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Hesitant Fuzzy Variable and Distribution

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Abstract: In recent decades, the hesitant fuzzy set theory has been used as a main tool to describe the hesitant fuzzy phenomenon, which usually exists in multiple attributes of decision making. However, in the general case concerning numerous decision-making problems, values of attributes are real numbers, and some decision makers are hesitant about these values. Consequently, the possibility of taking a number contains several possible values in the real number interval [0, 1]. As a result, the hesitant possibility of hesitant fuzzy events cannot be inferred from the given hesitant fuzzy set which only presents the hesitant membership degree with respect to an element belonging to this one. To address this problem, this paper explores the axiomatic system of the hesitant possibility measure from which the hesitant fuzzy theory is constructed. Firstly, a hesitant possibility measure from the pattern space to the power set of [0, 1] is defined, and some properties of this measure are discussed. Secondly, a hesitant fuzzy variable, which is a symmetric set-valued function on the hesitant possibility measure space, is proposed, and the distribution of this variable and one of its functions are studied. Finally, two examples are shown in order to explain the practical applications of the hesitant fuzzy variable in the hesitant fuzzy graph model and decision-making considering hesitant fuzzy attributes. The relevant research results of this paper provide an important mathematical tool for hesitant fuzzy information processing from another new angle different from the theory of hesitant fuzzy sets, and can be utilized to solve decision problems in light of the hesitant fuzzy value of multiple attributes.

Keywords: hesitant possibility measure; hesitant fuzzy variable; hesitant possibility distribution; hesitant fuzzy graph; hesitant credibility graph

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1. Introduction

The hesitant fuzzy set, which is an extended version of the fuzzy set [1], is characterized by hesitant membership grades that contain several possible real numbers in an interval [0, 1] [2], and is illustrated by decomposition theorems and extension principles concerning this concept [3]. In the 2010s, the theory and application in respect to the hesitant fuzzy set were investigated for making a group decision. For one thing, Xia et al. introduced some operators with respect to hesitant fuzzy sets and utilized these operators to deal with hesitant fuzzy information [4]. For another thing, on the basis of the assumption that elements of a hesitant fuzzy set are increasingly arranged and the length of two elements is the same, Xia et al. proposed some distance measures of hesitant fuzzy sets [5]. Consequently, they considered connections of these measures and presented several kinds of similarity measures for hesitant fuzzy sets [6]. After that, Xia et al. extended the preference relation to the hesitant fuzzy case and introduced some relevant properties [7]. Zhang et al. developed the normalization extension and group extension concerning the hesitant fuzzy preference relation for describing the consensus process in terms of hesitant fuzzy information [8,9]. Zhu et al. defined a consensus index for measuring the agreement between the group and arbitrary individual hesitant fuzzy preference relation considering hesitant fuzzy attributes [10]. Xu et al. introduced a consensus model with respect to the hesitant fuzzy preference relation and presented feedback mechanisms [11]. Zhang et al. established some consistency models in order to obtain missing elements for incomplete hesitant fuzzy preference relations. In addition, some aggregation operators have been introduced to fuse hesitant fuzzy sets in decision making [12–16]. These operators are able to provide powerful guarantees for decision makers to handle complex situations. In order to determine better group decisions, some methods based on hesitant fuzzy sets or the extended hesitant fuzzy sets have been introduced, for instance, the hesitant fuzzy TOPSIS [17], m-polar hesitant fuzzy TOPSIS approach [18], hesitant fuzzy PROMETHEE [19], hesitant fuzzy ELECTRE [20], hesitant fuzzy VIKOR [21], hesitant fuzzy TODIM [22], hesitant fuzzy QUALIFLEX [23], hesitant N-soft sets decision method [24], necessary and possible hesitant fuzzy sets method [25], dual extended hesitant fuzzy sets method [26], hesitant fuzzy N-soft sets method [27] and hesitant fuzzy LINMAP [28].

In recent years, the hesitant fuzzy set theory has been used as a main mathematical tool to describe the hesitant fuzzy phenomenon in the research of multiple attributes of decision making. However, a hesitant fuzzy set only presents several possible values on the unit interval [0, 1], which defines the hesitant membership of an element belonging to this hesitant fuzzy set, and the hesitant possibility of hesitant fuzzy events cannot be inferred from the given hesitant fuzzy sets [2]. Because it is the general case that the value of an attribute is a real number in multiple attributes of group decision making problems, decision makers are hesitant about this real number corresponding to the attribute, that is, the possibility of taking this real number contains several possible values in the interval [0, 1]. In this situation, the hesitant fuzzy model is more appropriate to describe hesitant fuzzy events. On the one hand, probability measures can explain the probability of the occurrence of random events and a random variable is essentially a measurable function from the probability measure space to the real value line [29]. On the other hand, as a pair of dual measures, the possibility measure and necessity measure can show the possibility and necessity of fuzzy events, and the basic concept of fuzzy theory is a fuzzy variable, which is a measurable function from measure space to the real set. Similarly, a credibility measure is the self-dual fuzzy measure, which can measure the credibility of fuzzy events [30–32]. Therefore, the axiomatic system with respect to the hesitant possibility of hesitant fuzzy events has important theoretical significance in the study of the hesitant fuzzy phenomenon; this axiomatic framework is based on the hesitant fuzzy variable, which is a symmetric set-valued function with respect to the union operation of the hesitant fuzzy variable, and can provide theoretical support for hesitant fuzzy information processing.

In this paper, the main contributions include several points: (1) Proposing a hesitant possibility measure, a hesitant necessary measure and a hesitant credibility measure and studying some properties with respect to these three set-valued measures. (2) Defining a hesitant fuzzy variable on the hesitant possibility measure space, and discussing the distribution of this variable and one of its functions and listing several common hesitant fuzzy variables, including triangle, trapezoid, normal and exponential types. (3) Introducing the concept of the hesitant fuzzy graph based on the proposed hesitant fuzzy variable. The rest of this paper is structured as follows: In Section 2, preliminaries are illustrated. Section 3 presents the axiomatic system with respect to the hesitant possibility of hesitant fuzzy events. Two applications are showed in Section 4. Some conclusions and future works are stated in the final section.

2. Preliminaries

Definition 1 [33]. *If Γ is a domain of discourse and I is any index set, an ample field $\mathcal{A}$ is a collection of subsets of Γ with the following conditions:*

(1) $\Gamma \in \mathcal{A}$;
(2) $A \in \mathcal{A} \Rightarrow \Gamma \backslash A \in \mathcal{A}$;
(3) $B_k \in \mathcal{A}, k \in I \Rightarrow \bigcup_{k \in I} B_k \in \mathcal{A}$.

$(\Gamma, \mathcal{A})$ is called an ample space. Evidently, $(\Gamma, \mathcal{P}(\Gamma))$ is a special ample space in which $\mathcal{P}(\Gamma)$ is the power set of Γ.

Definition 2 [34]. *Let $(\Gamma, \mathcal{A})$ be an ample space. A fuzzy measure on an ample field $\mathcal{A}$ is defined by a function μ from $\mathcal{A}$ to $[0, \infty]$ with two conditions:*

(1) $\mu(\phi) = 0$;
(2) $B_1 \in \mathcal{A}, B_2 \in \mathcal{A}, B_1 \subset B_2 \Rightarrow \mu(B_1) \leq \mu(B_2)$.

Particularly, $\mu : \mathcal{A} \to [0, \infty]$ is called the lower semicontinuous when:

$$B_n \in \mathcal{A}, n = 1, 2, \cdots, B_1 \subset B_2 \subset \cdots \Rightarrow \lim_{n \to \infty} \mu(B_n) = \mu(\cup_{n=1}^{\infty} B_n).$$

$\mu : \mathcal{A} \to [0, \infty]$ is called upper semicontinuous when:

$$B_n \in \mathcal{A}, n = 1, 2, \cdots, B_1 \supset B_2 \supset \cdots \Rightarrow \lim_{n \to \infty} \mu(B_n) = \mu(\cap_{n=1}^{\infty} B_n).$$

Definition 3 [35]. *Let $(\Gamma, \mathcal{A})$ be an ample space Γ and I be an arbitrary index set. A set function $Pos : \mathcal{A} \to [0, 1]$ is called the possibility measure on $\mathcal{A}$ when the following conditions hold:*

(1) $Pos(\phi) = 0, Pos(\Gamma) = 1$;
(2) $B_k \in \mathcal{A}, k \in I \Rightarrow Pos(\cup_{k \in I} B_k) = \sup_{k \in I} Pos(B_k)$.

The triple $(\Gamma, \mathcal{A}, Pos)$ is defined as a possibility measure space. Set functions $Nec : \mathcal{A} \to [0, 1]$ and $Cr : \mathcal{A} \to [0, 1]$ are, respectively, called the necessity measure and credibility measure on $\mathcal{A}$ if the following formulas hold for the arbitrary set $B \in \mathcal{A}$.

$$Nec(B) = 1 - Pos(B^c)$$

$$Cr(B) = \frac{1}{2}(Pos(B) + Nec(B))$$

Definition 4 [36]. *The T norm is a binary operation T from $[0, 1] \times [0, 1]$ to $[0, 1]$, and it satisfies the following formulas for three real numbers t_1, t_2 and t_3 on $[0, 1]$.*

(1) $T(t_1, t_2) = T(t_2, t_1)$;
(2) $T(t_1, T(t_2, t_3)) = T(T(t_1, t_2), t_3)$;
(3) $t_2 \leq t_3 \Rightarrow T(t_1, t_2) \leq T(t_1, t_3)$;
(4) $T(t_1, 1) = t_1, T(t_1, 0) = 0$.

Similarly, $S : [0, 1] \times [0, 1] \to [0, 1]$ is called the S norm if several conditions are satisfied for arbitrary $t_1, t_2, t_3 \in [0, 1]$.

(1) $S(t_1, t_2) = S(t_2, t_1)$;
(2) $S(t_1, S(t_2, t_3)) = S(S(t_1, t_2), t_3)$;
(3) $t_2 \leq t_3 \Rightarrow S(t_1, t_2) \leq S(t_1, t_3)$;
(4) $S(t_1, 1) = 1, S(t_1, 0) = t_1$.

3. Hesitant Fuzzy Variable and Its Distribution

In this section, the order relation " $\leq$ " on $\mathcal{P}([0, 1])$ is introduced. Let $A, B \in \mathcal{P}([0, 1])$, so $A \leq B$ means that:

(1) For $x_0 \in A$, $y_0 \in B$ exists with $x_0 \leq y_0$;
(2) For $y_0 \in B$, $x_0 \in A$ exists with $x_0 \leq y_0$.

When $A, B \in \Gamma$, operations $A + B$ and $A \times B$ are presented in the following formulas.

$$A + B = \{ S(t_1, t_2) | t_1 \in A, t_2 \in B \} \tag{1}$$

$$A \times B = \{ T(t_1, t_2) | t_1 \in A, t_2 \in B \} \tag{2}$$

3.1. Hesitant Possibility Measure Space

Definition 5. *Let $(\Gamma, \mathcal{A})$ be an ample space, $\mathcal{P}([0,1])$ be the power set of [0, 1] and I an index set. If the following conditions are satisfied, a set-valued set function $Pos_H : \mathcal{A} \to \mathcal{P}([0,1])$ is called the hesitant possibility measure on $\mathcal{A}$.*

(1) $Pos_H(\phi) = \{0\}, Pos_H(\Gamma) = \{1\}$;

(2) $B_k \in \mathcal{A}, k \in I \Rightarrow Pos_H(\cup_{k\in I} B_k) = \left\{ \sup_{k\in I}\{t_k \,|\, t_k \in Pos_H(B_k)\} \right\}$.

Any $B_k \in \mathcal{A}$ is a hesitant fuzzy event, and the triple $(\Gamma, \mathcal{A}, Pos_H)$ is a hesitant possibility measure space. In this paper, the finite real values of [0, 1] are discussed with respect to the hesitant possibility measure Pos_H, which is a symmetric mapping with respect to the union operation of subsets in an ample $\mathcal{A}$. Particularly,

(1) For two sets $B_1 \in \mathcal{A}$ and $B_2 \in \mathcal{A}$, the following formulation holds, where B_1 and B_2 are mutually independent:

$$Pos_H(B_1 \cap B_2) = \left\{ t_1 \wedge t_2 \middle| \begin{array}{l} t_1 \in Pos_H(B_1) \\ t_2 \in Pos_H(B_2) \end{array} \right\}$$

(2) A hesitant necessity measure Nec_H from $\mathcal{A}$ to $\mathcal{P}([0,1])$ is presented for arbitrary $B \in \mathcal{A}$:

$$Nec_H(B) = \left\{ 1 - t | t \in Pos_H(B^c) \right\}$$

(3) A hesitant credibility measure Cr_H from $\mathcal{A}$ to $\mathcal{P}([0,1])$ is defined for arbitrary $B \in \mathcal{A}$:

$$Cr_H(B) = \left\{ \frac{1}{2}(t_1 + t_2) \middle| t_1 \in Pos_H(B), t_2 \in Nec_H(B) \right\}$$

The triple $(\Gamma, \mathcal{A}, Cr_H)$ is a hesitant credibility measure space. By Definition 5, some properties of hesitant possibility measure were discussed on the basis of a theorem.

Theorem 6. *A hesitant possibility measure Pos_H on an ample space $(\Gamma, \mathcal{A})$ had the following properties:*

(1) *Monotonicity:*

$$B_1 \in \mathcal{A}, B_2 \in \mathcal{A} B_1 \subset B_2 \Rightarrow Pos_H(B_1) \le Pos_H(B_2).$$

(2) *Boundedness:*

$$B \in \mathcal{A} \Rightarrow \{0\} \le Pos_H(B) \le \{1\}.$$

(3) *Lower semicontinuity on a closed interval set sequence $\{Pos_H(B_n)\}$:*

$$B_n \in \mathcal{A}, n = 1, 2, \cdots, B_1 \subset B_2 \subset \cdots, \Rightarrow \lim_{n\to\infty} Pos_H(B_n) = Pos_H(\cup_{n=1}^{\infty} B_n).$$

(4) *Strong subadditivity:*

$$B_1 \in \mathcal{A}, B_2 \in \mathcal{A} \Rightarrow Pos_H(B_1 \cup B_2) + Pos_H(B_1 \cap B_2) \le Pos_H(B_1) + Pos_H(B_2).$$

Proof.

(1) Monotonicity.

$$Pos_H(B_1) = \{t_1 \,|\, t_1 \in Pos_H(B_1)\} \le \left\{ t_1 \vee t_2 \middle| \begin{array}{l} t_1 \in Pos_H(B_1) \\ t_2 \in Pos_H(B_2 - B_1) \end{array} \right\} = Pos_H(B_1 \cup (B_2 - B_1)) = Pos_H(B_2).$$

(2) Boundedness. According to (1), it was evident that the boundedness held.

(3) Lower semicontinuity. The following topology was introduced:

$$\lim_{n\to\infty}\sup Pos_H(B_n) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} Pos_H(B_k) = \left\{ t\in[0,1] \;\middle|\; \begin{array}{l} t=\lim\limits_{k\to\infty}(t_{n_k}) \\ t_{n_k}\in Pos_H(B_k) \end{array} \right\}$$

$$\lim_{n\to\infty}\inf Pos_H(B_n) = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} Pos_H(B_k) = \left\{ t\in[0,1] \;\middle|\; \begin{array}{l} t=\lim\limits_{n\to\infty}(t_n) \\ t_n\in Pos_H(B_n) \end{array} \right\}$$

Evidently,

$$\lim_{n\to\infty}\inf Pos_H(B_n) = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} Pos_H(B_k) \subset \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} Pos_H(B_k) = \lim_{n\to\infty}\sup Pos_H(B_n) \tag{3}$$

Therefore, the following aspects were discussed:

(a) $\bigcup\limits_{n=1}^{\infty}\bigcap\limits_{k=n}^{\infty} Pos_H(B_k) \le Pos_H\left(\bigcup\limits_{k=1}^{\infty} B_k\right)$.

By relation $B_n \in \mathcal{A}, n = 1,2,\cdots, B_1 \subset B_2 \subset \cdots$, we had $\lim\limits_{n\to\infty}(B_n) = \bigcup\limits_{k=1}^{\infty} B_k$, and according to the monotonicity of $Pos_H(B_k)$, the following relation was true.

$$Pos_H(B_1) \le Pos_H(B_2) \le \cdots \le Pos_H(B_k) \le \cdots$$

Therefore, for any $k \in N$, the inequation $Pos_H(B_k) \le Pos_H\left(\bigcup\limits_{k=1}^{\infty} B_k\right)$ held. Thus,

$$\begin{gathered} \bigcap_{k=n}^{\infty} Pos_H(B_k) \le Pos_H\left(\bigcup_{k=1}^{\infty} B_k\right), \\ \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} Pos_H(B_k) \le Pos_H\left(\bigcup_{k=1}^{\infty} B_k\right) \end{gathered} \tag{4}$$

(b) $Pos_H\left(\bigcup\limits_{k=1}^{\infty} B_k\right) \le \bigcap\limits_{n=1}^{\infty}\bigcup\limits_{k=n}^{\infty} Pos_H(B_k)$.

By the monotonicity of Pos_H and the condition

$$Pos_H\left(\bigcup_{k=1}^{\infty} B_k\right) = \left\{ \sup_k \{t_k \,|\, t_k \in Pos_H(B_k)\} \right\}$$

we obtained $y_{n_k} \in Pos_H(B_{n_k})$ with relation $y_{n_1} \le y_{n_2} \le \cdots \le y_{n_k}\cdots$, and $t_k \le y_{n_k}$ for $k \le n_k$. Thus, $\sup\limits_k\{t_k \,|\, t_k \in Pos_H(B_k)\} \le \lim\limits_{k\to\infty} y_{n_k}$. In other words, when $\forall t_0 \in Pos_H\left(\bigcup\limits_{k=1}^{\infty} B_k\right)$, $y_0 \in \bigcap\limits_{n=1}^{\infty}\bigcup\limits_{k=n}^{\infty} Pos_H(B_k)$ had to exist with $t_0 \le y_0$. Similarly, when $\forall y_0 \in \bigcap\limits_{n=1}^{\infty}\bigcup\limits_{k=n}^{\infty} Pos_H(B_k)$, $t_0 \in Pos_H\left(\bigcup\limits_{k=1}^{\infty} B_k\right)$ had to exist with $t_0 \le y_0$. Therefore

$$Pos_H\left(\bigcup_{k=1}^{\infty} B_k\right) \le \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} Pos_H(B_k) \tag{5}$$

(c) $\bigcup\limits_{n=1}^{\infty}\bigcap\limits_{k=n}^{\infty} Pos_H(B_k) = \bigcap\limits_{n=1}^{\infty}\bigcup\limits_{k=n}^{\infty} Pos_H(B_k)$.

According to Equation (1), we needed to prove $\bigcup\limits_{n=1}^{\infty}\bigcap\limits_{k=n}^{\infty} Pos_H(B_k) \supset \bigcap\limits_{n=1}^{\infty}\bigcup\limits_{k=n}^{\infty} Pos_H(B_k)$. Let $t \in \bigcap\limits_{n=1}^{\infty}\bigcup\limits_{k=n}^{\infty} Pos_H(B_k)$, that is $t = \lim\limits_{k\to\infty} t_{n_k}$, $t_{n_k} \in Pos_H(B_{n_k})$, $k \ge 1$. The constructive proof was used for $t \in \bigcup\limits_{n=1}^{\infty}\bigcap\limits_{k=n}^{\infty} Pos_H(B_k)$ in the following steps, that is, we chose $t_n \in Pos_H(B_n)$ in order to obtain $t = \lim\limits_{n\to\infty} t_n$.

Supposing $\left|t_{n_k} - t\right| < \varepsilon$ for $k \geq k_0$, when $n < n_{k_0}$, we selected an arbitrary $t_n \in Pos_H(B_n)$. When $n \geq n_{k_0}$, $k \geq k_0$; therefore we consider the two cases. For one thing, when $t_{n_k} \leq t_{n_{k+1}}$, $t_{n_k}, t_{n_k+1}, t_{n_k+2}, \cdots, t_{n_{k+1}}$ was complemented by t_{n_k} and the following method.

$$\begin{aligned} t_{n_k+1} &= \inf_y\{t_{n_k} \vee y \big| y \in Pos_H(B_{n_k+1})\}; \\ t_{n_k+2} &= \inf_y\{t_{n_k+1} \vee y \big| y \in Pos_H(B_{n_k+2})\}; \\ t_{n_k+3} &= \inf_y\{t_{n_k+2} \vee y \big| y \in Pos_H(B_{n_k+3})\}; \\ &\vdots \\ &\vdots \\ t_{n_{k+1}} &= \inf_y\{t_{n_{k+1}-1} \vee y \big| y \in Pos_H(B_{n_{k+1}})\}. \end{aligned}$$

Based on the conditions,

$$B_{n_k} \subset B_{n_k+1},\ Pos_H(B_{n_k+1}) = Pos_H(B_{n_k} \cup B_{n_k+1}) = \left\{ t \vee y \middle| \begin{array}{l} t \in Pos_H(B_{n_k}) \\ y \in Pos_H(B_{n_k+1}) \end{array} \right\}.$$

The above process was possible. For another thing, when $t_{n_k} > t_{n_{k+1}}$, we could obtain $y_{n_k} \in Pos_H(B_{n_k})$ with $y_{n_k} \leq t_{n_{k+1}}$ based on $Pos_H(B_{n_k}) \leq Pos_H(B_{n_k+1})$; therefore, we let $t_{n_k} = y_{n_k}$ and repeated the completion of the first case such that $t_{n_k}, t_{n_k+1}, t_{n_k+2}, \cdots, t_{n_{k+1}}$. From the above process of constructing $\{t_n\}$, we had $|t_n - t| < \varepsilon$ when $n \geq n_{k_0}$, that is,

$$t = \lim_{k\to\infty} t_{n_k} = \lim_{n\to\infty} t_n \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} Pos_H(B_k)$$

Thus, $\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} Pos_H(B_k) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} Pos_H(B_k)$. According to the above, the lower semicontinuity was true based on Equations (3)–(5).

(4) Strong subadditivity.

We let $u \in Pos_H(B_1 \cup B_2)$,$z \in Pos_H(B_1 \cap B_2)$, and from the definition of the hesitant possibility measure, we had $x_1, x_2 \in Pos_H(B_1)$, $y_1, y_2 \in Pos_H(B_2)$ with $u = x_1 \vee y_1$, $z = x_2 \wedge y_2$. By the order relation on the operation of Formula (1) and the definition of the S norm, the following result could be obtained.

$$S(u,z) = S(x_1 \vee y_1, x_2 \wedge y_2) = \begin{cases} S(x_1, x_2 \wedge y_2) \leq S(x_1, y_2) \\ S(y_1, x_2 \wedge y_2) \leq S(y_1, x_2) = S(x_2, y_1) \end{cases}$$

Thus, $Pos_H(B_1 \cup B_2) + Pos_H(B_1 \cap B_2) \leq Pos_H(B_1) + Pos_H(B_2)$ and the strong subadditivity were proved. □

Theorem 7. *A hesitant necessity measure Nec_H on an ample space $(\Gamma, \mathcal{A})$ has the following properties.*

(1) *Monotonicity:*

$$B_1 \in \mathcal{A},\ B_2 \in \mathcal{A}\ B_1 \subset B_2 \Rightarrow Nec_H(B_1) \leq Nec_H(B_2).$$

(2) *Boundedness:*

$$B \in \mathcal{A} \Rightarrow \{0\} \leq Nec_H(B) \leq \{1\}.$$

(3) *Upper semicontinuity:*

$$B_n \in \mathcal{A},\ n = 1,2,\cdots,\ B_1 \supset B_2 \supset \cdots, \Rightarrow \lim_{n\to\infty} Nec_H(B_n) = Nec_H(\cap_{n=1}^{\infty} B_n).$$

(4) *Weak superadditivity:*

$$B_1 \in \mathcal{A}, B_2 \in \mathcal{A} \Rightarrow Nec_H(B_1 \cup B_2) + Nec_H(B_1 \cap B_2) \geq Nec_H(B_1) + Nec_H(B_2)$$

Proof.

(1) Monotonicity.

We let $B_1 \in \mathcal{A}$, $B_2 \in \mathcal{A}$ and $B_1 \subset B_2$, which gave us

$$\begin{aligned} Nec_H(B_1) &= \{1-t\,|\,t \in Pos_H(B_1^c)\} = \{1-t\,|\,t \in Pos_H(B_2^c \cup (B_1^c - B_2^c))\} \\ &= \left\{1-(t_1 \vee t_2)\middle|\, \begin{array}{l} t_1 \in Pos_H(B_2^c) \\ t_2 \in Pos_H(B_1^c - B_2^c) \end{array}\right\} \leq \{1-t_1|t_1 \in Pos_H(B_2^c)\} = Nec_H(B_2) \end{aligned}$$

(2) Boundedness.

From the above (1), boundedness was easily proved.

(3) Upper semicontinuity.

According to $B_n \in \mathcal{A}, n = 1,2,\cdots, B_1 \supset B_2 \supset \cdots$, we had $\{B_n^c\} \subset \Gamma, B_1^c \supset B_2^c \supset \cdots$. Since the hesitant possibility measure Pos_H satisfied the lower semicontinuity, we had

$$\lim_{n\to\infty} Pos_H(B_n^c) = Pos_H(\cup_{n=1}^{\infty} B_n^c) = Pos_H\big((\cap_{n=1}^{\infty} B_n)^c\big)$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} Nec_H(B_n) &= \lim_{n\to\infty}\{1-t|t \in Pos_H(B_n^c)\} = \left\{1-t\middle|t \in \lim_{n\to\infty} Pos_H(B_n^c)\right\} \\ &= \left\{1-t\middle|t \in Pos_H\Big(\big(\cap_{n=1}^{\infty} B_n\big)^c\Big)\right\} = Nec_H(\cap_{n=1}^{\infty} B_n) \end{aligned}$$

(4) Weak superadditivity.

We let $u \in Nec_H(B_1 \cup B_2)$, $z \in Nec_H(B_1 \cap B_2)$, and by the definition of the hesitant necessity measure, $u_1 \in Pos_H(B_1^c \cap B_2^c)$, $x_2 \in Pos_H(B_1^c)$ and $y_2 \in Pos_H(B_2^c)$ existed, such that $u = 1 - u_1$, $z = x_2 \wedge y_2$. On the basis of the monotonicity hesitant possibility measure Pos_H, we obtained $Pos_H(B_1^c \cap B_2^c) \leq Pos_H(B_1^c)$ and $Pos_H(B_1^c \cap B_2^c) \leq Pos_H(B_2^c)$; therefore, $x_1 \in Pos_H(B_1^c)$ and $y_1 \in Pos_H(B_2^c)$ existed with $u_1 \leq x_1$, $u_1 \leq y_1$ and $u_1 \leq x_1 \wedge y_1$. From the order relation on $\mathcal{P}([0,1])$, Formula (1) and the definition of S norm, hawse had the following result.

$$\begin{aligned} &S(u,z) = S(1-u_1, 1-x_2 \vee y_2) \geq S(1-x_1 \wedge y_1, 1-x_2 \vee y_2) \\ &= \left\{ \begin{array}{l} S(1-x_1 \wedge y_1, 1-x_2) \geq S(1-y_1, 1-x_2) = S(1-x_2, 1-y_1) \\ S(1-x_1 \wedge y_1, 1-y_2) \geq S(1-x_1, 1-y_2) \end{array} \right. \end{aligned}$$

Thus, $Nec_H(B_1 \cup B_2) + Nec_H(B_1 \cap B_2) \geq Nec_H(B_1) + Nec_H(B_2)$. □

Theorem 8. *If* $(\Gamma, \mathcal{A}, Nec_H)$ *is a hesitant credibility measure space, the following results hold.*

(1) $Cr_H(\phi) = \{0\}, Cr_H(\Gamma) = \{1\}$.
(2) *Monotonicity:* $B_1 \in \mathcal{A}, B_2 \in \mathcal{A}\ B_1 \subset B_2 \Rightarrow Cr_H(B_1) \leq Cr_H(B_2)$.
(3) *Boundedness:* $B \in \mathcal{A} \Rightarrow \{0\} \leq Cr_H(B) \leq \{1\}$.
(4) *Weak duality:* $B \in \mathcal{A} \Rightarrow Cr_H(B_1) + Cr_H(B_1^c) \leq \{1\}$.
(5) $B_1 \in \mathcal{A}$, $B_2 \in \mathcal{A} \Rightarrow Cr_H(B_1 \cup B_2) \geq Cr_H(B_1) \times Cr_H(B_2)$.

Proof.

(1) According to the definition of the hesitant credibility measure Cr_H, we could easily obtain $Cr_H(\phi) = \{0\}, Cr_H(\Gamma) = \{1\}$.
(2) Monotonicity.

We assumed that $B_1 \in \mathcal{A}$, $B_2 \in \mathcal{A}$, $B_1 \subset B_2$ and $x_0 \in Cr_H(B_1)$, by the definition of $Cr_H(B_1)$, $t_1 \in Pos_H(B_1)$, $t_2 \in Nec_H(B_1)$ existed, such that $x_0 = \frac{1}{2}(t_1 + t_2)$. According to the monotonicity of Pos_H and Nec_H, $u \in Pos_H(B_2)$ and $v \in Nec_H(B_2)$ existed with $t_1 \leq u$ and $t_2 \leq v$. Thus, $x_0 = \frac{1}{2}(t_1 + t_2) \leq \frac{1}{2}(u + v) \in Cr_H(B_2)$ and we let $y_0 = \frac{1}{2}(u + v)$. Therefore, if $x_0 \in Cr_H(B_1)$, $y_0 \in Cr_H(B_2)$ existed, such that $x_0 \leq y_0$. Similarly, if $y_0 \in Cr_H(B_2)$,

$x_0 \in Cr_H(B_1)$ existed, such that $x_0 \le y_0$. As a result, $Cr_H(B_1) \le Cr_H(B_2)$ by the order relation " $\le$ " on power set $\mathcal{P}([0,1])$.

(3) Boundedness.

According to the above (2), boundedness could be easily proved.

(4) Weak duality.

According to Formula (1), $Cr_H(B) + Cr_H(B^c) = \{S(t_1,t_2) \,|\, t_1 \in Cr_H(B), t_2 \in Cr_H(B^c)\}$. Therefore, for any $x_0 \in Cr_H(B) + Cr_H(B^c)$, $t_1 \in Cr_H(B) t_2 \in Cr_H(B^c)$ existed with $x_0 = S(t_1,t_2)$. Moreover, from the definition of the hesitant credibility measure Cr_H, $x \in Pos_H(B)$, $y \in Nec_H(B)$, $u \in Pos_H(B^c)$ and $v \in Nec_H(B^c)$ existed, such that $t_1 = \frac{1}{2}(x+y)$ and $t_2 = \frac{1}{2}(u+v)$.

Thus, $x_0 = S(t_1,t_2) = S(\frac{1}{2}(x+y), \frac{1}{2}(u+v)) \le S(\frac{1}{2}(x+y), 1) = 1$.
Therefore, $Cr_H(B) + Cr_H(B^c) \le \{1\}$.

(4) We proved the fifth property.

According to $B_1 \in \mathcal{A}$, $B_2 \in \mathcal{A}$, $Cr_H(B_1 \cup B_2)$ and $Pos_H(B_1 \cup B_2)$, we had

$$Cr_H(B_1 \cup B_2) = \left\{ \frac{1}{2}(u_{B_1 \cup B_2} + v_{B_1 \cup B_2}) \middle| \begin{array}{l} u_{B_1 \cup B_2} \in Pos_H(B_1 \cup B_2) \\ v_{B_1 \cup B_2} \in Nec_H(B_1 \cup B_2) \end{array} \right\},$$
$$Pos_H(B_1 \cup B_2) = \left\{ u_A \vee u_B \middle| \begin{array}{l} u_A \in Pos_H(B_1) \\ u_B \in Pos_H(B_2) \end{array} \right\}.$$

From the monotonicity of the hesitant necessary measure Nec_H, we obtained

$$v_{B_1} \in Nec_H(B_1) \text{ and } v_B \in Nec_H(B_2),$$

such that $v_{B_1} \le v_{B_1 \cup B_2}$ and $v_{B_2} \le v_{B_1 \cup B_2}$. Thus,

$$\begin{aligned} \tfrac{1}{2}(u_{B_1 \cup B_2} + v_{B_1 \cup B_2}) &= \tfrac{1}{2}\big((u_{B_1} \vee u_{B_2}) + v_{B_1 \cup B_2}\big) = \tfrac{1}{2}\big(u_{B_1} + v_{B_1 \cup B_2}\big) \vee \tfrac{1}{2}\big(u_{B_2} + v_{B_1 \cup B_2}\big) \\ &\ge \tfrac{1}{2}\big(u_{B_1} + v_{B_1}\big) \vee \tfrac{1}{2}\big(u_{B_2} + v_{B_2}\big) \ge \tfrac{1}{2}\big(u_{B_1} + v_{B_1}\big) \wedge \tfrac{1}{2}\big(u_{B_2} + v_{B_2}\big) \\ &\ge T\Big(\tfrac{1}{2}\big(u_{B_1} + v_{B_1}\big), \tfrac{1}{2}\big(u_{B_2} + v_{B_2}\big)\Big) = Cr_H(B_1) \times Cr_H(B_2). \end{aligned}$$

Therefore, the proof of property (5) was completed. □

Example 1. *Let* $\Gamma = \{1,2,3\}$, $B_1 = \{1\}$, $B_2 = \{2\}$, $B_3 = \{3\}$, $B_4 = \{1,2\}$, $B_5 = \{1,3\}$, $B_6 = \{2,3\}$, $\mathcal{A} = \mathcal{P}(\Gamma) = \{\phi, B_1, B_2, B_3, B_4, B_5, B_6, \Gamma\}$, $Pos_H(\phi) = \{0\}$, $Pos_H(B_1) = Pos_H(B_4) = \{0.6, 0.7\}$ *and* $Pos_H(B_3) = Pos_H(B_5) = Pos_H(B_6) = Pos_H(\Gamma) = \{1\}$, $Pos_H(B_2) = \{0.3, 0.4\}$. *It is evident that this is a hesitant possibility measure, and has the following formulas.*

$$\begin{aligned} &Pos_H(B_4 \cap B_5) = Pos_H(B_1) = \{0.6, 0.7\} = \left\{ x \wedge y \middle| \begin{array}{l} x \in Pos_H(B_4) \\ y \in Pos_H(B_5) \end{array} \right\}; \\ &Pos_H(B_4 \cap B_6) = Pos_H(B_2) = \{0.3, 0.4\} \ne \left\{ x \wedge y \middle| \begin{array}{l} x \in Pos_H(B_4) \\ y \in Pos_H(B_6) \end{array} \right\} = \{0.6, 0.7\}; \\ &Pos_H(B_5 \cap B_6) = Pos_H(B_3) = \{1\} = \left\{ x \wedge y \middle| \begin{array}{l} x \in Pos_H(B_5) \\ y \in Pos_H(B_6) \end{array} \right\}. \end{aligned}$$

From the definition of independence among hesitant fuzzy events, B_4, B_5 is mutually independent, and B_5, B_6 is the same as B_4, B_5. However, B_4, B_6 is not mutually independent. In addition, from the definition of the hesitant necessity measure and hesitant credibility measure, we could have the following formulas:

$Nec_H(\Gamma) = \{1\}$, $Nec_H(B_3) = Nec_H(B_6) = \{0.3, 0.4\}$;
$Nec_H(B_1) = Nec_H(B_2) = Nec_H(B_4) = Nec_H(\phi) = \{0\}$, $Nec_H(B_5) = \{0.6, 0.7\}$, $Cr_H(\phi) = \{0\}$;
$Cr_H(\Gamma) = \{1\}, Cr_H(B_2) = \{0.15, 0.2\}$, $Cr_H(B_1) = Cr_H(B_4) = \{0.3, 0.35\}$;
$Cr_H(B_3) = Cr_H(B_6) = \{0.65, 0.7\}$, $Cr_H(B_5) = \{0.8, 0.85\}$.

3.2. Hesitant Fuzzy Variable

Definition 9. *Suppose that $(\Gamma, \mathcal{A}, Pos_H)$ is a hesitant possibility measure space, $\mathcal{R}$ is the real number set and $\mathcal{P}(\mathcal{R})$ is the power set of $\mathcal{R}$. A real-valued function $\xi : \Gamma \to \mathcal{R}$ is a hesitant fuzzy variable if, and only if, $\{\gamma | \xi(\gamma) \le t\} \in \mathcal{A}$ for any $t \in \mathcal{R}$. Especially, for any $B \in \mathcal{P}(\mathcal{R})$, a set-valued mapping is defined as follows:*

$$Pos_{H\xi}(B) = Pos_H\{\gamma \in \Gamma | \xi(\gamma) \in B\} \tag{6}$$

Therefore, we had the following theorem:

Theorem 10. *The set-valued mapping based on Equation (6) is a hesitant possibility measure on $\mathcal{P}(\mathcal{R})$.*

This result was proved if $Pos_{H\xi}(B)$ satisfied the conditions of the hesitant possibility measure.

Proof.

(1) The following formulas evidently held.

$$\begin{array}{l} Pos_{H\xi}(\phi) = Pos_H\{\gamma \in \Gamma | \xi(\gamma) \in \phi\} = \{0\}, \\ Pos_{H\xi}(\mathcal{R}) = Pos_H\{\gamma \in \Gamma | \xi(\gamma) \in \mathcal{R}\} = \{1\}. \end{array}$$

(2) For arbitrary $B_i \in \mathcal{P}(\mathcal{R}), i \in I$, we could obtain

$$\begin{aligned} Pos_{H\xi}(\cup_{i \in I} B_i) &= Pos_H\{\gamma \in \Gamma | \xi(\gamma) \in \cup_{i \in I} B_i\} \\ &= Pos_H(\cup_{i \in I}\{\gamma \in \Gamma | \xi(\gamma) \in B_i\}) \\ &= \left\{ \sup_{i \in I}\{x_i \,|\, x_i \in \{\gamma \in \Gamma | \xi(\gamma) \in B_i\}\} \right\} \\ &= \left\{ \sup_{i \in I}\{x_i | x_i \in Pos_{H\xi} B_i\} \right\} \end{aligned}$$

Therefore, $Pos_{H\xi} : \mathcal{P}(\mathcal{R}) \to \mathcal{P}([0, 1])$ is a hesitant possibility measure on $\mathcal{P}(\mathcal{R})$. When $t \in \mathcal{R}$, we defined the following function on the basis of the hesitant fuzzy variable $\xi : \Gamma \to \mathcal{R}$ with respect to the hesitant possibility measure space $(\Gamma, \mathcal{A}, Pos_H)$.

$$\mu_\xi(t) = Pos_H\{\gamma \in \Gamma | \xi(\gamma) = t\} \tag{7}$$

Thus, a hesitant possibility measure $Pos_{H\xi}$ on $\mathcal{P}(\mathcal{R})$ was determined by Equation (7) with the following formula:

$$Pos_{H\xi}(B) = \left\{ \sup_{t \in B}\{x_t \,|\, x_t \in \mu_\xi(t)\} \right\} \forall B \in \mathcal{P}(\mathcal{R}).$$

Therefore, from an original hesitant possibility measure space $(\Gamma, \mathcal{A}, Pos_H)$, a new hesitant possibility measure space $(\mathcal{R}, \mathcal{P}(\mathcal{R}), Pos_{H\xi})$ was induced by using the hesitant fuzzy variable ξ on the space $(\Gamma, \mathcal{A}, Pos_H)$. Moreover, the hesitant fuzzy variable could be studied according to the induced hesitant possibility measure space $(\mathcal{R}, \mathcal{P}(\mathcal{R}), Pos_{H\xi})$. As a result, we had the definition of the hesitant possibility distribution. □

Definition 11. *The set-valued function $\mu_\xi(t), t \in \mathcal{R}$ defined by Equation (7) is called the hesitant possibility distribution of the hesitant fuzzy variable ξ.*

If $\frac{\sum_{x\in\mu_\xi(t)} x}{\#\mu_\xi(t)}$ is a continuous real-valued function, a hesitant fuzzy variable ξ is continuous, where $\#\mu_\xi(t)$ on behalf of the number of elements in $\mu_\xi(t)$, and $\frac{\sum_{x\in\mu_\xi(t)} x}{\#\mu_\xi(t)}$ is symmetric with respect to elements x in $\mu_\xi(t)$ when t real variables are determined. For instance, let $\mu_\xi(t) = \{x_{t1}, x_{t2}, \cdots, x_{tn}\}(t \in \mathcal{R})$ and $\left\{x_{t(1)}, x_{t(2)}, \cdots, x_{t(n)}\right\}$ be a permutation of $\{x_{s1}, x_{s2}, \cdots, x_{sn}\}$; then, one would have the following formula.

$$\frac{\sum_{x\in\mu_\xi(t)} x}{\#\mu_\xi(t)} = \frac{x_{t1}+x_{t2}+\cdots+x_{tn}}{\#\mu_\xi(t)} = \frac{x_{t(1)}+x_{t(2)}+\cdots+x_{t(n)}}{\#\mu_\xi(t)}$$

ξ is discrete if $X = \{t_0, t_1, t_2 \cdots\}$ is the set of all values of the hesitant fuzzy variable ξ on the space $(\Gamma, \mathcal{A}, Pos_H)$, and let $\mu_i = Pos_H\{\xi = t_i\}(i = 0, 1, 2, \cdots)$.Thus, the hesitant possibility of hesitant fuzzy events related to a discrete hesitant fuzzy variable can directly be determined by the hesitant possibility, such as

$$Pos_{H\xi}\{a < \xi \le b\} = \left\{\sup_{i\in\{j|a<t_j\le b\}}\{x_i \,|\, x_i \in \mu_i\}\right\},$$
$$Pos_{H\xi}\{\xi > a\} = \left\{\sup_{i\in\{j|t_j>a\}}\{x_i \,|\, x_i \in \mu_i\}\right\}.$$

Similarly, for a continuous hesitant fuzzy variable ξ, we could obtain the same results by the hesitant possibility distribution μ_ξ, for instance

$$Pos_{H\xi}\{\xi \ge a\} = \left\{\sup_{u\ge a}\{x_u \,|\, x_u \in \mu_\xi(u)\}\right\},$$
$$Nec_{H\xi}\{\xi \ge a\} = \{1 - x | x \in Pos_{H\xi}\{\xi < a\}\} = \left\{1 - \sup_{u<a}\{x_u \,|\, x_u \in \mu_\xi(u)\}\right\},$$
$$Cr_{H\xi}\{\xi \ge a\} = \left\{\frac{1}{2}(x+y) \;\middle|\; \begin{array}{c} x \in \left\{\sup_{u\ge a}\{x_u \,|\, x_u \in \mu_\xi(u)\}\right\} \\ y \in \left\{1 - \sup_{u<a}\{x_u \,|\, x_u \in \mu_\xi(u)\}\right\} \end{array}\right\}.$$

where $Pos_{H\xi}\{\xi \ge a\}$, $Nec_{H\xi}\{\xi \ge a\}$ and $Cr_{H\xi}\{\xi \ge a\}$ are, respectively, called the hesitant possibility, hesitant necessity and hesitant credibility of the hesitant fuzzy event $\xi \ge a$.

3.3. Several Common Continuous Hesitant Fuzzy Variables and Their Distribution

Example 2. *The hesitant possibility distribution $\mu_\xi(t)$ of triangle hesitant fuzzy variable ξ.*

$$\frac{\sum_{x\in\mu_\xi(t)} x}{\#\mu_\xi(t)} = \begin{cases} \frac{t-t_1}{t_2-t_1} & t_1 \le t < t_2 \\ \frac{t_3-t}{t_3-t_2} & t_2 \le t < t_3 \\ 0 & else \end{cases}$$

where $t_1 < t_2 < t_3$, and is shorthand for $(t_1, t_2, t_3)_H$. For instance, the hesitant possibility distribution of $(2, 6, 8)_H$ is showed in Figure 1.

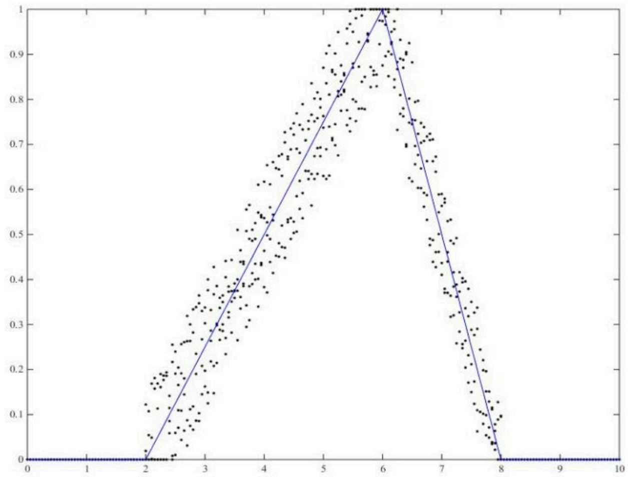

Figure 1. Hesitant possibility distribution map of $(2,6,8)_H$.

Example 3. *The hesitant possibility distribution $\mu_\xi(t)$ of the trapezoidal hesitant fuzzy variable ξ.*

$$\frac{\sum_{x\in\mu_\xi(t)} x}{\#\mu_\xi(t)} = \begin{cases} \frac{t-t_1}{t_2-t_1} & t_1 \le t < t_2 \\ 1 & t_2 \le t < t_3 \\ \frac{t_4-t}{t_4-t_3} & t_3 \le t < t_4 \\ 0 & else \end{cases}$$

where $t_1 < t_2 < t_3 < t_4$, and is shorthand for$(t_1, t_2, t_3, t_4)_H$. For instance, the hesitant possibility distribution of $(2,4,6,8)_H$ is showed in Figure 2.

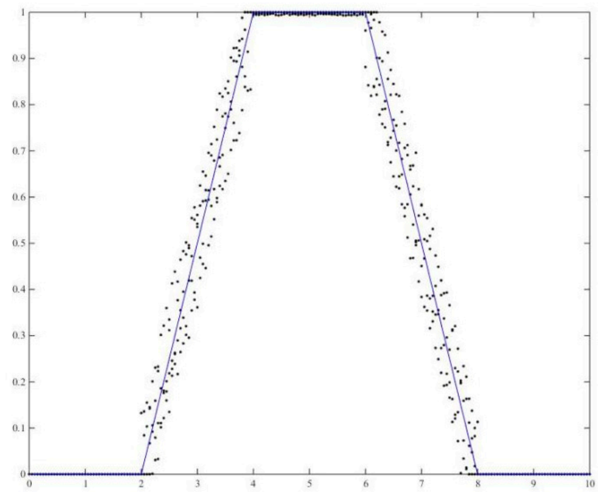

Figure 2. Hesitant possibility distribution map of $(2,4,6,8)_H$.

Example 4. *The hesitant possibility distribution $\mu_\xi(t)$ of the normal hesitant fuzzy variable ξ.*

$$\frac{\sum_{x\in\mu_\xi(t)} x}{\#\mu_\xi(t)} = e^{\{-\frac{(t-\alpha)^2}{\sigma^2}\}} = \exp\left\{-(t-\alpha)^2/\sigma^2\right\}\ t \in \mathcal{R}$$

where $\alpha \in \mathcal{R}, \sigma > 0$, and is shorthand for (α, σ^2). For instance, the hesitant possibility distribution of $(5, 3^2)$ is showed in Figure 3.

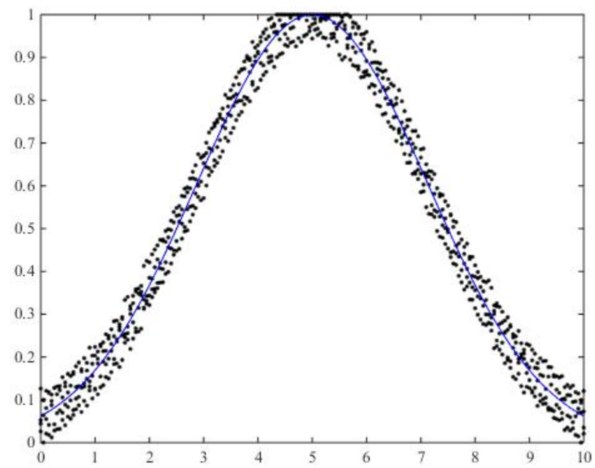

Figure 3. Hesitant possibility distribution map of $(5, 3^2)$.

Example 5. *The hesitant possibility distribution $\mu_\xi(t)$ of the exponential hesitant fuzzy variable ξ.*

$$\frac{\sum_{x\in\mu_\xi(t)} x}{\#\mu_\xi(t)} = e^{-\lambda t} = \exp\{-\lambda t\}\ t \in \mathcal{R}^+$$

where $\lambda > 0$, and is shorthand for λ. For instance, the hesitant possibility distribution of $\lambda = 2$ is showed in Figure 4.

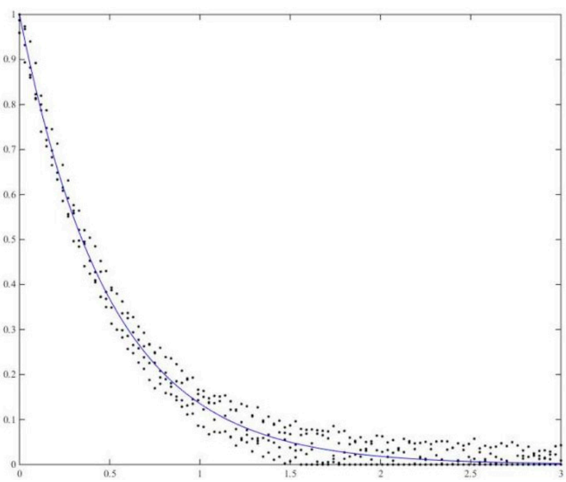

Figure 4. Hesitant possibility distribution map of $\lambda = 2$.

3.4. The Distribution of Functions of Hesitant Fuzzy Variable and the Distribution of Sum of Hesitant Fuzzy Variables

Theorem 12. *Provided that$(\Gamma, \mathcal{A}, Pos_H)$is a hesitant possibility measure space,ξ is a hesitant fuzzy variable, and g is a real-valued function; then, $g(\xi)$ is also a hesitant fuzzy variable, and its hesitant possibility distribution is presented as follows.*

$$\mu_{g(\xi)}(x) = \left\{ \sup_{u\in g^{-1}(x)} \{x_u\} \middle| x_u \in \mu_\xi(u), u \in g^{-1}(x) \right\}$$

Proof. From the above conditions, we had

$$g(\xi) : \Gamma \to \mathcal{R}\ , \text{ and } \{g(\xi) = x\} = \cup_{u\in g^{-1}(x)} \{\xi = u\Big\} \subset \mathcal{A}.$$

Therefore, $g(\xi)$ was a hesitant fuzzy variable; thus,

$$\begin{aligned}
&\mu_{g(\xi)}(x) = Pos_H\{g(\xi) = x\} = Pos_H\Big\{ \cup_{u\in g^{-1}(x)}\{\xi = u\} \Big\} \\
&= \left\{ \sup_{u\in g^{-1}(x)} \{x_u\} \middle| x_u \in Pos_H\{\xi = U\}, u \in g^{-1}(x) \right\} \\
&= \left\{ \sup_{u\in g^{-1}(x)} \{x_u\} \middle| x_u \in \mu_\xi(u), u \in g^{-1}(x) \right\}
\end{aligned}$$

□

Theorem 13. *Supposing that $(\Gamma, \mathcal{A}, Pos_H)$ is a hesitant possibility measure space, ξ and η are mutually independent hesitant fuzzy variables, then $\xi + \eta$ is also a hesitant fuzzy variable, and its hesitant possibility distribution is presented as follows.*

$$\mu_{\xi+\eta}(z) = \left\{ \sup_x\{u\} \middle| u \in \left\{ u_1 \wedge u_2 \middle| \begin{array}{l} u_1 \in \mu_\xi(x) \\ u_2 \in \mu_\eta(z-x) \end{array} \right\} \right\}$$

Proof. According to conditions, we obtained $\xi + \eta : \Gamma \to \mathcal{R}$ and

$$\{\xi + \eta = z\} = \{\{\xi + \eta = z\} \cap \Gamma\} = \Big\{ \{\xi + \eta = z\} \cap \bigcup_x\{\xi = x\} \Big\} = \bigcup_x\{\xi = x, \xi + \eta = z\} \subset \mathcal{A}.$$

Therefore, $\xi + \eta$ is a hesitant fuzzy variable; thus,

$$\begin{aligned}
Pos_H\{\xi + \eta = z\} &= Pos_H\{\{\xi + \eta = z\} \cap \Gamma\} = Pos_H\Big\{ \{\xi + \eta = z\} \cap \bigcup_x\{\xi = x\} \Big\} \\
&= Pos_H\Big\{ \bigcup_x\{\xi = x, \xi + \eta = z\} \Big\} = Pos_H\Big\{ \bigcup_x\{\xi = x, \eta = z - \xi\} \Big\} \\
&= \Big\{ \sup_x\{u\} | u \in Pos_H\{\xi = x, \eta = z - \xi\} \Big\} \\
&= \left\{ \sup_x\{u\} \middle| u \in \left\{ u_1 \wedge u_2 \middle| \begin{array}{l} u_1 \in Pos_H\{\xi = x\} \\ u_2 \in Pos_H\{\eta = z - \xi\} \end{array} \right\} \right\} \\
&= \left\{ \sup_x\{u\} \middle| u \in \left\{ u_1 \wedge u_2 \middle| \begin{array}{l} u_1 \in \mu_\xi(x) \\ u_2 \in \mu_\eta(z-x) \end{array} \right\} \right\}
\end{aligned}$$

□

4. Application of Hesitant Fuzzy Variables

In this section, we introduced two examples to show how hesitant fuzziness can be used to model for hesitant fuzzy graph and hesitant fuzzy group decision making.

4.1. Hesitant Fuzzy Graph Based on Hesitant Fuzzy Variable

The complexity of thinking and the cognitive differences among people have determined that it is difficult for people to reach an agreement on the same issue, they often have different views and, finally, they show a hesitant state. How can we more objectively reflect people's different preferences and hesitations, and how close can we come to the real thinking mode of human beings. References [37–40] utilize the hesitant fuzzy set to describe the degree of hesitation in the human thinking process, while this paper, from another perspective, shows people's hesitation more flexibly. This section mainly discussed the application of the hesitant fuzzy variable in simulating human subjective thinking and proposed a hesitant fuzzy graph according to the hesitant fuzzy variable.

Definition 14. *Suppose that $G = (V, E)$ is a graph, $V = \{v_1, v_2, \cdots, v_n\}$ is a vertex set of the graph G, $E = \{e_{ij}\}$ is an edge collection, $(V \times V, \mathcal{P}(V \times V), Pos_H)$ is the hesitant possibility measure space and ξ_{ij} is a hesitant fuzzy variable on $(V \times V, \mathcal{P}(V \times V), Pos_H)$. If edges e_{ij} of the* *graph $G = (V, E)$ are presented by ξ_{ij}, $G = (V, E)$is called the hesitant fuzzy graph, which is shorthand for $G = \left(V, E_{\xi}, Pos_H\right)$.*

From the above definition, the hesitant fuzzy variables show the existence and the hesitant fuzzy degree of vertexes and, in this sense, it can more objectively simulate the hesitant fuzzy degree in people's thinking process. When $\xi_{ij} = 1$, it explains that there is an edge e_{ij} and its hesitant possibility is $Pos_H\{\xi_{ij} = 1\} \in \mathcal{P}([0, 1])$; when $\xi_{ij} = 0$, it explains that there is no edge e_{ij}, and its hesitant possibility is $Pos_H\{\xi_{ij} = 0\} \in \mathcal{P}([0, 1])$.

Therefore, a hesitant fuzzy graph consists of two components, the first one being the topological structure of the graph and the other one being the distribution table of the hesitant fuzzy variables corresponding to edges. For instance, Figure 5 shows the topology of a hesitant fuzzy graph, and Table 1 shows the distribution table of the related hesitation possibility and hesitation credibility.

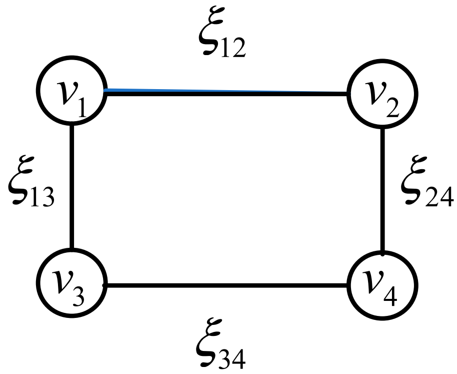

Figure 5. A hesitant fuzzy graph containing 4 vertexes.

Table 1. The possibility and credibility distribution table of hesitant fuzzy graph.

Hesitant Fuzzy Distribution	Value	ξ_{12}	ξ_{13}	ξ_{24}	ξ_{34}
Hesitant possibility distribution	1	{0.8,0.9}	{0.2,0.3}	{1}	{0.6,0.7}
	0	{1}	{1}	{0.3,0.4,0.45}	{1}
Hesitant credibility distribution	1	{0.4,0.45}	{0.1,0.15}	{0.7,0.8,0.85}	{0.3,0.35}
	0	{0.55,0.6}	{0.85,0.9}	{0.15,0.2,0.225}	{0.5,0.7}

4.2. Group Decision Making Based on Hesitant Fuzzy Variable

It could be potentially significant to describe a subjective opinion with hesitation by using hesitant fuzzy modelling. One of the most general situations is that a clear opinion of an individual is not necessarily on a practical question, but their hesitant feeling is stronger than others. In this sense, an individual's hesitant judgement about an issue may be characterized by a hesitant fuzzy variable. As an example, a company manager may want to determine a collective opinion of his subordinates in regards to the quantity of allocated money, which applies to research and development activities. Let ξ_i = {number of money the subordinate *i* should be put into research and development activities}. The hesitant possible distributions of $\xi_i (i = 1, 2, 3, 4, 5, 6)$ might be as follows.

$$\mu_{\xi_1}(x) = \begin{cases} \{0.05, 0.1\} & x \leq 200{,}000 \\ \{0.1, 0.2\} & 200{,}000 < x \leq 400{,}000 \\ \{0.2, 0.3, 0.4, \} & 400{,}000 < x \leq 550{,}000 \\ \{0.8, 0.9, 1.0\} & 550{,}000 < x \leq 700{,}000 \\ \{0.3, 0.4\} & 700{,}000 < x \leq 900{,}000 \\ \{0.0, 0.05\} & 900{,}000 < x \end{cases}$$

$$\mu_{\xi_2}(x) = \begin{cases} \{0.02, 0.05\} & x \leq 150,000 \\ \{0.2, 0.3\} & 150,000 < x \leq 350,000 \\ \{0.3, 0.4, 0.5, \} & 350,000 < x \leq 500,000 \\ \{0.7, 0.9, 1.0\} & 500,000 < x \leq 650,000 \\ \{0.2, 0.4, 0.5\} & 650,000 < x \leq 850,000 \\ \{0.0, 0.02\} & 850,000 < x \end{cases}$$

$$\mu_{\xi_3}(x) = \begin{cases} \{0.0, 0.05\} & x \leq 100,000 \\ \{0.1, 0.3\} & 100,000 < x \leq 200,000 \\ \{0.3, 0.5, \} & 200,000 < x \leq 400,000 \\ \{0.8, 1.0\} & 400,000 < x \leq 700,000 \\ \{0.1, 0.2, 0.4\} & 700,000 < x \leq 800,000 \\ \{0.0\} & 800,000 < x \end{cases}$$

$$\mu_{\xi_4}(x) = \begin{cases} \{0.05, 0.15\} & x \leq 250,000 \\ \{0.1, 0.3\} & 250,000 < x \leq 350,000 \\ \{0.2, 0.4, 0.5, \} & 350,000 < x \leq 550,000 \\ \{0.7, 0.8, 1.0\} & 550,000 < x \leq 750,000 \\ \{0.4, 0.5\} & 750,000 < x \leq 850,000 \\ \{0.02\} & 850,000 < x \end{cases}$$

$$\mu_{\xi_5}(x) = \begin{cases} \{0.05\} & x \leq 100,000 \\ \{0.2, 0.3\} & 100,000 < x \leq 250,000 \\ \{0.3, 0.4\} & 250,000 < x \leq 450,000 \\ \{0.7, 0.9\} & 450,000 < x \leq 650,000 \\ \{0.2, 0.3, 0.5\} & 650,000 < x \leq 800,000 \\ \{0.0, 0.02\} & 800,000 < x \end{cases}$$

$$\mu_{\xi_6}(x) = \begin{cases} \{0.05\} & x \leq 160,000 \\ \{0.2, 0.3\} & 160,000 < x \leq 280,000 \\ \{0.3, 0.4\} & 280,000 < x \leq 480,000 \\ \{0.7, 0.9\} & 480,000 < x \leq 680,000 \\ \{0.2, 0.3, 0.5\} & 680,000 < x \leq 880,000 \\ \{0.0, 0.02\} & 880,000 < x \end{cases}$$

Now, suppose that a manager obtained the hesitant possible distribution describing the opinion of each subordinate, which is represented by a hesitant fuzzy variable. The question we deal with is the following: is there a way of making an inference for the hesitant possible distribution of a hesitant fuzzy variable associated with the group opinion from the hesitant possible distributions of hesitant fuzzy variables corresponding to the issues of each subordinate? Where the group opinion is only modeled except for an interactive group decision, we could assume that there are six constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ with $\sum_{i=1}^{6} \alpha_i = 1$, and α_i represent the importance of the ith subordinate's opinion about the group opinion. For instance, when $\alpha_1 = 0.15$ and $\alpha_2 = 0.45$, the opinion of the second subordinate is three times more important than the first. We present that the hesitant fuzzy variable $\eta = \sum_{i=1}^{6} \alpha_i \xi_i$ is a reasonable explanation of the group opinion. As a result, the hesitant possible distribution of η would be inferred by the following steps based on the hesitant possible distributions of $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$.

(1) Calculate the hesitant possible distribution of $\alpha_1\xi_1$, denoted as $\mu_{\alpha_1\xi_1}(x)$.
(2) Compute the hesitant possible distribution of $\alpha_1\xi_1 + \alpha_2\xi_2$ by the given operation in Theorem 13.
(3) Sequentially perform the operation in the second step for the hesitant possible distribution of $\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 + \alpha_4\xi_4 + \alpha_5\xi_5 + \alpha_6\xi_6$.
(4) According to the above, Theorem 4.1 and from the hesitant possible distribution of η, the hesitant possibility of $\{r_1 < \eta \leq r_2\}$ can be easily inferred for any $-\infty <$

$r_1 < +\infty, -\infty < r_2 < +\infty$. where $\{r_1 < \eta \leq r_2\}$ may represent the final opinion of a manager.

Although the introduced example is only a rough description of an application of the hesitant fuzzy variable in a group decision-making framework, it showed that the hesitant fuzzy variable concept appears to be convenient for modelling hesitant fuzzy decision-making problems.

5. Conclusions

This paper introduced the hesitant possibility measure, hesitant necessity measure and hesitant credibility measure on an ample space $(\Gamma, \mathcal{A})$, and discussed some properties concerning these three set-valued measures, which, respectively, corresponded to the hesitant possibility measure space, hesitant necessity measure space and hesitant credibility measure space. On the hesitant fuzzy possibility measure space, a hesitant fuzzy variable was defined. Consequently, the distribution of this variable and one of its functions were introduced, including several common hesitant fuzzy variables such as the triangle hesitant fuzzy variable, trapezoid hesitant fuzzy variable, normal hesitant fuzzy variable and exponential hesitant fuzzy variable. As a result, the axiomatic system with regard to the hesitant possibility of hesitant fuzzy events was constructed. This axiomatic framework was based on the hesitant fuzzy variable and could provide theoretical support for hesitant fuzzy information processing, which was illustrated by two practical applications in the fourth section of this paper. In the future, according to obtained results of the hesitant fuzzy variable, the hesitant fuzzy theory could be enriched in digital numerical characteristics of the hesitant fuzzy variable such as the mathematical expectation, variance, correlation coefficient, covariance and all sorts of moments. Besides these notions, the multidimensional hesitant fuzzy variable and its distribution are also directions of further research in the future.

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