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On Strengthened Extragradient Methods Non-Convex Combination with Adaptive Step Sizes Rule for Equilibrium Problems

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Abstract: Symmetries play a vital role in the study of physical phenomena in diverse areas such as dynamic systems, optimization, physics, scientific computing, engineering, mathematical biology, chemistry, and medicine, to mention a few. These phenomena specialize mostly in solving equilibria-like problems in abstract spaces. Motivated by these facts, this research provides two innovative modifying extragradient strategies for solving pseudomonotone equilibria problems in real Hilbert space with the Lipschitz-like bifunction constraint. Such strategies make use of multiple step-size concepts that are modified after each iteration and are reliant on prior iterations. The excellence of these strategies comes from the fact that they were developed with no prior knowledge of Lipschitz-type parameters or any line search strategy. Mild assumptions are required to prove strong convergence theorems for proposed strategies. Various numerical tests have been reported to demonstrate the numerical behavior of the techniques and then contrast them with others.

Keywords: Lipschitz-like conditions; equilibrium problem; strong convergence theorems; variational inequality problems; fixed-point problem

MSC: 47J25; 47H09; 47H06; 47J05

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1. Introduction

Consider that Σ is a nonempty, convex, and closed subset of a real Hilbert space Π. The inner product and norm are indicated with $\langle ., . \rangle$ and $\|.\|$, respectively. Furthermore, $\mathbb{R}$ and $\mathbb{N}$ symbolize the set of real numbers and the set of natural numbers, respectively. Assume that $\mathcal{R} : \Pi \times \Pi \to \mathbb{R}$ is indeed a bifunction with the equilibrium problem solution set $EP(\mathcal{R}, \Sigma)$. Let

$$s^* = P_{EP(\mathcal{R},\Sigma)},$$

whereas θ represents a zero element in Π. In this case, Σ characterizes the subset of a Hilbert space Π and $\mathcal{R}$ as follows: $\mathcal{R} : \Pi \times \Pi \to \mathbb{R}$ is a bifunction through $\mathcal{R}(r_1, r_1) = 0$, for all $r_1 \in \Sigma$. The *equilibrium problem* [1,2] for $\mathcal{R}$ on Σ is to:

$$\text{Find } s^* \in \Sigma \text{ such that } \mathcal{R}(s^*, r_1) \geq 0, \ \forall r_1 \in \Sigma. \tag{1}$$

The above-mentioned framework is an appropriate mathematical framework that incorporates a variety of problems, including vector and scalar minimization problems, saddle point problems, variational inequality problems, complementarity problems, Nash equilibrium problems in non-cooperative games, and inverse optimization problems [1,3,4]. This issue is primarily connected to Ky Fan inequity on the grounds of his prior contributions to the field [2]. It is also important to consider an approximate solution if the problem does not have an exact solution or is difficult to calculate. Several methodologies have been proposed and tested to tackle various types of equilibrium problems (1). Many successful algorithmic techniques, as well as theoretical characteristics, have already been proposed to solve the (1) issue in both finite- and infinite-dimensional spaces.

The regularization technique is the most significant method for dealing with many ill-posed problems in various subfields of applied and pure mathematics. The regularization approach is distinguished by the use of monotone equilibrium problems to convert the original problem into a strongly monotone equilibrium subproblem. As a result, each computationally productive subproblem is strongly monotone and has a unique solution. The discovered subproblem, for example, may be more successfully resolved than the initial problem, and the regularization solutions may lead to some solution to the basic problem once the regularization variables look to have an adequate limit. The two most prevalent regularization methods are the proximal point and Tikhonov's regularized approaches. These approaches were recently extended to equilibrium problems [5–13]. A few techniques to address non-monotone equilibrium problems can be found in [14–26].

The proximal method [27] is indeed an innovative approach for determining equilibrium problems that are founded on minimization problems. Along with Korpelevich's contribution [28] technique to addressing the saddle point problem, this procedure has also been known as the two-step extragradient method in [29]. Tran et al. [29] constructed an iterative sequence of $\{s_k\}$ in the following manner:

$$\begin{cases} s_1 \in \Sigma, \\ m_k = \arg\min_{v \in \Sigma}\{\lambda \mathcal{R}(s_k, v) + \frac{1}{2}\|s_k - v\|^2\}, \\ s_{k+1} = \arg\min_{v \in \Sigma}\{\lambda \mathcal{R}(m_k, v) + \frac{1}{2}\|s_k - v\|^2\}, \end{cases}$$

where $0 < \lambda < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. The iterative sequence created by the aforementioned approach exhibits weak convergence, and prior knowledge of Lipschitz-type variables is necessary in order to use it. Lipschitz-type parameters are frequently unknown or difficult to calculate. To address this issue, Hieu et al. [30] introduced the following adaptation of the approach in [31] for equilibrium: Let $[t]_+ = \max\{t, 0\}$ and select $s_1 \in \Sigma$, $\mu \in (0,1)$ with $\lambda_0 > 0$, such that

$$\begin{cases} m_k = \arg\min_{v \in \Sigma}\{\lambda_k \mathcal{R}(s_k, v) + \frac{1}{2}\|s_k - v\|^2\}, \\ s_{k+1} = \arg\min_{v \in \Sigma}\{\lambda_k \mathcal{R}(m_k, v) + \frac{1}{2}\|s_k - v\|^2\}, \end{cases}$$

along with

$$\lambda_{k+1} = \min\left\{\lambda_k, \frac{\mu(\|s_k - m_k\|^2 + \|s_{k+1} - m_k\|^2)}{2[\mathcal{R}(s_k, s_{k+1}) - \mathcal{R}(s_k, m_k) - \mathcal{R}(m_k, s_{k+1})]_+}\right\}.$$

To solve a pseudomonotone equilibrium problem, the authors have suggested a non-convex combination iterative technique in [32]. The availability of a strong convergence iterative sequence without the need for hybrid projection or viscosity techniques is the main contribution. The details of the algorithm are as follows: Choose $0 < \lambda_k < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\delta_k \subset [\delta, 1)$ with $\delta > 0$ and ϕ_k such that

$$\lim_{k \to +\infty} \phi_k = 0 \text{ and } \sum_{k=1}^{+\infty} \phi_k = +\infty.$$

$$\begin{cases} m_k = \arg\min_{v \in \Sigma}\{\lambda_k \mathcal{R}(s_k, v) + \frac{1}{2}\|s_k - v\|^2\}, \\ r_k = \arg\min_{v \in \Sigma}\{\lambda_k \mathcal{R}(m_k, v) + \frac{1}{2}\|s_k - v\|^2\}, \end{cases}$$

and

$$s_{k+1} = P_\Sigma\big[\phi_k s_k + (1 - \phi_k) r_k - \phi_k \delta_k s_k\big].$$

The main objective of this study is to focus on using well-known projection algorithms that are, in general, easier to apply due to their efficient and easy mathematical computation. We design and adapt an explicit subgradient extragradient method to solve the problem of pseudomonotone equilibrium and other specific classes of variational inequality problems and fixed-point problems, inspired by the works of [30,33]. Our techniques are a variation on the approaches described in [32]. Strong convergence results matching the sequence of the two methods are achieved under specific, moderate circumstances. Some applications of variational inequality and fixed-point problems are given. Consequently, experimental investigations have shown that the proposed strategy is more successful than the current one [32].

The rest of the article is organized as follows: Section 2 includes basic definitions and lemmas. Section 3 proposes new methods and their convergence analysis theorems. Section 4 contains several applications of our findings to variational inequality and fixed-point problems. Section 5 contains numerical tests to demonstrate the computational effectiveness of our proposed methods.

2. Preliminaries

Suppose that a convex function $\Im : \Sigma \to \mathbb{R}$ and *subdifferential of* $\Im$ at $r_1 \in \Sigma$ is expressed as follows:

$$\partial\Im(r_1) = \{r_3 \in \Pi : \Im(r_2) - \Im(r_1) \geq \langle r_3, r_2 - r_1\rangle, \; \forall r_2 \in \Sigma\}.$$

A *normal cone of* Σ at $r_1 \in \Sigma$ is expressed as follows:

$$N_\Sigma(r_1) = \{r_3 \in \Pi : \langle r_3, r_2 - r_1\rangle \leq 0, \; \forall r_2 \in \Sigma\}.$$

Lemma 1. ([34]) *Suppose that a convex function $\Im : \Sigma \to \mathbb{R}$ is subdifferentiable and lower semicontinuous upon Σ. Then $r_1 \in \Sigma$ is a minimizer of a function $\Im$ if and only if*

$$0 \in \partial\Im(r_1) + N_\Sigma(r_1),$$

where $\partial\Im(r_1)$ and $N_\Sigma(r_1)$ denotes the subdifferential of $\Im$ at $r_1 \in \Sigma$ and the normal cone of Σ at r_1, respectively.

Definition 1. ([35]) *A metric projection $P_\Sigma(r_1)$ for $r_1 \in \Pi$ onto a convex and closed subset Σ of Π is stated as follows:*

$$P_\Sigma(r_1) = \arg\min\{\|r_2 - r_1\| : r_2 \in \Sigma\}.$$

Lemma 2. ([36]) *Consider that a metric projection $P_\Sigma : \Pi \to \Sigma$. Then*

(i) *For some $r_2 \in \Sigma$ and $r_1 \in \Pi$ in order that*

$$\|r_1 - P_\Sigma(r_1)\| \leq \|r_1 - r_2\|^2.$$

(ii) $r_3 = P_\Sigma(r_1)$ *if and only if*

$$\langle r_1 - r_3, r_2 - r_3\rangle \leq 0, \; \forall r_2 \in \Sigma.$$

Lemma 3. ([37]) *For some $r_1, r_2 \in \Pi$ and $\chi \in \mathbb{R}$. Then*
(i)
$$\|\chi r_1 + (1-\chi) r_2\|^2 = \chi \|r_1\|^2 + (1-\chi)\|r_2\|^2 - \chi(1-\chi)\|r_1 - r_2\|^2;$$
(ii)
$$\|r_1 + r_2\|^2 \leq \|r_1\|^2 + 2\langle r_2, r_1 + r_2 \rangle.$$

Lemma 4. ([38]) *Consider a sequence of non-negative real numbers $\{\chi_k\}$ such that*
$$\chi_{k+1} \leq (1-\tau_k)\chi_k + \tau_k \delta_k, \ \forall\, k \in \mathbb{N},$$
while $\{\tau_k\} \subset (0,1)$ and $\{\delta_k\} \subset \mathbb{R}$ conforming to the following parameters:
$$\lim_{k\to+\infty} \tau_k = 0, \ \sum_{k=1}^{+\infty} \tau_k = +\infty, \ \text{and} \ \limsup_{k\to+\infty} \delta_k \leq 0.$$

Thus, $\lim_{k\to+\infty} \chi_k = 0$.

Lemma 5. ([39]) *Assume that $\{\chi_k\}$ is a sequence of real numbers namely that there exists a subsequence $\{k_i\}$ of $\{k\}$ such that*
$$\chi_{k_i} < \chi_{k_{i+1}}, \ \text{for all } i \in \mathbb{N}.$$

Then, there would be a nondecreasing sequence $\{e_k\} \subset \mathbb{N}$, namely that $e_k \to +\infty$ as $k \to +\infty$, and the following criteria are fulfilled by all (sufficiently big) integers $k \in \mathbb{N}$:
$$\chi_{e_k} \leq \chi_{m_{k+1}} \ \text{and} \ \chi_k \leq \chi_{m_{k+1}}.$$

In fact, $e_k = \max\{j \leq k : \chi_j \leq \chi_{j+1}\}$.

Now, we consider the following bifunction monotonicity notions (for more information, see [1,40]). A bifunction $\mathcal{R} : \Pi \times \Pi \to \mathbb{R}$ on Σ for $\xi > 0$ such that
(1) *strongly monotone* if
$$\mathcal{R}(r_1, r_2) + \mathcal{R}(r_2, r_1) \leq -\xi\|r_1 - r_2\|^2, \ \forall\, r_1, r_2 \in \Sigma;$$
(2) *monotone* if
$$\mathcal{R}(r_1, r_2) + \mathcal{R}(r_2, r_1) \leq 0, \ \forall\, r_1, r_2 \in \Sigma;$$
(3) *strongly pseudomonotone* if
$$\mathcal{R}(r_1, r_2) \geq 0 \Longrightarrow \mathcal{R}(r_2, r_1) \leq -\xi\|r_1 - r_2\|^2, \ \forall\, r_1, r_2 \in \Sigma;$$
(4) *pseudomonotone* if
$$\mathcal{R}(r_1, r_2) \geq 0 \Longrightarrow \mathcal{R}(r_2, r_1) \leq 0, \ \forall\, r_1, r_2 \in \Sigma.$$

Suppose that $\mathcal{R} : \Pi \times \Pi \to \mathbb{R}$ meets the Lipschitz-type condition [41] over Σ if $c_1, c_2 > 0$, such that
$$\mathcal{R}(r_1, r_3) \leq \mathcal{R}(r_1, r_2) + \mathcal{R}(r_2, r_3) + c_1\|r_1 - r_2\|^2 + c_2\|r_2 - r_3\|^2, \ \forall\, r_1, r_2, r_3 \in \Sigma.$$

We shall presume that the requirements listed below have been satisfied. A bifunction $\mathcal{R}$ meets the following criteria:
($\mathcal{R}$1) $\mathcal{R}(r_2, r_2) = 0$ for all $r_2 \in \Sigma$ and $\mathcal{R}$ is pseudomonotone on feasible set Σ;
($\mathcal{R}$2) $\mathcal{R}$ meet the Lipschitz-type condition on Π with constants c_1 and c_2;
($\mathcal{R}$3) $\mathcal{R}(r_1, r_2)$ is jointly weakly continuous on $\Pi \times \Pi$;
($\mathcal{R}$4) $\mathcal{R}(r_1, .)$ need to be convex and subdifferentiable over Π for each $r_1 \in \Pi$.

3. Main Results

We add a method and have strong convergence results for that method. The following is a detailed algorithm:

The following lemma can be used to demonstrate that the step-size sequence λ_k generated by the previous formula decreases monotonically and is bounded, as required for iterative sequence convergence.

Lemma 6. *A sequence* $\{\lambda_k\}$ *is decreasing monotonically with lower bound* $\min\left\{\frac{\mu}{2\max\{c_1,c_2\}}, \lambda_0\right\}$ *and converge to* $\lambda > 0$.

Proof. It is straightforward that $\{\lambda_k\}$ decreases monotonically. Let $\mathcal{R}(s_k, r_k) - \mathcal{R}(s_k, m_k) - \mathcal{R}(m_k, r_k) > 0$, such that

$$\begin{aligned} &\frac{\mu(\|s_k - m_k\|^2 + \|r_k - m_k\|^2)}{2[\mathcal{R}(s_k, r_k) - \mathcal{R}(s_k, m_k) - \mathcal{R}(m_k, r_k)]} \\ &\geq \frac{\mu(\|s_k - m_k\|^2 + \|r_k - m_k\|^2)}{2[c_1\|s_k - m_k\|^2 + c_2\|r_k - m_k\|^2]} \geq \frac{\mu}{2\max\{c_1, c_2\}}. \end{aligned}$$

Thus, sequence $\{\lambda_k\}$ has the lower bound $\min\left\{\frac{\mu}{2\max\{c_1,c_2\}}, \lambda_0\right\}$. Thus, there exists a real number $\lambda > 0$, to ensure that $\lim_{k\to+\infty} \lambda_k = \lambda$. □

The following lemma can be used to verify the boundedness of an iterative sequence.

Lemma 7. *Let* $\mathcal{R} : \Pi \times \Pi \to \mathbb{R}$ *be a bifunction that satisfies the conditions* $(\mathcal{R}1)$–$(\mathcal{R}4)$. *For any* $s^* \in EP(\mathcal{R}, \Sigma) \neq \emptyset$, *we have*

$$\|r_k - s^*\|^2 \leq \|s_k - s^*\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|s_k - m_k\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|r_k - m_k\|^2.$$

Proof. By the value r_k and Lemma 1, we obtain

$$\lambda_k\mathcal{R}(m_k, y) - \lambda_k\mathcal{R}(m_k, r_k) \geq \langle s_k - r_k, y - r_k\rangle, \ \forall y \in \Pi_k. \quad (2)$$

From definition of Π_k, we have

$$\lambda_k\mathcal{R}(s_k, r_k) - \lambda_k\mathcal{R}(s_k, m_k) \geq \langle s_k - m_k, r_k - m_k\rangle. \quad (3)$$

Using the value of λ_{k+1}, we can write

$$\mathcal{R}(s_k, r_k) - \mathcal{R}(s_k, m_k) - \mathcal{R}(m_k, r_k) \leq \frac{\mu(\|s_k - m_k\|^2 + \|r_k - m_k\|^2)}{2\lambda_{k+1}}. \quad (4)$$

Expressions (2)–(4) imply that (see Lemma 3.3 in [42]):

$$\|r_k - s^*\|^2 \leq \|s_k - s^*\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|s_k - m_k\|^2 - \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right)\|r_k - m_k\|^2.$$

□

The strong convergence analysis for Algorithm 1 is presented in the following theorem. The details of the convergence theorems are given below.

Algorithm 1 Self-Adaptive Explicit Extragradient Method with Non-Convex Combination

Step 0: Let $s_1 \in \Pi$, $\lambda_0 > 0$, $\mu \in (0,1)$, $\delta_k \subset [\delta, 1)$ through $\delta > 0$ and $\phi_k \subset (0,1)$ such that

$$\lim_{k \to +\infty} \phi_k = 0 \text{ and } \sum_{k=1}^{+\infty} \phi_k = +\infty.$$

Step 1: Compute

$$m_k = \arg\min_{v \in \Sigma} \{\lambda_k \mathcal{R}(s_k, v) + \frac{1}{2}\|s_k - v\|^2\}.$$

In the case that $m_k = s_k$, stop and $s_k \in EP(\mathcal{R}, \Sigma)$. Otherwise, go to the next step.
Step 2: First, choose $\omega_k \in \partial\mathcal{R}(s_k, m_k)$ satisfying $s_k - \lambda_k \omega_k - m_k \in N_{\mathcal{K}}(m_k)$ and generate a half-space

$$\Pi_k = \{z \in \Pi : \langle s_k - \lambda_k \omega_k - m_k, z - m_k \rangle \leq 0\}.$$

Solve $r_k = \arg\min_{v \in \Pi_k} \{\lambda_k \mathcal{R}(m_k, v) + \frac{1}{2}\|s_k - v\|^2\}$.
Step 3: Compute

$$s_{k+1} = P_\Sigma[\phi_k s_k + (1 - \phi_k) r_k - \phi_k \delta_k s_k].$$

Step 4: Revise the step size as follows and continue:

$$\lambda_{k+1} = \begin{cases} \min\left\{\lambda_k, \frac{\mu(\|s_k - m_k\|^2 + \|r_k - m_k\|^2)}{2[\mathcal{R}(s_k, r_k) - \mathcal{R}(s_k, m_k) - \mathcal{R}(m_k, r_k)]}\right\} \\ \quad \text{if} \quad \mathcal{R}(s_k, r_k) - \mathcal{R}(s_k, m_k) - \mathcal{R}(m_k, r_k) > 0 \\ \lambda_k & \text{otherwise.} \end{cases}$$

Set $k := k + 1$ and move back to **Step 1**.

Theorem 1. *Let a sequence $\{s_k\}$ be generated by Algorithm 1. Then, sequence $\{s_k\}$ converges strongly to $s^* \in EP(\mathcal{R}, \Sigma)$.*

Proof. Given that $\lambda_k \to \lambda$, then $\epsilon \in (0, 1 - \mu)$, is a number such that

$$\lim_{k \to +\infty} \left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right) = 1 - \mu > \epsilon > 0.$$

As a result, there exists a finite number $k_1 \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\right) > \epsilon > 0, \ \forall k \geq k_1. \tag{5}$$

Using Lemma 7, we have

$$\|r_k - s^*\|^2 \leq \|s_k - s^*\|^2, \ \forall k \geq k_1. \tag{6}$$

We derive using Lemma 3 (i) for any $k \geq k_1$, such that

$$\begin{aligned}
\|s_{k+1} - s^*\|^2 &= \|P_\Sigma\big[\phi_k(1-\delta_k)s_k + (1-\phi_k)r_k\big] - P_\Sigma(s^*)\|^2 \\
&\leq \|\phi_k(1-\delta_k)s_k + (1-\phi_k)r_k - s^*\|^2 \\
&= \|\phi_k\big[(1-\delta_k)s_k - s^*\big] + (1-\phi_k)(r_k - s^*)\|^2 \\
&\leq \phi_k\|(1-\delta_k)s_k - s^*\|^2 + (1-\phi_k)\|r_k - s^*\|^2 \\
&\leq \phi_k\big[\|(1-\delta_k)(s_k - s^*) + \delta_k s^*\|^2\big] + (1-\phi_k)\|s_k - s^*\|^2 \\
&\quad - (1-\phi_k)\Big[\Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big)\|s_k - m_k\|^2 + \Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big)\|r_k - m_k\|^2\Big] \\
&\leq \phi_k\big[(1-\delta_k)\|s_k - s^*\|^2 + \delta_k\|s^*\|^2\big] + (1-\phi_k)\|s_k - s^*\|^2 \\
&\quad - (1-\phi_k)\big[\epsilon\|s_k - m_k\|^2 + \epsilon\|r_k - m_k\|^2\big] \\
&= (1-\phi_k\delta_k)\|s_k - s^*\|^2 + \phi_k\delta_k\|s^*\|^2 \\
&\quad - \epsilon(1-\phi_k)\big[\|s_k - m_k\|^2 + \|r_k - m_k\|^2\big] \qquad (7) \\
&\leq \max\{\|s_k - s^*\|^2, \|s^*\|^2\} \\
&\leq \max\{\|s_{k_1} - s^*\|^2, \|s^*\|^2\}. \qquad (8)
\end{aligned}$$

It is deduced that sequence $\{s_k\}$ is a bounded sequence. Let $q_k = \phi_k s_k + (1-\phi_k)r_k$, for any $k \in \mathbb{N}$. By Lemma 3 (i), we have

$$\|q_k - s^*\|^2 = \|\phi_k s_k + (1-\phi_k)r_k - s^*\|^2 \leq \|s_k - s^*\|^2, \ \forall\, k \geq k_1. \tag{9}$$

Notice that there is

$$s_{k+1} = P_\Sigma(q_k - \phi_k\delta_k s_k) = P_\Sigma\big[(1-\phi_k\delta_k)q_k + \phi_k\delta_k(1-\phi_k)(r_k - s_k)\big]. \tag{10}$$

By Lemma 3 (ii) and (9), (10) implies that (see Equation (3.6) [32])

$$\begin{aligned}
&\|s_{k+1} - s^*\|^2 \\
&= \|P_\Sigma\big[(1-\phi_k\delta_k)q_k + \phi_k\delta_k(1-\phi_k)(r_k - s_k)\big] - P_\Sigma(s^*)\|^2 \\
&\leq (1-\phi_k\delta_k)\|s_k - s^*\|^2 + 2\phi_k\delta_k(1-\phi_k) \\
&\quad \big\langle r_k - s_k, (1-\phi_k\delta_k)q_k + \phi_k\delta_k(1-\phi_k)(r_k - s_k) - s^*\big\rangle \\
&\quad + 2\phi_k\delta_k(1-\phi_k)\big\langle -s^*, r_k - s_k\big\rangle + 2\phi_k\delta_k\big\langle -s^*, s_k - s^*\big\rangle + 2\phi_k^2\delta_k^2\langle s^*, s_k\rangle. \qquad (11)
\end{aligned}$$

The remains of the proof can be split into two parts:

Case 1: Let $k_2 \in \mathbb{N}$ $(k_2 \geq k_1)$ such that

$$\|s_{k+1} - s^*\| \leq \|s_k - s^*\|, \ \forall\, k \geq k_2.$$

Thus, $\lim_{k\to+\infty}\|s_k - s^*\|$, exists and let $\lim_{k\to+\infty}\|s_k - s^*\| = l$. By relationship (7), we have

$$\begin{aligned}
\epsilon\big[\|s_k - m_k\|^2 + \|r_k - m_k\|^2\big] &\leq \|s_k - s^*\|^2 - \|s_{k+1} - s^*\|^2 + \phi_k\delta_k\|s^*\|^2 \\
&\quad + \epsilon\phi_k\big[\|s_k - m_k\|^2 + \|r_k - m_k\|^2\big], \ \forall\, k \geq k_2. \qquad (12)
\end{aligned}$$

The existence of $\lim_{k\to+\infty}\|s_k - s^*\| = l$, provides that

$$\lim_{k\to+\infty}\|s_k - m_k\| = \lim_{k\to+\infty}\|r_k - m_k\| = 0, \tag{13}$$

and accordingly

$$\lim_{k\to+\infty}\|s_k - r_k\| \leq \lim_{k\to+\infty}\|s_k - m_k\| + \lim_{k\to+\infty}\|m_k - r_k\| = 0. \tag{14}$$

Thus, the sequence $\{s_k\}$ is a bounded sequence. Hence, we may select a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ such that $\{s_{k_j}\}$ converges weakly to a certain $s \in \Sigma$ such that

$$\limsup_{k\to+\infty}\langle -s^*, s_k - s^*\rangle = \limsup_{j\to+\infty}\langle -s^*, s_{k_j} - s^*\rangle = \langle -s^*, s - s^*\rangle. \tag{15}$$

From (13) the subsequence $\{m_{k_j}\}$ also converges weakly to s as $j \to +\infty$. Due to the expression (3), we obtain

$$\lambda_{k_j}\mathcal{R}(s_{k_j}, y) - \lambda_{k_j}\mathcal{R}(s_{k_j}, m_{k_j}) \geq \langle s_{k_j} - m_{k_j}, y - m_{k_j}\rangle, \ \forall\ y \in \Sigma. \tag{16}$$

Allowing $j \to +\infty$ entails that

$$\mathcal{R}(s, y) \geq 0, \ \forall\, y \in \Sigma. \tag{17}$$

As a result, $s \in EP(\mathcal{R}, \Sigma)$. Eventually, using (15) and Lemma 2 (ii), we derive

$$\begin{aligned}\limsup_{k\to+\infty}\langle -s^*, s_k - s^*\rangle &= \limsup_{j\to+\infty}\langle -s^*, s_{k_j} - s^*\rangle \\ &= \langle -s^*, s - s^*\rangle. \\ &= \langle \theta - P_{EP(\mathcal{R},\Sigma)}, s - P_{EP(\mathcal{R},\Sigma)}\rangle. \\ &\leq 0. \end{aligned} \tag{18}$$

We have the desired results from of the assertion on ϕ_k, δ_k, (11), (13), (14), (18) and Lemma 4.

Case 2: Assume that there exists a subsequence $\{k_i\}$ of $\{k\}$ such that

$$\|s_{k_i} - s^*\| \leq \|s_{k_{i+1}} - s^*\|, \ \forall\, i \in \mathbb{N}.$$

Consequently, according to Lemma 5, there is indeed a sequence $\{n_j\} \subset \mathbb{N}$ such that $n_j \to +\infty$, we have

$$\|s_{n_j} - s^*\| \leq \|s_{m_{j+1}} - s^*\| \ \text{ and } \ \|s_j - s^*\| \leq \|s_{m_{j+1}} - s^*\|, \ \text{for all } j \in \mathbb{N}. \tag{19}$$

By the expression (7), we have

$$\begin{aligned}\epsilon\big[\|s_{n_j} - m_{n_j}\|^2 + \|r_{n_j} - m_{n_j}\|^2\big] &\leq \|s_{n_j} - s^*\|^2 - \|s_{n_j+1} - s^*\|^2 + \phi_{n_j}\delta_{n_j}\|s^*\|^2 \\ &\quad + \epsilon\phi_{n_j}\big[\|s_{n_j} - m_{n_j}\|^2 + \|r_{n_j} - m_{n_j}\|^2\big], \ \forall\, n_j \geq k_1. \end{aligned} \tag{20}$$

The above expressions imply that

$$\lim_{j\to+\infty}\|s_{n_j} - m_{n_j}\| = \lim_{j\to+\infty}\|r_{n_j} - m_{n_j}\| = 0, \tag{21}$$

thus

$$\lim_{j\to+\infty}\|s_{n_j} - r_{n_j}\| \leq \lim_{j\to+\infty}\|s_{n_j} - m_{n_j}\| + \lim_{j\to+\infty}\|m_{n_j} - r_{n_j}\| = 0. \tag{22}$$

By statements identical to those in expression (18), we have

$$\limsup_{j\to+\infty}\langle -s^*, s_{n_j} - s^*\rangle \leq 0. \tag{23}$$

From expression (11), we obtain

$$\begin{aligned}
&\|s_{n_j+1} - s^*\|^2 \\
&\le (1 - \phi_{n_j}\delta_{n_j})\|s_{n_j} - s^*\|^2 + 2\phi_{n_j}\delta_{n_j}(1 - \phi_{n_j}) \\
&\quad \langle r_{n_j} - s_{n_j}, (1 - \phi_{n_j}\delta_{n_j})q_{n_j} + \phi_{n_j}\delta_{n_j}(1 - \phi_{n_j})(r_{n_j} - s_{n_j}) - s^* \rangle \\
&\quad + 2\phi_{n_j}\delta_{n_j}(1 - \phi_{n_j})\langle -s^*, r_{n_j} - s_{n_j}\rangle + 2\phi_{n_j}\delta_{n_j}\langle -s^*, s_{n_j} - s^*\rangle + 2\phi_{n_j}^2\delta_{n_j}^2\langle s^*, s_{n_j}\rangle.
\end{aligned} \tag{24}$$

It is given that $\|s_{n_j} - s^*\| \le \|s_{m_{j+1}} - s^*\|$, implies that

$$\begin{aligned}
&\|s_{n_j+1} - s^*\|^2 \\
&\le (1 - \phi_{n_j}\delta_{n_j})\|s_{m_{j+1}} - s^*\|^2 + 2\phi_{n_j}\delta_{n_j}(1 - \phi_{n_j}) \\
&\quad \langle r_{n_j} - s_{n_j}, (1 - \phi_{n_j}\delta_{n_j})q_{n_j} + \phi_{n_j}\delta_{n_j}(1 - \phi_{n_j})(r_{n_j} - s_{n_j}) - s^* \rangle \\
&\quad + 2\phi_{n_j}\delta_{n_j}(1 - \phi_{n_j})\langle -s^*, r_{n_j} - s_{n_j}\rangle + 2\phi_{n_j}\delta_{n_j}\langle -s^*, s_{n_j} - s^*\rangle + 2\phi_{n_j}^2\delta_{n_j}^2\langle s^*, s_{n_j}\rangle.
\end{aligned} \tag{25}$$

The expression (19) and (25) implies that

$$\begin{aligned}
&\|s_j - s^*\|^2 \le \|s_{n_j+1} - s^*\|^2 \\
&\le 2(1 - \phi_{n_j})\langle r_{n_j} - s_{n_j}, (1 - \phi_{n_j}\delta_{n_j})q_{n_j} + \phi_{n_j}\delta_{n_j}(1 - \phi_{n_j})(r_{n_j} - s_{n_j}) - s^* \rangle \\
&\quad + 2(1 - \phi_{n_j})\langle -s^*, r_{n_j} - s_{n_j}\rangle + 2\langle -s^*, s_{n_j} - s^*\rangle + 2\phi_{n_j}\delta_{n_j}\langle s^*, s_{n_j}\rangle, \ \forall k \ge k_1.
\end{aligned} \tag{26}$$

Because $\phi_{n_j} \to 0$, it derives via expressions (21), (22) such that

$$\lim_{k \to +\infty} \|s_j - s^*\|^2 \le \lim_{k \to +\infty} \|s_{n_j+1} - s^*\|^2 \le 0. \tag{27}$$

Consequently, $s_k \to s^*$. This is the required result. □

Now, a modification of Algorithm 1 proves a strong convergence theorem for it. For the purpose of simplicity, we will adopt the notation $[t]_+ = \max\{0, t\}$ and the conventional $\frac{0}{0} = +\infty$ and $\frac{a}{0} = +\infty$ $(a \neq 0)$. The following is a more detailed algorithm:

Lemma 8. *Let $\mathcal{R} : \Pi \times \Pi \to \mathbb{R}$ be a bifunction satisfies the conditions ($\mathcal{R}$1)–($\mathcal{R}$4). For any $s^* \in EP(\mathcal{R}, \Sigma) \neq \emptyset$, we have*

$$\|r_k - s^*\|^2 \le \|P_\Sigma(s_k) - s^*\|^2 - \Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big)\|P_\Sigma(s_k) - m_k\|^2 - \Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big)\|r_k - m_k\|^2.$$

The strong convergence analysis for Algorithm 2 is presented in the following theorem. The details of the convergence theorems are given below.

Algorithm 2 Modified Self-Adaptive Explicit Extragradient Method with Non-Convex Combination

Step 0: Let $s_1 \in \Pi$, $\lambda_0 > 0$, $\mu \in (0,1)$, $\delta_k \subset [\delta, 1)$ with $\delta > 0$ and $\phi_k, \varphi_k \subset (0,1)$ such that

$$\lim_{k \to +\infty} \phi_k = 0, \quad \sum_{k=1}^{+\infty} \phi_k = +\infty \quad \text{and} \quad \liminf_{k \to +\infty} \varphi_k(1 - \varphi_k) > 0.$$

Step 1: Compute

$$m_k = \arg\min_{v \in \Sigma}\{\lambda_k \mathcal{R}(P_\Sigma(s_k), v) + \frac{1}{2}\|P_\Sigma(s_k) - v\|^2\}.$$

If $m_k = s_k$, then s_k is the solution of problem (EP). Otherwise, go to next step.

Algorithm 2 *Cont.*

Step 2: First, choose $\omega_k \in \partial\mathcal{R}(P_\Sigma(s_k), m_k)$ satisfying $P_\Sigma(s_k) - \lambda_k\omega_k - m_k \in N_\mathcal{K}(m_k)$ and generate a half-space

$$\Pi_k = \{z \in \Pi : \langle P_\Sigma(s_k) - \lambda_k\omega_k - m_k, z - m_k\rangle \leq 0\}.$$

Solve

$$r_k = \underset{v\in\Pi_k}{\arg\min}\{\lambda_k\mathcal{R}(m_k, v) + \frac{1}{2}\|P_\Sigma(s_k) - v\|^2\}.$$

Step 3: Compute

$$s_{k+1} = \phi_k(1-\delta_k)s_k + (1-\phi_k)\big[\varphi_k r_k + (1-\varphi_k)s_k\big].$$

Step 4: Modify step size as follows:

$$\lambda_{k+1} = \min\left\{\lambda_k, \frac{\mu(\|P_\Sigma(s_k) - m_k\|^2 + \|r_k - m_k\|^2)}{2[\mathcal{R}(P_\Sigma(s_k), r_k) - \mathcal{R}(P_\Sigma(s_k), m_k) - \mathcal{R}(m_k, r_k)]_+}\right\}.$$

Set $k := k+1$ and go back to **Step 1**.

Theorem 2. *Let a sequence $\{s_k\}$ be generated by Algorithm 2 and satisfy the conditions ($\mathcal{R}1$)–($\mathcal{R}4$). Then, a sequence $\{s_k\}$ is strongly convergent to an element s^* of $EP(\mathcal{R}, \Sigma)$.*

Proof. Using Lemma 8, we have

$$\begin{aligned}
&\|s_{k+1} - s^*\|^2 \\
&= \|\phi_k(1-\delta_k)s_k + (1-\phi_k)\big[\varphi_k r_k + (1-\varphi_k)s_k\big] - s^*\|^2 \\
&= \|\phi_k\big[(1-\delta_k)s_k - s^*\big] + (1-\phi_k)\big[\varphi_k(r_k - s^*) + (1-\varphi_k)(s_k - s^*)\big]\|^2 \\
&\leq \phi_k\|(1-\delta_k)s_k - s^*\|^2 + (1-\phi_k)\|\varphi_k(r_k - s^*) + (1-\varphi_k)(s_k - s^*)\|^2 \\
&\leq \phi_k\|(1-\delta_k)(s_k - s^*) + \delta_k s^*\|^2 \\
&\quad + (1-\phi_k)\big[\varphi_k\|r_k - s^*\|^2 + (1-\varphi_k)\|s_k - s^*\|^2 - \varphi_k(1-\varphi_k)\|r_k - s_k\|^2\big] \\
&\leq \phi_k\big[(1-\delta_k)\|s_k - s^*\|^2 + \delta_k\|s^*\|^2\big] + (1-\phi_k)\Big[\|s_k - s^*\|^2 \\
&\quad - \varphi_k\Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big)\|P_\Sigma(s_k) - m_k\|^2 - \varphi_k\Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big)\|r_k - m_k\|^2 - \varphi_k(1-\varphi_k)\|r_k - s_k\|^2\Big] \\
&\leq (1-\phi_k\delta_k)\|s_k - s^*\|^2 + \phi_k\delta_k\|s^*\|^2 - (1-\phi_k)\Big[\varphi_k\Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big)\|P_\Sigma(s_k) - m_k\|^2 \\
&\quad + \varphi_k\Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big)\|r_k - m_k\|^2 + \varphi_k(1-\varphi_k)\|r_k - s_k\|^2\Big].
\end{aligned} \tag{28}$$

It is given that $\lambda_k \to \lambda$, there exists a fixed number $\epsilon_0 \in (0, 1-\mu)$, which is indeed a specific number such that

$$\lim_{k\to+\infty}\Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big) = 1 - \mu > \epsilon_0 > 0.$$

Thus, there exists a fixed number $m_1 \in \mathbb{N}$ such that

$$\Big(1 - \frac{\mu\lambda_k}{\lambda_{k+1}}\Big) > \epsilon_0 > 0,\ \forall\, k \geq m_1. \tag{29}$$

Combining the expression (28) and (29), we obtain

$$\|s_{k+1}-s^*\|^2 \leq \max\{\|s_k-s^*\|^2, \|s^*\|^2\} \leq \max\{\|s_{m_1}-s^*\|^2, \|s^*\|^2\}. \tag{30}$$

The value of s_{k+1} with Lemma 3 provides (see Equation (3.17) [32])

$$\begin{aligned}\|s_{k+1}-s^*\|^2 \leq (1-\phi_k\delta_k)\|s_k-s^*\|^2 + 2\phi_k\delta_k(1-\phi_k)\varphi_k\langle r_k-s_k, s_{k+1}-s^*\rangle \\ + 2\phi_k\delta_k\langle -s^*, s_{k+1}-s^*\rangle.\end{aligned} \tag{31}$$

The rest of the discussion will be divided into two parts:
Case 1: Assume that there exists an integer $m_2 \in \mathbb{N}$ $(m_2 \geq m_1)$ such that

$$\|s_{k+1}-s^*\| \leq \|s_k-s^*\|, \ \forall k \geq m_2. \tag{32}$$

Thus, the $\lim_{k\to+\infty}\|s_k-s^*\|$ exists. By expression (28), we have

$$\begin{aligned}&\epsilon_0\varphi_k\big[\|P_\Sigma(s_k)-m_k\|^2+\|r_k-m_k\|^2\big]+\varphi_k(1-\varphi_k)\|r_k-s_k\|^2 \\ &\leq \|s_k-s^*\|^2-\|s_{k+1}-s^*\|^2+\phi_k\delta_k\|s^*\|^2 \\ &\quad + \epsilon_0\phi_k\varphi_k\big[\|P_\Sigma(s_k)-m_k\|^2+\|r_k-m_k\|^2\big]+\phi_k\varphi_k(1-\varphi_k)\|r_k-s_k\|^2.\end{aligned} \tag{33}$$

The above, together with the assumptions on λ_k, ϕ_k and φ_k, yields that

$$\lim_{k\to+\infty}\|P_\Sigma(s_k)-m_k\| = \lim_{k\to+\infty}\|r_k-m_k\| = 0 = \lim_{k\to+\infty}\|s_k-r_k\| = 0. \tag{34}$$

As a result, $\{s_k\}$ is bounded, and we may choose a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ such that $\{s_{k_j}\}$ converges weakly to $s \in \Sigma$ and

$$\limsup_{k\to+\infty}\big\langle -s^*, s_k-s^*\big\rangle = \limsup_{j\to+\infty}\big\langle -s^*, s_{k_j}-s^*\big\rangle = \big\langle -s^*, s-s^*\big\rangle. \tag{35}$$

As with expression (3) with (34), we have

$$\lambda_k\mathcal{R}(P_\Sigma(s_{k_j}), y) - \lambda_{k_j}\mathcal{R}(P_\Sigma(s_{k_j}), m_k) \geq \big\langle P_\Sigma(s_{k_j})-m_{k_j}, y-m_{k_j}\big\rangle, \ \forall y \in \Sigma. \tag{36}$$

Allowing $j \to +\infty$, indicates that $\mathcal{R}(s,y) \geq 0, \ \forall y \in \Sigma$. It continues that $s \in EP(\mathcal{R},\Sigma)$. In the end, by expression (35) and Lemma 2, we may obtain

$$\begin{aligned}\limsup_{k\to+\infty}\langle -s^*, s_k-s^*\rangle &= \limsup_{j\to+\infty}\langle -s^*, s_{k_j}-s^*\rangle \\ &= \langle -s^*, s-s^*\rangle. \\ &= \langle \theta - P_{EP(\mathcal{R},\Sigma)}, s-P_{EP(\mathcal{R},\Sigma)}\rangle. \\ &\leq 0.\end{aligned} \tag{37}$$

The needed result is obtained using Equation (31) and the Lemma 4.
Case 2: Assume that a subsequence $\{k_i\}$ of $\{k\}$ such that

$$\|s_{k_i}-s^*\| \leq \|s_{k_{i+1}}-s^*\|, \ \forall i \in \mathbb{N}.$$

Thus, by Lemma 5 there exists a nondecreasing sequence $\{n_j\} \subset \mathbb{N}$ such that $\{n_j\} \to +\infty$, which gives

$$\|s_{n_j}-s^*\| \leq \|s_{m_{j+1}}-s^*\| \ \text{ and } \ \|s_j-s^*\| \leq \|s_{m_{j+1}}-s^*\|, \ \text{for all } j \in \mathbb{N}. \tag{38}$$

Using expression (31), we have

$$\begin{aligned}
&\|s_{n_j+1} - s^*\|^2 \\
&\leq (1 - \phi_{n_j}\delta_{n_j})\|s_{n_j} - s^*\|^2 + 2\phi_{n_j}\delta_{n_j}(1 - \phi_{n_j})\varphi_{n_j}\langle r_{n_j} - s_{n_j}, s_{n_j+1} - s^*\rangle \\
&\quad + 2\phi_{n_j}\delta_{n_j}\langle -s^*, s_{n_j+1} - s^*\rangle
\end{aligned} \tag{39}$$

The remaining proof is analogous to Case 2 in Theorem 1. □

4. Applications

In this section, we derive our main results, which are used to solve fixed-point and variational inequality problems. An operator $\mathcal{T} : \Sigma \subset \Pi \to \Sigma$ is said to be

(i) κ-*strict pseudocontraction* [43] on Σ if

$$\|\mathcal{T}r_1 - \mathcal{T}r_2\|^2 \leq \|r_1 - r_2\|^2 + \kappa\|(r_1 - \mathcal{T}r_1) - (r_2 - \mathcal{T}r_2)\|^2, \ \forall r_1, r_2 \in \Sigma;$$

which is equivalent to

$$\langle \mathcal{T}r_1 - \mathcal{T}r_2, r_1 - r_2\rangle \leq \|r_1 - r_2\|^2 - \frac{1-\kappa}{2}\|(r_1 - \mathcal{T}r_1) - (r_2 - \mathcal{T}r_2)\|^2, \ \forall r_1, r_2 \in \Sigma.$$

(ii) *Weakly sequentially continuous* on Σ if

$$\mathcal{T}(s_k) \rightharpoonup \mathcal{T}(s^*) \text{ as each sequence in } \Sigma \text{ satisfying } s_k \rightharpoonup s^*.$$

Note: If we take $\mathcal{R}(x, y) = \langle x - Tx, y - x\rangle$, $\forall x, y \in \Sigma$, the equilibrium problem converts into to the fixed-point problem through $2c_1 = 2c_2 = \frac{3-2\kappa}{1-\kappa}$. The algorithm's m_k and r_k values become (for more information, see [32]):

$$\begin{cases}
m_k = \arg\min_{v\in\Sigma}\{\lambda_k\mathcal{R}(s_k, v) + \frac{1}{2}\|s_k - v\|^2\} = (1 - \lambda_k)s_k + \lambda_k\mathcal{T}(s_k), \\
r_k = \arg\min_{v\in\Pi_k}\{\lambda_k\mathcal{R}(m_k, v) + \frac{1}{2}\|s_k - v\|^2\} = P_\Sigma\big[s_k - \lambda_k(m_k - \mathcal{T}(m_k))\big].
\end{cases} \tag{40}$$

The following fixed-point theorems are derived from the results in Section 3.

Corollary 1. *Suppose that Σ is a nonempty closed and convex subset of a Hilbert space Π. Let $\mathcal{T} : \Sigma \to \Sigma$ is a weakly continuous and κ-strict pseudocontraction with $Fix(\mathcal{T}) \neq \emptyset$. Let $s_1 \in \Sigma$, $\lambda_0 > 0$, $\mu \in (0, 1)$, $\delta_k \subset [\delta, 1)$ with $\delta > 0$ and $\phi_k \subset (0, 1)$*

$$\lim_{k\to+\infty}\phi_k = 0 \ \text{ and } \ \sum_{k=1}^{+\infty}\phi_k = +\infty.$$

Additionally, the sequence $\{s_k\}$ is created as follows:

$$\begin{cases}
m_k = (1 - \lambda_k)s_k + \lambda_k\mathcal{T}(s_k), \\
r_k = P_{\Pi_k}\big[s_k - \lambda_k(m_k - \mathcal{T}(m_k))\big], \\
s_{k+1} = P_\Sigma\big[\phi_k s_k + (1 - \phi_k)r_k - \phi_k\delta_k s_k\big],
\end{cases}$$

where

$$\Pi_k = \{z \in \Pi : \langle s_k - \lambda_k\mathcal{T}(s_k) - m_k, z - m_k\rangle \leq 0\}.$$

The relevant step-size λ_{k+1} is obtained:

$$\lambda_{k+1} = \min\left\{\lambda_k, \frac{\mu(\|s_k - m_k\|^2 + \|r_k - m_k\|^2)}{2\big[\langle(s_k - m_k) - (\mathcal{T}(s_k) - \mathcal{T}(m_k)), r_k - m_k\rangle\big]_+}\right\}.$$

Thus, the sequence $\{s_k\}$ *strongly converges to* $s^* = P_{Fix(\mathcal{T})}(\theta)$.

Corollary 2. *Suppose that* Σ *is a nonempty closed and convex subset of a Hilbert space* Π. *Let* $\mathcal{T} : \Sigma \to \Sigma$ *is a weakly continuous and* κ*-strict pseudocontraction with* $Fix(\mathcal{T}) \neq \emptyset$. *Let* $s_1 \in \Pi$, $\lambda_0 > 0$, $\mu \in (0,1)$, $\delta_k \subset [\delta, 1)$ *with* $\delta > 0$ *and* $\phi_k, \varphi_k \subset (0,1)$ *such that*

$$\lim_{k\to+\infty} \phi_k = 0, \ \sum_{k=1}^{+\infty} \phi_k = +\infty \quad \text{and} \quad \liminf_{k\to+\infty} \varphi_k(1-\varphi_k) > 0.$$

Additionally, the sequence $\{s_k\}$ *is created as follows:*

$$\begin{cases} m_k = (1-\lambda_k)P_\Sigma(s_k) + \lambda_k \mathcal{T}(P_\Sigma(s_k)), \\ r_k = P_{\Pi_k}\big[s_k - \lambda_k(m_k - \mathcal{T}(m_k))\big], \\ s_{k+1} = \phi_k(1-\delta_k)s_k + (1-\phi_k)\big[\varphi_k r_k + (1-\varphi_k)s_k\big], \end{cases}$$

where

$$\Pi_k = \{z \in \Pi : \langle P_\Sigma(s_k) - \lambda_k \mathcal{T}(P_\Sigma(s_k)) - m_k, z - m_k\rangle \leq 0\}.$$

The relevant step size λ_{k+1} *is obtained as follows:*

$$\lambda_{k+1} = \min\left\{\lambda_k, \frac{\mu(\|P_\Sigma(s_k) - m_k\|^2 + \|r_k - m_k\|^2)}{2\big[\langle (P_\Sigma(s_k) - m_k) - (\mathcal{T}(P_\Sigma(s_k)) - \mathcal{T}(m_k)), r_k - m_k\rangle\big]_+}\right\}.$$

Thus, the sequence $\{s_k\}$ *strongly converges to* $s^* = P_{Fix(\mathcal{T})}(\theta)$.

The variational inequality problem is presented as follows:

$$\text{Find } s^* \in \Sigma \text{ such that } \big\langle G(s^*), y - s^*\big\rangle \geq 0, \ \forall y \in \Sigma.$$

An operator $G : \Pi \to \Pi$ is said to be

(i) *L-Lipschitz continuous* on Σ if

$$\|G(r_1) - G(r_2)\| \leq L\|r_1 - r_2\|, \ \forall r_1, r_2 \in \Sigma;$$

(ii) *pseudomonotone* on Σ if

$$\big\langle G(r_1), r_2 - r_1\big\rangle \geq 0 \Longrightarrow \big\langle G(r_2), r_1 - r_2\big\rangle \leq 0, \ \forall r_1, r_2 \in \Sigma.$$

Note: If $\mathcal{R}(x,y) := \big\langle G(x), y - x\big\rangle$ for all $x, y \in \Sigma$, the equilibrium problem converts into a variational inequality problem via $L = 2c_1 = 2c_2$ (for more information, see [44]). By the value of m_k and r_k in Algorithm 1, we derived

$$\begin{cases} m_k = \arg\min_{v\in\Sigma}\{\lambda_k \mathcal{R}(s_k, v) + \frac{1}{2}\|s_k - v\|^2\} = P_\Sigma\big[s_k - \lambda_k G(s_k)\big], \\ r_k = \arg\min_{v\in\Pi_k}\{\lambda_k \mathcal{R}(m_k, v) + \frac{1}{2}\|s_k - v\|^2\} = P_{\Pi_k}\big[s_k - \lambda_k G(m_k)\big]. \end{cases} \tag{41}$$

Due to $\omega_k \in \partial\mathcal{R}(s_k, m_k)$, we obtain

$$\begin{aligned} \langle \omega_k, z - m_k\rangle &\leq \langle G(s_k), z - s_k\rangle - \langle G(s_k), m_k - s_k\rangle, \ \forall z \in \Pi \\ &= \langle G(s_k), z - m_k\rangle, \ \forall z \in \Pi, \end{aligned} \tag{42}$$

and consequently $0 \leq \langle G(s_k) - \omega_k, z - m_k \rangle, \forall z \in \Pi$. It implies that

$$\begin{aligned}&\langle s_k - \lambda_k G(s_k) - m_k, z - m_k \rangle \\ &\leq \langle s_k - \lambda_k G(s_k) - m_k, z - m_k \rangle + \lambda_k \langle G(s_k) - \omega_k, z - m_k \rangle \\ &= \langle s_k - \lambda_k \omega_k - m_k, z - m_k \rangle. \end{aligned} \tag{43}$$

Assumption 1. *Assume that G fulfills the following conditions:*

(i) *An operator G is pseudomonotone upon Σ and $VI(G,\Sigma)$ is nonempty;*
(ii) *G is L-Lipschitz continuous on Σ with $L > 0$;*
(iii) $\limsup_{k\to+\infty}\langle G(s_k), y - s_k \rangle \leq \langle G(s^*, y - s^* \rangle$ *for any $y \in \Sigma$ and $\{s_k\} \subset \Sigma$ meet $s_k \rightharpoonup s^*$.*

Corollary 3. *Let $G : \Sigma \to \Pi$ be an operator and satisfies Assumption 1. Assume that sequence $\{s_k\}$ is generated as follows: Let $s_1 \in \Pi$, $\lambda_0 > 0$, $\mu \in (0,1)$, $\delta_k \subset [\delta, 1)$ with $\delta > 0$ and $\phi_k \subset (0,1)$ such that*

$$\lim_{k\to+\infty} \phi_k = 0 \ \textit{and} \ \sum_{k=1}^{+\infty} \phi_k = +\infty.$$

Moreover, sequence $\{s_k\}$ is generated as follows:

$$\begin{cases} m_k = P_\Sigma\big[s_k - \lambda_k G(s_k)\big], \\ r_k = P_{\Pi_k}\big[s_k - \lambda_k G(m_k)\big], \\ s_{k+1} = P_\Sigma\big[\phi_k s_k + (1-\phi_k) r_k - \phi_k \delta_k s_k\big], \end{cases}$$

where

$$\Pi_k = \{z \in \Pi : \langle s_k - \lambda_k G(s_k) - m_k, z - m_k \rangle \leq 0\}.$$

Next, step size λ_{k+1} is obtained as follows:

$$\lambda_{k+1} = \min\left\{\lambda_k, \frac{\mu(\|s_k - m_k\|^2 + \|r_k - m_k\|^2)}{2\big[\langle F(s_k) - F(m_k), r_k - m_k \rangle\big]_+}\right\}.$$

Then, sequence $\{s_k\}$ strongly converges to the solution $s^* \in VI(G,\Sigma)$.

Corollary 4. *Let $G : \Sigma \to \Pi$ be an operator that satisfies Assumption 1. Assume that $\{s_k\}$, is generated as follows: Let $s_1 \in \Pi$, $\lambda_0 > 0$, $\mu \in (0,1)$, $\delta_k \subset [\delta, 1)$ with $\delta > 0$ and $\phi_k, \varphi_k \subset (0,1)$ such that*

$$\lim_{k\to+\infty} \phi_k = 0, \ \sum_{k=1}^{+\infty} \phi_k = +\infty \quad \textit{and} \quad \liminf_{k\to+\infty} \varphi_k(1-\varphi_k) > 0.$$

Moreover, the sequence $\{s_k\}$ generated as follows:

$$\begin{cases} m_k = P_\Sigma\big[P_\Sigma(s_k) - \lambda_k G(P_\Sigma(s_k))\big], \\ r_k = P_{\Pi_k}\big[P_\Sigma(s_k) - \lambda_k G(m_k)\big], \\ s_{k+1} = \phi_k(1-\delta_k)s_k + (1-\phi_k)\big[\varphi_k r_k + (1-\varphi_k)s_k\big], \end{cases}$$

where

$$\Pi_k = \{z \in \Pi : \langle P_\Sigma(s_k) - \lambda_k G(P_\Sigma(s_k)) - m_k, z - m_k \rangle \leq 0\}.$$

Next step-size λ_{k+1} is obtained as follows:

$$\lambda_{k+1} = \min\left\{\lambda_k, \frac{\mu(\|P_\Sigma(s_k) - m_k\|^2 + \|r_k - m_k\|^2)}{2\big[\langle F(P_\Sigma(s_k)) - F(m_k), r_k - m_k \rangle\big]_+}\right\}.$$

Then, sequence $\{s_k\}$ strongly converges to the solution $s^* \in VI(G, \Sigma)$.

5. Numerical Illustration

The computational results in this section show that our proposed algorithms are more efficient than Algorithms 3.1 and 3.2 in [32]. The MATLAB program was executed in MATLAB version 9.5 on a PC (with Intel(R) Core(TM)i3-4010U CPU @ 1.70 GHz 1.70 GHz, RAM 4.00 GB) (R2018b). In all our algorithms, we used the built-in MATLAB fmincon function to solve the minimization problems. (i) The setting for design variables for Algorithm 3.1 (Algo. 3.1) and Algorithm 3.2 (Algo. 3.2) in [32] possess different values that are given in all examples.

$$\phi_k = \frac{1}{40k}, \delta_k = \frac{1}{10} + \frac{1}{10k}, \lambda_k = \frac{k}{3+2c_1}, \varphi_k = \frac{1}{4} + \frac{1}{4n} \text{ and } D_k = \|s_k - m_k\| \leq \epsilon.$$

(ii) The settings for the design variables for Algorithm 1 (Algo. 1) and Algorithm 2 (Algo. 2) are

$$\phi_k = \frac{1}{40k}, \delta_k = \frac{1}{10} + \frac{1}{10k}, \varphi_k = \frac{1}{4} + \frac{1}{4k}, D_k = \|s_k - m_k\| \leq \epsilon \text{ and for different } \lambda_0.$$

Example 1. *Let us consider a bifunction $\mathcal{R} : \Sigma \times \Sigma \to \mathbb{R}$, which is represented as follows:*

$$\mathcal{R}(s, m) = \sum_{i=2}^{5} (m_i - s_i)\|s\|, \; \forall s, m \in \mathbb{R}^5.$$

In addition, the convex set is defined as follows:

$$\Sigma = \{(s_1, \cdots, s_5) : s_1 \geq -1, s_i \geq 1, i = 2, \cdots, 5\}.$$

Consequently, $\mathcal{R}$ is Lipschitz-type continuous across $c_1 = c_2 = 2$ and meets the condition ($\mathcal{R}1$)–($\mathcal{R}4$). The obtained simulations are shown in Figures 1 and 2 and Tables 1 and 2 by using $s_1 = (2, 3, 2, 5, 5)$ and $\epsilon = 10^{-4}$.

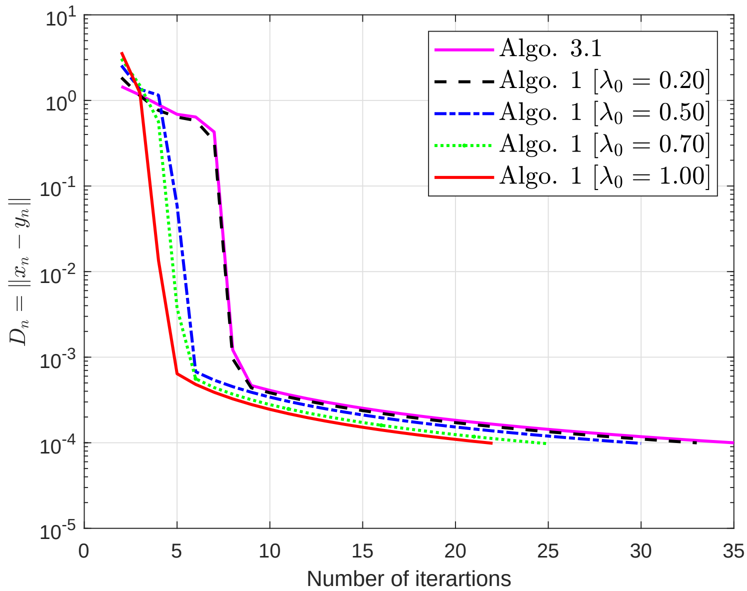

Figure 1. Algorithm 1 is compared to Algorithm 3.1 in [32].

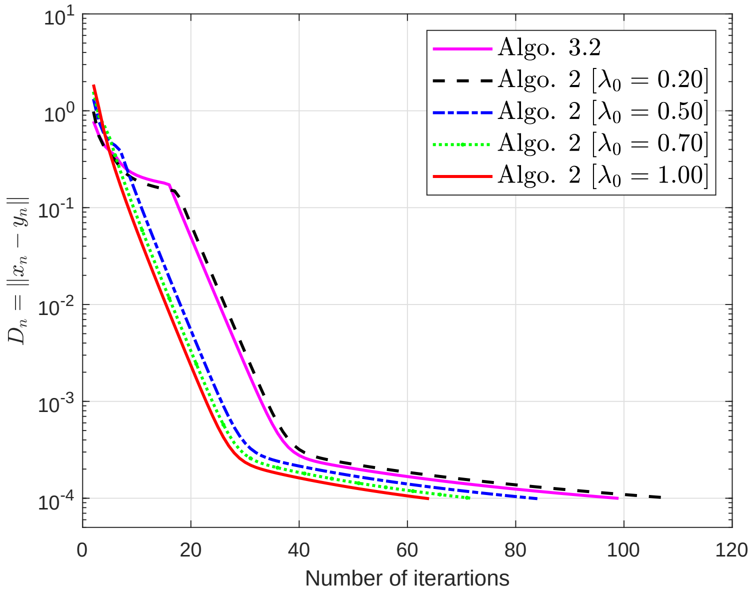

Figure 2. Algorithm 2 is compared to Algorithm 3.2 in [32].

Table 1. Algorithm 1 is compared to Algorithm 3.1 in [32].

	Number of Iterations		CPU Time in Seconds	
λ_0	**Algo. 3.1**	**Algo. 1**	**Algo. 3.1**	**Algo. 1**
0.20	35	33	1.6799	1.6119
0.50	-	30	-	1.4787
0.70	-	25	-	1.1520
1.00	-	22	-	1.0100

Table 2. Algorithm 2 is compared to Algorithm 3.2 in [32].

	Number of Iterations		CPU Time in Seconds	
λ_0	**Algo. 3.2**	**Algo. 2**	**Algo. 3.2**	**Algo. 2**
0.20	99	109	4.9391	5.3081
0.50	-	84	-	4.0511
0.70	-	72	-	3.2269
1.00	-	64	-	2.9225

Example 2. *According to the articles [29], the bifunction $\mathcal{R}$ might be written as follows:*

$$\mathcal{R}(s, m) = \langle As + Bm + c, m - s \rangle,$$

where $c \in \mathbb{R}^5$ and A, B are

$$A = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

The Lipschitz parameters are also $c_1 = c_2 = \frac{1}{2}\|A - B\|$ (see [29]). The possible set Σ and its subset $\mathbb{R}^5$ are given as

$$\Sigma := \{s \in \mathbb{R}^5 : -5 \le s_i \le 5\}.$$

Figures 3 and 4 and Tables 3 and 4 display the numeric effects with $s_1 = (1, \cdots, 1)$ *and* $\epsilon = 10^{-6}$.

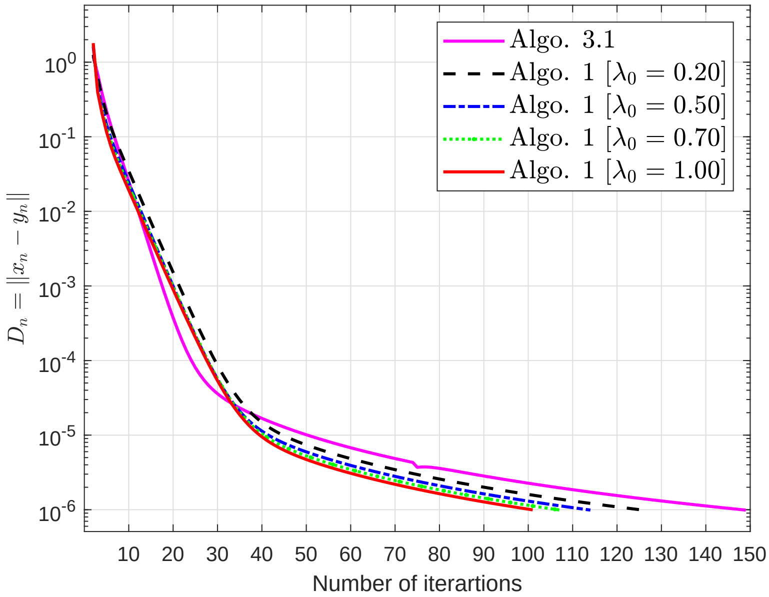

Figure 3. Algorithm 1 is compared to Algorithm 3.1 in [32].

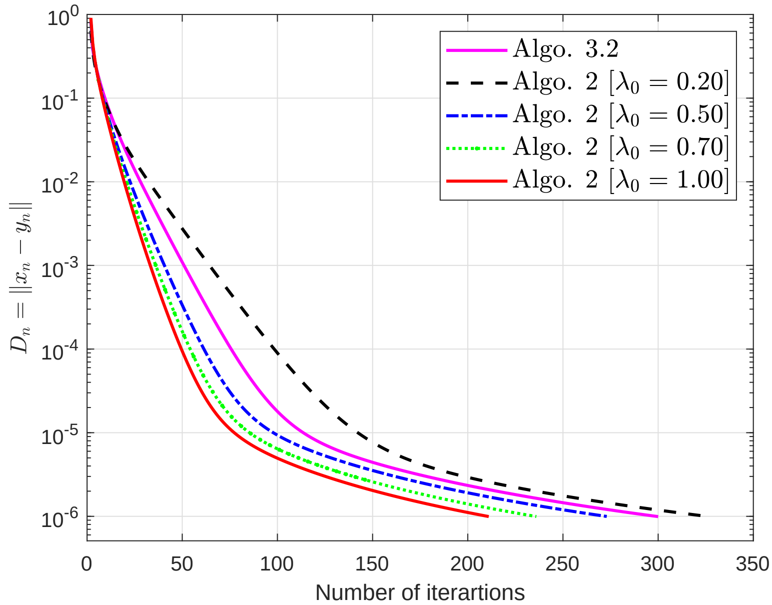

Figure 4. Algorithm 2 is compared to Algorithm 3.2 in [32].

Table 3. Algorithm 1 is compared to Algorithm 3.1 in [32].

	Number of Iterations		CPU Time in Seconds	
λ_0	**Algo. 3.1**	**Algo. 1**	**Algo. 3.1**	**Algo. 1**
0.20	149	126	7.0363	5.7321
0.50	-	114	-	5.0527
0.70	-	107	-	4.8159
1.00	-	101	-	4.5495

Table 4. Algorithm 2 is compared to Algorithm 3.2 in [32].

	Number of Iterations		CPU Time in Seconds	
λ_0	**Algo. 3.2**	**Algo. 2**	**Algo. 3.2**	**Algo. 2**
0.20	300	326	14.7791	14.2233
0.50	-	273	-	13.7107
0.70	-	236	-	11.7754
1.00	-	211	-	11.2054

Example 3. *Consider that* $\Pi = L^2([0,1])$ *is indeed a Hilbert space with*

$$\|s\| = \sqrt{\int_0^1 |s(t)|^2 dt},$$

where the internal product

$$\langle s, m \rangle = \int_0^1 s(t)m(t)dt, \ \forall s, m \in \Pi.$$

Suppose that unit ball is $\Sigma := \{s \in L^2([0,1]) : \|s\| \leq 1\}$. *Let us begin by defining an operator*

$$G(s)(t) = \int_0^1 \big(s(t) - H(t,s)\mathcal{R}(s(s))\big)ds + g(t),$$

where

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \quad \mathcal{R}(s) = cosx, \quad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

As illustrated in [45], G is monotone and L-Lipschitz-continuous via $L = 2$. *Figures 5 and 6 and Tables 5 and 6 illustrate the numerical results with* $s_1 = t$ *and* $\epsilon = 10^{-6}$.

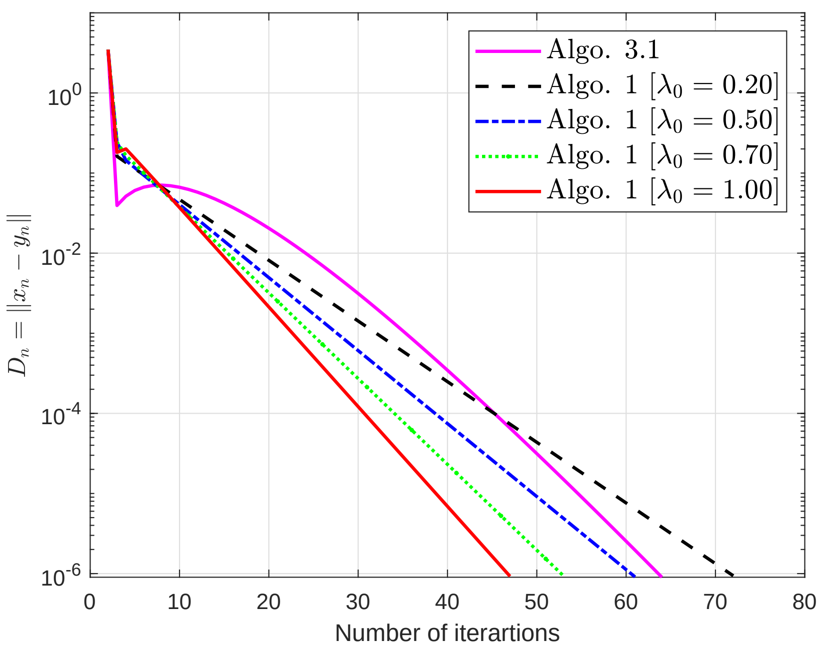

Figure 5. Algorithm 1 is compared to Algorithm 3.1 in [32].

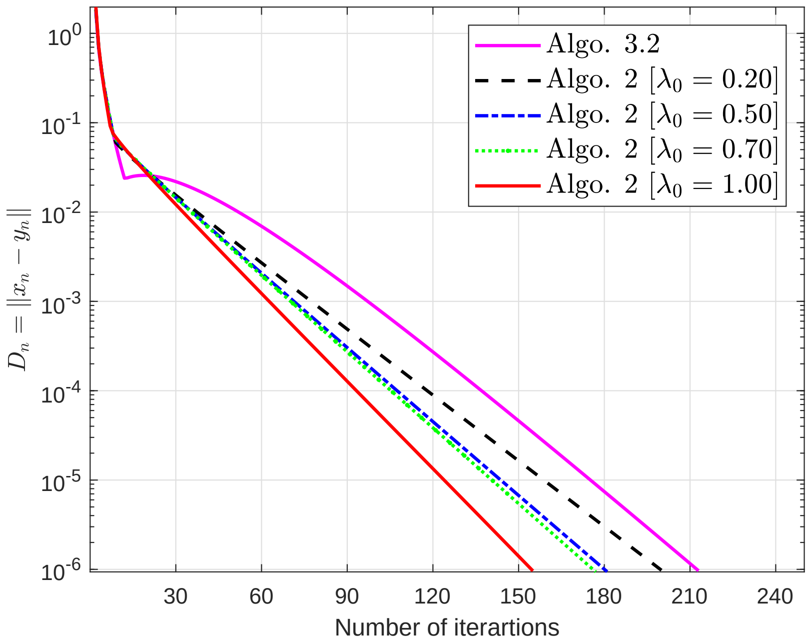

Figure 6. Algorithm 2 is compared to Algorithm 3.2 in [32].

Table 5. Algorithm 1 is compared to Algorithm 3.1 in [32].

	Number of Iterations		CPU Time in Seconds	
λ_0	**Algo. 3.1**	**Algo. 1**	**Algo. 3.1**	**Algo. 1**
0.20	64	72	0.0174	0.0331
0.50	–	61	–	0.0295
0.70	–	53	–	0.0273
1.00	–	47	–	0.0265

Table 6. Algorithm 2 is compared to Algorithm 3.2 in [32].

	Number of Iterations		CPU Time in Seconds	
λ_0	**Algo. 3.2**	**Algo. 2**	**Algo. 3.2**	**Algo. 2**
0.20	213	200	0.0313	0.0500
0.50	–	181	–	0.0460
0.70	–	177	–	0.0352
1.00	–	155	–	0.0260

Discussion About Numerical Experiments: The following conclusions may be drawn from the numerical experiments outlined above: (i) Examples 1–3 have reported data for numerous methods in both finite- and infinite-dimensional domains. It is apparent that the given algorithms outperformed in terms of number of iterations and elapsed time in practically all circumstances. All trials demonstrate that the suggested algorithms outperform the previously available techniques. (ii) Examples 1–3 have reported results for several methods in finite and infinite-dimensional domains. In most cases, we can observe that the scale of the problem and the relative standard deviation used impact the algorithm's effectiveness. (iii) The development of an inappropriate variable step size generates a hump in the graph of algorithms in all examples. It has no impact on the effectiveness of the algorithms. (iv) For large-dimensional problems, all approaches typically took longer and showed significant variation in execution time. The number of iterations, on the other hand, changes slightly less.

6. Conclusions

The paper provides two explicit extragradient-like approaches for solving an equilibrium problem involving a pseudomonotone and a Lipschitz-type bifunction in a real Hilbert space. A new step-size rule has been presented that does not rely on Lipschitz-type constant information. The algorithm's convergence has been established. Several tests are presented to show the numerical behavior of our two algorithms and to compare them to others that are well known in the literature.

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