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Continued Fraction Expansions of Stable Discrete-Time Systems of Difference Equations

Ziad Zahreddine

Mathematics Division, College of Engineering, Kuwait College of Science and Technology, Doha Area, 7th Ring Road, Al Jahraa 13133, Kuwait; z.zahreddine@kcst.edu.kw or ziadzahr@hotmail.com

Abstract: We provide a systematic procedure for generating the coefficients of the continued fraction expansion of the test function associated with the characteristic polynomial of a stable system of difference equations. We illustrate the feasibility of the procedure, and we provide an application on the stability of two-dimensional digital filters.

Keywords: Schur–Cohn stability; discrete-time systems of difference equations; Schur polynomials; stability of digital filters

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1. Introduction

The subject of continued fractions and its applications to both continuous and discrete systems have a long history and continue to impact research in various stability contexts [1–9]. The discovery of interesting relationships between continued fraction expansions and Schur polynomials, i.e., polynomials having their zeros inside the unit circle [1], led to several investigations into the stability of polynomials via continued fractions.

In a recent work [10], the symmetric properties between Routh–Hurwitz and Schur–Cohn stability types were highlighted. In [10], the test functions of each of these stability types were expanded in continued fraction forms satisfying specific conditions.

In the Routh–Hurwitz case; Theorem 3 of [10], the coefficients of the continued fraction expansion of the test function associated with the characteristic polynomial of the system are relatively easy to obtain using sequential long division. In the Schur–Cohn case; Theorem 4 of [10], generating such coefficients is far from trivial. The aim of this paper is to develop a systematic procedure for generating the coefficients of the continued fraction expansion in the Schur case and to illustrate the procedure through an engineering application to the stability of two-dimensional digital filters.

It should be noted that assessing the stability of 2-D digital filters requires one to check the location of the zeros of complex polynomials [11]. Hence, all polynomials considered in this paper have complex coefficients.

In Section 2, we lay out some definitions, notations, and the required results from [10]. In Section 3, we provide a systematic procedure to generate the coefficients of the continued fraction expansion associated with a Schur stable polynomial. The feasibility of the procedure is illustrated in Section 4. An application to the stability of 2-D digital filters is advanced in Section 5 to verify the proposed method.

2. Definitions and Notations

A reminder of the required definitions and results established in [10].

Definition 1. *A linear discrete-time system of difference equations is stable if and only if all its eigenvalues lie inside the unit disc. If*

$$g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n \tag{1}$$

is the characteristic polynomial of the system, then the system is stable if all zeros of g(z) lie inside the unit disc. Such polynomials are said to be Schur stable.

Definition 2. *The reciprocal of g is defined by* $g^{\tau}(z) = z^n \overline{g(1/\overline{z})}$*. Then, Then,* g^{τ} *can be written as* $g^{\tau}(z) = \overline{a}_n + \overline{a}_{n-1}z + \overline{a}_{n-2}z^2 + \cdots + \overline{a}_0 z^n$ *where* $\overline{a}_k$ *denotes the complex conjugate of* a_k *for* $k = 0, 1, \ldots, n$*.*

Definition 3. *The test function of the given discrete-time system is defined by*

$$\Psi(z) = \frac{g(z) - g^{\tau}(z)}{g(z) + g^{\tau}(z)} \tag{2}$$

Theorem 1 ([10] Theorem 4)**.** *The linear discrete-time system of difference equations characterized by (1) is stable if and only if the test function Ψ(z) defined by (2) can be written in the continued fraction expansion*

$$\Psi(z) = h_0 \frac{z-1}{z+1} + k_0 + \cfrac{1}{h_1 \frac{z-1}{z+1} + k_1 + \cfrac{}{\ddots + \cfrac{1}{h_n \frac{z-1}{z+1} + k_n}}} \tag{3}$$

where $h_0 \geq 0,\ h_1 > 0, \ldots, h_n > 0$ *and* k_j*are imaginary or zero for* $0 \leq j \leq n$*. This expansion in* $\frac{z-1}{z+1}$ *is known as the bilinear transformation.*

3. A Procedure to Generate the Coefficients of $\Psi(z)$

Motivated by [12], we would like to substitute the variable $\frac{z-1}{z+1}$ in (3) by $\frac{s-1}{s}$. Solving $\frac{z-1}{z+1} = \frac{s-1}{s}$ for z leads to $z = 2s - 1$.

Therefore, we define the function $T(s)$ in the following way:

$$T(s) = \Psi(2s - 1).$$

$T(s)$ can now be written as

$$T(s) = h_0 \frac{s-1}{s} + k_0 + \cfrac{1}{h_1 \frac{s-1}{s} + k_1 + \cfrac{}{\ddots + \cfrac{1}{h_n \frac{s-1}{s} + k_n}}}$$

This expansion in $\frac{s-1}{s}$ is known as the backward difference transform. We begin by breaking up the above form of $T(s)$ in the following way. Define

$$T(s) = T_0(s) = h_0 \frac{s-1}{s} + k_0 + \frac{1}{T_1(s)},$$

$$T_1(s) = h_1 \frac{s-1}{s} + k_1 + \frac{1}{T_2(s)}.$$

In general, define

$$T_j(s) = h_j \frac{s-1}{s} + k_j + \frac{1}{T_{j+1}(s)} \text{ for } 0 \leq j \leq n-1 \tag{4}$$

Finally,

$$T_n(s) = h_n \frac{s-1}{s} + k_n.$$

Since each $T_j(s)$ is a rational function, write it in the form :

$$T_j(s) = \frac{\sum_{l=0}^{n-j} a_{j,j+l} s^l}{\sum_{l=1}^{n-j} b_{j,j+l} s^l} \tag{5}$$

The following theorem determines the values of the coefficients h_j of (3).

Theorem 2.

$$h_j = -\frac{a_{j,j}}{b_{j,j+1}} \; for \; 0 \leq j \leq n.$$

Proof.

$$\text{Multiply } T_j(s) \text{ of } (5) \text{ by } \frac{s}{s-1}.$$

$$\begin{aligned} & T_j(s) \cdot \tfrac{s}{s-1} = \tfrac{\sum_{l=0}^{n-j} a_{j,j+l} s^l}{\sum_{l=1}^{n-j} b_{j,j+l} s^l} \cdot \tfrac{s}{s-1} \\ & = \tfrac{\sum_{l=0}^{n-j} a_{j,j+l} s^{l+1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^l \cdot (s-1)} \\ & = \tfrac{\sum_{l=0}^{n-j} a_{j,j+l} s^{l+1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1} - \sum_{l=1}^{n-j} b_{j,j+l} s^l}. \end{aligned}$$

By changing the indices, the last form can be written as:

$$T_j(s) \cdot \frac{s}{s-1} = \frac{\sum_{l=0}^{n-j} a_{j,j+l} s^{l+1}}{-b_{j,j+1} s + \sum_{l=1}^{n-j-1} (b_{j,j+l} - b_{j,j+l+1}) s^{l+1} + b_{j,n} s^{n-j+1}}.$$

By a simple application of L'Hopital rule, we get:

$$\lim_{s \to 0} \left[T_j(s) \cdot \frac{s}{s-1} \right] = -\frac{a_{j,j}}{b_{j,j+1}}.$$

Going back to the expression of $T_j(s)$ in (4), $T_j(s) = h_j \frac{s-1}{s} + k_j + \frac{1}{T_{j+1}(s)}$

$$\text{Multiply both sides by } \frac{s}{s-1} \text{ to get}$$

$$\begin{aligned} & T_j(s) \cdot \tfrac{s}{s-1} = h_j \tfrac{s-1}{s} \cdot \tfrac{s}{s-1} + k_j \cdot \tfrac{s}{s-1} + \tfrac{1}{T_{j+1}(s)} \cdot \tfrac{s}{s-1} \\ & = h_j + k_j \cdot \tfrac{s}{s-1} + \tfrac{1}{T_{j+1}(s)} \cdot \tfrac{s}{s-1}. \end{aligned}$$

Hence,

$$\lim_{s \to 0} \left[T_j(s) \cdot \frac{s}{s-1} \right] = \lim_{s \to 0} \left[h_j + k_j \cdot \frac{s}{s-1} + \frac{1}{T_{j+1}(s)} \cdot \frac{s}{s-1} \right] = h_j.$$

That leads to the desired conclusion:

$$h_j = -\frac{a_{j,j}}{b_{j,j+1}} \text{ for } 0 \leq j \leq n. \qquad \square$$

The next theorem determines the values of the coefficients k_j of (3) in addition to some important relations which will prove useful in generating the required coefficients.

Theorem 3.

$$k_j = \frac{a_{j,j+1} - h_j (b_{j,j+1} - b_{j,j+2})}{b_{j,j+1}} \; for \; 0 \leq j \leq n,$$

where $h_j$as determined in Theorem 2.

In addition, the following two relations hold:

$$a_{j+1,j+l} = b_{j,j+l} \text{ for } 0 \le j \le n \text{ and } 1 \le l \le n-j,$$

and

$$b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l} \text{ for } 0 \le j \le n \text{ and } 2 \le l \le n-j.$$

Proof. Consider the expression of $T_j(s)$ as in (5) from which we subtract $h_j \cdot \frac{s-1}{s}$,

$$\begin{aligned} T_j(s) - h_j \cdot \tfrac{s-1}{s} &= \tfrac{\sum_{l=0}^{n-j} a_{j,j+l} s^l}{\sum_{l=1}^{n-j} b_{j,j+l} s^l} - h_j \cdot \tfrac{s-1}{s} \\ &= \tfrac{\sum_{l=0}^{n-j} a_{j,j+l} s^{l+1} - h_j (s-1)\left(\sum_{l=1}^{n-j} b_{j,j+l} s^l\right)}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}} \\ &= \tfrac{\sum_{l=0}^{n-j} a_{j,j+l} s^{l+1} + h_j\left(\sum_{l=1}^{n-j} b_{j,j+l} s^l\right) - h_j\left(\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}\right)}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}} \end{aligned}$$

Isolate the first term in the first two summations in the numerator of the last expression,

$$\begin{aligned} &T_j(s) - h_j \cdot \tfrac{s-1}{s} \\ &= \tfrac{\left(a_{j,j} + h_j \cdot b_{j,j+1}\right)s + \sum_{l=1}^{n-j} a_{j,j+l} s^{l+1} + h_j\left(\sum_{l=2}^{n-j} b_{j,j+l} s^l\right) - h_j\left(\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}\right)}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}} \end{aligned}$$

By Theorem 1, $h_j = -\frac{a_{j,j}}{b_{j,j+1}}$. That leads to $a_{j,j} + h_j \cdot b_{j,j+1} = 0$. Therefore,

$$T_j(s) - h_j \cdot \frac{s-1}{s} = \frac{\sum_{l=1}^{n-j} a_{j,j+l} s^{l+1} + h_j\left(\sum_{l=2}^{n-j} b_{j,j+l} s^l\right) - h_j\left(\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}\right)}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}}$$

By changing indices in the second summation in the numerator, we get

$$T_j(s) - h_j \cdot \frac{s-1}{s} = \frac{\sum_{l=1}^{n-j} a_{j,j+l} s^{l+1} + h_j\left(\sum_{l=1}^{n-j-1} b_{j,j+l+1} s^{l+1}\right) - h_j\left(\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}\right)}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}}$$

Combining like terms leads to

$$T_j(s) - h_j \cdot \frac{s-1}{s} = \frac{\sum_{l=1}^{n-j-1}\left[a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right)\right] s^{l+1} + \left(a_{j,n} - h_j b_{j,n}\right) s^{n-j+1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l+1}}$$

Taking s^2 as a common factor in both the numerator and denominator leads to:

$$\begin{aligned} T_j(s) - h_j \cdot \tfrac{s-1}{s} &= \tfrac{s^2\left\{\sum_{l=1}^{n-j-1}\left[a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right)\right] s^{l-1} + \left(a_{j,n} - h_j b_{j,n}\right) s^{n-j-1}\right\}}{s^2\left[\sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}\right]} \\ &= \tfrac{\sum_{l=1}^{n-j-1}\left[a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right)\right] s^{l-1} + \left(a_{j,n} - h_j b_{j,n}\right) s^{n-j-1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}}. \end{aligned}$$

Therefore,

$$T_j(s) - h_j \cdot \frac{s-1}{s} = \frac{\sum_{l=1}^{n-j-1}\left[a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right)\right] s^{l-1} + \left(a_{j,n} - h_j b_{j,n}\right) s^{n-j-1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}} \tag{6}$$

On the other hand, returning to the expression of $T_j(s)$ as in (4),

$$T_j(s) = h_j \frac{s-1}{s} + k_j + \frac{1}{T_{j+1}(s)},$$

We get,

$$\frac{1}{T_{j+1}(s)} = T_j(s) - h_j \frac{s-1}{s} - k_j \tag{7}$$

In addition, using the expression of $T_j(s)$ in (5), we can write $T_{j+1}(s)$ in the form

$$T_{j+1}(s) = \frac{\sum_{l=0}^{n-j-1} a_{j+1,j+l+1} s^l}{\sum_{l=1}^{n-j-1} b_{j+1,j+l+1} s^l}$$

which is inverted to,

$$\frac{1}{T_{j+1}(s)} = \frac{\sum_{l=1}^{n-j-1} b_{j+1,j+l+1} s^l}{\sum_{l=0}^{n-j-1} a_{j+1,j+l+1} s^l} \tag{8}$$

By comparing the two forms of $\frac{1}{T_{j+1}(s)}$ (7) and (8), we get

$$T_j(s) - h_j \frac{s-1}{s} - k_j = \frac{\sum_{l=1}^{n-j-1} b_{j+1,j+l+1} s^l}{\sum_{l=0}^{n-j-1} a_{j+1,j+l+1} s^l}.$$

We already proved relation (6), which is

$$T_j(s) - h_j \cdot \frac{s-1}{s} = \frac{\sum_{l=1}^{n-j-1} [a_{j,j+l} - h_j \left(b_{j,j+l} - b_{j,j+l+1} \right)] s^{l-1} + \left(a_{j,n-} h_j b_{j,n} \right) s^{n-j-1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}}$$

hence,

$$\frac{\sum_{l=1}^{n-j-1} [a_{j,j+l} - h_j \left(b_{j,j+l} - b_{j,j+l+1} \right)] s^{l-1} + \left(a_{j,n-} h_j b_{j,n} \right) s^{n-j-1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}} - k_j = \frac{\sum_{l=1}^{n-j-1} b_{j+1,j+l+1} s^l}{\sum_{l=0}^{n-j-1} a_{j+1,j+l+1} s^l}.$$

Therefore,

$$\frac{\sum_{l=1}^{n-j-1} [a_{j,j+l} - h_j \left(b_{j,j+l} - b_{j,j+l+1} \right)] s^{l-1} + \left(a_{j,n-} h_j b_{j,n} \right) s^{n-j-1} - k_j \sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}}$$
$$= \frac{\sum_{l=1}^{n-j-1} b_{j+1,j+l+1} s^l}{\sum_{l=0}^{n-j-1} a_{j+1,j+l+1} s^l}.$$

We write it as

$$\frac{\sum_{l=1}^{n-j-1} \left[a_{j,j+l} - h_j \left(b_{j,j+l} - b_{j,j+l+1} \right) - k_j b_{j,j+l} \right] s^{l-1} + \left[\left(a_{j,n-} h_j b_{j,n} \right) - k_j b_{j,n} \right] s^{n-j-1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}}.$$
$$= \frac{\sum_{l=1}^{n-j-1} b_{j+1,j+l+1} s^l}{\sum_{l=0}^{n-j-1} a_{j+1,j+l+1} s^l}.$$

By changing indices in the numerator and denominator of the right-hand side, we get

$$\frac{\sum_{l=1}^{n-j-1} \left[a_{j,j+l} - h_j \left(b_{j,j+l} - b_{j,j+l+1} \right) - k_j b_{j,j+l} \right] s^{l-1} + \left[\left(a_{j,n-} h_j b_{j,n} \right) - k_j b_{j,n} \right] s^{n-j-1}}{\sum_{l=1}^{n-j} b_{j,j+l} s^{l-1}}$$
$$= \frac{\sum_{l=2}^{n-j} b_{j+1,j+l} s^{l-1}}{\sum_{l=1}^{n-j} a_{j+1,j+l} s^{l-1}}.$$

The above equation leads to the following three conclusions:

1. $a_{j,j+1} - h_j(b_{j,j+1} - b_{j,j+2}) - k_j b_{j,j+1} = 0$.
Hence,

$$k_j = \frac{a_{j,j+1} - h_j(b_{j,j+1} - b_{j,j+2})}{b_{j,j+1}} \text{ for } 0 \le j \le n, \text{ as desired.}$$

In addition,
2. $a_{j+1,j+l} = b_{j,j+l}$ for $0 \le j \le n$ and $1 \le l \le n-j$, and
3. $b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l}$ for $0 \le j \le n$ and $2 \le l \le n-j$.□

4. Feasibility of the Procedure

We shall illustrate the feasibility of the above procedure by applying it to a 3rd degree Schur polynomial.

We reconsider the same example we addressed in Example 2 of [10], but instead of the trial-and-error approach we used there, we shall apply the above systematic procedure to obtain the coefficients of the continued fraction expansion.

Consider the Schur polynomial

$$g(z) = 4z^3 - 6z^2 + 4z - 1$$

whose zeros are

$$\frac{1}{2}, \ \frac{1}{2} + \frac{1}{2}i, \ \frac{1}{2} - \frac{1}{2}i$$

all lying inside the unit disc.

The reciprocal of g is

$$g^{\tau}(z) = z^n\overline{g(1/\bar{z})} = -z^3 + 4z^2 - 6z + 4$$

Therefore, the test function can be written as

$$\Psi(z) = \frac{g(z) - g^{\tau}(z)}{g(z) + g^{\tau}(z)} = \frac{5z^3 - 10z^2 + 10z - 5}{3z^3 - 2z^2 - 2z + 3}$$

Apply the transformation

$$T(s) = \Psi(2s-1).$$

Then,

$$T(s) = \Psi(2s-1) = \frac{5(2s-1)^3 - 10(2s-1)^2 + 10(2s-1) - 5}{3(2s-1)^3 - 2(2s-1)^2 - 2(2s-1) + 3}.$$

$T(s)$ can now be written as

$$T(s) = \frac{20s^3 - 50s^2 + 45s - 15}{12s^3 - 22s^2 + 11s}.$$

We would like to expand $T(s)$ in the form

$$T(s) = h_0\frac{s-1}{s} + k_0 + \cfrac{1}{h_1\frac{s-1}{s} + k_1 + \cfrac{1}{h_2\frac{s-1}{s} + k_2}}$$

We seek the values of $h_0, \ k_0, \ h_1, k_1, h_2, k_2$ using the above procedure.
By (5), we have $T_j(s) = \frac{\sum_{l=0}^{n-j} a_{j,j+l}s^l}{\sum_{l=1}^{n-j} b_{j,j+l}s^l}$.
By (4), $T(s) = T_0(s)$, so

$$T(s) = T_0(s) = \frac{\sum_{l=0}^{3} a_{0,l}s^l}{\sum_{l=1}^{3} b_{0,l}s^l} = \frac{a_{0,0} + a_{0,1}s + a_{0,2}s^2 + a_{0,3}s^3}{b_{0,1}s + b_{0,2}s^2 + b_{0,3}s^3},$$

which compared with $T(s) = \frac{20s^3-50s^2+45s-15}{12s^3-22s^2+11s}$ leads to

$$a_{0,0} = -15,\ a_{0,1} = 45,\ a_{0,2} = -50,\ a_{0,3} = 20,$$

$$b_{0,1} = 11,\ b_{0,2} = -22,\ b_{0,3} = 12.$$

Calculation of h_0 and k_0 :
By Theorem 1, $h_0 = -\frac{a_{0,0}}{b_{0,1}} = \frac{15}{11}$.
By Theorem 2, $k_0 = \frac{a_{0,1}-h_0(b_{0,1}-b_{0,2})}{b_{0,1}} = \frac{45-\frac{15}{11}(11+22)}{b_{0,1}} = 0$.
Calculation of h_1 and k_1 :
Again by Theorem 1, $h_1 = -\frac{a_{1,1}}{b_{1,2}}$.
By the formulas of Theorem 3, namely

$$a_{j+1,j+l} = b_{j,j+l},$$

and

$$b_{j+1,j+l} = a_{j,j+l} - h_j\Big(b_{j,j+l} - b_{j,j+l+1}\Big) - k_j b_{j,j+l},$$

We get, $a_{1,1} = b_{0,1} = 11$.

$$b_{1,2} = a_{0,2} - h_0(b_{0,2} - b_{0,3}) - k_0 b_{0,2} = -50 - \frac{15}{11}(-22-12) - 0 = -\frac{40}{11}.$$

Then,

$$h_1 = -\frac{a_{1,1}}{b_{1,2}} = -\frac{11}{-\frac{40}{11}} = \frac{121}{40}.$$

In addition, by Theorem 2, $k_1 = \frac{a_{1,2}-h_1(b_{1,2}-b_{1,3})}{b_{1,2}}$. $b_{1,2} = -\frac{40}{11}$, already calculated.

$$a_{j+1,j+l} = b_{j,j+l} \text{ leads to } a_{1,2} = b_{0,2} = -22$$

$$\begin{aligned} b_{j+1,j+l} = a_{j,j+l} - h_j\Big(b_{j,j+l} - b_{j,j+l+1}\Big) - k_j b_{j,j+l} &\text{ leads to } b_{1,3} \\ = a_{0,3} - h_0(b_{0,3} - b_{0,4}) - k_0 b_{0,3} \end{aligned}$$

$$b_{1,3} = 20 - \frac{15}{11}(12-0) - 0 \cdot b_{0,3} = \frac{40}{11}.$$

Therefore, $k_1 = \frac{-22-\frac{121}{40}\left(-\frac{40}{11}-\frac{40}{11}\right)}{b_{1,2}} = 0$.
Calculation of h_2 and k_2 :
Now, $h_2 = -\frac{a_{2,2}}{b_{2,3}}$, where

$$a_{2,2} = b_{1,2} = -\frac{40}{11}.$$

$$b_{j+1,j+l} = a_{j,j+l} - h_j\Big(b_{j,j+l} - b_{j,j+l+1}\Big) - k_j b_{j,j+l} \text{ leads to } b_{2,3} = a_{1,3} - h_1(b_{1,3} - b_{1,4}) - k_1 b_{1,3}$$

$$a_{1,3} = b_{0,3} = 12.$$

$$b_{1,3} = \frac{40}{11} \text{ already calculated.}$$

$$\begin{aligned} b_{j+1,j+l} = a_{j,j+l} - h_j\Big(b_{j,j+l} - b_{j,j+l+1}\Big) - k_j b_{j,j+l} &\text{ implies } b_{1,4} \\ = a_{0,4} - h_0(b_{0,4} - b_{0,5}) - 0 \cdot b_{1,3} \end{aligned}$$

$$b_{1,4} = a_{0,4} - h_0(b_{0,4} - b_{0,5}) - 0 \cdot b_{1,3} = 0 - \frac{15}{11}(0-0) = 0.$$

Then, $b_{2,3} = 12 - \frac{121}{40} \cdot \frac{40}{11} = 1$.

Therefore, $h_2 = -\frac{a_{2,2}}{b_{2,3}} = -\frac{-\frac{40}{11}}{1} = \frac{40}{11}$.

$$k_2 = \frac{a_{2,3} - h_2(b_{2,3} - b_{2,4})}{b_{2,3}}.$$

We know that, $b_{2,3} = 1$.

$$a_{2,3} = b_{1,3} = \frac{40}{11} \text{ was already calculated.}$$

$$b_{j+1,j+l} = a_{j,j+l} - h_j\left(b_{j,j+l} - b_{j,j+l+1}\right) - k_j b_{j,j+l} \text{ implies } b_{2,4} = a_{1,4} - h_1(b_{1,4} - b_{1,5}) - k_1 b_{1,4} = 0.$$

$$k_2 = \frac{a_{2,3} - h_2(b_{2,3} - b_{2,4})}{b_{2,3}} = \frac{\frac{40}{11} - \frac{40}{11}(1-0)}{1} = 0.$$

Therefore,

$$T(s) = \frac{15}{11}\frac{s-1}{s} + 0 + \frac{1}{\frac{121}{40}\frac{s-1}{s} + 0 + \frac{1}{\frac{40}{11}\frac{s-1}{s} + 0}}$$

or

$$\Psi(z) = \frac{15}{11}\left(\frac{z-1}{z+1}\right) + \frac{1}{\frac{121}{40}\left(\frac{z-1}{z+1}\right) + \frac{1}{\frac{40}{11}\left(\frac{z-1}{z+1}\right)}}$$

It is no surprise that the ks are all zeros since by Theorem 1 they are either 0 or pure imaginary. So, they must be zero because the given polynomial is of real coefficients. We wanted to show via calculations that the k's are zero to match the application with the theory.

5. Application of the Procedure

In this section, we present an application of the above results to test the bounded input bounded output (BIBO) stability of two-dimensional digital filters. Specifically, we shall prove the relationship between BIBO stability and the procedure introduced in Section 3.

BIBO stability has been characterized in several equivalent ways of which we shall mention the following two.

Theorem 4 [13]. *The general two-dimensional complex digital filter* $F(z_1, z_2) = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{jk} z_1^j z_2^k$ *is BIBO stable if and only if* $F(z_1, z_2) \neq 0$ *for all* $\{(z_1, z_2) : |z_1| \leq 1,\ |z_2| \leq 1\}$.

Based on the work of Ansell [14], Huang [15] proved an equivalent result to that of Justice and Shanks which states the following:

Theorem 5 [15]. *The digital filter* $F(z_1, z_2)$ *is BIBO stable if and only if the following two conditions hold:*

$$F(z_1, 0) \neq 0 \text{ for } |z_1| \leq 1, \tag{9}$$

and

$$F(z_1, z_2) \neq 0 \text{ for } |z_1| = 1, \text{ and } |z_2| \leq 1. \tag{10}$$

Other equivalent conditions for BIBO stability can also be found in Strintzis [16] and DeCarlo et al. [17].

In this section, we shall follow the conditions of Huang [15] to assess BIBO stability.

Consider the two-dimensional digital filter characterized by

$$F(z_1,z_2)=\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}z_1^j z_2^k,\ \text{where } b_{00}=4,\ b_{10}=-6,\ b_{20}=4,\ b_{30}=-1. \tag{11}$$

Theorem 6. *The system* $F(z_1,z_2)=\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}z_1^j z_2^k$ *where* $b_{00}=4,\ b_{10}=-6,\ b_{20}=4,$ $b_{30}=-1$ *satisfies condition* (9).

Proof. Note that $F(z_1,0)=b_{00}+b_{10}z_1+b_{20}z_1^2+b_{30}z_1^3=4-6z_1+4z_1^2-z_1^3$.

It can be verified that the zeros of $F(z_1,0)$ are $1-i,\ 1+i$ and 2 which all lie outside the closed unit disc.

Therefore, $F(z_1,0)\neq 0$ for $|z_1|\leq 1$, and condition (9) is verified. □

The following theorem illustrates the relationship between condition (9) of BIBO stability, and the procedure introduced in Section 3.

Theorem 7. *Condition (9) of BIBO stability of the system*

$$F(z_1,z_2)=\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}z_1^j z_2^k\ \text{where } b_{00}=4,\ b_{10}=-6,\ b_{20}=4,\ b_{30}=-1,$$

is satisfied if and only if the test function associated with the function $F(z_1,0)$ *can be expanded in the continued fraction form*

$$\Psi(z)=\frac{15}{11}\left(\frac{z-1}{z+1}\right)+\cfrac{1}{\frac{121}{40}\left(\frac{z-1}{z+1}\right)+\cfrac{1}{\frac{40}{11}\left(\frac{z-1}{z+1}\right)}}.$$

Proof. We need to realize that the function $F(z_1,0)$ is the reciprocal of the polynomial

$$g(z_1)=4z_1^3-6z_1^2+4z_1-1$$

introduced in Section 4.

Following the steps in the procedure of Section 4, the theorem is established. □

Theorem 8. $F(z_1,z_2)$ *as defined in* (11) *has no zeros for* $|z_1|=1$, *and* $|z_2|\leq 1$ *if and only if the test function* $\Psi(z)$ *defined in (2) is a complex discrete reactance function.*

Proof. We apply the procedure used by Reddy and Rajan [11]. On top of page 1691 in [11], the functions $B_{*o}\left(e^{i\theta_1},z_2\right)$ and $B_{*e}\left(e^{i\theta_1},z_2\right)$ which Reddy and Rajan call the para-odd and para-even parts are exactly the numerator and denominator, respectively, of our test function (2) defined in Section 2, and therefore their function, $F_o\left(e^{i\theta_1},z_2\right)=\frac{B_{*e}\left(e^{i\theta_1},z_2\right)}{B_{*o}\left(e^{i\theta_1},z_2\right)}$ is exactly our test function $\Psi(z)$.

Since $\Psi(z)$ is already a complex discrete reactance function, condition (10) is thus satisfied, and that proves the theorem. □

Now, system (11) satisfies both conditions (9) and (10) and is therefore BIBO stable.

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