}_{\varsigma\varrho}} \succcurlyeq \varphi^+_{\mathcal{M}''_{\varsigma\varrho}}$ *and* $\mathcal{N}^-_{\varsigma\varrho} \preccurlyeq \mathcal{N}''^-_{\varsigma\varrho}, \varphi^-_{\mathcal{N}_{\varsigma\varrho}} \preccurlyeq \varphi^-_{\mathcal{N}''_{\varsigma\varrho}}, \mathcal{N}^+_{\varsigma\varrho} \preccurlyeq \mathcal{N}''^+_{\varsigma\varrho}, \varphi^+_{\mathcal{N}_{\varsigma\varrho}} \preccurlyeq \varphi^+_{\mathcal{N}''_{\varsigma\varrho}}$, *then:*

$$CIVq - ROFHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu) \succcurlyeq CIVq - ROFHM^{(\eta)}(\mathfrak{k}''_1, \mathfrak{k}''_2, \ldots, \mathfrak{k}''_\mu) \tag{15}$$

Proof of this Property 2 is given in Appendix C.

Proposition 3 (Boundedness). *When* $\mathfrak{k}^+ = \left(\begin{array}{c} [\mathcal{M}^{\mp}_{max}(\eta), \mathcal{M}^{++}_{max}(\eta)] e^{\varsigma 2\pi [\varphi^-_{\mathcal{M}^+_{max}}(\eta), \varphi^+_{\mathcal{M}^+_{max}}(\eta)]}, \\ [\mathcal{N}^{-+}_{min}(\eta), \mathcal{N}^{++}_{min}(\eta)] e^{\varsigma 2\pi [\varphi^-_{\mathcal{N}^+_{min}}(\eta), \varphi^+_{\mathcal{N}^+_{min}}(\eta)]} \end{array} \right)$,
$(\varsigma = 1, 2, \ldots, \mu)$ *and* $\mathfrak{k}^- = \left(\begin{array}{c} [\mathcal{M}^{--}_{min}(\eta), \mathcal{M}^{\mp}_{min}(\eta)] e^{\varsigma 2\pi [\varphi^-_{\mathcal{M}^-_{min}}(\eta), \varphi^+_{\mathcal{M}^-_{min}}(\eta)]}, \\ [\mathcal{N}^{--}_{max}(\eta), \mathcal{N}^{\mp}_{max}(\eta)] e^{\varsigma 2\pi [\varphi^-_{\mathcal{N}^-_{max}}(\eta), \varphi^+_{\mathcal{N}^-_{max}}(\eta)]} \end{array} \right)$, $(\varsigma = 1, 2, \ldots, \mu)$, *then*:

$$\mathfrak{k}^- \prec CIVq - ROFHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu) \prec \mathfrak{k}^+ \ \ (16) \tag{16}$$

Proof of this Property 3 is given in Appendix D.

Proposition 4 (Commutativity). *Assume* $\mathfrak{k}_\varsigma = \left(\left[\mathcal{M}^-_{\varsigma\varrho}(\eta), \mathcal{M}^+_{\varsigma\varrho}(\eta) \right] e^{\varsigma 2\pi [\varphi^-_{\mathcal{M}_{\varsigma\varrho}}(\eta), \varphi^+_{\mathcal{M}_{\varsigma\varrho}}(\eta)]}, \right.$
$\left. \left[\mathcal{N}^-_{\varsigma\varrho}(\eta), \mathcal{N}^+_{\varsigma\varrho}(\eta) \right] e^{\varsigma 2\pi [\varphi^-_{\mathcal{N}_{\varsigma\varrho}}(\eta), \varphi^+_{\mathcal{N}_{\varsigma\varrho}}(\eta)]} \right)$, $(\varsigma = 1, 2, \ldots, \mu)$ *and* $\mathfrak{k}''_\varsigma = \left(\left[\mathcal{M}''^-_{\varsigma\varrho}(\eta), \mathcal{M}''^+_{\varsigma\varrho}(\eta) \right] \right.$
$\left. e^{\varsigma 2\pi [\varphi^-_{\mathcal{M}''_{\varsigma\varrho}}(\eta), \varphi^+_{\mathcal{M}''_{\varsigma\varrho}}(\eta)]}, \left[\mathcal{N}''^-_{\varsigma\varrho}(\eta), \mathcal{N}''^+_{\varsigma\varrho}(\eta) \right] e^{\varsigma 2\pi [\varphi^-_{\mathcal{N}''_{\varsigma\varrho}}(\eta), \varphi^+_{\mathcal{N}''_{\varsigma\varrho}}(\eta)]} \right)$,
$(\varsigma = 1, 2, \ldots, \mu)$, *if* $(\mathfrak{k}''_1, \mathfrak{k}''_2, \ldots, \mathfrak{k}''_\mu)$ *is any permutation of* $(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu)$, *then:*

$$CIVq - ROFHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu) = CIVq - ROFHM^{(\eta)}(\mathfrak{k}''_1, \mathfrak{k}''_2, \ldots, \mathfrak{k}''_\mu) \tag{17}$$

Proof of this Property 4 is given in Appendix E. In the presence of the η, we diagnose various specific cases.

Case 1: When $\eta = 1$ in Cq-ROFHM operator, then we get:

$$CIVq-ROFHM^{(1)}\left(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu\right)$$

$$=\left(\begin{array}{c}\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\mathcal{M}^{-\mathfrak{O}}_{\varsigma}\right)\right)^{\frac{1}{\mu}}\right)^{\frac{1}{\mathfrak{O}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\mathcal{M}^{+\mathfrak{O}}_{\varsigma}\right)\right)^{\frac{1}{\mu}}\right)^{\frac{1}{\mathfrak{O}}}\right]\\ e^{\varsigma 2\pi[(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-\varphi^{-\mathfrak{O}}_{\mathcal{M}_\varsigma}))^{\frac{1}{\mu}})^{\frac{1}{\mathfrak{O}}},(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-\varphi^{+\mathfrak{O}}_{\mathcal{M}_\varsigma}))^{\frac{1}{\mu}})^{\frac{1}{\mathfrak{O}}}]},\\ \left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(1-\left(\mathcal{N}^-_\varsigma\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\mathfrak{O}}}\right)^{\frac{1}{\mu}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(1-\left(\mathcal{N}^+_\varsigma\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\mathfrak{O}}}\right)^{\frac{1}{\mu}}\right]\\ e^{\varsigma 2\pi[(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-(1-(\varphi^-_{\mathcal{N}_\varsigma})^{\mathfrak{O}}))^{\frac{1}{\mathfrak{O}}})^{\frac{1}{\mu}},(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-(1-(\varphi^+_{\mathcal{N}_\varsigma})^{\mathfrak{O}}))^{\frac{1}{\mathfrak{O}}})^{\frac{1}{\mu}}]}\end{array}\right)$$

$$=\frac{1}{\mu}\oplus^{\mu}_{\varsigma}\mathfrak{k}_\varsigma$$

Case 2: When $\eta=\mu$ in Cq-ROFHM operator, then we get:

$$CIVq-ROFHM^{(1)}\left(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu\right)$$

$$=\left(\begin{array}{c}\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^-_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{C^\eta_\mu}}\right)^{\frac{1}{\mathfrak{O}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^+_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{C^\eta_\mu}}\right)^{\frac{1}{\mathfrak{O}}}\right]\\ e^{\varsigma 2\pi[(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi^-_{\mathcal{M}_{\varsigma_\varrho}})^{\frac{1}{\eta}})^{\mathfrak{O}}))^{\frac{1}{C^\eta_\mu}})^{\frac{1}{\mathfrak{O}}},(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi^+_{\mathcal{M}_{\varsigma_\varrho}})^{\frac{1}{\eta}})^{\mathfrak{O}}))^{\frac{1}{C^\eta_\mu}})^{\frac{1}{\mathfrak{O}}}]},\\ \left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^-_{\varsigma_\varrho}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{\frac{1}{C^\eta_\mu}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^+_{\varsigma_\varrho}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{\frac{1}{C^\eta_\mu}}\right]\\ e^{\varsigma 2\pi[(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi^-_{\mathcal{N}_{\varsigma_\varrho}})^{\mathfrak{O}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{O}}})^{\frac{1}{C^\eta_\mu}},(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi^+_{\mathcal{N}_{\varsigma_\varrho}})^{\mathfrak{O}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{O}}})^{\frac{1}{C^\eta_\mu}}]}\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\left(\prod_{\varsigma=1}^{\mu}\mathcal{M}^-_\varsigma\right)^{\frac{1}{\mu}},\left(\prod_{\varsigma=1}^{\mu}\mathcal{M}^+_\varsigma\right)^{\frac{1}{\mu}}\right]e^{\varsigma 2\pi[(\prod_{\varsigma=1}^{\mu}\varphi^-_{\mathcal{M}_\varsigma})^{\frac{1}{\mu}},(\prod_{\varsigma=1}^{\mu}\varphi^+_{\mathcal{M}_\varsigma})^{\frac{1}{\mu}}]},\\ \left[\left(1-\left(\prod_{\varsigma=1}^{\mu}\left(1-\left(\mathcal{N}^-_\varsigma\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\mu}}\right)^{\frac{1}{\mathfrak{O}}},\left(1-\left(\prod_{\varsigma=1}^{\mu}\left(1-\left(\mathcal{N}^+_\varsigma\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\mu}}\right)^{\frac{1}{\mathfrak{O}}}\right]\\ e^{\varsigma 2\pi[(1-(\prod_{\varsigma=1}^{\mu}(1-(\varphi^-_{\mathcal{N}_\varsigma})^{\mathfrak{O}}))^{\frac{1}{\mu}})^{\frac{1}{\mathfrak{O}}},(1-(\prod_{\varsigma=1}^{\mu}(1-(\varphi^+_{\mathcal{N}_\varsigma})^{\mathfrak{O}}))^{\frac{1}{\mu}})^{\frac{1}{\mathfrak{O}}}]}\end{array}\right)=\frac{1}{\mu}\oplus^{\mu}_{\varsigma}\mathfrak{k}_\varsigma$$

Definition 5. *The CIVq-ROFWHM operator is particularized by:*

$$CIVq-ROFWHM^{(\eta)}\left(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu\right)=\begin{cases}\dfrac{\oplus_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}}{C^\eta_\mu} & 1\preccurlyeq\eta\prec\mu\\ \otimes_{\varsigma=1}^{\eta}\mathfrak{k}_\varsigma^{\frac{1-\tilde{\omega}_\varsigma}{\mu-1}} & \eta=\mu\end{cases}\tag{18}$$

where $\eta=1,2,\ldots,\mu,\varsigma_1,\varsigma_2,\ldots,\varsigma_\eta$, *stated positive integers, in the form of the parameter of* $\{1,2,\ldots,\mu\}$ *of* μ *positive integers with* $C^\eta_\mu=\frac{\mu!}{\eta!(\mu-\eta)!}$ *, stated the BC with weight vectors* $\sum_{\varrho=1}^{\eta}\tilde{\omega}_\varrho=1,\tilde{\omega}_\varrho\in[0,1],\varrho=1,2,3,\ldots,n.$

Theorem 2. *Suggested in the occurrence of Equation (18), we get:*

$$
\begin{aligned}
&CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu)=\frac{\oplus_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)\left(\otimes_{\varrho=1}^{\eta}(\mathfrak{k}_{\varsigma_\varrho})\right)^{\frac{1}{\eta}}}{C_\mu^\eta}\\
&=\left(\begin{array}{c}
\left[\begin{array}{c}
\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^-_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}},\\
\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^+_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}}
\end{array}\right]\\
e^{\varsigma 2\pi\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^-_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^+_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right]},\\
\left[\begin{array}{c}
\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^-_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_\mu^\eta}},\\
\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^+_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_\mu^\eta}}
\end{array}\right]\\
e^{\varsigma 2\pi\left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^-_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_\mu^\eta}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^+_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_\mu^\eta}}\right]}
\end{array}\right)
\end{aligned}
\tag{19}
$$

Proof of this Theorem 2 is given in Appendix F.

Proposition 5 (Idempotency). *When $\mathfrak{k}=\mathfrak{k}_\varsigma$, then:*

$$CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu)=\mathfrak{k} \tag{20}$$

Proof of this Property 5 is given in Appendix G.

Proposition 6 (Monotonicity). *Suggested $\mathfrak{k}_\varsigma=\left(\left[\mathcal{M}^-_{\varsigma\varrho}(\eta),\mathcal{M}^+_{\varsigma\varrho}(\eta)\right]e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}_{\varsigma\varrho}}(\eta),\varphi^+_{\mathcal{M}_{\varsigma\varrho}}(\eta)]},\left[\mathcal{N}^-_{\varsigma\varrho}(\eta),\mathcal{N}^+_{\varsigma\varrho}(\eta)\right]e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}_{\varsigma\varrho}}(\eta),\varphi^+_{\mathcal{N}_{\varsigma\varrho}}(\eta)]}\right),(\varsigma=1,2,\ldots,\mu)$ and $\mathfrak{k}''_\varsigma=\left(\left[\mathcal{M}''^-_{\varsigma\varrho}(\eta),\mathcal{M}''^+_{\varsigma\varrho}(\eta)\right]e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}''_{\varsigma\varrho}}(\eta),\varphi^+_{\mathcal{M}''_{\varsigma\varrho}}(\eta)]},\left[\mathcal{N}''^-_{\varsigma\varrho}(\eta),\mathcal{N}''^+_{\varsigma\varrho}(\eta)\right]e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}''_{\varsigma\varrho}}(\eta),\varphi^+_{\mathcal{N}''_{\varsigma\varrho}}(\eta)]}\right),(\varsigma=1,2,\ldots,\mu)$, if $\mathcal{M}^-_{\varsigma\varrho}\succcurlyeq\mathcal{M}''^-_{\varsigma\varrho},\varphi^-_{\mathcal{M}_{\varsigma\varrho}}\succcurlyeq\varphi^-_{\mathcal{M}''_{\varsigma\varrho}},\mathcal{M}^+_{\varsigma\varrho}\succcurlyeq\mathcal{M}''^+_{\varsigma\varrho},\varphi^+_{\mathcal{M}_{\varsigma\varrho}}\succcurlyeq\varphi^+_{\mathcal{M}''_{\varsigma\varrho}}$ and $\mathcal{N}^-_{\varsigma\varrho}\preccurlyeq\mathcal{N}''^-_{\varsigma\varrho},\varphi^-_{\mathcal{N}_{\varsigma\varrho}}\preccurlyeq\varphi^-_{\mathcal{N}''_{\varsigma\varrho}},\mathcal{N}^+_{\varsigma\varrho}\preccurlyeq\mathcal{N}''^+_{\varsigma\varrho},\varphi^+_{\mathcal{N}_{\varsigma\varrho}}\preccurlyeq\varphi^+_{\mathcal{N}''_{\varsigma\varrho}}$, then:*

$$CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu)\succcurlyeq CIVq-ROFWHM^{(\eta)}(\mathfrak{k}''_1,\mathfrak{k}''_2,\ldots,\mathfrak{k}''_\mu) \tag{21}$$

Proof of this Property 6 is given in Appendix H.

Proposition 7 (Boundedness). *When* $\mathfrak{k}^{+} = \begin{pmatrix} [\mathcal{M}^{\mp}_{max}(\eta), \mathcal{M}^{++}_{max}(\eta)]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{M}^{+}_{max}}(\eta), \varphi^{+}_{\mathcal{M}^{+}_{max}}(\eta)]}, \\ [\mathcal{N}^{-+}_{min}(\eta), \mathcal{N}^{++}_{min}(\eta)]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{N}^{+}_{min}}(\eta), \varphi^{+}_{\mathcal{N}^{+}_{min}}(\eta)]} \end{pmatrix}$, $(\varsigma = 1, 2, \ldots, \mu)$ *and* $\mathfrak{k}^{-} = \begin{pmatrix} [\mathcal{M}^{--}_{min}(\eta), \mathcal{M}^{\mp}_{min}(\eta)]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{M}^{-}_{min}}(\eta), \varphi^{+}_{\mathcal{M}^{-}_{min}}(\eta)]}, \\ [\mathcal{N}^{--}_{max}(\eta), \mathcal{N}^{\mp}_{max}(\eta)]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{N}^{-}_{max}}(\eta), \varphi^{+}_{\mathcal{N}^{-}_{max}}(\eta)]} \end{pmatrix}$, $(\varsigma = 1, 2, \ldots, \mu)$, *then*:

$$\mathfrak{k}^{-} \prec CIVq - ROFWHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_{\mu}) \prec \mathfrak{k}^{+} \tag{22}$$

Proof of this Property 7 is given in Appendix I.

Proposition 8 (Commutativity). *Assumed* $\mathfrak{k}_{\varsigma} = \Big(\Big[\mathcal{M}^{-}_{\varsigma\varrho}(\eta), \mathcal{M}^{+}_{\varsigma\varrho}(\eta)\Big]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{M}\varsigma\varrho}(\eta), \varphi^{+}_{\mathcal{M}\varsigma\varrho}(\eta)]}, \Big[\mathcal{N}^{-}_{\varsigma\varrho}(\eta), \mathcal{N}^{+}_{\varsigma\varrho}(\eta)\Big]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{N}\varsigma\varrho}(\eta), \varphi^{+}_{\mathcal{N}\varsigma\varrho}(\eta)]}\Big)$, $(\varsigma = 1, 2, \ldots, \mu)$ *and* $\mathfrak{k}''_{\varsigma} = \Big(\Big[\mathcal{M}''^{-}_{\varsigma\varrho}(\eta), \mathcal{M}''^{+}_{\varsigma\varrho}(\eta)\Big] e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{M}''_{\varsigma\varrho}}(\eta), \varphi^{+}_{\mathcal{M}''_{\varsigma\varrho}}(\eta)]}, \Big[\mathcal{N}''^{-}_{\varsigma\varrho}(\eta), \mathcal{N}''^{+}_{\varsigma\varrho}(\eta)\Big]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{N}''_{\varsigma\varrho}}(\eta), \varphi^{+}_{\mathcal{N}''_{\varsigma\varrho}}(\eta)]}\Big)$, $(\varsigma = 1, 2, \ldots, \mu)$, *if* $(\mathfrak{k}''_1, \mathfrak{k}''_2, \ldots, \mathfrak{k}''_{\mu})$ *is any permutation of* $(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_{\mu})$, *then:*

$$CIVq - ROFWHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_{\mu}) = CIVq - ROFWHM^{(\eta)}(\mathfrak{k}''_1, \mathfrak{k}''_2, \ldots, \mathfrak{k}''_{\mu}) \tag{23}$$

Proof of this Property 8 is given in Appendix J: Yet, we find some specific cases, using η.

Case 1: When $\eta = 1$ in Cq-ROFWHM operator, then we get:

$$CIVq - ROFWHM^{(1)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_{\mu})$$

$$= \begin{pmatrix} \left[\left(1 - \left(\prod_{1 \preccurlyeq \varsigma_1 \preccurlyeq \mu}\left(1 - \mathcal{M}^{-\mathfrak{D}}_{\varsigma}\right)^{1-\tilde{\omega}_{\varsigma}}\right)^{\frac{1}{\mu-1}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(\prod_{1 \preccurlyeq \varsigma_1 \preccurlyeq \mu}\left(1 - \mathcal{M}^{+\mathfrak{D}}_{\varsigma}\right)^{1-\tilde{\omega}_{\varsigma}}\right)^{\frac{1}{\mu-1}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\varsigma 2\pi[(1-(\prod_{1 \preccurlyeq \varsigma_1 \preccurlyeq \mu}(1-\varphi^{-\mathfrak{D}}_{\mathcal{M}_{\varsigma}})^{1-\tilde{\omega}_{\varsigma}})^{\frac{1}{\mu-1}})^{\frac{1}{\mathfrak{D}}}, (1-(\prod_{1 \preccurlyeq \varsigma_1 \preccurlyeq \mu}(1-\varphi^{+\mathfrak{D}}_{\mathcal{M}_{\varsigma}})^{1-\tilde{\omega}_{\varsigma}})^{\frac{1}{\mu-1}})^{\frac{1}{\mathfrak{D}}}]}, \\ \left[\left(\prod_{1 \preccurlyeq \varsigma_1 \preccurlyeq \mu}\left(\mathcal{N}^{-}_{\varsigma}\right)^{1-\tilde{\omega}_{\varsigma}}\right)^{\frac{1}{\mu-1}}, \left(\prod_{1 \preccurlyeq \varsigma_1 \preccurlyeq \mu}\left(\mathcal{N}^{+}_{\varsigma}\right)^{1-\tilde{\omega}_{\varsigma}}\right)^{\frac{1}{\mu-1}}\right] \\ e^{\varsigma 2\pi[(\prod_{1 \preccurlyeq \varsigma_1 \preccurlyeq \mu}(\varphi^{-}_{\mathcal{N}_{\varsigma}})^{1-\tilde{\omega}_{\varsigma}})^{\frac{1}{\mu-1}}, (\prod_{1 \preccurlyeq \varsigma_1 \preccurlyeq \mu}(\varphi^{+}_{\mathcal{N}_{\varsigma}})^{1-\tilde{\omega}_{\varsigma}})^{\frac{1}{\mu-1}}]} \end{pmatrix}$$

Case 2: When $\eta = \mu$ in Cq-ROFWHM operator, then we get:

$$CIVq - ROFWHM^{(1)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_{\mu})$$

$$= \otimes_{\varsigma=1}^{\eta} \mathfrak{k}_{\varsigma}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}} = \begin{pmatrix} \left[\prod_{\varsigma=1}^{\mu} \mathcal{M}^{-\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}_{\varsigma}, \prod_{\varsigma=1}^{\mu} \mathcal{M}^{+\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}_{\varsigma}\right] e^{\varsigma 2\pi([\prod_{\varsigma=1}^{\mu} \varphi^{-\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}_{\mathcal{M}_{\varsigma}}, \prod_{\varsigma=1}^{\mu} \varphi^{+\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}_{\mathcal{M}_{\varsigma}}])}, \\ \left[\left(1 - \prod_{\varsigma=1}^{\mu}\left(1 - \left(\mathcal{N}^{-}_{\varsigma}\right)^{\mathfrak{D}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \prod_{\varsigma=1}^{\mu}\left(1 - \left(\mathcal{N}^{+}_{\varsigma}\right)^{\mathfrak{D}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\varsigma 2\pi[(1-\prod_{\varsigma=1}^{\mu}(1-(\varphi^{-}_{\mathcal{N}_{\varsigma}})^{\mathfrak{D}})^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}})^{\frac{1}{\mathfrak{D}}}, (1-\prod_{\varsigma=1}^{\mu}(1-(\varphi^{+}_{\mathcal{N}_{\varsigma}})^{\mathfrak{D}})^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}})^{\frac{1}{\mathfrak{D}}}]} \end{pmatrix}$$

4. Application (Brief Discussion of the MADM Technique)

MADM scenario refers to a tool considered to obtain a complete and feasible assessment of decisions in the availability of many co-attributes' alternatives. As research has become more advanced, a various number of decision-making techniques have occurred and are widely employed in economics, engineering, pattern recognition, and network systems. One of the most common dilemmas, given in the form of ambiguity and inconsistency, has involved every field of life and affected the resultant values during the decision-making process. To reduce the affected ratio of data from ambiguity, in this scenario, we deliberated the novel procedure of the MADM tool in the occurrence of the invented approaches. The MADM technique, obtained from the decision-making process which can help experts, is used to determine the beneficial required optimal.

All experts have an idea about MADM, which is a beneficial and dominant procedure for determining the optimal form of the family of alternatives. MADM tool includes various steps, used for finding the beneficial optimal. For this, we have needed the family of alternatives and their attributes such as $X = \{\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_m\}$ and $C = \{C_1, C_2, \ldots, C_n\}$. We have also needed some weights which have been provided by an expert in the form of their opinion, stated by: $\tilde{\omega} = \left\{\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n\right\}^T, \tilde{\omega}_\varsigma \in [0,1]$, and satisfied the tool: $\sum_{\varsigma=1}^{n} \tilde{\omega}_\varsigma = 1$. To employ the above mathematical tools in the region of MADM technique, we needed to make a matrix whose every term is in the form of CIVq-ROF numbers e.g., $\mathfrak{k}_\varsigma = \left(\left[\mathcal{M}_\varsigma^-, \mathcal{M}_\varsigma^+\right].e^{\varsigma 2\pi[\varphi_{\mathcal{M}_\varsigma}^-, \varphi_{\mathcal{M}_\varsigma}^+]}, \left[\mathcal{N}_\varsigma^-, \mathcal{N}_\varsigma^+\right] e^{\varsigma 2\pi[\varphi_{\mathcal{N}_\varsigma}^-, \varphi_{\mathcal{N}_\varsigma}^+]}\right)$, $(\varsigma = 1, 2, \ldots, n)$. By following the below procedure, we obtain our theme.

To resolve ambiguity and complications which are involved in every day-to-day life, we constructed some procedures to obtain our major analysis, and the stages are diagnosed here:

Stage 1: To evaluate the decision-making procedure, we needed to compute a new decision-making matrix by including various alternatives and their attributes. The constructed matrix includes information in the form of CIVq-ROF number provided by experts given in matrix, whose mathematical shape: $M = \left(\mathfrak{k}_{\varsigma j}\right)$.

Stage 2: Further, to resolve the information given in stage one, we try to aggregate it with the help of the CIVq-ROFWHM operator into a single set, called the CIVq-ROF number. This procedure will help us find the score values of the CIVq-ROF numbers.

Stage 3: This time, we can easily determine the score value of the information obtained from using the CIVq-ROFWHM operator. With the help of this technique, we can easily rank the values for finding the beneficial optimal.

Stage 4: After following a long procedure, we can finally find the beneficial optimal.

Using the well-known procedure of the MADM technique, we suggest some practical data and try to resolve it with the help of invented operators to enhance the quality of the proposed work. Moreover, in the availability of the above study, we explained some examples and tried to employ them in the region of the MADM technique.

Example 1. *This study intrudes the rationality of the invented MADM tool via a case whose important results are moreover compared with various existing drawbacks. The main aspect of the dilemma is as follows:*

The main theme of this example is to resolve some practical life issues. Here, we have considered the problem that occurred in the State Bank of Pakistan (SBP) related to some personal issues. The main owner of SBP wants to buy petrol and diesel industrial cars with air conditions for the consideration of SBP local and non-local branches. For this, the main panel of SBP gives this project to some companies for finding the best way. SBP is a well-known and important financial procedure or way for providing for the government, having its main office in Islamabad. SBP welcomed proposals for buying petrol and diesel industrial cars with air conditions for the consideration of SBP local and non-local branches. The engineer should be recommended by the main office of Pakistan, Government of Pakistan/state/any other public organization and must know about the provision of vehicles for the last few years. The consideration committee members have considered some cars based on the following data, expressing the attributes:

1. C_1: Comfortability;
2. C_2: Maximum speed;
3. C_3: Price;
4. C_4: Maximum payload.

For this, the expert has visited different sorts of car enterprises, which stated in the form of five alternatives $\mathfrak{k}_\varsigma, i = 1,2,3,4,5$, whose attributes are given above. For the above four attributes, the experts have given their decision in the form: 0.3, 0.3, 0.3, and 0.1, which expressed the weight vector.

To resolving ambiguity and complications which are involved in every day-to-day life, we constructed some procedures to make our major analysis, and the stages are diagnosed here:

Stage 1: Arranging the preference to evaluate the decision-making procedure, we needed to compute a new decision-making matrix by including various alternatives and their attributes. The constructed matrix includes information in the form of CIVq-ROF number provided by experts given in matrix, whose mathematical shape: $M = \left(\mathfrak{k}_{\varsigma j}\right)$, described in Table 1.

Table 1. Complex interval-valued intuitionistic fuzzy data.

	C_1	C_2
$\mathfrak{k}_1$	$\left([0.2, 0.3]e^{\varsigma 2\pi[0.3,0.4]}, [0.1, 0.2].e^{\varsigma 2\pi[0.2,0.3]}\right)$	$\left([0.21, 0.31]e^{\varsigma 2\pi[0.31,0.41]}, [0.11, 0.21].e^{\varsigma 2\pi[0.21,0.31]}\right)$
$\mathfrak{k}_2$	$\left([0.3, 0.4]e^{\varsigma 2\pi[0.1,0.5]}, [0.2, 0.3].e^{\varsigma 2\pi[0.1,0.2]}\right)$	$\left([0.31, 0.41]e^{\varsigma 2\pi[0.11,0.51]}, [0.21, 0.31].e^{\varsigma 2\pi[0.11,0.21]}\right)$
$\mathfrak{k}_3$	$\left([0.3, 0.5]e^{\varsigma 2\pi[0.1,0.6]}, [0.3, 0.4].e^{\varsigma 2\pi[0.1,0.2]}\right)$	$\left([0.31, 0.51]e^{\varsigma 2\pi[0.11,0.61]}, [0.31, 0.41].e^{\varsigma 2\pi[0.11,0.21]}\right)$
$\mathfrak{k}_4$	$\left([0.4, 0.6]e^{\varsigma 2\pi[0.2,0.3]}, [0.2, 0.3].e^{\varsigma 2\pi[0.2,0.3]}\right)$	$\left([0.41, 0.61]e^{\varsigma 2\pi[0.21,0.31]}, [0.21, 0.31].e^{\varsigma 2\pi[0.21,0.31]}\right)$
$\mathfrak{k}_5$	$\left([0.5, 0.7]e^{\varsigma 2\pi[0.6,0.8]}, [0.1, 0.2].e^{\varsigma 2\pi[0.1,0.4]}\right)$	$\left([0.51, 0.71]e^{\varsigma 2\pi[0.61,0.81]}, [0.11, 0.21].e^{\varsigma 2\pi[0.11,0.41]}\right)$
	C_3	C_4
$\mathfrak{k}_1$	$\left([0.22, 0.32]e^{\varsigma 2\pi[0.32,0.42]}, [0.12, 0.22].e^{\varsigma 2\pi[0.22,0.32]}\right)$	$\left([0.23, 0.33]e^{\varsigma 2\pi[0.33,0.43]}, [0.13, 0.23].e^{\varsigma 2\pi[0.23,0.33]}\right)$
$\mathfrak{k}_2$	$\left([0.32, 0.42]e^{\varsigma 2\pi[0.12,0.52]}, [0.22, 0.32].e^{\varsigma 2\pi[0.12,0.22]}\right)$	$\left([0.33, 0.43]e^{\varsigma 2\pi[0.13,0.53]}, [0.23, 0.33].e^{\varsigma 2\pi[0.13,0.23]}\right)$
$\mathfrak{k}_3$	$\left([0.32, 0.52]e^{\varsigma 2\pi[0.12,0.62]}, [0.32, 0.42].e^{\varsigma 2\pi[0.12,0.22]}\right)$	$\left([0.33, 0.53]e^{\varsigma 2\pi[0.13,0.63]}, [0.33, 0.43].e^{\varsigma 2\pi[0.13,0.23]}\right)$
$\mathfrak{k}_4$	$\left([0.42, 0.62]e^{\varsigma 2\pi[0.22,0.32]}, [0.22, 0.32].e^{\varsigma 2\pi[0.22,0.32]}\right)$	$\left([0.43, 0.63]e^{\varsigma 2\pi[0.23,0.33]}, [0.23, 0.33].e^{\varsigma 2\pi[0.23,0.33]}\right)$
$\mathfrak{k}_5$	$\left([0.52, 0.72]e^{\varsigma 2\pi[0.62,0.82]}, [0.12, 0.22].e^{\varsigma 2\pi[0.12,0.42]}\right)$	$\left([0.53, 0.73]e^{\varsigma 2\pi[0.63,0.83]}, [0.13, 0.23].e^{\varsigma 2\pi[0.13,0.43]}\right)$

Stage 2: Further, to resolve the information given in stage one, we try to aggregate it with the help of the CIVq-ROFWHM operator into a single set, called the CIVq-ROF number. This procedure will help us find the score values of the CIVq-ROF numbers, stated in Table 2.

Stage 3: This time, we can easily determine the score value of the information obtained from using the CIVq-ROFWHM operator. With the help of this technique, we can easily rank the values to find the beneficial optimal, stated in Table 3.

Table 2. Aggregated numbers from Table 1 for $\mathfrak{O} = 3$.

	CIVq-ROFWHM Operator
$\mathfrak{k}_1$	$\left([0.02898, 0.04314]e^{\varsigma 2\pi[0.04314,0.05962]}, [0.42312, 0.52179].e^{\varsigma 2\pi[0.52179,0.59563]}\right)$
$\mathfrak{k}_2$	$\left([0.04314, 0.05962]e^{\varsigma 2\pi[0.01648,0.07928]}, [0.52179, 0.59563].e^{\varsigma 2\pi[0.42312,0.52179]}\right)$
$\mathfrak{k}_3$	$\left([0.04314, 0.07928]e^{\varsigma 2\pi[0.01648,0.10352]}, [0.59563, 0.65677].e^{\varsigma 2\pi[0.42312,0.52179]}\right)$
$\mathfrak{k}_4$	$\left([0.05962, 0.10352]e^{\varsigma 2\pi[0.02898,0.04314]}, [0.52179, 0.59563].e^{\varsigma 2\pi[0.52179,0.59563]}\right)$
$\mathfrak{k}_5$	$\left([0.07928, 0.13495]e^{\varsigma 2\pi[0.10352,0.17962]}, [0.42312, 0.52179].e^{\varsigma 2\pi[0.42312,0.65677]}\right)$

Table 3. Score values from Table 2.

	Score Values
$\mathfrak{k}_1$	0.1427
$\mathfrak{k}_2$	0.1426
$\mathfrak{k}_3$	0.17768
$\mathfrak{k}_4$	0.17634
$\mathfrak{k}_5$	0.14175

Stage 4: After following a long procedure, we can finally find the beneficial optimal. Thus, we rank the alternatives in order to diagnose the beneficial required optimal.

$$\mathfrak{k}_3 \geq \mathfrak{k}_4 \geq \mathfrak{k}_1 \geq \mathfrak{k}_2 \geq \mathfrak{k}_5$$

Clearly, that we obtain the beneficial optimal in the form of $\mathfrak{k}_3$.

One of the most important questions is, what happens if we choose the CIVPFS? To resolving ambiguity and complications which are involved in every day-to-day life, we constructed some procedures to make our major analysis, and the stages are diagnosed here:

Stage 1: Arranging the preference to evaluate the decision-making procedure, we needed to compute a new decision-making matrix by including various alternatives and their attributes. The constructed matrix includes information in the form of CIVq-ROF number provided by experts given in matrix, whose mathematical shape: $M = \left(\mathfrak{k}_{\varsigma j}\right)$, described in Table 4.

Stage 2: Further, to resolve the information given in stage one, we try to aggregate it with the help of the CIVq-ROFWHM operator into a single set, called the CIVq-ROF number. This procedure will help us find the score values of the CIVq-ROF numbers, stated in Table 5.

Stage 3: This time, we can easily determine the score value of the information obtained from using the CIVq-ROFWHM operator. With the help of this technique, we can easily rank the values to find the beneficial optimal, stated in Table 6.

Stage 4: After following a long procedure, we can finally find the beneficial optimal. Thus, we rank the alternatives to diagnose the beneficial required optimal.

$$\mathfrak{k}_4 \geq \mathfrak{k}_3 \geq \mathfrak{k}_2 \geq \mathfrak{k}_1 \geq \mathfrak{k}_5$$

It is clear that we obtain the beneficial optimal in the form of $\mathfrak{k}_4$.

Table 4. Complex interval-valued Pythagorean fuzzy data.

	C_1	C_2
$\mathfrak{k}_1$	$\left([0.2, 0.8]e^{\varsigma 2\pi[0.3,0.7]}, [0.1, 0.2].e^{\varsigma 2\pi[0.2,0.3]}\right)$	$\left([0.21, 0.81]e^{\varsigma 2\pi[0.31,0.71]}, [0.11, 0.21].e^{\varsigma 2\pi[0.21,0.31]}\right)$
$\mathfrak{k}_2$	$\left([0.3, 0.7]e^{\varsigma 2\pi[0.1,0.8]}, [0.2, 0.3].e^{\varsigma 2\pi[0.1,0.2]}\right)$	$\left([0.31, 0.71]e^{\varsigma 2\pi[0.11,0.81]}, [0.21, 0.31].e^{\varsigma 2\pi[0.11,0.21]}\right)$
$\mathfrak{k}_3$	$\left([0.3, 0.6]e^{\varsigma 2\pi[0.1,0.8]}, [0.3, 0.4].e^{\varsigma 2\pi[0.1,0.2]}\right)$	$\left([0.31, 0.61]e^{\varsigma 2\pi[0.11,0.81]}, [0.31, 0.41].e^{\varsigma 2\pi[0.11,0.21]}\right)$
$\mathfrak{k}_4$	$\left([0.4, 0.6]e^{\varsigma 2\pi[0.2,0.3]}, [0.3, 0.5].e^{\varsigma 2\pi[0.2,0.3]}\right)$	$\left([0.41, 0.61]e^{\varsigma 2\pi[0.21,0.31]}, [0.31, 0.51].e^{\varsigma 2\pi[0.21,0.31]}\right)$
$\mathfrak{k}_5$	$\left([0.5, 0.7]e^{\varsigma 2\pi[0.6,0.8]}, [0.1, 0.2].e^{\varsigma 2\pi[0.1,0.4]}\right)$	$\left([0.51, 0.71]e^{\varsigma 2\pi[0.61,0.81]}, [0.11, 0.21].e^{\varsigma 2\pi[0.11,0.41]}\right)$
	C_3	C_4
$\mathfrak{k}_1$	$\left([0.22, 0.82]e^{\varsigma 2\pi[0.32,0.72]}, [0.12, 0.22].e^{\varsigma 2\pi[0.22,0.32]}\right)$	$\left([0.23, 0.83]e^{\varsigma 2\pi[0.33,0.73]}, [0.13, 0.23].e^{\varsigma 2\pi[0.23,0.33]}\right)$
$\mathfrak{k}_2$	$\left([0.32, 0.72]e^{\varsigma 2\pi[0.12,0.82]}, [0.22, 0.32].e^{\varsigma 2\pi[0.12,0.22]}\right)$	$\left([0.33, 0.73]e^{\varsigma 2\pi[0.13,0.83]}, [0.23, 0.33].e^{\varsigma 2\pi[0.13,0.23]}\right)$
$\mathfrak{k}_3$	$\left([0.32, 0.62]e^{\varsigma 2\pi[0.12,0.82]}, [0.32, 0.42].e^{\varsigma 2\pi[0.12,0.22]}\right)$	$\left([0.33, 0.63]e^{\varsigma 2\pi[0.13,0.83]}, [0.33, 0.43].e^{\varsigma 2\pi[0.13,0.23]}\right)$
$\mathfrak{k}_4$	$\left([0.42, 0.62]e^{\varsigma 2\pi[0.22,0.32]}, [0.32, 0.52].e^{\varsigma 2\pi[0.22,0.32]}\right)$	$\left([0.43, 0.63]e^{\varsigma 2\pi[0.23,0.33]}, [0.33, 0.53].e^{\varsigma 2\pi[0.23,0.33]}\right)$
$\mathfrak{k}_5$	$\left([0.52, 0.72]e^{\varsigma 2\pi[0.62,0.82]}, [0.12, 0.22].e^{\varsigma 2\pi[0.12,0.42]}\right)$	$\left([0.53, 0.73]e^{\varsigma 2\pi[0.63,0.83]}, [0.13, 0.23].e^{\varsigma 2\pi[0.13,0.43]}\right)$

Table 5. Aggregated numbers from Table 4 for $\mathfrak{O} = 2$.

	CIVq-ROFWHM Operator
$\mathfrak{k}_1$	$\left([0.07442, 0.30444]e^{\varsigma 2\pi[0.10099,0.243605]}, [0.39937, 0.49336]e^{\varsigma 2\pi[0.49336,0.56448]}\right)$
$\mathfrak{k}_2$	$\left([0.10099, 0.243605]e^{\varsigma 2\pi[0.04831,0.30444]}, [0.49336, 0.56448]e^{\varsigma 2\pi[0.39937,0.49336]}\right)$
$\mathfrak{k}_3$	$\left([0.10099, 0.19829]e^{\varsigma 2\pi[0.04831,0.30444]}, [0.56448, 0.62423]e^{\varsigma 2\pi[0.39937,0.49336]}\right)$
$\mathfrak{k}_4$	$\left([0.12951, 0.19829]e^{\varsigma 2\pi[0.07442,0.10099]}, [0.56448, 0.67754]e^{\varsigma 2\pi[0.49336,0.56448]}\right)$
$\mathfrak{k}_5$	$\left([0.16133, 0.24360]e^{\varsigma 2\pi[0.19828,0.30444]}, [0.39937, 0.493362]e^{\varsigma 2\pi[0.39937,0.62423]}\right)$

Table 6. Score values from Table 5.

	Score Values
$\mathfrak{k}_1$	0.19929
$\mathfrak{k}_2$	0.20009
$\mathfrak{k}_3$	0.24167
$\mathfrak{k}_4$	0.31698
$\mathfrak{k}_5$	0.18367

Again, what happens if we choose the CIVq-ROFS? To resolving ambiguity and complications which are involved in every day-to-day life, we constructed some procedures to make our major analysis, and the stages are diagnosed here:

Stage 1: Arranging the preference to evaluate the decision-making procedure, we needed to compute a new decision-making matrix by including various alternatives and their attributes. The constructed matrix includes information in the form of CIVq-ROF number provided by experts given in matrix, whose mathematical shape: $M = \left(\mathfrak{k}_{\varsigma j}\right)$, described in Table 7.

Table 7. Complex interval-valued q-rung orthopair fuzzy data.

	C_1	C_2
$\mathfrak{k}_1$	$\left([0.2, 0.8]e^{\varsigma 2\pi[0.3,0.7]}, [0.1, 0.7].e^{\varsigma 2\pi[0.2,0.3]}\right)$	$\left([0.21, 0.81]e^{\varsigma 2\pi[0.31,0.71]}, [0.11, 0.71].e^{\varsigma 2\pi[0.21,0.31]}\right)$
$\mathfrak{k}_2$	$\left([0.3, 0.9]e^{\varsigma 2\pi[0.1,0.8]}, [0.2, 0.8].e^{\varsigma 2\pi[0.1,0.2]}\right)$	$\left([0.31, 0.91]e^{\varsigma 2\pi[0.11,0.81]}, [0.21, 0.81].e^{\varsigma 2\pi[0.11,0.21]}\right)$
$\mathfrak{k}_3$	$\left([0.3, 0.8]e^{\varsigma 2\pi[0.1,0.8]}, [0.3, 0.6].e^{\varsigma 2\pi[0.1,0.2]}\right)$	$\left([0.31, 0.81]e^{\varsigma 2\pi[0.11,0.81]}, [0.31, 0.61].e^{\varsigma 2\pi[0.11,0.21]}\right)$
$\mathfrak{k}_4$	$\left([0.4, 0.9]e^{\varsigma 2\pi[0.2,0.3]}, [0.3, 0.9].e^{\varsigma 2\pi[0.2,0.3]}\right)$	$\left([0.41, 0.91]e^{\varsigma 2\pi[0.21,0.31]}, [0.31, 0.91].e^{\varsigma 2\pi[0.21,0.31]}\right)$
$\mathfrak{k}_5$	$\left([0.5, 0.7]e^{\varsigma 2\pi[0.6,0.8]}, [0.1, 0.7].e^{\varsigma 2\pi[0.1,0.4]}\right)$	$\left([0.51, 0.71]e^{\varsigma 2\pi[0.61,0.81]}, [0.11, 0.71].e^{\varsigma 2\pi[0.11,0.41]}\right)$
	C_3	C_4
$\mathfrak{k}_1$	$\left([0.22, 0.82]e^{\varsigma 2\pi[0.32,0.72]}, [0.12, 0.72].e^{\varsigma 2\pi[0.22,0.32]}\right)$	$\left([0.23, 0.83]e^{\varsigma 2\pi[0.33,0.73]}, [0.13, 0.73].e^{\varsigma 2\pi[0.23,0.33]}\right)$
$\mathfrak{k}_2$	$\left([0.32, 0.92]e^{\varsigma 2\pi[0.12,0.82]}, [0.22, 0.82].e^{\varsigma 2\pi[0.12,0.22]}\right)$	$\left([0.33, 0.93]e^{\varsigma 2\pi[0.13,0.83]}, [0.23, 0.83].e^{\varsigma 2\pi[0.13,0.23]}\right)$
$\mathfrak{k}_3$	$\left([0.32, 0.82]e^{\varsigma 2\pi[0.12,0.82]}, [0.32, 0.62].e^{\varsigma 2\pi[0.12,0.22]}\right)$	$\left([0.33, 0.83]e^{\varsigma 2\pi[0.13,0.83]}, [0.33, 0.63].e^{\varsigma 2\pi[0.13,0.23]}\right)$
$\mathfrak{k}_4$	$\left([0.42, 0.92]e^{\varsigma 2\pi[0.22,0.32]}, [0.32, 0.92].e^{\varsigma 2\pi[0.22,0.32]}\right)$	$\left([0.43, 0.93]e^{\varsigma 2\pi[0.23,0.33]}, [0.33, 0.93].e^{\varsigma 2\pi[0.23,0.33]}\right)$
$\mathfrak{k}_5$	$\left([0.52, 0.72]e^{\varsigma 2\pi[0.62,0.82]}, [0.12, 0.72].e^{\varsigma 2\pi[0.12,0.42]}\right)$	$\left([0.53, 0.73]e^{\varsigma 2\pi[0.63,0.83]}, [0.13, 0.73].e^{\varsigma 2\pi[0.13,0.43]}\right)$

Stage 2: Further, to resolve the information given in stage one, we try to aggregate it with the help of the CIVq-ROFWHM operator into a single set, called the CIVq-ROF number. This procedure will help us find the score values of the CIVq-ROF numbers, stated in Table 8.

Table 8. Aggregated values from Table 7 for $\mathfrak{D} = 8$.

	CIVq-ROFWHM Operator
$\mathfrak{k}_1$	$\left([0.00126, 0.0376]e^{\varsigma 2\pi[0.00277,0.02284]}, [0.4556, 0.8542]e^{\varsigma 2\pi[0.5612,0.6399]}\right)$
$\mathfrak{k}_2$	$\left([0.00277, 0.06927]e^{\varsigma 2\pi[0.00039,0.0376]}, [0.5612, 0.8967]e^{\varsigma 2\pi[0.4556,0.5612]}\right)$
$\mathfrak{k}_3$	$\left([0.00277, 0.0376]e^{\varsigma 2\pi[0.00039,0.0376]}, [0.6399, 0.8092]e^{\varsigma 2\pi[0.4556,0.5612]}\right)$
$\mathfrak{k}_4$	$\left([0.00513, 0.06927]e^{\varsigma 2\pi[0.00126,0.00277]}, [0.6399, 0.9403]e^{\varsigma 2\pi[0.5612,0.6399]}\right)$
$\mathfrak{k}_5$	$\left([0.00872, 0.02284]e^{\varsigma 2\pi[0.0142,0.0376]}, [0.4556, 0.8542]e^{\varsigma 2\pi[0.4556,0.7045]}\right)$

Stage 3: This time, we can easily determine the score value of the information obtained from using the CIVq-ROFWHM operator. With the help of this technique, we can easily rank the values to find the beneficial optimal, stated in Table 9.

Table 9. Score values from Table 8.

	Score Values
$\mathfrak{k}_1$	0.08083
$\mathfrak{k}_2$	0.10987
$\mathfrak{k}_3$	0.05596
$\mathfrak{k}_4$	0.1693
$\mathfrak{k}_5$	0.08697

Stage 4: After following a long procedure, we can finally find the beneficial optimal. Thus, we rank the alternatives to diagnose the beneficial required optimal.

$$\mathfrak{k}_4 \geq \mathfrak{k}_2 \geq \mathfrak{k}_5 \geq \mathfrak{k}_1 \geq \mathfrak{k}_3$$

It is clear that we obtain the beneficial optimal in the form of $\mathfrak{k}_4$.

The main problem with existing data is that, if experts provide data in the form of CIVIFS, then the prevailing CIVPFS and proposed the idea of CIVq-ROF information are easily resolved. However, if an expert gives information, for example in the shape of CIVPF numbers, then the prevailing CIVIFS has been unsuitable, but the proposed CIVq-ROF information can easily evaluate it. Finally, the main advantage of the proposed work is that, if an expert gives information in the shape of CIvq-ROF information, the existing theories such as CIVIFS and CIVPFSs have failed. Therefore, the invented work is more powerful than the existing work. To explain how the proposed work is more powerful is briefly discussed below.

Furthermore, in the light of expressing the efficiency and competence of the invented approach, the discissions based on the exposed method are compared here with existing CIVIFSs, CIVPFSs, and CIVq-ROFS studies. Before carrying out comparative analysis, we develop the list of the existing theories and then identify the best optimal by using the simplest ways. Some existing theories are as follows: Garg [11] invented the accuracy value for IVPFSs, Joshi et al. [17] diagnosed the IVq-ROFSs, Garg and Rani [27] exposed the aggregation operators for CIVIFSs, Ali et al. [28] initiated the Einstein geometric aggregation operators for CIVPFSs, and Garg et al. [36] invented the Muirhead mean operator for CIVq-ROFSs, and exposed operators. Table 10 includes sensitive analysis, using data in Table 1.

Table 10. Some diagnosed existing works using data in Table 1.

Methods	Score Values	Ranking Values
Garg [11]	Not Justified	Not Justified
Joshi et al. [17]	Not Justified	Not Justified
Garg and Rani [27]	$0.09724, 0.08673, 0.09081, 0.09501, 0.1049$	$\mathfrak{k}_5 \geq \mathfrak{k}_1 \geq \mathfrak{k}_4 \geq \mathfrak{k}_3 \geq \mathfrak{k}_2$
Ali et al. [28]	$0.23056, 0.2274, 0.2583, 0.2631, 0.1837$	$\mathfrak{k}_4 \geq \mathfrak{k}_3 \geq \mathfrak{k}_1 \geq \mathfrak{k}_2 \geq \mathfrak{k}_5$
Garg et al. [36]	$0.3416, 0.3385, 0.3694, 0.3742, 0.2948$	$\mathfrak{k}_4 \geq \mathfrak{k}_3 \geq \mathfrak{k}_1 \geq \mathfrak{k}_2 \geq \mathfrak{k}_5$
CIVq-ROFWHM operator	$0.1427, 0.1426, 0.1776, 0.1763, 0.1417$	$\mathfrak{k}_3 \geq \mathfrak{k}_4 \geq \mathfrak{k}_1 \geq \mathfrak{k}_2 \geq \mathfrak{k}_5$

We know that, in Table 1, we suggested information in the shape of CIVIFS and the proposed work based on CIVq-ROF information easily resolved it. Thus, only the information given in [11,17] are not able to cope with it, because they cannot deal with information which contains two-dimensional information in the shape of singleton sets. However, the theories given in [27,28,36] are easily resolved. From Table 10, we obtain the most useful and dominant results in the form of $\mathfrak{k}_3$, $\mathfrak{k}_4$, and $\mathfrak{k}_5$. Table 11 includes sensitive analysis, using data in Table 4.

We know that, in Table 4, we suggested information in the shape of CIVPFS and the proposed work based on CIVq-ROF information easily resolved it. So, only the information given in [11,17] are not able to cope with it, because they cannot deal with information which contains two-dimensional information in the shape of singleton sets and the information given in [27] only deal with intuitionistic types of information, so it is also neglected. However, the theory given in [28,36] is easily resolved. From Table 11, we obtain the most useful and dominant result in the form of $\mathfrak{k}_4$. Table 12 includes sensitive analysis, using data in Table 5.

Table 11. Some diagnosed existing works using data in Table 4.

Methods	Score Values	Ranking Values
Garg [11]	Not Justified	Not Justified
Joshi et al. [17]	Not Justified	Not Justified
Garg and Rani [27]	Not Justified	Not Justified
Ali et al. [28]	$0.2882, 0.3001, 0.3306, 0.406,\ 0.2726$	$\mathfrak{k}_4 \geq \mathfrak{k}_3 \geq \mathfrak{k}_2 \geq \mathfrak{k}_1 \geq \mathfrak{k}_5$
Garg et al. [36]	$0.3771, 0.4112, 0.4315, 0.5151,\ 0.3617$	$\mathfrak{k}_4 \geq \mathfrak{k}_3 \geq \mathfrak{k}_2 \geq \mathfrak{k}_1 \geq \mathfrak{k}_5$
CIVq-ROFWHM operator	$0.1993, 0.2001, 0.2417, 0.317,\ 0.1837$	$\mathfrak{k}_4 \geq \mathfrak{k}_3 \geq \mathfrak{k}_2 \geq \mathfrak{k}_1 \geq \mathfrak{k}_5$

Table 12. Some diagnosed existing works, using data in Table 7.

Methods	Score Values	Ranking Values
Garg [11]	Not Justified	Not Justified
Joshi et al. [17]	Not Justified	Not Justified
Garg and Rani [27]	Not Justified	Not Justified
Ali et al. [28]	Not Justified	Not Justified
Garg et al. [36]	$0.17174, 0.20996, 0.14685, 0.2594, 0.18796$	$\mathfrak{k}_4 \geq \mathfrak{k}_2 \geq \mathfrak{k}_5 \geq \mathfrak{k}_1 \geq \mathfrak{k}_3$
CIVq-ROFWHM operator	$0.08083, 0.10987, 0.05596, 0.1693, 0.09697$	$\mathfrak{k}_4 \geq \mathfrak{k}_2 \geq \mathfrak{k}_5 \geq \mathfrak{k}_1 \geq \mathfrak{k}_3$

We know that, in Table 7, we suggested information in the shape of CIVq-ROFS and the proposed work based on CIVq-ROF information easily resolved it. So, only the information given in [11,17] are not able to cope with it, because they cannot deal with information which contains two-dimensional information in the shape of singleton sets and the information given in [27,28] only deal with intuitionistic and Pythagorean types of information, so they are also neglected. But the theory given in [36] is easily resolved. From Table 11, we obtain the most useful and dominant result in the form of $\mathfrak{k}_4$.

Information in Tables 11 and 12 is obtained based on the information given in Tables 1, 4 and 7. The information given in Table 1 contained CIVIFSs, Table 4 contained CIVPFSs, and the information contained in Table 7 in the form of CIVq-ROFSs. We already explained what the difference is between this information in the introduction section. In special cases, we mentioned that if we choose the value of parameter $\eta = \mu$, then the proposed work is converted to Ref. [36] which is the special case of the proposed work. The proposed work is more generalized than the extending work. Therefore, in the occurrence of the works cited above, we find that the invented work is a massive tool compared to existing works.

5. Conclusions

The major findings and objectives of this analysis are described below:

1. We know that the theory of CIVq-ROF information plays an important role in the fields of decision-making theory. The main theme of this theory is to diagnose the fundamental concept of the CIVq-ROFHM operator and Cq-ROFWHM operator and describe their well-known results and various properties.
2. Furthermore, in the consideration of the above-cited theory, we investigated the decision-making dilemmas based on CIVq-ROF information to determine the beneficial term from the finite group of alternatives with the help of evaluating several examples.
3. Finally, based on the evaluating examples, we try to discover the sensitive analysis and supremacy of the invented operators to find the flexibility and dominancy of the diagnosed approaches.

In the future, based on the existing theories such as spherical fuzzy sets [43], complex spherical fuzzy sets [44], and Linear Diophantine fuzzy sets [45], we aim to diagnose various important techniques in order to evaluate/resolve genuine life troubles and also try to utilize various aggregation operators, new methods, and many techniques to enhance the worth of the prevailing works.

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Appendix A

Proof of this Theorem 1. Using Equation (12), we have:

$$\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_{\varrho}}\right)=\left(\begin{array}{c}\left[\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_{\varrho}}^{-},\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_{\varrho}}^{+}\right].e^{\varsigma 2\pi([\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_{\varrho}}}^{-},\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_{\varrho}}}^{+}])},\\ \left[\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_{\varrho}}^{-}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_{\varrho}}^{+}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\varsigma 2\pi[(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_{\varrho}}}^{-})^{\mathfrak{D}})))^{\frac{1}{\mathfrak{D}}},(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_{\varrho}}}^{+})^{\mathfrak{D}})))^{\frac{1}{\mathfrak{D}}}]}\end{array}\right)$$

$$\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_{\varrho}}\right)\right)^{\frac{1}{\eta}}=\left(\begin{array}{c}\left[\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_{\varrho}}^{-}\right)^{\frac{1}{\eta}},\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_{\varrho}}^{+}\right)^{\frac{1}{\eta}}\right].e^{\varsigma 2\pi[(\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_{\varrho}}}^{-})^{\frac{1}{\eta}},(\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_{\varrho}}}^{+})^{\frac{1}{\eta}}]},\\ \left[\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_{\varrho}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_{\varrho}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\varsigma 2\pi[(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_{\varrho}}}^{-})^{\mathfrak{D}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{D}}},(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_{\varrho}}}^{+})^{\mathfrak{D}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{D}}}]}\end{array}\right)$$

$$\begin{array}{c}\oplus_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_{\varrho}}\right)\right)^{\frac{1}{\eta}}\\ =\left(\begin{array}{c}\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_{\varrho}}^{-}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_{\varrho}}^{+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)\right)^{\frac{1}{\mathfrak{D}}}\right]\\ e^{\varsigma 2\pi[(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_{\varrho}}}^{-})^{\frac{1}{\eta}})^{\mathfrak{D}})))^{\frac{1}{\mathfrak{D}}},(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_{\varrho}}}^{+})^{\frac{1}{\eta}})^{\mathfrak{D}})))^{\frac{1}{\mathfrak{D}}}]},\\ \left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_{\varrho}}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right),\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_{\varrho}}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)\right]\\ e^{\varsigma 2\pi[(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_{\varrho}}}^{-})^{\mathfrak{D}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{D}}}),(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_{\varrho}}}^{+})^{\mathfrak{D}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{D}}})]}\end{array}\right)\end{array}$$

Then:

$$CIVq-ROFHM^{(\eta)}(ꝁ_1,ꝁ_2,\ldots,ꝁ_\mu)=\frac{\oplus_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\otimes_{\varrho=1}^{\eta}\left(ꝁ_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}}{C^{\eta}_{\mu}}$$

$$=\left(\begin{array}{c}\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{-}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\varsigma 2\pi\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{-}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{+}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{-}_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{+}_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right] \\ e^{\varsigma 2\pi\left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{-}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{+}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right]}\end{array}\right)$$

□

Appendix B

Proof of this Property 1. Suggested $ꝁ=ꝁ_\varsigma$, then:

$$CIVq-ROFHM^{(\eta)}(ꝁ_1,ꝁ_2,\ldots,ꝁ_\mu)$$

$$=\left(\begin{array}{c}\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{-}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\varsigma 2\pi\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{-}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{+}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{-}_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{+}_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right] \\ e^{\varsigma 2\pi\left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{-}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{+}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right]}\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\mathcal{M}^{-\mathfrak{D}}_{\varsigma}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\mathcal{M}^{+\mathfrak{D}}_{\varsigma}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}}\right] \\ e^{\varsigma 2\pi\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\varphi^{-\mathfrak{D}}_{\mathcal{M}_{\varsigma}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\varphi^{+\mathfrak{D}}_{\mathcal{M}_{\varsigma}}\right)\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(1-\left(\mathcal{N}^{-}_{\varsigma}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(1-\left(\mathcal{N}^{+}_{\varsigma}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right] \\ e^{\varsigma 2\pi\left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(1-\varphi^{-\mathfrak{D}}_{\mathcal{N}_{\varsigma}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(1-\varphi^{+\mathfrak{D}}_{\mathcal{N}_{\varsigma}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right]}\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\left(1-\left(1-\mathcal{M}^{-\mathfrak{D}}_{\varsigma}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\mathcal{M}^{+\mathfrak{D}}_{\varsigma}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]e^{\varsigma 2\pi\left[\left(1-\left(1-\varphi^{-\mathfrak{D}}_{\mathcal{M}_{\varsigma}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\varphi^{+\mathfrak{D}}_{\mathcal{M}_{\varsigma}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\left(1-\left(1-\left(\mathcal{N}^{-}_{\varsigma}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\mathcal{N}^{+}_{\varsigma}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]e^{\varsigma 2\pi\left[\left(1-\left(1-\varphi^{-\mathfrak{D}}_{\mathcal{N}_{\varsigma}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\varphi^{+\mathfrak{D}}_{\mathcal{N}_{\varsigma}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\left(1-\left(1-\mathcal{M}^{-\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\mathcal{M}^{+\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]e^{\varsigma 2\pi\left[\left(1-\left(1-\varphi^{-\mathfrak{D}}_{\mathcal{M}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\varphi^{+\mathfrak{D}}_{\mathcal{M}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\left(1-\left(1-\left(\mathcal{N}^{-}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\left(\mathcal{N}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]e^{\varsigma 2\pi\left[\left(1-\left(1-\varphi^{-\mathfrak{D}}_{\mathcal{N}}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(1-\varphi^{+\mathfrak{D}}_{\mathcal{N}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]}\end{array}\right)$$

$$=\left([\mathcal{M}^{-}(\eta),\mathcal{M}^{+}(\eta)]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{M}}(\eta),\varphi^{+}_{\mathcal{M}}(\eta)]},[\mathcal{N}^{-}(\eta),\mathcal{N}^{+}(\eta)]e^{\varsigma 2\pi[\varphi^{-}_{\mathcal{N}}(\eta),\varphi^{+}_{\mathcal{N}}(\eta)]}\right)=ꝁ.$$

□

Appendix C

Proof of this Property 2. Suggested $\mathcal{M}_{\varsigma\varrho}^{-} \succcurlyeq \mathcal{M}_{\varsigma\varrho}^{\prime\prime-}, \varphi_{\mathcal{M}_{\varsigma\varrho}}^{-} \succcurlyeq \varphi_{\mathcal{M}_{\varsigma\varrho}^{\prime\prime}}^{-}, \mathcal{M}_{\varsigma\varrho}^{+} \succcurlyeq \mathcal{M}_{\varsigma\varrho}^{\prime\prime+}$, $\varphi_{\mathcal{M}_{\varsigma\varrho}}^{+} \succcurlyeq \varphi_{\mathcal{M}_{\varsigma\varrho}^{\prime\prime}}^{+}$ and $\mathcal{N}_{\varsigma\varrho}^{-} \preccurlyeq \mathcal{N}_{\varsigma\varrho}^{\prime\prime-}, \varphi_{\mathcal{N}_{\varsigma\varrho}}^{-} \preccurlyeq \varphi_{\mathcal{N}_{\varsigma\varrho}^{\prime\prime}}^{-}, \mathcal{N}_{\varsigma\varrho}^{+} \preccurlyeq \mathcal{N}_{\varsigma\varrho}^{\prime\prime+}, \varphi_{\mathcal{N}_{\varsigma\varrho}}^{+} \preccurlyeq \varphi_{\mathcal{N}_{\varsigma\varrho}^{\prime\prime}}^{+}$, implies that:

$$\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}} \succcurlyeq \left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{\prime\prime+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}} \Rightarrow 1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}} \preccurlyeq 1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{\prime\prime+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}$$

$$\Rightarrow \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_\mu^\eta}} \preccurlyeq \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{\prime\prime+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_\mu^\eta}}$$

$$\Rightarrow \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}} \succcurlyeq \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{\prime\prime+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}}$$

$$\Rightarrow \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}} .e^{\varsigma 2\pi(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_\varrho}}^{+})^{\frac{1}{\eta}})^{\mathfrak{D}}))^{\frac{1}{C_\mu^\eta}})^{\frac{1}{\mathfrak{D}}}}$$

$$\succcurlyeq \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_\varrho}^{\prime\prime+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}} .e^{\varsigma 2\pi(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_\varrho}^{\prime\prime}}^{+})^{\frac{1}{\eta}})^{\mathfrak{D}}))^{\frac{1}{C_\mu^\eta}})^{\frac{1}{\mathfrak{D}}}}$$

$$\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_\varrho}^{+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_\mu^\eta}} .e^{\varsigma 2\pi(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_\varrho}}^{+})^{\mathfrak{D}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{D}}})^{\frac{1}{C_\mu^\eta}}}$$

$$\preccurlyeq \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_\varrho}^{\prime\prime+}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\frac{1}{C_\mu^\eta}} .e^{\varsigma 2\pi(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_\varrho}^{\prime\prime}}^{+})^{\mathfrak{D}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{D}}})^{\frac{1}{C_\mu^\eta}}}$$

By using the SV, we obtained that:

(1) If $\dot{\check{S}}(\mathfrak{k}_\varsigma) > \dot{\check{S}}(\mathfrak{k}_\varsigma^{\prime\prime})$, then:

$$CIVq-ROF\mathrm{HM}^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu) \succcurlyeq CIVq-ROF\mathrm{HM}^{(\eta)}(\mathfrak{k}_1^{\prime\prime},\mathfrak{k}_2^{\prime\prime},\ldots,\mathfrak{k}_\mu^{\prime\prime})$$

(2) If $\dot{\check{S}}(\mathfrak{k}_\varsigma) = \dot{\check{S}}(\mathfrak{k}_\varsigma^{\prime\prime})$, then $H(\mathfrak{k}_\varsigma) = H(\mathfrak{k}_\varsigma^{\prime\prime})$, then:

$$CIVq-ROF\mathrm{HM}^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu) = CIVq-ROF\mathrm{HM}^{(\eta)}(\mathfrak{k}_1^{\prime\prime},\mathfrak{k}_2^{\prime\prime},\ldots,\mathfrak{k}_\mu^{\prime\prime})$$

□

Appendix D

Proof of this Property 3. Based on Proposition 1 and 2, then:

$$CIVq-ROF\mathrm{HM}^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu) \succcurlyeq CIVq-RF\mathrm{HM}^{(\eta)}\left(\mathfrak{k}_1^{-},\mathfrak{k}_2^{-},\ldots,\mathfrak{k}_\mu^{-}\right) = \mathfrak{k}^{-}$$

$$CIVq-ROF\mathrm{HM}^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu) \preccurlyeq CIVq-ROF\mathrm{HM}^{(\eta)}\left(\mathfrak{k}_1^{+},\mathfrak{k}_2^{+},\ldots,\mathfrak{k}_\mu^{+}\right) = \mathfrak{k}^{+}$$

Finally, we have:

$$\mathfrak{k}^{-} \prec CIVq - ROF\mathrm{HM}^{(\eta)}\left(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu\right) \prec \mathfrak{k}^{+}.$$

□

Appendix E

Proof of this Property 4. From above, we know that $\left(\mathfrak{k}''_1, \mathfrak{k}''_2, \ldots, \mathfrak{k}''_\mu\right)$ is a permutation of $\left(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu\right)$, then:

$$\frac{\bigoplus_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu}\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}}{C_\mu^\eta} = \frac{\bigoplus_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu}\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}''_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}}{C_\mu^\eta}$$

Then:

$$CIVq - ROF\mathrm{HM}^{(\eta)}\left(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu\right) = CIVq - ROF\mathrm{HM}^{(\eta)}\left(\mathfrak{k}''_1, \mathfrak{k}''_2, \ldots, \mathfrak{k}''_\mu\right) \;.$$

□

Appendix F

Proof of this Theorem 2. Considered the Def. (1), such that:

$$\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}} = \left(\begin{array}{c} \left[\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{-}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}, \left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right] e^{\varsigma 2\pi\left[\left(\prod_{\varrho=1}^{\eta}\varphi^{-}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}, \left(\prod_{\varrho=1}^{\eta}\varphi^{+}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right]}, \\ \left[\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{-}_{\varsigma_\varrho}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}, \left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{+}_{\varsigma_\varrho}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right] \\ e^{\varsigma 2\pi\left[\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{-}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}, \left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{+}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right]} \end{array}\right)$$

$$\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}$$

$$= \left(\begin{array}{c} \left[\left(1-\left(\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{-}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)\right)^{\frac{1}{\mathfrak{O}}}, \left(1-\left(\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)\right)^{\frac{1}{\mathfrak{O}}}\right] \\ e^{\varsigma 2\pi\left[\left(1-\left(\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{-}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)\right)^{\frac{1}{\mathfrak{O}}}, \left(1-\left(\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{+}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)\right)^{\frac{1}{\mathfrak{O}}}\right]}, \\ \left[\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{-}_{\varsigma_\varrho}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}, \left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{+}_{\varsigma_\varrho}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right] \\ e^{\varsigma 2\pi\left[\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{-}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}, \left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{+}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right]} \end{array}\right)$$

$$\oplus_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{-}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)\right)^{\frac{1}{\mathfrak{D}}}, \\ \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)\right)^{\frac{1}{\mathfrak{D}}} \end{array}\right] \\ e^{\varsigma 2\pi\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{-}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{+}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\begin{array}{c} \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{-}_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right), \\ \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{+}_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right) \end{array}\right] \\ e^{\varsigma 2\pi\left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{-}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right),\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{+}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)\right]} \end{array}\right)$$

$$CIVq-ROFWHM^{(\eta)}\left(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu\right)=\frac{\oplus_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}}{C_{\mu}^{\eta}}$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{-}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}}, \\ \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}} \end{array}\right] \\ e^{\varsigma 2\pi\left[\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{-}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}},\left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\varphi^{+}_{\mathcal{M}_{\varsigma_\varrho}}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)^{\frac{1}{C_\mu^\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right]}, \\ \left[\begin{array}{c} \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{-}_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)^{\frac{1}{C_\mu^\eta}}, \\ \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{+}_{\varsigma_\varrho}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)^{\frac{1}{C_\mu^\eta}} \end{array}\right] \\ e^{\varsigma 2\pi\left[\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{-}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)^{\frac{1}{C_\mu^\eta}},\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\varphi^{+}_{\mathcal{N}_{\varsigma_\varrho}}\right)^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)}\right)^{\frac{1}{C_\mu^\eta}}\right]} \end{array}\right)$$

For $\eta=\mu$, then:

$$\mathfrak{k}_{\varsigma}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}} = \begin{pmatrix} \left[\mathcal{M}^{-}{}_{\varsigma}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}, \mathcal{M}^{+}{}_{\varsigma}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right] e^{\varsigma 2\pi\left(\left[\varphi^{-}{}_{\mathcal{M}_{\varsigma}}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}, \varphi^{+}{}_{\mathcal{M}_{\varsigma}}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right]\right)}, \\ \left[\left(1-\left(1-\left(\mathcal{N}_{\varsigma}^{-}\right)^{\mathfrak{O}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{O}}}, \left(1-\left(1-\left(\mathcal{N}_{\varsigma}^{+}\right)^{\mathfrak{O}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{O}}}\right] e^{\varsigma 2\pi\left[\left(1-\left(1-\left(\varphi_{\mathcal{N}_{\varsigma}}^{-}\right)^{\mathfrak{O}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{O}}}, \left(1-\left(1-\left(\varphi_{\mathcal{N}_{\varsigma}}^{+}\right)^{\mathfrak{O}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{O}}}\right]} \end{pmatrix}$$

$$\otimes_{\varsigma=1}^{\eta} \mathfrak{k}_{\varsigma}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}} = \begin{pmatrix} \left[\prod_{\varsigma=1}^{\mu} \mathcal{M}^{-}{}_{\varsigma}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}, \prod_{\varsigma=1}^{\mu} \mathcal{M}^{+}{}_{\varsigma}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right] e^{\varsigma 2\pi\left(\left[\prod_{\varsigma=1}^{\mu} \varphi^{-}{}_{\mathcal{M}_{\varsigma}}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}, \prod_{\varsigma=1}^{\mu} \varphi^{+}{}_{\mathcal{M}_{\varsigma}}^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right]\right)}, \\ \left[\left(1-\prod_{\varsigma=1}^{\mu}\left(1-\left(\mathcal{N}_{\varsigma}^{-}\right)^{\mathfrak{O}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{O}}}, \left(1-\prod_{\varsigma=1}^{\mu}\left(1-\left(\mathcal{N}_{\varsigma}^{+}\right)^{\mathfrak{O}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{O}}}\right] \\ e^{\varsigma 2\pi\left[\left(1-\prod_{\varsigma=1}^{\mu}\left(1-\left(\varphi_{\mathcal{N}_{\varsigma}}^{-}\right)^{\mathfrak{O}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{O}}}, \left(1-\prod_{\varsigma=1}^{\mu}\left(1-\left(\varphi_{\mathcal{N}_{\varsigma}}^{+}\right)^{\mathfrak{O}}\right)^{\frac{1-\tilde{\omega}_{\varsigma}}{\mu-1}}\right)^{\frac{1}{\mathfrak{O}}}\right]} \end{pmatrix}$$

$\square$

Appendix G

Proof of this Property 5. Suggested $\mathfrak{k} = \mathfrak{k}_{\varsigma}$, then for $1 \preccurlyeq \eta \prec \mu$, we have:

$$CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_{\mu})$$

$$= \begin{pmatrix} \begin{bmatrix} \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_{\eta}\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_{\varrho}}^{-}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_{\varrho}})}\right)^{\frac{1}{C_{\mu}^{\eta}}}\right)^{\frac{1}{\mathfrak{O}}}, \\ \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_{\eta}\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}_{\varsigma_{\varrho}}^{+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_{\varrho}})}\right)^{\frac{1}{C_{\mu}^{\eta}}}\right)^{\frac{1}{\mathfrak{O}}} \end{bmatrix} \\ e^{\varsigma 2\pi \begin{bmatrix} (1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_{\eta}\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_{\varrho}}}^{-})^{\frac{1}{\eta}})^{\mathfrak{O}})^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_{\varrho}})})^{\frac{1}{C_{\mu}^{\eta}}})^{\frac{1}{\mathfrak{O}}}, \\ (1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_{\eta}\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi_{\mathcal{M}_{\varsigma_{\varrho}}}^{+})^{\frac{1}{\eta}})^{\mathfrak{O}})^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_{\varrho}})})^{\frac{1}{C_{\mu}^{\eta}}})^{\frac{1}{\mathfrak{O}}} \end{bmatrix}}, \\ \begin{bmatrix} \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_{\eta}\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_{\varrho}}^{-}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_{\varrho}})}\right)^{\frac{1}{C_{\mu}^{\eta}}}, \\ \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_{\eta}\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}_{\varsigma_{\varrho}}^{+}\right)^{\mathfrak{O}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_{\varrho}})}\right)^{\frac{1}{C_{\mu}^{\eta}}} \end{bmatrix} \\ e^{\varsigma 2\pi \begin{bmatrix} (\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_{\eta}\preccurlyeq\mu}((1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_{\varrho}}}^{-})^{\mathfrak{O}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{O}}})^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_{\varrho}})})^{\frac{1}{C_{\mu}^{\eta}}}, \\ (\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_{\eta}\preccurlyeq\mu}((1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}_{\varsigma_{\varrho}}}^{+})^{\mathfrak{O}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{O}}})^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_{\varrho}})})^{\frac{1}{C_{\mu}^{\eta}}} \end{bmatrix}} \end{pmatrix}$$

$$
= \begin{pmatrix}
\left[\begin{array}{c}
\left(1 - \left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\left(\prod_{\varrho=1}^{\eta} \mathcal{M}^{-}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{(1-\sum_{\varrho=1}^{\eta} \tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}\right)^{\frac{1}{\mathfrak{D}}}, \\
\left(1 - \left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(1 - \left(\left(\prod_{\varrho=1}^{\eta} \mathcal{M}^{+}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{D}}\right)^{(1-\sum_{\varrho=1}^{\eta} \tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}\right)^{\frac{1}{\mathfrak{D}}}
\end{array} \right] \\
e^{\varsigma 2\pi[(1-(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu}(1-((\prod_{\varrho=1}^{\eta} \varphi_{\mathcal{M}}^{-})^{\frac{1}{\eta}})^{\mathfrak{D}})^{(1-\sum_{\varrho=1}^{\eta} \tilde{\omega}_{\varsigma_\varrho})})^{\frac{1}{C_{\mu-1}^{\eta}}})^{\frac{1}{\mathfrak{D}}}, (1-(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu}(1-((\prod_{\varrho=1}^{\eta} \varphi_{\mathcal{M}}^{+})^{\frac{1}{\eta}})^{\mathfrak{D}})^{(1-\sum_{\varrho=1}^{\eta} \tilde{\omega}_{\varsigma_\varrho})})^{\frac{1}{C_{\mu-1}^{\eta}}})^{\frac{1}{\mathfrak{D}}}]}, \\
\left[\begin{array}{c}
\left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(\left(1 - \left(\prod_{\varrho=1}^{\eta} \left(1 - (\mathcal{N}^{-})^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{(1-\sum_{\varrho=1}^{\eta} \tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}, \\
\left(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu} \left(\left(1 - \left(\prod_{\varrho=1}^{\eta} \left(1 - (\mathcal{N}^{+})^{\mathfrak{D}}\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{D}}}\right)^{(1-\sum_{\varrho=1}^{\eta} \tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}
\end{array} \right] \\
e^{\varsigma 2\pi[(\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu}((1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}}^{-})^{\mathfrak{D}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{D}}})^{(1-\sum_{\varrho=1}^{\eta} \tilde{\omega}_{\varsigma_\varrho})})^{\frac{1}{C_{\mu-1}^{\eta}}}, (\prod_{1 \preccurlyeq \varsigma_1 \prec, \ldots, \prec \varsigma_\eta \preccurlyeq \mu}((1-(\prod_{\varrho=1}^{\eta}(1-(\varphi_{\mathcal{N}}^{+})^{\mathfrak{D}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{D}}})^{(1-\sum_{\varrho=1}^{\eta} \tilde{\omega}_{\varsigma_\varrho})})^{\frac{1}{C_{\mu-1}^{\eta}}}]}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\left[\left(1 - \left(\left(1 - (\mathcal{M}^{-})^{\mathfrak{D}}\right)^{(C_{\mu}^{\eta} - \sum_{\varsigma=1}^{\eta} C_{\mu-1}^{\eta-1} \tilde{\omega}_{\varsigma})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(\left(1 - (\mathcal{M}^{+})^{\mathfrak{D}}\right)^{(C_{\mu}^{\eta} - \sum_{\varsigma=1}^{\eta} C_{\mu-1}^{\eta-1} \tilde{\omega}_{\varsigma})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}\right)^{\frac{1}{\mathfrak{D}}} \right] \\
e^{\varsigma 2\pi[(1-((1-(\varphi_{\mathcal{M}}^{-})^{\mathfrak{D}})^{(C_{\mu}^{\eta} - \sum_{\varsigma=1}^{\eta} C_{\mu-1}^{\eta-1} \tilde{\omega}_{\varsigma})})^{\frac{1}{C_{\mu-1}^{\eta}}})^{\frac{1}{\mathfrak{D}}}, (1-((1-(\varphi_{\mathcal{M}}^{+})^{\mathfrak{D}})^{(C_{\mu}^{\eta} - \sum_{\varsigma=1}^{\eta} C_{\mu-1}^{\eta-1} \tilde{\omega}_{\varsigma})})^{\frac{1}{C_{\mu-1}^{\eta}}})^{\frac{1}{\mathfrak{D}}}]}, \\
\left[\left(\left(\left(1 - \left(1 - (\mathcal{N}^{-})^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{(C_{\mu}^{\eta} - \sum_{\varsigma=1}^{\eta} C_{\mu-1}^{\eta-1} \tilde{\omega}_{\varsigma})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}, \left(\left(\left(1 - \left(1 - (\mathcal{N}^{+})^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{(C_{\mu}^{\eta} - \sum_{\varsigma=1}^{\eta} C_{\mu-1}^{\eta-1} \tilde{\omega}_{\varsigma})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}} \right] \\
e^{\varsigma 2\pi[(((1-(1-(\varphi_{\mathcal{N}}^{-})^{\mathfrak{D}}))^{\frac{1}{\mathfrak{D}}})^{(C_{\mu}^{\eta} - \sum_{\varsigma=1}^{\eta} C_{\mu-1}^{\eta-1} \tilde{\omega}_{\varsigma})})^{\frac{1}{C_{\mu-1}^{\eta}}}, (((1-(1-(\varphi_{\mathcal{N}}^{+})^{\mathfrak{D}}))^{\frac{1}{\mathfrak{D}}})^{(C_{\mu}^{\eta} - \sum_{\varsigma=1}^{\eta} C_{\mu-1}^{\eta-1} \tilde{\omega}_{\varsigma})})^{\frac{1}{C_{\mu-1}^{\eta}}}]}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\left[\left(1 - \left(\left(1 - (\mathcal{M}^{-})^{\mathfrak{D}}\right)^{(C_{\mu}^{\eta} - C_{\mu-1}^{\eta-1} \sum_{\varsigma=1}^{\eta} \tilde{\omega}_{\varsigma})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1 - \left(\left(1 - (\mathcal{M}^{+})^{\mathfrak{D}}\right)^{(C_{\mu}^{\eta} - C_{\mu-1}^{\eta-1} \sum_{\varsigma=1}^{\eta} \tilde{\omega}_{\varsigma})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}\right)^{\frac{1}{\mathfrak{D}}} \right] \\
e^{\varsigma 2\pi[(1-((1-(\varphi_{\mathcal{M}}^{-})^{\mathfrak{D}})^{(C_{\mu}^{\eta} - C_{\mu-1}^{\eta-1} \sum_{\varsigma=1}^{\eta} \tilde{\omega}_{\varsigma})})^{\frac{1}{C_{\mu-1}^{\eta}}})^{\frac{1}{\mathfrak{D}}}, (1-((1-(\varphi_{\mathcal{M}}^{+})^{\mathfrak{D}})^{(C_{\mu}^{\eta} - C_{\mu-1}^{\eta-1} \sum_{\varsigma=1}^{\eta} \tilde{\omega}_{\varsigma})})^{\frac{1}{C_{\mu-1}^{\eta}}})^{\frac{1}{\mathfrak{D}}}]}, \\
\left[\left(\left(\left(1 - \left(1 - (\mathcal{N}^{-})^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{(C_{\mu}^{\eta} - C_{\mu-1}^{\eta-1} \sum_{\varsigma=1}^{\eta} \tilde{\omega}_{\varsigma})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}}, \left(\left(\left(1 - \left(1 - (\mathcal{N}^{+})^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{(C_{\mu}^{\eta} - C_{\mu-1}^{\eta-1} \sum_{\varsigma=1}^{\eta} \tilde{\omega}_{\varsigma})}\right)^{\frac{1}{C_{\mu-1}^{\eta}}} \right] \\
e^{\varsigma 2\pi[(((1-(1-(\varphi_{\mathcal{N}}^{-})^{\mathfrak{D}}))^{\frac{1}{\mathfrak{D}}})^{(C_{\mu}^{\eta} - C_{\mu-1}^{\eta-1} \sum_{\varsigma=1}^{\eta} \tilde{\omega}_{\varsigma})})^{\frac{1}{C_{\mu-1}^{\eta}}}, (((1-(1-(\varphi_{\mathcal{N}}^{+})^{\mathfrak{D}}))^{\frac{1}{\mathfrak{D}}})^{(C_{\mu}^{\eta} - C_{\mu-1}^{\eta-1} \sum_{\varsigma=1}^{\eta} \tilde{\omega}_{\varsigma})})^{\frac{1}{C_{\mu-1}^{\eta}}}]}
\end{pmatrix}
$$

$$= \left(\begin{array}{c} \left[\left(1-\left(\left(1-(\mathcal{M}^-)^{\mathfrak{D}}\right)^{\left(\frac{(\mu-1)!}{\eta!(\mu-1-\eta)!}\right)}\right)^{\frac{\eta!(\mu-1-\eta)!}{(\mu-1)!}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(\left(1-(\mathcal{M}^+)^{\mathfrak{D}}\right)^{\left(\frac{(\mu-1)!}{\eta!(\mu-1-\eta)!}\right)}\right)^{\frac{\eta!(\mu-1-\eta)!}{(\mu-1)!}}\right)^{\frac{1}{\mathfrak{D}}} \right] \\ e^{\varsigma 2\pi[(1-((1-(\varphi^-_{\mathcal{M}})^{\mathfrak{D}})^{(\frac{(\mu-1)!}{\eta!(\mu-1-\eta)!})})^{\frac{\eta!(\mu-1-\eta)!}{(\mu-1)!}})^{\frac{1}{\mathfrak{D}}},(1-((1-(\varphi^+_{\mathcal{M}})^{\mathfrak{D}})^{(\frac{(\mu-1)!}{\eta!(\mu-1-\eta)!})})^{\frac{\eta!(\mu-1-\eta)!}{(\mu-1)!}})^{\frac{1}{\mathfrak{D}}}]}, \\ \left[\left(\left(\left(1-\left(1-(\mathcal{N}^-)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(\frac{(\mu-1)!}{\eta!(\mu-1-\eta)!}\right)}\right)^{\frac{\eta!(\mu-1-\eta)!}{(\mu-1)!}}, \left(\left(\left(1-\left(1-(\mathcal{N}^+)^{\mathfrak{D}}\right)\right)^{\frac{1}{\mathfrak{D}}}\right)^{\left(\frac{(\mu-1)!}{\eta!(\mu-1-\eta)!}\right)}\right)^{\frac{\eta!(\mu-1-\eta)!}{(\mu-1)!}} \right] \\ e^{\varsigma 2\pi[(((1-(1-(\varphi^-_{\mathcal{N}})^{\mathfrak{D}}))^{\frac{1}{\mathfrak{D}}})^{(\frac{(\mu-1)!}{\eta!(\mu-1-\eta)!})})^{\frac{\eta!(\mu-1-\eta)!}{(\mu-1)!}},(((1-(1-(\varphi^+_{\mathcal{N}})^{\mathfrak{D}}))^{\frac{1}{\mathfrak{D}}})^{(\frac{(\mu-1)!}{\eta!(\mu-1-\eta)!})})^{\frac{\eta!(\mu-1-\eta)!}{(\mu-1)!}}]} \end{array} \right)$$

$$= \left([\mathcal{M}^-(\eta), \mathcal{M}^+(\eta)]e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}}(\eta),\varphi^+_{\mathcal{M}}(\eta)]}, [\mathcal{N}^-(\eta), \mathcal{N}^+(\eta)]e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}}(\eta),\varphi^+_{\mathcal{N}}(\eta)]}\right) = \mathfrak{k}$$

For $\eta = \mu$, then:

$$CIVq - ROFWHM^{(\eta)}(\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_\mu)$$

$$= \left(\begin{array}{c} \left[\prod_{\varsigma=1}^{\mu} \mathcal{M}^{-\frac{1-\tilde{\omega}_\varsigma}{\mu-1}}_{\varsigma}, \prod_{\varsigma=1}^{\mu} \mathcal{M}^{+\frac{1-\tilde{\omega}_\varsigma}{\mu-1}}_{\varsigma} \right] e^{\varsigma 2\pi([\prod_{\varsigma=1}^{\mu} \varphi^{-\frac{1-\tilde{\omega}_\varsigma}{\mu-1}}_{\mathcal{M}_\varsigma}, \prod_{\varsigma=1}^{\mu} \varphi^{+\frac{1-\tilde{\omega}_\varsigma}{\mu-1}}_{\mathcal{M}_\varsigma}])}, \\ \left[\left(1-\prod_{\varsigma=1}^{\mu}\left(1-\left(\mathcal{N}^-_\varsigma\right)^{\mathfrak{D}}\right)^{\frac{1-\tilde{\omega}_\varsigma}{\mu-1}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\prod_{\varsigma=1}^{\mu}\left(1-\left(\mathcal{N}^+_\varsigma\right)^{\mathfrak{D}}\right)^{\frac{1-\tilde{\omega}_\varsigma}{\mu-1}}\right)^{\frac{1}{\mathfrak{D}}} \right] \\ e^{\varsigma 2\pi[(1-\prod_{\varsigma=1}^{\mu}(1-(\varphi^-_{\mathcal{N}_\varsigma})^{\mathfrak{D}})^{\frac{1-\tilde{\omega}_\varsigma}{\mu-1}})^{\frac{1}{\mathfrak{D}}},(1-\prod_{\varsigma=1}^{\mu}(1-(\varphi^+_{\mathcal{N}_\varsigma})^{\mathfrak{D}})^{\frac{1-\tilde{\omega}_\varsigma}{\mu-1}})^{\frac{1}{\mathfrak{D}}}]} \end{array} \right)$$

where $\sum_{\varsigma=1}^{\eta} \tilde{\omega}_\varsigma = 1$

$$= \left(\begin{array}{c} \left[\mathcal{M}^{-\frac{\mu-1}{\mu-1}}, \mathcal{M}^{+\frac{\mu-1}{\mu-1}}\right] e^{\varsigma 2\pi([\varphi^{-\frac{\mu-1}{\mu-1}}_{\mathcal{M}}, \varphi^{+\frac{\mu-1}{\mu-1}}_{\mathcal{M}}])}, \\ \left[\left(1-\left(1-(\mathcal{N}^-)^{\mathfrak{D}}\right)^{\frac{\mu-1}{\mu-1}}\right)^{\frac{1}{\mathfrak{D}}}, \left(1-\left(1-(\mathcal{N}^+)^{\mathfrak{D}}\right)^{\frac{\mu-1}{\mu-1}}\right)^{\frac{1}{\mathfrak{D}}} \right] e^{\varsigma 2\pi[(1-(1-(\varphi^-_{\mathcal{N}})^{\mathfrak{D}})^{\frac{\mu-1}{\mu-1}})^{\frac{1}{\mathfrak{D}}},(1-(1-(\varphi^+_{\mathcal{N}})^{\mathfrak{D}})^{\frac{\mu-1}{\mu-1}})^{\frac{1}{\mathfrak{D}}}]} \end{array} \right)$$

$$= \left([\mathcal{M}^-(\eta), \mathcal{M}^+(\eta)]e^{\varsigma 2\pi[\varphi^-_{\mathcal{M}}(\eta),\varphi^+_{\mathcal{M}}(\eta)]}, [\mathcal{N}^-(\eta), \mathcal{N}^+(\eta)]e^{\varsigma 2\pi[\varphi^-_{\mathcal{N}}(\eta),\varphi^+_{\mathcal{N}}(\eta)]}\right) = \mathfrak{k}.$$

□

Appendix H

Proof of this Property 6. When $\mathcal{M}^-_{\varsigma\varrho} \succcurlyeq \mathcal{M}''^-_{\varsigma\varrho}, \varphi^-_{\mathcal{M}_{\varsigma\varrho}} \succcurlyeq \varphi^-_{\mathcal{M}''_{\varsigma\varrho}}$, $\mathcal{M}^+_{\varsigma\varrho} \succcurlyeq \mathcal{M}''^+_{\varsigma\varrho}$, $\varphi^+_{\mathcal{M}_{\varsigma\varrho}} \succcurlyeq \varphi^+_{\mathcal{M}''_{\varsigma\varrho}}$ and $\mathcal{N}^-_{\varsigma\varrho} \preccurlyeq \mathcal{N}''^-_{\varsigma\varrho}, \varphi^-_{\mathcal{N}_{\varsigma\varrho}} \preccurlyeq \varphi^-_{\mathcal{N}''_{\varsigma\varrho}}, \mathcal{N}^+_{\varsigma\varrho} \preccurlyeq \mathcal{N}''^+_{\varsigma\varrho}, \varphi^+_{\mathcal{N}_{\varsigma\varrho}} \preccurlyeq \varphi^+_{\mathcal{N}''_{\varsigma\varrho}}$, then:

$$\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}} \succcurlyeq \left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}''^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}} \Rightarrow 1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}} \preccurlyeq 1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}''^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}$$

$$\Rightarrow \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C^{\eta}_{\mu}}} \preccurlyeq \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}''^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C^{\eta}_{\mu}}}$$

$$\Rightarrow \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{O}}}$$

$$\succcurlyeq \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}''^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{O}}}$$

$$\Rightarrow \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{O}}} .e^{\varsigma 2\pi(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi^{+}_{\mathcal{M}_{\varsigma_\varrho}})^{\frac{1}{\eta}})^{\mathfrak{O}})^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})})^{\frac{1}{C^{\eta}_{\mu}}})^{\frac{1}{\mathfrak{O}}}}$$

$$\succcurlyeq \left(1-\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\left(\left(\prod_{\varrho=1}^{\eta}\mathcal{M}''^{+}_{\varsigma_\varrho}\right)^{\frac{1}{\eta}}\right)^{\mathfrak{O}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})}\right)^{\frac{1}{C^{\eta}_{\mu}}}\right)^{\frac{1}{\mathfrak{O}}} .e^{\varsigma 2\pi(1-(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}(1-((\prod_{\varrho=1}^{\eta}\varphi^{+}_{\mathcal{M}''_{\varsigma_\varrho}})^{\frac{1}{\eta}})^{\mathfrak{O}})^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})})^{\frac{1}{C^{\eta}_{\mu}}})^{\frac{1}{\mathfrak{O}}}}$$

$$\left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}^{+}_{\varsigma_\varrho}\right)^{\mathfrak{O}}\right)\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})\frac{1}{C^{\eta}_{\mu}}} .e^{\varsigma 2\pi(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}((1-(\prod_{\varrho=1}^{\eta}(1-(\varphi^{+}_{\mathcal{N}_{\varsigma_\varrho}})^{\mathfrak{O}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{O}}})^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})})^{\frac{1}{C^{\eta}_{\mu}}}}$$

$$\preccurlyeq \left(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(\left(1-\left(\prod_{\varrho=1}^{\eta}\left(1-\left(\mathcal{N}''^{+}_{\varsigma_\varrho}\right)^{\mathfrak{O}}\right)\right)\right)^{\frac{1}{\eta}}\right)^{\frac{1}{\mathfrak{O}}}\right)^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})\frac{1}{C^{\eta}_{\mu}}} .e^{\varsigma 2\pi(\prod_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}((1-(\prod_{\varrho=1}^{\eta}(1-(\varphi^{+}_{\mathcal{N}''_{\varsigma_\varrho}})^{\mathfrak{O}}))^{\frac{1}{\eta}})^{\frac{1}{\mathfrak{O}}})^{(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho})})^{\frac{1}{C^{\eta}_{\mu}}}}$$

Using the tool of SV and AV, then:

1. If $\dot{\check{S}}(\mathfrak{k}_\varsigma) > \dot{\check{S}}(\mathfrak{k}''_\varsigma)$, then:

$$CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_k) \succcurlyeq CIVq-ROFWHM^{(\eta)}\left(\mathfrak{k}''_1,\mathfrak{k}''_2,\ldots,\mathfrak{k}''_k\right)$$

2. If $\dot{\check{S}}(\mathfrak{k}_\varsigma) = \dot{\check{S}}(\mathfrak{k}''_\varsigma)$, then if $H(\mathfrak{k}_\varsigma) = H(\mathfrak{k}''_\varsigma)$, then:

$$CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_k) = CIVq-ROFWHM^{(\eta)}\left(\mathfrak{k}''_1,\mathfrak{k}''_2,\ldots,\mathfrak{k}''_\mu\right)$$

For $\eta=\mu$, the proof is similar. □

Appendix I

Proof of this Property 7. Based on Proposition 5 and 6, such that:

$$CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu) \succcurlyeq CIVq-RFWHM^{(\eta)}\left(\mathfrak{k}^{-}_1,\mathfrak{k}^{-}_2,\ldots,\mathfrak{k}^{-}_\mu\right) = \mathfrak{k}^{-}$$

$$CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu)\preccurlyeq CIVq-ROFWHM^{(\eta)}\left(\mathfrak{k}_1^+,\mathfrak{k}_2^+,\ldots,\mathfrak{k}_\mu^+\right)=\mathfrak{k}^+$$

Then:

$$\mathfrak{k}^-\prec CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu)\prec\mathfrak{k}^+$$

For $\eta=\mu$, the proof is similar. □

Appendix J

Proof of this Property 8. By hypothesis, we obtained:

$$\frac{\oplus_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}}{C_\mu^\eta}=\frac{\oplus_{1\preccurlyeq\varsigma_1\prec,\ldots,\prec\varsigma_\eta\preccurlyeq\mu}\left(1-\sum_{\varrho=1}^{\eta}\tilde{\omega}_{\varsigma_\varrho}\right)\left(\otimes_{\varrho=1}^{\eta}\left(\mathfrak{k}''_{\varsigma_\varrho}\right)\right)^{\frac{1}{\eta}}}{C_\mu^\eta}$$

Therefore, $CIVq-ROFWHM^{(\eta)}(\mathfrak{k}_1,\mathfrak{k}_2,\ldots,\mathfrak{k}_\mu)=CIVq-ROFWHM^{(\eta)}(\mathfrak{k}''_1,\mathfrak{k}''_2,\ldots,\mathfrak{k}''_\mu)$.

□

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