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The Solvability of a System of Quaternion Matrix Equations Involving ϕ-Skew-Hermicity

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Abstract: Let $\mathbb{H}$ be the real quaternion algebra and $\mathbb{H}^{m\times n}$ denote the set of all $m \times n$ matrices over $\mathbb{H}$. For $A \in \mathbb{H}^{m\times n}$, we denote by A_ϕ the $n \times m$ matrix obtained by applying ϕ entrywise to the transposed matrix A^T, where ϕ is a non-standard involution of $\mathbb{H}$. $A \in \mathbb{H}^{n\times n}$ is said to be ϕ-skew-Hermicity if $A = -A_\phi$. In this paper, we provide some necessary and sufficient conditions for the existence of a ϕ-skew-Hermitian solution to the system of quaternion matrix equations with four unknowns $A_iX_i(A_i)_\phi + B_iX_{i+1}(B_i)_\phi = C_i,\ (i = 1,2,3),\ A_4X_4(A_4)_\phi = C_4$.

Keywords: quaternion algebra; matrix decompositions; matrix equations; ϕ-skew-Hermicity

1. Introduction

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{H}$ be a four-dimensional vector space over $\mathbb{R}$ with an ordered basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. A real quaternion, simply called quaternion, is a vector $x = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}$ with real coefficients a_0, a_1, a_2, a_3. Moreover, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfies

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Let $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ stand, respectively, for the real number field and the set of all $m \times n$ matrices over the real quaternion algebra

$$\mathbb{H} = \{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R}\}.$$

The definitions of ϕ-skew-Hermitian quaternion matrices were first introduced by Rodman (Definition 3.6.1 in [1]). For $A \in \mathbb{H}^{m\times n}$, we denote by A_ϕ the $n \times m$ matrix obtained by applying ϕ entrywise to the transposed matrix A^T, where ϕ is a non-standard involution of $\mathbb{H}$ (see Definition 1). $A \in \mathbb{H}^{n\times n}$ is said to be ϕ-skew-Hermicity if $A = -A_\phi$.

The decompositions of the quaternion matrices have applications in many fields, such as color image processing(e.g., [2,3]), quantum mechanics [4], signal processing [5], and so on. Research on quaternion matrix theories (e.g., [6–18]) and equations (e.g., [13,19–27]) is ongoing.

The quaternion matrix equation involving Hermicity is one of the active research topics in the matrix field and its applications. Wang and Zhang [28] provided necessary and sufficient conditions for the existence and expression of the Re-nonnegative definite solution to the system

$$AXA^* + BYB^* = C$$

over $\mathbb{H}$ by using the decomposition of pairwise matrices, where $*$ stands for conjugate transpose. Wang and Jiang [29] further studied the extreme ranks of the (skew-)Hermitian

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solutions to the quaternion matrix equation. He [30] investigated the system of coupled real quaternion matrix equations involving η-Hermicity

$$A_iX_iA_i^{\eta*} + B_iX_{i+1}B_i^{\eta*} = C_i, \quad (i = 1, 2, 3),$$

where $A_i \in \mathbb{H}^{p_i \times t_i}$, $B_i \in \mathbb{H}^{p_i \times t_{i+1}}$, $C_i \in \mathbb{H}^{p_i \times p_i}$, and C_i are η-Hermitian matrices. He gave the solvability conditions, general solutions, and the rank bounds of the general η-Hermitian solutions. Some researchers have considered the ϕ-skew-Hermitian solution to some quaternion matrix equations. For example, He [31] derived some necessary and sufficient conditions for the existence of a ϕ-skew-Hermitian solution to the following system of quaternion matrix equations involving ϕ-skew-Hermicity

$$\begin{cases} BXB_\phi + CYC_\phi = A, \\ EZE_\phi + DYD_\phi = F, \end{cases} \quad X = -X_\phi,\ Y = -Y_\phi,\ Z = -Z_\phi,$$

where A, B, C, D, E, F are the given quaternion matrices.

To our knowledge, there is little information on the system of quaternion matrix equations involving ϕ-skew-Hermicity with four unknowns

$$\begin{cases} A_1X_1(A_1)_\phi + B_1X_2(B_1)_\phi = C_1, \\ A_2X_2(A_2)_\phi + B_2X_3(B_2)_\phi = C_2, \\ A_3X_3(A_3)_\phi + B_3X_4(B_3)_\phi = C_3, \\ \qquad A_4X_4(A_4)_\phi = C_4, \end{cases} \quad X_i = -(X_i)_\phi, \tag{1}$$

where $A_i \in \mathbb{H}^{p_i \times t_i}$, $B_i \in \mathbb{H}^{p_i \times t_{i+1}}$, $C_i \in \mathbb{H}^{p_i \times p_i}$, and C_i are ϕ-skew-Hermitian matrices. Using the simultaneous decomposition of a set of seven real quaternion matrices

$$\begin{pmatrix} A_1 & B_1 & & \\ & A_2 & B_2 & \\ & & A_3 & B_3 \\ & & & A_4 \end{pmatrix}, \tag{2}$$

we provide some necessary and sufficient conditions for the existence of a ϕ-skew-Hermitian solution to the system (1).

The remainder of this paper is organized as follows. In Section 2, we review the definitions of the non-standard involution ϕ and the ϕ-skew-Hermitian quaternion matrix; we also provide a simultaneous decomposition for a set of eleven real quaternion matrices involving ϕ-skew-Hermicity and present a canonical form of the system of the quaternion matrix, Equation (1). In Section 3, we provide some necessary and sufficient conditions for the existence of a ϕ-skew-Hermitian solution to the system (1).

2. A Canonical Form of the System of the Quaternion Matrix Equation

In this section, we investigate the structure of a simultaneous decomposition for the matrix array (2) and provide a canonical form of the system of the quaternion matrix Equations (1). First, we review the definitions of non-standard involution ϕ and ϕ-skew-Hermitian matrix.

Definition 1 (Non-standard involution [1]). *Let ϕ be an anti-endomorphism of $\mathbb{H}$. Assume that ϕ does not map $\mathbb{H}$ into zero. Then, ϕ is one-to-one and onto $\mathbb{H}$; thus, ϕ is an anti-automorphism. Moreover, ϕ is real linear and can be represented as a 4×4 real matrix with respect to the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Then, ϕ is a non-standard involution if and only if*

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

where T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$.

Definition 2 (**ϕ-skew-Hermitian** [1])**.** *$A \in \mathbb{H}^{n\times n}$ is said to be ϕ-skew-Hermitian if $A = -(A)_\phi$, where ϕ is a non-standard involution.*

The following Theorem presents the equivalence canonical form of the set of seven real quaternion matrices (2).

Using the results of [31,32], we can obtain the following result.

Lemma 1. *Consider a set of seven matrices (2); there exists a unitary matrix $\widehat{T}_1 \in \mathbb{H}^{t_1\times t_1}$, nonsingular matrices $\widehat{P}_i \in \mathbb{H}^{p_i\times p_i}$, $(i = 1, 2, 3)$, $\widehat{T}_2 \in \mathbb{H}^{t_2\times t_2}$, $\widehat{T}_3 \in \mathbb{H}^{t_3\times t_3}$, $\widehat{T}_4 \in \mathbb{H}^{t_4\times t_4}$, $\widehat{P}_4 \in \mathbb{H}^{p_4\times p_4}$ such that*

$$\widehat{P}_i A_i \widehat{T}_i = S_{a_i},\ \ \widehat{P}_i B_i \widehat{T_{i+1}} = S_{b_i},\ \ \widehat{P}_4 A_4 \widehat{T}_4 = S_{a_4}$$

where

$$
\left(\begin{array}{cccc}
S_{a_1} & S_{b_1} & & \\ \hline
 & S_{a_2} & S_{b_2} & \\ \hline
 & & S_{a_3} & S_{b_3} \\ \hline
 & & & S_{a_4}
\end{array}\right)
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\end{matrix}
\end{array}\right). \tag{3}
$$

It follows from Lemma 1 that the system (1) becomes

$$\begin{cases} \widehat{P}_1 A_1 \widehat{T}_1 \widehat{T}_1^{-1} X_1 (\widehat{T}_1)_\phi^{-1} (\widehat{T}_1)_\phi (A_1)_\phi (\widehat{P}_1)_\phi + \widehat{P}_1 B_1 \widehat{T}_2 \widehat{T}_2^{-1} X_2 (\widehat{T}_2)_\phi^{-1} (\widehat{T}_2)_\phi (B_1)_\phi (\widehat{P}_1)_\phi = \widehat{P}_1 C_1 (\widehat{P}_1)_\phi, \\ \widehat{P}_2 A_2 \widehat{T}_2 \widehat{T}_2^{-1} X_2 (\widehat{T}_2)_\phi^{-1} (\widehat{T}_2)_\phi (A_2)_\phi (\widehat{P}_2)_\phi + \widehat{P}_2 B_2 \widehat{T}_3 \widehat{T}_3^{-1} X_3 (\widehat{T}_3)_\phi^{-1} (\widehat{T}_3)_\phi (B_2)_\phi (\widehat{P}_2)_\phi = \widehat{P}_2 C_2 (\widehat{P}_2)_\phi, \\ \widehat{P}_3 A_3 \widehat{T}_3 \widehat{T}_3^{-1} X_3 (\widehat{T}_3)_\phi^{-1} (\widehat{T}_3)_\phi (A_3)_\phi (\widehat{P}_3)_\phi + \widehat{P}_3 B_3 \widehat{T}_4 \widehat{T}_4^{-1} X_4 (\widehat{T}_4)_\phi^{-1} (\widehat{T}_4)_\phi (B_3)_\phi (\widehat{P}_3)_\phi = \widehat{P}_3 C_3 (\widehat{P}_3)_\phi, \\ \widehat{P}_4 A_4 \widehat{T}_4 \widehat{T}_4^{-1} X_4 (\widehat{T}_4)_\phi^{-1} (\widehat{T}_4)_\phi (A_4)_\phi (\widehat{P}_4)_\phi = \widehat{P}_4 C_4 (\widehat{P}_4)_\phi, \end{cases}$$

where $X_i = -(X_i)_\phi$.

Put

$$\widehat{X_1} = \widehat{T}_1^{-1} X_1 (\widehat{T}_1)_\phi^{-1},\ \widehat{X_2} = \widehat{T}_2^{-1} X_2 (\widehat{T}_2)_\phi^{-1},\ \widehat{X_3} = \widehat{T}_3^{-1} X_3 (\widehat{T}_3)_\phi^{-1},\ \widehat{X_4} = \widehat{T}_4^{-1} X_4 (\widehat{T}_4)_\phi^{-1},$$

$$D_{ij}^{(1)} = \widehat{P}_1 C_1 (\widehat{P}_1)_\phi,\ D_{ij}^{(2)} = \widehat{P}_2 C_2 (\widehat{P}_2)_\phi,\ D_{ij}^{(3)} = \widehat{P}_3 C_3 (\widehat{P}_3)_\phi,\ D_{ij}^{(4)} = \widehat{P}_4 C_4 (\widehat{P}_4)_\phi.$$

According to Lemma 1, the system (1) is equivalent to the following system:

$$\begin{cases} S_{a_1} \widehat{X_1} (S_{a_1})_\phi + S_{b_1} \widehat{X_2} (S_{b_1})_\phi = D_{ij}^{(1)}, \\ S_{a_2} \widehat{X_2} (S_{a_2})_\phi + S_{b_2} \widehat{X_3} (S_{b_2})_\phi = D_{ij}^{(2)}, \\ S_{a_3} \widehat{X_3} (S_{a_3})_\phi + S_{b_3} \widehat{X_4} (S_{b_3})_\phi = D_{ij}^{(3)}, \\ S_{a_4} \widehat{X_4} (S_{a_4})_\phi = D_{ij}^{(4)}, \end{cases} \tag{4}$$

where $X_i = -(X_i)_\phi$. In the next section, we will consider the system (4).

3. Solvability Conditions for the Quaternion Matrix Equation to Possess a ϕ-Skew-Hermitian Solution

In this section, we provide some necessary and sufficient conditions for the existence of a ϕ-skew-Hermitian solution to the system (1). In order to solve the system of quaternion matrix Equation (1), we need to solve the system of quaternion matrix Equation (4).

First, let matrices $\widehat{X_1}$, $\widehat{X_2}$, $\widehat{X_3}$, $\widehat{X_4}$ have the following forms:

$$\widehat{X_1} = -(\widehat{X_1})_\phi = \begin{pmatrix} X_{11}^{(1)} & X_{12}^{(1)} & \cdots & X_{18}^{(1)} \\ -(X_{12}^{(1)})_\phi & X_{22}^{(1)} & \cdots & X_{28}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ -(X_{18}^{(1)})_\phi & -(X_{28}^{(1)})_\phi & \cdots & X_{88}^{(1)} \end{pmatrix},$$

$$\widehat{X_2} = -(\widehat{X_2})_\phi = \begin{pmatrix} X_{11}^{(2)} & X_{12}^{(2)} & \cdots & X_{1,18}^{(2)} \\ -(X_{12}^{(2)})_\phi & X_{22}^{(2)} & \cdots & X_{2,18}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ -(X_{1,18}^{(2)})_\phi & -(X_{2,18}^{(2)})_\phi & \cdots & X_{18,18}^{(2)} \end{pmatrix},$$

$$\widehat{X_3} = -(\widehat{X_3})_\phi = \begin{pmatrix} X_{11}^{(3)} & X_{12}^{(3)} & \cdots & X_{1,20}^{(3)} \\ -(X_{12}^{(3)})_\phi & X_{22}^{(3)} & \cdots & X_{2,20}^{(3)} \\ \vdots & \vdots & \cdots & \vdots \\ -(X_{1,20}^{(3)})_\phi & -(X_{2,20}^{(3)})_\phi & \cdots & X_{20,20}^{(3)} \end{pmatrix},$$

$$\widehat{X_4} = -(\widehat{X_4})_\phi = \begin{pmatrix} X_{11}^{(4)} & X_{12}^{(4)} & \cdots & X_{1,14}^{(4)} \\ -(X_{12}^{(4)})_\phi & X_{22}^{(4)} & \cdots & X_{2,14}^{(4)} \\ \vdots & \vdots & \cdots & \vdots \\ -(X_{1,14}^{(4)})_\phi & -(X_{2,14}^{(4)})_\phi & \cdots & X_{14,14}^{(4)} \end{pmatrix}.$$

Then, substituting $\widehat{X_1}$ and $\widehat{X_2}$ into the first equation in (4) yields

$$(D_{ij}^{(1)})_{14\times 14} = (D_1^{(1)}, D_2^{(1)}), \tag{5}$$

where

$$D_1^{(1)} = \begin{pmatrix} X_{11}^{(1)}+X_{11}^{(2)} & X_{12}^{(1)}+X_{12}^{(2)} & X_{13}^{(1)}+X_{13}^{(2)} & X_{14}^{(1)}+X_{14}^{(2)} & X_{15}^{(1)}+X_{15}^{(2)} \\ -(X_{12}^{(1)}+X_{12}^{(2)})_\phi & X_{22}^{(1)}+X_{22}^{(2)} & X_{23}^{(1)}+X_{23}^{(2)} & X_{24}^{(1)}+X_{24}^{(2)} & X_{25}^{(1)}+X_{25}^{(2)} \\ -(X_{13}^{(1)}+X_{13}^{(2)})_\phi & -(X_{23}^{(1)}+X_{23}^{(2)})_\phi & X_{33}^{(1)}+X_{33}^{(2)} & X_{34}^{(1)}+X_{34}^{(2)} & X_{35}^{(1)}+X_{35}^{(2)} \\ -(X_{14}^{(1)}+X_{14}^{(2)})_\phi & -(X_{24}^{(1)}+X_{24}^{(2)})_\phi & -(X_{34}^{(1)}+X_{34}^{(2)})_\phi & X_{44}^{(1)}+X_{44}^{(2)} & X_{45}^{(1)}+X_{45}^{(2)} \\ -(X_{15}^{(1)}+X_{15}^{(2)})_\phi & -(X_{25}^{(1)}+X_{25}^{(2)})_\phi & -(X_{35}^{(1)}+X_{35}^{(2)})_\phi & -(X_{45}^{(1)}+X_{45}^{(2)})_\phi & X_{55}^{(1)}+X_{55}^{(2)} \\ -(X_{16}^{(1)}+X_{16}^{(2)})_\phi & -(X_{26}^{(1)}+X_{26}^{(2)})_\phi & -(X_{36}^{(1)}+X_{36}^{(2)})_\phi & -(X_{46}^{(1)}+X_{46}^{(2)})_\phi & -(X_{56}^{(1)}+X_{56}^{(2)})_\phi \\ -(X_{17}^{(1)})_\phi & -(X_{27}^{(1)})_\phi & -(X_{37}^{(1)})_\phi & -(X_{47}^{(1)})_\phi & -(X_{57}^{(1)})_\phi \\ -(X_{17}^{(2)})_\phi & -(X_{27}^{(2)})_\phi & -(X_{37}^{(2)})_\phi & -(X_{47}^{(2)})_\phi & -(X_{57}^{(2)})_\phi \\ -(X_{18}^{(2)})_\phi & -(X_{28}^{(2)})_\phi & -(X_{38}^{(2)})_\phi & -(X_{48}^{(2)})_\phi & -(X_{58}^{(2)})_\phi \\ -(X_{19}^{(2)})_\phi & -(X_{29}^{(2)})_\phi & -(X_{39}^{(2)})_\phi & -(X_{49}^{(2)})_\phi & -(X_{59}^{(2)})_\phi \\ -(X_{1,10}^{(2)})_\phi & -(X_{2,10}^{(2)})_\phi & -(X_{3,10}^{(2)})_\phi & -(X_{4,10}^{(2)})_\phi & -(X_{5,10}^{(2)})_\phi \\ -(X_{1,11}^{(2)})_\phi & -(X_{2,11}^{(2)})_\phi & -(X_{3,11}^{(2)})_\phi & -(X_{4,11}^{(2)})_\phi & -(X_{5,11}^{(2)})_\phi \\ -(X_{1,12}^{(2)})_\phi & -(X_{2,12}^{(2)})_\phi & -(X_{3,12}^{(2)})_\phi & -(X_{4,12}^{(2)})_\phi & -(X_{5,12}^{(2)})_\phi \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_2^{(1)} = \begin{pmatrix} X_{16}^{(1)}+X_{16}^{(2)} & X_{17}^{(1)} & X_{17}^{(2)} & X_{18}^{(2)} & X_{19}^{(2)} & X_{1,10}^{(2)} & X_{1,11}^{(2)} & X_{1,12}^{(2)} & 0 \\ X_{26}^{(1)}+X_{26}^{(2)} & X_{27}^{(1)} & X_{27}^{(2)} & X_{28}^{(2)} & X_{29}^{(2)} & X_{2,10}^{(2)} & X_{2,11}^{(2)} & X_{2,12}^{(2)} & 0 \\ X_{36}^{(1)}+X_{36}^{(2)} & X_{37}^{(1)} & X_{37}^{(2)} & X_{38}^{(2)} & X_{39}^{(2)} & X_{3,10}^{(2)} & X_{3,11}^{(2)} & X_{3,12}^{(2)} & 0 \\ X_{46}^{(1)}+X_{46}^{(2)} & X_{47}^{(1)} & X_{47}^{(2)} & X_{48}^{(2)} & X_{49}^{(2)} & X_{4,10}^{(2)} & X_{4,11}^{(2)} & X_{4,12}^{(2)} & 0 \\ X_{56}^{(1)}+X_{56}^{(2)} & X_{57}^{(1)} & X_{57}^{(2)} & X_{58}^{(2)} & X_{59}^{(2)} & X_{5,10}^{(2)} & X_{5,11}^{(2)} & X_{5,12}^{(2)} & 0 \\ X_{66}^{(1)}+X_{66}^{(2)} & X_{67}^{(1)} & X_{67}^{(2)} & X_{68}^{(2)} & X_{69}^{(2)} & X_{6,10}^{(2)} & X_{6,11}^{(2)} & X_{6,12}^{(2)} & 0 \\ -(X_{67}^{(1)})_\phi & X_{77}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(X_{67}^{(2)})_\phi & 0 & X_{77}^{(2)} & X_{78}^{(2)} & X_{79}^{(2)} & X_{7,10}^{(2)} & X_{7,11}^{(2)} & X_{7,12}^{(2)} & 0 \\ -(X_{68}^{(2)})_\phi & 0 & -(X_{78}^{(2)})_\phi & X_{88}^{(2)} & X_{89}^{(2)} & X_{8,10}^{(2)} & X_{8,11}^{(2)} & X_{8,12}^{(2)} & 0 \\ -(X_{69}^{(2)})_\phi & 0 & -(X_{79}^{(2)})_\phi & -(X_{89}^{(2)})_\phi & X_{9,9}^{(2)} & X_{9,10}^{(2)} & X_{9,11}^{(2)} & X_{9,12}^{(2)} & 0 \\ -(X_{6,10}^{(2)})_\phi & 0 & -(X_{7,10}^{(2)})_\phi & -(X_{8,10}^{(2)})_\phi & -(X_{9,10}^{(2)})_\phi & X_{10,10}^{(2)} & X_{10,11}^{(2)} & X_{10,12}^{(2)} & 0 \\ -(X_{6,11}^{(2)})_\phi & 0 & -(X_{7,11}^{(2)})_\phi & -(X_{8,11}^{(2)})_\phi & -(X_{9,11}^{(2)})_\phi & -(X_{10,11}^{(2)})_\phi & X_{11,11}^{(2)} & X_{11,12}^{(2)} & 0 \\ -(X_{6,12}^{(2)})_\phi & 0 & -(X_{7,12}^{(2)})_\phi & -(X_{8,12}^{(2)})_\phi & -(X_{9,12}^{(2)})_\phi & -(X_{10,12}^{(2)})_\phi & -(X_{11,12}^{(2)})_\phi & X_{12,12}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Substituting $\widehat{X_2}$ and $\widehat{X_3}$ into the second Equation in (4) yields

$$(D_{ij}^{(2)})_{20\times 20} = (D_1^{(2)}, D_2^{(2)}, D_3^{(2)}), \tag{6}$$

where

$$D_1^{(2)} = \begin{pmatrix}
X_{11}^{(2)} + X_{11}^{(3)} & X_{12}^{(2)} + X_{12}^{(3)} & X_{13}^{(2)} + X_{13}^{(3)} & X_{14}^{(2)} + X_{14}^{(3)} & X_{15}^{(2)} & X_{17}^{(2)} + X_{15}^{(3)} \\
-(X_{12}^{(2)} + X_{12}^{(3)})_\phi & X_{22}^{(2)} + X_{22}^{(3)} & X_{23}^{(2)} + X_{23}^{(3)} & X_{24}^{(2)} + X_{24}^{(3)} & X_{25}^{(2)} & X_{27}^{(2)} + X_{25}^{(3)} \\
-(X_{13}^{(2)} + X_{13}^{(3)})_\phi & -(X_{23}^{(2)} + X_{23}^{(3)})_\phi & X_{33}^{(2)} + X_{33}^{(3)} & X_{34}^{(2)} + X_{34}^{(3)} & X_{35}^{(2)} & X_{37}^{(2)} + X_{35}^{(3)} \\
-(X_{14}^{(2)} + X_{14}^{(3)})_\phi & -(X_{24}^{(2)} + X_{24}^{(3)})_\phi & -(X_{34}^{(2)} + X_{34}^{(3)})_\phi & X_{44}^{(2)} + X_{44}^{(3)} & X_{45}^{(2)} & X_{47}^{(2)} + X_{45}^{(3)} \\
-(X_{15}^{(2)})_\phi & -(X_{25}^{(2)})_\phi & -(X_{35}^{(2)})_\phi & -(X_{45}^{(2)})_\phi & X_{55}^{(2)} & X_{57}^{(2)} \\
-(X_{17}^{(2)} + X_{15}^{(3)})_\phi & -(X_{27}^{(2)} + X_{25}^{(3)})_\phi & -(X_{37}^{(2)} + X_{35}^{(3)})_\phi & -(X_{47}^{(2)} + X_{45}^{(3)})_\phi & -(X_{57}^{(2)})_\phi & X_{77}^{(2)} + X_{55}^{(3)} \\
-(X_{18}^{(2)} + X_{16}^{(3)})_\phi & -(X_{28}^{(2)} + X_{26}^{(3)})_\phi & -(X_{38}^{(2)} + X_{36}^{(3)})_\phi & -(X_{48}^{(2)} + X_{46}^{(3)})_\phi & -(X_{58}^{(2)})_\phi & -(X_{78}^{(2)} + X_{56}^{(3)})_\phi \\
-(X_{19}^{(2)} + X_{17}^{(3)})_\phi & -(X_{29}^{(2)} + X_{27}^{(3)})_\phi & -(X_{39}^{(2)} + X_{37}^{(3)})_\phi & -(X_{49}^{(2)} + X_{47}^{(3)})_\phi & -(X_{59}^{(2)})_\phi & -(X_{79}^{(2)} + X_{57}^{(3)})_\phi \\
-(X_{1,10}^{(2)} + X_{18}^{(3)})_\phi & -(X_{2,10}^{(2)} + X_{28}^{(3)})_\phi & -(X_{3,10}^{(2)} + X_{38}^{(3)})_\phi & -(X_{4,10}^{(2)} + X_{48}^{(3)})_\phi & -(X_{5,10}^{(2)})_\phi & -(X_{7,10}^{(2)} + X_{58}^{(3)})_\phi \\
-(X_{1,11}^{(2)})_\phi & -(X_{2,11}^{(2)})_\phi & -(X_{3,11}^{(2)})_\phi & -(X_{4,11}^{(2)})_\phi & -(X_{5,11}^{(2)})_\phi & -(X_{7,11}^{(2)})_\phi \\
-(X_{1,13}^{(2)} + X_{19}^{(3)})_\phi & -(X_{2,13}^{(2)} + X_{29}^{(3)})_\phi & -(X_{3,13}^{(2)} + X_{39}^{(3)})_\phi & -(X_{4,13}^{(2)} + X_{49}^{(3)})_\phi & -(X_{5,13}^{(2)})_\phi & -(X_{7,13}^{(2)} + X_{59}^{(3)})_\phi \\
-(X_{1,14}^{(2)} + X_{1,10}^{(3)})_\phi & -(X_{2,14}^{(2)} + X_{2,10}^{(3)})_\phi & -(X_{3,14}^{(2)} + X_{3,10}^{(3)})_\phi & -(X_{4,14}^{(2)} + X_{4,10}^{(3)})_\phi & -(X_{5,14}^{(2)})_\phi & -(X_{7,14}^{(2)} + X_{5,10}^{(3)})_\phi \\
-(X_{1,15}^{(2)} + X_{1,11}^{(3)})_\phi & -(X_{2,15}^{(2)} + X_{2,11}^{(3)})_\phi & -(X_{3,15}^{(2)} + X_{3,11}^{(3)})_\phi & -(X_{4,15}^{(2)} + X_{4,11}^{(3)})_\phi & -(X_{5,15}^{(2)})_\phi & -(X_{7,15}^{(2)} + X_{5,11}^{(3)})_\phi \\
-(X_{1,16}^{(2)} + X_{1,12}^{(3)})_\phi & -(X_{2,16}^{(2)} + X_{2,12}^{(3)})_\phi & -(X_{3,16}^{(2)} + X_{3,12}^{(3)})_\phi & -(X_{4,16}^{(2)} + X_{4,12}^{(3)})_\phi & -(X_{5,16}^{(2)})_\phi & -(X_{7,16}^{(2)} + X_{5,12}^{(3)})_\phi \\
-(X_{1,17}^{(2)})_\phi & -(X_{2,17}^{(2)})_\phi & -(X_{3,17}^{(2)})_\phi & -(X_{4,17}^{(2)})_\phi & -(X_{5,17}^{(2)})_\phi & -(X_{7,17}^{(2)})_\phi \\
-(X_{1,13}^{(3)})_\phi & -(X_{2,13}^{(3)})_\phi & -(X_{3,13}^{(3)})_\phi & -(X_{4,13}^{(3)})_\phi & 0 & -(X_{5,13}^{(3)})_\phi \\
-(X_{1,14}^{(3)})_\phi & -(X_{2,14}^{(3)})_\phi & -(X_{3,14}^{(3)})_\phi & -(X_{4,14}^{(3)})_\phi & 0 & -(X_{5,14}^{(3)})_\phi \\
-(X_{1,15}^{(3)})_\phi & -(X_{2,15}^{(3)})_\phi & -(X_{3,15}^{(3)})_\phi & -(X_{4,15}^{(3)})_\phi & 0 & -(X_{5,15}^{(3)})_\phi \\
-(X_{1,16}^{(3)})_\phi & -(X_{2,16}^{(3)})_\phi & -(X_{3,16}^{(3)})_\phi & -(X_{4,16}^{(3)})_\phi & 0 & -(X_{5,16}^{(3)})_\phi \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$D_2^{(2)} = \begin{pmatrix}
X_{18}^{(2)}+X_{16}^{(3)} & X_{19}^{(2)}+X_{17}^{(3)} & X_{1,10}^{(2)}+X_{18}^{(3)} & X_{1,11}^{(2)} & X_{1,13}^{(2)}+X_{19}^{(3)} & X_{1,14}^{(2)}+X_{1,10}^{(3)} \\
X_{28}^{(2)}+X_{26}^{(3)} & X_{29}^{(2)}+X_{27}^{(3)} & X_{2,10}^{(2)}+X_{28}^{(3)} & X_{2,11}^{(2)} & X_{2,13}^{(2)}+X_{29}^{(3)} & X_{2,14}^{(2)}+X_{2,10}^{(3)} \\
X_{38}^{(2)}+X_{36}^{(3)} & X_{39}^{(2)}+X_{37}^{(3)} & X_{3,10}^{(2)}+X_{38}^{(3)} & X_{3,11}^{(2)} & X_{3,13}^{(2)}+X_{39}^{(3)} & X_{3,14}^{(2)}+X_{3,10}^{(3)} \\
X_{48}^{(2)}+X_{46}^{(3)} & X_{49}^{(2)}+X_{47}^{(3)} & X_{4,10}^{(2)}+X_{48}^{(3)} & X_{4,11}^{(2)} & X_{4,13}^{(2)}+X_{49}^{(3)} & X_{4,14}^{(2)}+X_{4,10}^{(3)} \\
X_{58}^{(2)} & X_{59}^{(2)} & X_{5,10}^{(2)} & X_{5,11}^{(2)} & X_{5,13}^{(2)} & X_{5,14}^{(2)} \\
X_{78}^{(2)}+X_{56}^{(3)} & X_{79}^{(2)}+X_{57}^{(3)} & X_{7,10}^{(2)}+X_{58}^{(3)} & X_{7,11}^{(2)} & X_{7,13}^{(2)}+X_{59}^{(3)} & X_{7,14}^{(2)}+X_{5,10}^{(3)} \\
X_{88}^{(2)}+X_{66}^{(3)} & X_{89}^{(2)}+X_{67}^{(3)} & X_{8,10}^{(2)}+X_{68}^{(3)} & X_{8,11}^{(2)} & X_{8,13}^{(2)}+X_{69}^{(3)} & X_{8,14}^{(2)}+X_{6,10}^{(3)} \\
-(X_{89}^{(2)}+X_{67}^{(3)})_\phi & X_{99}^{(2)}+X_{77}^{(3)} & X_{9,10}^{(2)}+X_{78}^{(3)} & X_{9,11}^{(2)} & X_{9,13}^{(2)}+X_{79}^{(3)} & X_{9,14}^{(2)}+X_{7,10}^{(3)} \\
-(X_{8,10}^{(2)}+X_{68}^{(3)})_\phi & -(X_{9,10}^{(2)}+X_{78}^{(3)})_\phi & X_{10,10}^{(2)}+X_{88}^{(3)} & X_{10,11}^{(2)} & X_{10,13}^{(2)}+X_{89}^{(3)} & X_{10,14}^{(2)}+X_{8,10}^{(3)} \\
-(X_{8,11}^{(2)})_\phi & -(X_{9,11}^{(2)})_\phi & -(X_{10,11}^{(2)})_\phi & X_{11,11}^{(2)} & X_{11,13}^{(2)} & X_{11,14}^{(2)} \\
-(X_{8,13}^{(2)}+X_{69}^{(3)})_\phi & -(X_{9,13}^{(2)}+X_{79}^{(3)})_\phi & -(X_{10,13}^{(2)}+X_{89}^{(3)})_\phi & -(X_{11,13}^{(2)})_\phi & X_{13,13}^{(2)}+X_{99}^{(3)} & X_{13,14}^{(2)}+X_{9,10}^{(3)} \\
-(X_{8,14}^{(2)}+X_{6,10}^{(3)})_\phi & -(X_{9,14}^{(2)}+X_{7,10}^{(3)})_\phi & -(X_{10,14}^{(2)}+X_{8,10}^{(3)})_\phi & -(X_{11,14}^{(2)})_\phi & -(X_{13,14}^{(2)}+X_{9,10}^{(3)})_\phi & X_{14,14}^{(2)}+X_{10,10}^{(3)} \\
-(X_{8,15}^{(2)}+X_{6,11}^{(3)})_\phi & -(X_{9,15}^{(2)}+X_{7,11}^{(3)})_\phi & -(X_{10,15}^{(2)}+X_{8,11}^{(3)})_\phi & -(X_{11,15}^{(2)})_\phi & -(X_{13,15}^{(2)}+X_{9,11}^{(3)})_\phi & -(X_{14,15}^{(2)}+X_{10,11}^{(3)})_\phi \\
-(X_{8,16}^{(2)}+X_{6,12}^{(3)})_\phi & -(X_{9,16}^{(2)}+X_{7,12}^{(3)})_\phi & -(X_{10,16}^{(2)}+X_{8,12}^{(3)})_\phi & -(X_{11,16}^{(2)})_\phi & -(X_{13,16}^{(2)}+X_{9,12}^{(3)})_\phi & -(X_{14,16}^{(2)}+X_{10,12}^{(3)})_\phi \\
-(X_{8,17}^{(2)})_\phi & -(X_{9,17}^{(2)})_\phi & -(X_{10,17}^{(2)})_\phi & -(X_{11,17}^{(2)})_\phi & -(X_{13,17}^{(2)})_\phi & -(X_{14,17}^{(2)})_\phi \\
-(X_{6,13}^{(3)})_\phi & -(X_{7,13}^{(3)})_\phi & -(X_{8,13}^{(3)})_\phi & 0 & -(X_{9,13}^{(3)})_\phi & -(X_{10,13}^{(3)})_\phi \\
-(X_{6,14}^{(3)})_\phi & -(X_{7,14}^{(3)})_\phi & -(X_{8,14}^{(3)})_\phi & 0 & -(X_{9,14}^{(3)})_\phi & -(X_{10,14}^{(3)})_\phi \\
-(X_{6,15}^{(3)})_\phi & -(X_{7,15}^{(3)})_\phi & -(X_{8,15}^{(3)})_\phi & 0 & -(X_{9,15}^{(3)})_\phi & -(X_{10,15}^{(3)})_\phi \\
-(X_{6,16}^{(3)})_\phi & -(X_{7,16}^{(3)})_\phi & -(X_{8,16}^{(3)})_\phi & 0 & -(X_{9,16}^{(3)})_\phi & -(X_{10,16}^{(3)})_\phi \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$D_3^{(2)} = \begin{pmatrix}
X^{(2)}_{1,15} + X^{(3)}_{1,11} & X^{(2)}_{1,16} + X^{(3)}_{1,12} & X^{(2)}_{1,17} & X^{(3)}_{1,13} & X^{(3)}_{1,14} & X^{(3)}_{1,15} & X^{(3)}_{1,16} & 0 \\
X^{(2)}_{2,15} + X^{(3)}_{2,11} & X^{(2)}_{2,16} + X^{(3)}_{2,12} & X^{(2)}_{2,17} & X^{(3)}_{2,13} & X^{(3)}_{2,14} & X^{(3)}_{2,15} & X^{(3)}_{2,16} & 0 \\
X^{(2)}_{3,15} + X^{(3)}_{3,11} & X^{(2)}_{3,16} + X^{(3)}_{3,12} & X^{(2)}_{3,17} & X^{(3)}_{3,13} & X^{(3)}_{3,14} & X^{(3)}_{3,15} & X^{(3)}_{3,16} & 0 \\
X^{(2)}_{4,15} + X^{(3)}_{4,11} & X^{(2)}_{4,16} + X^{(3)}_{4,12} & X^{(2)}_{4,17} & X^{(3)}_{4,13} & X^{(3)}_{4,14} & X^{(3)}_{4,15} & X^{(3)}_{4,16} & 0 \\
X^{(2)}_{5,15} & X^{(2)}_{5,16} & X^{(2)}_{5,17} & 0 & 0 & 0 & 0 & 0 \\
X^{(2)}_{7,15} + X^{(3)}_{5,11} & X^{(2)}_{7,16} + X^{(3)}_{5,12} & X^{(2)}_{7,17} & X^{(3)}_{5,13} & X^{(3)}_{5,14} & X^{(3)}_{5,15} & X^{(3)}_{5,16} & 0 \\
X^{(2)}_{8,15} + X^{(3)}_{6,11} & X^{(2)}_{8,16} + X^{(3)}_{6,12} & X^{(2)}_{8,17} & X^{(3)}_{6,13} & X^{(3)}_{6,14} & X^{(3)}_{6,15} & X^{(3)}_{6,16} & 0 \\
X^{(2)}_{9,15} + X^{(3)}_{7,11} & X^{(2)}_{9,16} + X^{(3)}_{7,12} & X^{(2)}_{9,17} & X^{(3)}_{7,13} & X^{(3)}_{7,14} & X^{(3)}_{7,15} & X^{(3)}_{7,16} & 0 \\
X^{(2)}_{10,15} + X^{(3)}_{8,11} & X^{(2)}_{10,16} + X^{(3)}_{8,12} & X^{(2)}_{10,17} & X^{(3)}_{8,13} & X^{(3)}_{8,14} & X^{(3)}_{8,15} & X^{(3)}_{8,16} & 0 \\
X^{(2)}_{11,15} & X^{(2)}_{11,16} & X^{(2)}_{11,17} & 0 & 0 & 0 & 0 & 0 \\
X^{(2)}_{13,15} + X^{(3)}_{9,11} & X^{(2)}_{13,16} + X^{(3)}_{9,12} & X^{(2)}_{13,17} & X^{(3)}_{9,13} & X^{(3)}_{9,14} & X^{(3)}_{9,15} & X^{(3)}_{9,16} & 0 \\
X^{(2)}_{14,15} + X^{(3)}_{10,11} & X^{(2)}_{14,16} + X^{(3)}_{10,12} & X^{(2)}_{14,17} & X^{(3)}_{10,13} & X^{(3)}_{10,14} & X^{(3)}_{10,15} & X^{(3)}_{10,16} & 0 \\
X^{(2)}_{15,15} + X^{(3)}_{11,11} & X^{(2)}_{15,16} + X^{(3)}_{11,12} & X^{(2)}_{15,17} & X^{(3)}_{11,13} & X^{(3)}_{11,14} & X^{(3)}_{11,15} & X^{(3)}_{11,16} & 0 \\
-(X^{(2)}_{15,16} + X^{(3)}_{11,12})_\phi & X^{(2)}_{16,16} + X^{(3)}_{12,12} & X^{(2)}_{16,17} & X^{(3)}_{12,13} & X^{(3)}_{12,14} & X^{(3)}_{12,15} & X^{(3)}_{12,16} & 0 \\
-(X^{(2)}_{15,17})_\phi & -(X^{(2)}_{16,17})_\phi & X^{(2)}_{17,17} & 0 & 0 & 0 & 0 & 0 \\
-(X^{(3)}_{11,13})_\phi & -(X^{(3)}_{12,13})_\phi & 0 & X^{(3)}_{13,13} & X^{(3)}_{13,14} & X^{(3)}_{13,15} & X^{(3)}_{13,16} & 0 \\
-(X^{(3)}_{11,14})_\phi & -(X^{(3)}_{12,14})_\phi & 0 & -(X^{(3)}_{13,14})_\phi & X^{(3)}_{14,14} & X^{(3)}_{14,15} & X^{(3)}_{14,16} & 0 \\
-(X^{(3)}_{11,15})_\phi & -(X^{(3)}_{12,15})_\phi & 0 & -(X^{(3)}_{13,15})_\phi & -(X^{(3)}_{14,15})_\phi & X^{(3)}_{15,15} & X^{(3)}_{15,16} & 0 \\
-(X^{(3)}_{11,16})_\phi & -(X^{(3)}_{12,16})_\phi & 0 & -(X^{(3)}_{13,16})_\phi & -(X^{(3)}_{14,16})_\phi & -(X^{(3)}_{15,16})_\phi & X^{(3)}_{16,16} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Substituting $\widehat{X_3}$ and $\widehat{X_4}$ into the third equation in (4) yields

$$(D^{(3)}_{ij})_{18\times 18} = (D^{(3)}_1, D^{(3)}_2, D^{(3)}_3), \tag{7}$$

where

$$D_1^{(3)} = \begin{pmatrix}
X^{(3)}_{11} + X^{(4)}_{11} & X^{(3)}_{12} + X^{(4)}_{12} & X^{(3)}_{13} & X^{(3)}_{15} + X^{(4)}_{13} & X^{(3)}_{16} + X^{(4)}_{14} & X^{(3)}_{17} \\
-(X^{(3)}_{12} + X^{(4)}_{12})_\phi & X^{(3)}_{22} + X^{(4)}_{22} & X^{(3)}_{23} & X^{(3)}_{25} + X^{(4)}_{23} & X^{(3)}_{26} + X^{(4)}_{24} & X^{(3)}_{27} \\
-(X^{(3)}_{13})_\phi & -(X^{(3)}_{23})_\phi & X^{(3)}_{33} & X^{(3)}_{35} & X^{(3)}_{36} & X^{(3)}_{37} \\
-(X^{(3)}_{15} + X^{(4)}_{13})_\phi & -(X^{(3)}_{25} + X^{(4)}_{23})_\phi & -(X^{(3)}_{35})_\phi & X^{(3)}_{55} + X^{(4)}_{33} & X^{(3)}_{56} + X^{(4)}_{34} & X^{(3)}_{57} \\
-(X^{(3)}_{16} + X^{(4)}_{14})_\phi & -(X^{(3)}_{26} + X^{(4)}_{24})_\phi & -(X^{(3)}_{36})_\phi & -(X^{(3)}_{56} + X^{(4)}_{34})_\phi & X^{(3)}_{66} + X^{(4)}_{44} & X^{(3)}_{67} \\
-(X^{(3)}_{17})_\phi & -(X^{(3)}_{27})_\phi & -(X^{(3)}_{37})_\phi & -(X^{(3)}_{57})_\phi & -(X^{(3)}_{67})_\phi & X^{(3)}_{77} \\
-(X^{(3)}_{19} + X^{(4)}_{15})_\phi & -(X^{(3)}_{29} + X^{(4)}_{25})_\phi & -(X^{(3)}_{39})_\phi & -(X^{(3)}_{59} + X^{(4)}_{35})_\phi & -(X^{(3)}_{69} + X^{(4)}_{45})_\phi & -(X^{(3)}_{79})_\phi \\
-(X^{(3)}_{1,10} + X^{(4)}_{16})_\phi & -(X^{(3)}_{2,10} + X^{(4)}_{26})_\phi & -(X^{(3)}_{3,10})_\phi & -(X^{(3)}_{5,10} + X^{(4)}_{36})_\phi & -(X^{(3)}_{6,10} + X^{(4)}_{46})_\phi & -(X^{(3)}_{7,10})_\phi \\
-(X^{(3)}_{1,11})_\phi & -(X^{(3)}_{2,11})_\phi & -(X^{(3)}_{3,11})_\phi & -(X^{(3)}_{5,11})_\phi & -(X^{(3)}_{6,11})_\phi & -(X^{(3)}_{7,11})_\phi \\
-(X^{(3)}_{1,13} + X^{(4)}_{17})_\phi & -(X^{(3)}_{2,13} + X^{(4)}_{27})_\phi & -(X^{(3)}_{3,13})_\phi & -(X^{(3)}_{5,13} + X^{(4)}_{37})_\phi & -(X^{(3)}_{6,13} + X^{(4)}_{47})_\phi & -(X^{(3)}_{7,13})_\phi \\
-(X^{(3)}_{1,14} + X^{(4)}_{18})_\phi & -(X^{(3)}_{2,14} + X^{(4)}_{28})_\phi & -(X^{(3)}_{3,14})_\phi & -(X^{(3)}_{5,14} + X^{(4)}_{38})_\phi & -(X^{(3)}_{6,14} + X^{(4)}_{48})_\phi & -(X^{(3)}_{7,14})_\phi \\
-(X^{(3)}_{1,15})_\phi & -(X^{(3)}_{2,15})_\phi & -(X^{(3)}_{3,15})_\phi & -(X^{(3)}_{5,15})_\phi & -(X^{(3)}_{6,15})_\phi & -(X^{(3)}_{7,15})_\phi \\
-(X^{(3)}_{1,17} + X^{(4)}_{19})_\phi & -(X^{(3)}_{2,17} + X^{(4)}_{29})_\phi & -(X^{(3)}_{3,17})_\phi & -(X^{(3)}_{5,17} + X^{(4)}_{39})_\phi & -(X^{(3)}_{6,17} + X^{(4)}_{49})_\phi & -(X^{(3)}_{7,17})_\phi \\
-(X^{(3)}_{1,18} + X^{(4)}_{1,10})_\phi & -(X^{(3)}_{2,18} + X^{(4)}_{2,10})_\phi & -(X^{(3)}_{3,18})_\phi & -(X^{(3)}_{5,18} + X^{(4)}_{3,10})_\phi & -(X^{(3)}_{6,18} + X^{(4)}_{4,10})_\phi & -(X^{(3)}_{7,18})_\phi \\
-(X^{(3)}_{1,19})_\phi & -(X^{(3)}_{2,19})_\phi & -(X^{(3)}_{3,19})_\phi & -(X^{(3)}_{5,19})_\phi & -(X^{(3)}_{6,19})_\phi & -(X^{(3)}_{7,19})_\phi \\
-(X^{(4)}_{1,11})_\phi & -(X^{(4)}_{2,11})_\phi & 0 & -(X^{(4)}_{3,11})_\phi & -(X^{(4)}_{4,11})_\phi & 0 \\
-(X^{(4)}_{1,12})_\phi & -(X^{(4)}_{2,12})_\phi & 0 & -(X^{(4)}_{3,12})_\phi & -(X^{(4)}_{4,12})_\phi & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$D_2^{(3)} = \begin{pmatrix} X_{19}^{(3)} + X_{15}^{(4)} & X_{1,10}^{(3)} + X_{16}^{(4)} & X_{1,11}^{(3)} & X_{1,13}^{(3)} + X_{17}^{(4)} & X_{1,14}^{(3)} + X_{18}^{(4)} & X_{1,15}^{(3)} \\ X_{29}^{(3)} + X_{25}^{(4)} & X_{2,10}^{(3)} + X_{26}^{(4)} & X_{2,11}^{(3)} & X_{2,13}^{(3)} + X_{27}^{(4)} & X_{2,14}^{(3)} + X_{28}^{(4)} & X_{2,15}^{(3)} \\ X_{39}^{(3)} & X_{3,10}^{(3)} & X_{3,11}^{(3)} & X_{3,13}^{(3)} & X_{3,14}^{(3)} & X_{3,15}^{(3)} \\ X_{59}^{(3)} + X_{35}^{(4)} & X_{5,10}^{(3)} + X_{36}^{(4)} & X_{5,11}^{(3)} & X_{5,13}^{(3)} + X_{37}^{(4)} & X_{5,14}^{(3)} + X_{38}^{(4)} & X_{5,15}^{(3)} \\ X_{69}^{(3)} + X_{45}^{(4)} & X_{6,10}^{(3)} + X_{46}^{(4)} & X_{6,11}^{(3)} & X_{6,13}^{(3)} + X_{47}^{(4)} & X_{6,14}^{(3)} + X_{48}^{(4)} & X_{6,15}^{(3)} \\ X_{79}^{(3)} & X_{7,10}^{(3)} & X_{7,11}^{(3)} & X_{7,13}^{(3)} & X_{7,14}^{(3)} & X_{7,15}^{(3)} \\ X_{99}^{(3)} + X_{55}^{(4)} & X_{9,10}^{(3)} + X_{56}^{(4)} & X_{9,11}^{(3)} & X_{9,13}^{(3)} + X_{57}^{(4)} & X_{9,14}^{(3)} + X_{58}^{(4)} & X_{9,15}^{(3)} \\ -(X_{9,10}^{(3)} + X_{56}^{(4)})_\phi & X_{10,10}^{(3)} + X_{66}^{(4)} & X_{10,11}^{(3)} & X_{10,13}^{(3)} + X_{67}^{(4)} & X_{10,14}^{(3)} + X_{68}^{(4)} & X_{10,15}^{(3)} \\ -(X_{9,11}^{(3)})_\phi & -(X_{10,11}^{(3)})_\phi & X_{11,11}^{(3)} & X_{11,13}^{(3)} & X_{11,14}^{(3)} & X_{11,15}^{(3)} \\ -(X_{9,13}^{(3)} + X_{57}^{(4)})_\phi & -(X_{10,13}^{(3)} + X_{67}^{(4)})_\phi & -(X_{11,13}^{(3)})_\phi & X_{13,13}^{(3)} + X_{77}^{(4)} & X_{13,14}^{(3)} + X_{78}^{(4)} & X_{13,15}^{(3)} \\ -(X_{9,14}^{(3)} + X_{58}^{(4)})_\phi & -(X_{10,14}^{(3)} + X_{68}^{(4)})_\phi & -(X_{11,14}^{(3)})_\phi & -(X_{13,14}^{(3)} + X_{78}^{(4)})_\phi & X_{14,14}^{(3)} + X_{88}^{(4)} & X_{14,15}^{(3)} \\ -(X_{9,15}^{(3)})_\phi & -(X_{10,15}^{(3)})_\phi & -(X_{11,15}^{(3)})_\phi & -(X_{13,15}^{(3)})_\phi & -(X_{14,15}^{(3)})_\phi & X_{15,15}^{(3)} \\ -(X_{9,17}^{(3)} + X_{59}^{(4)})_\phi & -(X_{10,17}^{(3)} + X_{69}^{(4)})_\phi & -(X_{11,17}^{(3)})_\phi & -(X_{13,17}^{(3)} + X_{79}^{(4)})_\phi & -(X_{14,17}^{(3)} + X_{89}^{(4)})_\phi & -(X_{15,17}^{(3)})_\phi \\ -(X_{9,18}^{(3)} + X_{5,10}^{(4)})_\phi & -(X_{10,18}^{(3)} + X_{6,10}^{(4)})_\phi & -(X_{11,18}^{(3)})_\phi & -(X_{13,18}^{(3)} + X_{7,10}^{(4)})_\phi & -(X_{14,18}^{(3)} + X_{8,10}^{(4)})_\phi & -(X_{15,18}^{(3)})_\phi \\ -(X_{9,19}^{(3)})_\phi & -(X_{10,19}^{(3)})_\phi & -(X_{11,19}^{(3)})_\phi & -(X_{13,19}^{(3)})_\phi & -(X_{14,19}^{(3)})_\phi & -(X_{15,19}^{(3)})_\phi \\ -(X_{5,11}^{(4)})_\phi & -(X_{6,11}^{(4)})_\phi & 0 & -(X_{7,11}^{(4)})_\phi & -(X_{8,11}^{(4)})_\phi & 0 \\ -(X_{5,12}^{(4)})_\phi & -(X_{6,12}^{(4)})_\phi & 0 & -(X_{7,12}^{(4)})_\phi & -(X_{8,12}^{(4)})_\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_3^{(3)} = \begin{pmatrix} X_{1,17}^{(3)} + X_{19}^{(4)} & X_{1,18}^{(3)} + X_{1,10}^{(4)} & X_{1,19}^{(3)} & X_{1,11}^{(4)} & X_{1,12}^{(4)} & 0 \\ X_{2,17}^{(3)} + X_{29}^{(4)} & X_{2,18}^{(3)} + X_{2,10}^{(4)} & X_{2,19}^{(3)} & X_{2,11}^{(4)} & X_{2,12}^{(4)} & 0 \\ X_{3,17}^{(3)} & X_{3,18}^{(3)} & X_{3,19}^{(3)} & 0 & 0 & 0 \\ X_{5,17}^{(3)} + X_{39}^{(4)} & X_{5,18}^{(3)} + X_{3,10}^{(4)} & X_{5,19}^{(3)} & X_{3,11}^{(4)} & X_{3,12}^{(4)} & 0 \\ X_{6,17}^{(3)} + X_{49}^{(4)} & X_{6,18}^{(3)} + X_{4,10}^{(4)} & X_{6,19}^{(3)} & X_{4,11}^{(4)} & X_{4,12}^{(4)} & 0 \\ X_{7,17}^{(3)} & X_{7,18}^{(3)} & X_{7,19}^{(3)} & 0 & 0 & 0 \\ X_{9,17}^{(3)} + X_{59}^{(4)} & X_{9,18}^{(3)} + X_{5,10}^{(4)} & X_{9,19}^{(3)} & X_{5,11}^{(4)} & X_{5,12}^{(4)} & 0 \\ X_{10,17}^{(3)} + X_{69}^{(4)} & X_{10,18}^{(3)} + X_{6,10}^{(4)} & X_{10,19}^{(3)} & X_{6,11}^{(4)} & X_{6,12}^{(4)} & 0 \\ X_{11,17}^{(3)} & X_{11,18}^{(3)} & X_{11,19}^{(3)} & 0 & 0 & 0 \\ X_{13,17}^{(3)} + X_{79}^{(4)} & X_{13,18}^{(3)} + X_{7,10}^{(4)} & X_{13,19}^{(3)} & X_{7,11}^{(4)} & X_{7,12}^{(4)} & 0 \\ X_{14,17}^{(3)} + X_{89}^{(4)} & X_{14,18}^{(3)} + X_{8,10}^{(4)} & X_{14,19}^{(3)} & X_{8,11}^{(4)} & X_{8,12}^{(4)} & 0 \\ X_{15,17}^{(3)} & X_{15,18}^{(3)} & X_{15,19}^{(3)} & 0 & 0 & 0 \\ X_{17,17}^{(3)} + X_{99}^{(4)} & X_{17,18}^{(3)} + X_{9,10}^{(4)} & X_{17,19}^{(3)} & X_{9,11}^{(4)} & X_{9,12}^{(4)} & 0 \\ -(X_{17,18}^{(3)} + X_{9,10}^{(4)})_\phi & X_{18,18}^{(3)} + X_{10,10}^{(4)} & X_{18,19}^{(3)} & X_{10,11}^{(4)} & X_{10,12}^{(4)} & 0 \\ -(X_{17,19}^{(3)})_\phi & -(X_{18,19}^{(3)})_\phi & X_{19,19}^{(3)} & 0 & 0 & 0 \\ -(X_{9,11}^{(4)})_\phi & -(X_{10,11}^{(4)})_\phi & 0 & X_{11,11}^{(4)} & X_{11,12}^{(4)} & 0 \\ -(X_{9,12}^{(4)})_\phi & -(X_{10,12}^{(4)})_\phi & 0 & -(X_{11,12}^{(4)})_\phi & X_{12,12}^{(4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Substituting $\widehat{X_4}$ into the fourth equation in (4) yields

$$(D_{ij}^{(4)})_{8\times 8} = \begin{pmatrix} X_{11}^{(4)} & X_{13}^{(4)} & X_{15}^{(4)} & X_{17}^{(4)} & X_{19}^{(4)} & X_{1,11}^{(4)} & X_{1,13}^{(4)} & 0 \\ -(X_{13}^{(4)})_\phi & X_{33}^{(4)} & X_{35}^{(4)} & X_{37}^{(4)} & X_{39}^{(4)} & X_{3,11}^{(4)} & X_{3,13}^{(4)} & 0 \\ -(X_{15}^{(4)})_\phi & -(X_{35}^{(4)})_\phi & X_{55}^{(4)} & X_{57}^{(4)} & X_{59}^{(4)} & X_{5,11}^{(4)} & X_{5,13}^{(4)} & 0 \\ -(X_{17}^{(4)})_\phi & -(X_{37}^{(4)})_\phi & -(X_{57}^{(4)})_\phi & X_{77}^{(4)} & X_{79}^{(4)} & X_{7,11}^{(4)} & X_{7,13}^{(4)} & 0 \\ -(X_{19}^{(4)})_\phi & -(X_{39}^{(4)})_\phi & -(X_{59}^{(4)})_\phi & -(X_{79}^{(4)})_\phi & X_{99}^{(4)} & X_{9,11}^{(4)} & X_{9,13}^{(4)} & 0 \\ -(X_{1,11}^{(4)})_\phi & -(X_{3,11}^{(4)})_\phi & -(X_{5,11}^{(4)})_\phi & -(X_{7,11}^{(4)})_\phi & -(X_{9,11}^{(4)})_\phi & X_{11,11}^{(4)} & X_{11,13}^{(4)} & 0 \\ -(X_{1,13}^{(4)})_\phi & -(X_{3,13}^{(4)})_\phi & -(X_{5,13}^{(4)})_\phi & -(X_{7,13}^{(4)})_\phi & -(X_{9,13}^{(4)})_\phi & -(X_{11,13}^{(4)})_\phi & X_{13,13}^{(4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (8)$$

Hence, the system of (1) has a ϕ-skew-Hermitian solution (X_1, X_2, X_3, X_4) if and only if Equation (4) has a ϕ-skew-Hermitian solution. Note that (5)–(8) are consistent if and only if

$$\begin{pmatrix} D^{(1)}_{14,1} \\ D^{(1)}_{14,2} \\ \vdots \\ D^{(1)}_{14,14} \end{pmatrix} = 0, \begin{pmatrix} D^{(2)}_{20,1} \\ D^{(2)}_{20,2} \\ \vdots \\ D^{(2)}_{20,20} \end{pmatrix} = 0, \begin{pmatrix} D^{(3)}_{18,1} \\ D^{(3)}_{18,2} \\ \vdots \\ D^{(3)}_{18,18} \end{pmatrix} = 0, \begin{pmatrix} D^{(4)}_{81} \\ D^{(4)}_{82} \\ \vdots \\ D^{(4)}_{88} \end{pmatrix} = 0, \tag{9}$$

$$\begin{pmatrix} D^{(1)}_{1,14} \\ D^{(1)}_{2,14} \\ \vdots \\ D^{(1)}_{14,14} \end{pmatrix} = 0, \begin{pmatrix} D^{(2)}_{1,20} \\ D^{(2)}_{2,20} \\ \vdots \\ D^{(2)}_{20,20} \end{pmatrix} = 0, \begin{pmatrix} D^{(3)}_{1,18} \\ D^{(3)}_{2,18} \\ \vdots \\ D^{(3)}_{18,18} \end{pmatrix} = 0, \begin{pmatrix} D^{(4)}_{18} \\ D^{(4)}_{28} \\ \vdots \\ D^{(4)}_{88} \end{pmatrix} = 0, \tag{10}$$

$$\begin{pmatrix} D^{(1)}_{87} \\ D^{(1)}_{97} \\ \vdots \\ D^{(1)}_{14,7} \end{pmatrix} = 0, \tag{11}$$

$$\begin{pmatrix} D^{(2)}_{16,5} \\ D^{(2)}_{17,5} \\ \vdots \\ D^{(2)}_{20,5} \end{pmatrix} = 0, \begin{pmatrix} D^{(2)}_{16,10} \\ D^{(2)}_{17,10} \\ \vdots \\ D^{(2)}_{20,10} \end{pmatrix} = 0, \begin{pmatrix} D^{(2)}_{16,15} \\ D^{(2)}_{17,15} \\ \vdots \\ D^{(2)}_{20,15} \end{pmatrix} = 0, \tag{12}$$

$$\begin{pmatrix} D^{(3)}_{16,3} \\ D^{(3)}_{17,3} \\ D^{(3)}_{18,3} \end{pmatrix} = 0, \begin{pmatrix} D^{(3)}_{16,6} \\ D^{(3)}_{17,6} \\ D^{(3)}_{18,6} \end{pmatrix} = 0, \begin{pmatrix} D^{(3)}_{16,9} \\ D^{(3)}_{17,9} \\ D^{(3)}_{18,9} \end{pmatrix} = 0, \begin{pmatrix} D^{(3)}_{16,12} \\ D^{(3)}_{17,12} \\ D^{(3)}_{18,12} \end{pmatrix} = 0, \begin{pmatrix} D^{(3)}_{16,15} \\ D^{(3)}_{17,15} \\ D^{(3)}_{18,15} \end{pmatrix} = 0, \tag{13}$$

$$D^{(1)}_{12,1} = D^{(2)}_{10,1}, D^{(1)}_{12,2} = D^{(2)}_{10,2}, D^{(1)}_{12,3} = D^{(2)}_{10,3}, D^{(1)}_{12,4} = D^{(2)}_{10,4}, D^{(1)}_{12,5} = D^{(2)}_{10,5},$$

$$D^{(1)}_{12,8} = D^{(2)}_{10,6}, D^{(1)}_{12,9} = D^{(2)}_{10,7}, D^{(1)}_{12,10} = D^{(2)}_{10,8}, D^{(1)}_{12,11} = D^{(2)}_{10,9}, D^{(1)}_{12,12} = D^{(2)}_{10,10}, \tag{14}$$

$$D^{(2)}_{18,1} = D^{(3)}_{12,1}, D^{(2)}_{18,2} = D^{(3)}_{12,2}, D^{(2)}_{18,3} = D^{(3)}_{12,3}, D^{(2)}_{18,6} = D^{(3)}_{12,4}, D^{(2)}_{18,7} = D^{(3)}_{12,5}, D^{(2)}_{18,8} = D^{(3)}_{12,6},$$

$$D^{(2)}_{18,11} = D^{(3)}_{12,7}, D^{(2)}_{18,12} = D^{(3)}_{12,8}, D^{(2)}_{18,13} = D^{(3)}_{12,9},$$

$$D^{(2)}_{18,16} = D^{(3)}_{12,10}, D^{(2)}_{18,17} = D^{(3)}_{12,11}, D^{(2)}_{18,18} = D^{(3)}_{12,12}, \tag{15}$$

$$D^{(1)}_{85} = D^{(2)}_{65}, D^{(1)}_{95} = D^{(2)}_{75}, D^{(1)}_{10,5} = D^{(2)}_{85}, D^{(1)}_{11,5} = D^{(2)}_{95}, D^{(1)}_{12,5} = D^{(2)}_{10,5},$$

$$D^{(1)}_{8,12} = D^{(2)}_{6,10}, D^{(1)}_{9,12} = D^{(2)}_{7,10}, D^{(1)}_{10,12} = D^{(2)}_{8,10}, D^{(1)}_{11,12} = D^{(2)}_{9,10}, D^{(1)}_{12,12} = D^{(2)}_{10,10}, \tag{16}$$

$$D^{(2)}_{16,3} = D^{(3)}_{10,3},\ D^{(2)}_{17,3} = D^{(3)}_{11,3},\ D^{(2)}_{18,3} = D^{(3)}_{12,3},\ D^{(2)}_{16,8} = D^{(3)}_{10,6},\ D^{(2)}_{17,8} = D^{(3)}_{11,6},\ D^{(2)}_{18,8} = D^{(3)}_{12,6},$$

$$D^{(2)}_{16,13} = D^{(3)}_{10,9},\ D^{(2)}_{17,13} = D^{(3)}_{11,9},\ D^{(2)}_{18,13} = D^{(3)}_{12,9},$$

$$D^{(2)}_{16,18} = D^{(3)}_{10,12},\ D^{(2)}_{17,18} = D^{(3)}_{11,12},\ D^{(2)}_{18,18} = D^{(3)}_{12,12}, \tag{17}$$

$$D^{(3)}_{16,1} = D^{(4)}_{61},\ D^{(3)}_{16,4} = D^{(4)}_{62},\ D^{(3)}_{16,7} = D^{(4)}_{63},\ D^{(3)}_{16,10} = D^{(4)}_{64},\ D^{(3)}_{16,13} = D^{(4)}_{65},\ D^{(3)}_{16,16} = D^{(4)}_{66}, \tag{18}$$

$$D^{(1)}_{10,1} + D^{(3)}_{61} = D^{(2)}_{81},\ D^{(1)}_{10,2} + D^{(3)}_{62} = D^{(2)}_{82},\ D^{(1)}_{10,3} + D^{(3)}_{63} = D^{(2)}_{83},$$

$$D^{(1)}_{10,8} + D^{(3)}_{64} = D^{(2)}_{86},\ D^{(1)}_{10,9} + D^{(3)}_{65} = D^{(2)}_{87},\ D^{(1)}_{10,10} + D^{(3)}_{66} = D^{(2)}_{88}, \tag{19}$$

$$D^{(1)}_{83} + D^{(3)}_{43} = D^{(2)}_{63},\ D^{(1)}_{93} + D^{(3)}_{53} = D^{(2)}_{73},\ D^{(1)}_{8,10} + D^{(3)}_{46} = D^{(2)}_{68},\ D^{(1)}_{9,10} + D^{(3)}_{56} = D^{(2)}_{78}, \tag{20}$$

$$D^{(2)}_{16,1} + D^{(4)}_{41} = D^{(3)}_{10,1},\ D^{(2)}_{16,6} + D^{(4)}_{42} = D^{(3)}_{10,4},$$

$$D^{(2)}_{16,11} + D^{(4)}_{43} = D^{(3)}_{10,7},\ D^{(2)}_{16,16} + D^{(4)}_{44} = D^{(3)}_{10,10}, \tag{21}$$

$$D^{(1)}_{81} + D^{(3)}_{41} = D^{(2)}_{61} + D^{(4)}_{21},\ D^{(1)}_{88} + D^{(3)}_{44} = D^{(2)}_{66} + D^{(4)}_{22}. \tag{22}$$

Based on the above analysis, we have the following conclusions:

Theorem 1. *The system (1) has a ϕ-skew-Hermitian solution (X_1, X_2, X_3, X_4) if and only if the Equations (9)–(22) hold.*

The following theorem presents the solvability conditions to the system (1) in terms of rank.

Theorem 2. *The system (1) has a ϕ-skew-Hermitian solution (X_1, X_2, X_3, X_4) if and only if the ranks satisfy:*

$$r(A_i,\ C_i,\ B_i) = r(A_i,\ B_i),\ (i = 1, 2, 3). \tag{23}$$

$$r(A_4,\ C_4) = r(A_4). \tag{24}$$

$$r\begin{pmatrix} A_i & C_i \\ 0 & (B_i)_\phi \end{pmatrix} = r(A_i) + r(B_i),\ (i = 1, 2, 3). \tag{25}$$

$$r\begin{pmatrix} A_j & C_j & B_j & 0 & 0 \\ 0 & (B_j)_\phi & 0 & (A_{j+1})_\phi & 0 \\ 0 & 0 & A_{j+1} & -C_{j+1} & B_{j+1} \end{pmatrix} = r\begin{pmatrix} A_j & B_j & 0 \\ 0 & A_{j+1} & B_{j+1} \end{pmatrix} + r\begin{pmatrix} B_j \\ A_{j+1} \end{pmatrix}\ (j = 1, 2). \tag{26}$$

$$r\begin{pmatrix} A_j & C_j & B_j & 0 \\ 0 & (B_j)_\phi & 0 & (A_{j+1})_\phi \\ 0 & 0 & A_{j+1} & -C_{j+1} \\ 0 & 0 & 0 & (B_{j+1})_\phi \end{pmatrix} = r\begin{pmatrix} A_j & B_j \\ 0 & A_{j+1} \end{pmatrix} + r\begin{pmatrix} B_j & 0 \\ A_{j+1} & B_{j+1} \end{pmatrix} \; (j = 1, 2). \quad (27)$$

$$r\begin{pmatrix} A_3 & C_3 & B_3 & 0 \\ 0 & (B_3)_\phi & 0 & (A_4)_\phi \\ 0 & 0 & A_4 & -C_4 \end{pmatrix} = r\begin{pmatrix} A_3 & B_3 \\ 0 & A_4 \end{pmatrix} + r\begin{pmatrix} B_3 \\ A_4 \end{pmatrix}. \quad (28)$$

$$r\begin{pmatrix} A_1 & C_1 & B_1 & 0 & 0 & 0 & 0 \\ 0 & (B_1)_\phi & 0 & (A_2)_\phi & 0 & 0 & 0 \\ 0 & 0 & A_2 & -C_2 & B_2 & 0 & 0 \\ 0 & 0 & 0 & (B_2)_\phi & 0 & (A_3)_\phi & 0 \\ 0 & 0 & 0 & 0 & A_3 & C_3 & B_3 \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix} + r\begin{pmatrix} B_1 & 0 \\ A_2 & B_2 \\ 0 & A_3 \end{pmatrix}. \quad (29)$$

$$r\begin{pmatrix} A_1 & C_1 & B_1 & 0 & 0 & 0 \\ 0 & (B_1)_\phi & 0 & (A_2)_\phi & 0 & 0 \\ 0 & 0 & A_2 & -C_2 & B_2 & 0 \\ 0 & 0 & 0 & (B_2)_\phi & 0 & (A_3)_\phi \\ 0 & 0 & 0 & 0 & A_3 & C_3 \\ 0 & 0 & 0 & 0 & 0 & (B_3)_\phi \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 & 0 \\ 0 & A_2 & B_2 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix}. \quad (30)$$

$$r\begin{pmatrix} A_2 & C_2 & B_2 & 0 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & (A_3)_\phi & 0 & 0 \\ 0 & 0 & A_3 & -C_3 & B_3 & 0 \\ 0 & 0 & 0 & (B_3)_\phi & 0 & (A_4)_\phi \\ 0 & 0 & 0 & 0 & A_4 & C_4 \end{pmatrix} = r\begin{pmatrix} A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \\ 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ A_3 & B_3 \\ 0 & A_4 \end{pmatrix}. \quad (31)$$

$$r\begin{pmatrix} A_1 & C_1 & B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (B_1)_\phi & 0 & (A_2)_\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & -C_2 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & (B_2)_\phi & 0 & (A_3)_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & A_3 & C_3 & B_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & (B_3)_\phi & 0 & (A_4)_\phi \\ 0 & 0 & 0 & 0 & 0 & 0 & A_4 & -C_4 \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \\ 0 & 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \\ 0 & 0 & A_4 \end{pmatrix}. \quad (32)$$

Proof. According to the structure of the matrix, Lemma 1, Theorem 1, and the elementary transformation of the row and column of the matrix, we have

$$\begin{aligned} & r(A_i,\ C_i,\ B_i) = r(A_i,\ B_i) \\ \Leftrightarrow\ & r(\widehat{P}_i A_i \widehat{T}_i,\ \widehat{P}_i C_i (\widehat{P}_i)_\phi,\ \widehat{P}_i B_i \widehat{T_{i+1}}) = r(\widehat{P}_i A_i \widehat{T}_i,\ \widehat{P}_i B_i \widehat{T_{i+1}}) \\ \Leftrightarrow\ & r(S_{a_i},\ D^{(i)},\ S_{b_i}) = r(S_{a_i},\ S_{b_i}) \\ \Leftrightarrow\ & D^{(1)}_{14,j} = 0,\ (j = 1, \cdots, 14),\ D^{(2)}_{20,j} = 0,\ (j = 1, \cdots, 20),\ D^{(3)}_{18,j} = 0,\ (j = 1, \cdots, 18). \end{aligned}$$

$$\begin{aligned} & r(A_4,\ C_4) = r(A_4) \\ \Leftrightarrow\ & r(\widehat{P}_4 A_4 \widehat{T}_4,\ \widehat{P}_4 C_4 (\widehat{P}_4)_\phi) = r(\widehat{P}_4 A_4 \widehat{T}_4) \\ \Leftrightarrow\ & r(S_{a_4},\ D^{(4)}) = r(S_{a_4}) \\ \Leftrightarrow\ & D^{(4)}_{8j} = 0,\ (j = 1, \cdots, 8). \end{aligned}$$

$$r\begin{pmatrix} A_1 & C_1 \\ 0 & (B_1)_\phi \end{pmatrix} = r(A_1) + r(B_1)$$

$$\Leftrightarrow r\begin{pmatrix} \widehat{P}_1 A_1 \widehat{T}_1 & \widehat{P}_1 C_1 (\widehat{P}_1)_\phi \\ 0 & (\widehat{T}_2)_\phi (B_1)_\phi (\widehat{P}_1)_\phi \end{pmatrix} = r(\widehat{P}_1 A_1 \widehat{T}_1) + r(\widehat{P}_1 B_1 \widehat{T}_2)$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_1} & D^{(1)} \\ 0 & (S_{b_1})_\phi \end{pmatrix} = r(S_{a_1}) + r(S_{b_1})$$

$$\Leftrightarrow D^{(1)}_{i,7} = 0,\ (i = 8, \cdots, 14),\ D^{(1)}_{i,14} = 0,\ (i = 8, \cdots, 14).$$

$$r\begin{pmatrix} A_2 & C_2 \\ 0 & (B_2)_\phi \end{pmatrix} = r(A_2) + r(B_2)$$

$$\Leftrightarrow r\begin{pmatrix} \widehat{P}_2 A_2 \widehat{T}_2 & \widehat{P}_2 C_2 (\widehat{P}_2)_\phi \\ 0 & (\widehat{T}_3)_\phi (B_2)_\phi (\widehat{P}_2)_\phi \end{pmatrix} = r(\widehat{P}_2 A_2 \widehat{T}_2) + r(\widehat{P}_2 B_2 \widehat{T}_3)$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_2} & D^{(2)} \\ 0 & (S_{b_2})_\phi \end{pmatrix} = r(S_{a_2}) + r(S_{b_2})$$

$$\Leftrightarrow D^{(2)}_{i,5} = 0,\ D^{(2)}_{i,10} = 0,\ D^{(2)}_{i,15} = 0,\ D^{(2)}_{i,20} = 0,\ (i = 16, \cdots, 20).$$

$$r\begin{pmatrix} A_3 & C_3 \\ 0 & (B_3)_\phi \end{pmatrix} = r(A_3) + r(B_3)$$

$$\Leftrightarrow r\begin{pmatrix} \widehat{P}_3 A_3 \widehat{T}_3 & \widehat{P}_3 C_3 (\widehat{P}_3)_\phi \\ 0 & (\widehat{T}_4)_\phi (B_3)_\phi (\widehat{P}_3)_\phi \end{pmatrix} = r(\widehat{P}_3 A_3 \widehat{T}_3) + r(\widehat{P}_3 B_3 \widehat{T}_4)$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_3} & D^{(3)} \\ 0 & (S_{b_3})_\phi \end{pmatrix} = r(S_{a_3}) + r(S_{b_3})$$

$$\Leftrightarrow D^{(3)}_{i,3} = 0,\ D^{(3)}_{i,6} = 0,\ D^{(3)}_{i,9} = 0,$$

$$D^{(3)}_{i,12} = 0,\ D^{(3)}_{i,15} = 0,\ D^{(3)}_{i,18} = 0,\ (i = 16,\ 17,\ 18).$$

$$r\begin{pmatrix} A_1 & C_1 & B_1 & 0 & 0 \\ 0 & (B_1)_\phi & 0 & (A_2)_\phi & 0 \\ 0 & 0 & A_2 & -C_2 & B_2 \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 & 0 \\ 0 & A_2 & B_2 \end{pmatrix} + r\begin{pmatrix} B_1 \\ A_2 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_1} & D^{(1)} & S_{b_1} & 0 & 0 \\ 0 & (S_{b_1})_\phi & 0 & (S_{a_2})_\phi & 0 \\ 0 & 0 & S_{a_2} & -D^{(2)} & S_{b_2} \end{pmatrix} = r\begin{pmatrix} S_{a_1} & S_{b_1} & 0 \\ 0 & S_{a_2} & S_{b_2} \end{pmatrix} + r\begin{pmatrix} S_{b_1} \\ S_{a_2} \end{pmatrix}$$

$$\Leftrightarrow D^{(1)}_{14,j} = 0,\ (j = 1, \ldots, 14),\ D^{(1)}_{12,7} = 0,\ D^{(1)}_{12,14} = 0,$$

$$D^{(2)}_{20,j} = 0,\ (j = 1, \ldots, 10, 16, \ldots, 20),\ D^{(2)}_{10,j} = 0,\ (j = 16, \ldots, 20),$$

$$D^{(1)}_{12,1} = D^{(2)}_{10,1},\ D^{(1)}_{12,2} = D^{(2)}_{10,2},\ D^{(1)}_{12,3} = D^{(2)}_{10,3},\ D^{(1)}_{12,4} = D^{(2)}_{10,4},\ D^{(1)}_{12,5} = D^{(2)}_{10,5},$$

$$D^{(1)}_{12,8} = D^{(2)}_{10,6},\ D^{(1)}_{12,9} = D^{(2)}_{10,7},\ D^{(1)}_{12,10} = D^{(2)}_{10,8},\ D^{(1)}_{12,11} = D^{(2)}_{10,9},\ D^{(1)}_{12,12} = D^{(2)}_{10,10}.$$

$$r\begin{pmatrix} A_2 & C_2 & B_2 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & (A_3)_\phi & 0 \\ 0 & 0 & A_3 & -C_3 & B_3 \end{pmatrix} = r\begin{pmatrix} A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix} + r\begin{pmatrix} B_2 \\ A_3 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_2} & D^{(2)} & S_{b_2} & 0 & 0 \\ 0 & (S_{b_2})_\phi & 0 & (S_{a_3})_\phi & 0 \\ 0 & 0 & S_{a_3} & -D^{(3)} & S_{b_3} \end{pmatrix} = r\begin{pmatrix} S_{a_2} & S_{b_2} & 0 \\ 0 & S_{a_3} & S_{b_3} \end{pmatrix} + r\begin{pmatrix} S_{b_2} \\ S_{a_3} \end{pmatrix}$$

$$\Leftrightarrow D^{(2)}_{20,j} = 0,\ (j = 1,2,3,5,6,7,8,10,11,12,13,15,16,17,18,20),$$

$$D^{(3)}_{18,j} = 0,\ (j = 1,\ldots,12,16,17,18),$$

$$D^{(2)}_{18,5} = 0,\ D^{(2)}_{18,10} = 0,\ D^{(2)}_{18,15} = 0,\ D^{(2)}_{18,18} = 0,$$

$$D^{(3)}_{12,16} = 0,\ D^{(3)}_{12,17} = 0,\ D^{(3)}_{12,18} = 0,$$

$$D^{(2)}_{18,1} = D^{(3)}_{12,1},\ D^{(2)}_{18,2} = D^{(3)}_{12,2},\ D^{(2)}_{18,3} = D^{(3)}_{12,3},$$

$$D^{(2)}_{18,6} = D^{(3)}_{12,4},\ D^{(2)}_{18,7} = D^{(3)}_{12,5},\ D^{(2)}_{18,8} = D^{(3)}_{12,6},$$

$$D^{(2)}_{18,11} = D^{(3)}_{12,7},\ D^{(2)}_{18,12} = D^{(3)}_{12,8},\ D^{(2)}_{18,13} = D^{(3)}_{12,9},$$

$$D^{(2)}_{18,16} = D^{(3)}_{12,10},\ D^{(2)}_{18,17} = D^{(3)}_{12,11},\ D^{(2)}_{18,18} = D^{(3)}_{12,12}.$$

$$r\begin{pmatrix} A_1 & C_1 & B_1 & 0 \\ 0 & (B_1)_\phi & 0 & (A_2)_\phi \\ 0 & 0 & A_2 & -C_2 \\ 0 & 0 & 0 & (B_2)_\phi \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 \\ 0 & A_2 \end{pmatrix} + r\begin{pmatrix} B_1 & 0 \\ A_2 & B_2 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_1} & D^{(1)} & S_{b_1} & 0 \\ 0 & (S_{b_1})_\phi & 0 & (S_{a_2})_\phi \\ 0 & 0 & S_{a_2} & -D^{(2)} \\ 0 & 0 & 0 & (S_{b_2})_\phi \end{pmatrix} = r\begin{pmatrix} S_{a_1} & S_{b_1} \\ 0 & S_{a_2} \end{pmatrix} + r\begin{pmatrix} S_{b_1} & 0 \\ S_{a_2} & S_{b_2} \end{pmatrix}$$

$$\Leftrightarrow D^{(1)}_{i,14} = 0,\ (i = 8,\ldots,14),\ D^{(2)}_{i,20} = 0,\ (i = 6,\ldots,10,16,\ldots,20),$$

$$D^{(1)}_{i,7} = 0,\ (i = 8,\ldots,14),\ D^{(1)}_{14,5} = 0,\ D^{(1)}_{14,12} = 0,$$

$$D^{(2)}_{i,5} = 0,\ D^{(2)}_{i,10} = 0,\ (i = 16,\ldots,20),$$

$$D^{(1)}_{85} = D^{(2)}_{65},\ D^{(1)}_{95} = D^{(2)}_{75},\ D^{(1)}_{10,5} = D^{(2)}_{85},\ D^{(1)}_{11,5} = D^{(2)}_{95},\ D^{(1)}_{12,5} = D^{(2)}_{10,5},$$

$$D^{(1)}_{8,12} = D^{(2)}_{6,10},\ D^{(1)}_{9,12} = D^{(2)}_{7,10},\ D^{(1)}_{10,12} = D^{(2)}_{8,10},\ D^{(1)}_{11,12} = D^{(2)}_{9,10},\ D^{(1)}_{12,12} = D^{(2)}_{10,10}.$$

$$r\begin{pmatrix} A_2 & C_2 & B_2 & 0 \\ 0 & (B_2)_\phi & 0 & (A_3)_\phi \\ 0 & 0 & A_3 & -C_3 \\ 0 & 0 & 0 & (B_3)_\phi \end{pmatrix} = r\begin{pmatrix} A_2 & B_2 \\ 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ A_3 & B_3 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_2} & D^{(2)} & S_{b_2} & 0 \\ 0 & (S_{b_2})_\phi & 0 & (S_{a_3})_\phi \\ 0 & 0 & S_{a_3} & -D^{(3)} \\ 0 & 0 & 0 & (S_{b_3})_\phi \end{pmatrix} = r\begin{pmatrix} S_{a_2} & S_{b_2} \\ 0 & S_{a_3} \end{pmatrix} + r\begin{pmatrix} S_{b_2} & 0 \\ S_{a_3} & S_{b_3} \end{pmatrix}$$

$$\Leftrightarrow D^{(2)}_{i,5} = 0,\ D^{(2)}_{i,10} = 0,\ D^{(2)}_{i,15} = 0,\ D^{(2)}_{i,20} = 0,\ (i = 16,17,18,20),$$

$$D^{(3)}_{i,3} = 0,\ D^{(3)}_{i,6} = 0,\ D^{(3)}_{i,9} = 0,$$

$$D^{(3)}_{i,12} = 0,\ (i = 16,17,18),\ D^{(3)}_{i,18} = 0,\ (i = 10,11,12,16,17,18),$$

$$D^{(2)}_{16,3} = D^{(3)}_{10,3},\ D^{(2)}_{17,3} = D^{(3)}_{11,3},\ D^{(2)}_{18,3} = D^{(3)}_{12,3},$$
$$D^{(2)}_{16,8} = D^{(3)}_{10,6},\ D^{(2)}_{17,8} = D^{(3)}_{11,6},\ D^{(2)}_{18,8} = D^{(3)}_{12,6},$$
$$D^{(2)}_{16,13} = D^{(3)}_{10,9},\ D^{(2)}_{17,13} = D^{(3)}_{11,9},\ D^{(2)}_{18,13} = D^{(3)}_{12,9},$$
$$D^{(2)}_{16,18} = D^{(3)}_{10,12},\ D^{(2)}_{17,18} = D^{(3)}_{11,12},\ D^{(2)}_{18,18} = D^{(3)}_{12,12}.$$

$$r\begin{pmatrix} A_3 & C_3 & B_3 & 0 \\ 0 & (B_3)_\phi & 0 & (A_4)_\phi \\ 0 & 0 & A_4 & -C_4 \end{pmatrix} = r\begin{pmatrix} A_3 & B_3 \\ 0 & A_4 \end{pmatrix} + r\begin{pmatrix} B_3 \\ A_4 \end{pmatrix}$$
$$\Leftrightarrow r\begin{pmatrix} S_{a_3} & D^{(3)} & S_{b_3} & 0 \\ 0 & (S_{b_3})_\phi & 0 & (S_{a_4})_\phi \\ 0 & 0 & S_{a_4} & -D^{(4)} \end{pmatrix} = r\begin{pmatrix} S_{a_3} & S_{b_3} \\ 0 & S_{a_4} \end{pmatrix} + r\begin{pmatrix} S_{b_3} \\ S_{a_4} \end{pmatrix}$$
$$\Leftrightarrow D^{(3)}_{16,j} = 0,\ (j = 3, 6, 9, 12, 15, 18),$$
$$D^{(3)}_{18,j} = 0,\ (j = 1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18),$$
$$D^{(4)}_{8j} = 0,\ (j = 1, \ldots, 6, 8),\ D^{(4)}_{68} = 0,$$
$$D^{(3)}_{16,1} = D^{(4)}_{61},\ D^{(3)}_{16,4} = D^{(4)}_{62},\ D^{(3)}_{16,7} = D^{(4)}_{63},$$
$$D^{(3)}_{16,10} = D^{(4)}_{64},\ D^{(3)}_{16,13} = D^{(4)}_{65},\ D^{(3)}_{16,16} = D^{(4)}_{66}.$$

$$r\begin{pmatrix} A_1 & C_1 & B_1 & 0 & 0 & 0 & 0 \\ 0 & (B_1)_\phi & 0 & (A_2)_\phi & 0 & 0 & 0 \\ 0 & 0 & A_2 & -C_2 & B_2 & 0 & 0 \\ 0 & 0 & 0 & (B_2)_\phi & 0 & (A_3)_\phi & 0 \\ 0 & 0 & 0 & 0 & A_3 & C_3 & B_3 \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \end{pmatrix} + r\begin{pmatrix} B_1 & 0 \\ A_2 & B_2 \\ 0 & A_3 \end{pmatrix}$$
$$\Leftrightarrow r\begin{pmatrix} S_{a_1} & D^{(1)} & S_{b_1} & 0 & 0 & 0 & 0 \\ 0 & (S_{b_1})_\phi & 0 & (S_{a_2})_\phi & 0 & 0 & 0 \\ 0 & 0 & S_{a_2} & -D^{(2)} & S_{a_2} & 0 & 0 \\ 0 & 0 & 0 & (S_{b_2})_\phi & 0 & (S_{a_3})_\phi & 0 \\ 0 & 0 & 0 & 0 & S_{a_3} & D^{(3)} & S_{b_3} \end{pmatrix}$$
$$= r\begin{pmatrix} S_{a_1} & S_{b_1} & 0 & 0 \\ 0 & S_{a_2} & S_{b_2} & 0 \\ 0 & 0 & S_{a_3} & S_{b_3} \end{pmatrix} + r\begin{pmatrix} S_{b_1} & 0 \\ S_{a_2} & S_{b_2} \\ 0 & S_{a_3} \end{pmatrix}$$
$$\Leftrightarrow D^{(1)}_{14,j} = 0,\ (j = 1, 2, 3, 5, 7, 8, 9, 10, 12, 14),\ D^{(1)}_{10,7} = 0,$$
$$D^{(1)}_{12,7} = 0,\ D^{(1)}_{10,14} = 0,\ D^{(1)}_{12,14} = 0,$$
$$D^{(2)}_{20,j} = 0,\ (j = 1, 2, 3, 5, 6, 7, 8, 10, 16, 17, 18, 20),\ D^{(2)}_{8,20} = 0,\ D^{(2)}_{10,20} = 0,\ D^{(2)}_{18,20} = 0,$$
$$D^{(2)}_{10,j} = 0,\ (j = 16, 17, 18),\ D^{(2)}_{18,5} = 0,\ D^{(2)}_{18,10} = 0,$$
$$D^{(3)}_{18,j} = 0,\ (j = 1, \ldots, 6, 10, 11, 12, 16, 17, 18),$$
$$D^{(3)}_{6,j} = 0,\ (j = 16, 17, 18),\ D^{(3)}_{12,j} = 0,\ (j = 16, 17, 18),$$
$$D^{(1)}_{12,1} = D^{(2)}_{10,1},\ D^{(1)}_{12,2} = D^{(2)}_{10,2},\ D^{(1)}_{12,3} = D^{(2)}_{10,3},\ D^{(1)}_{12,5} = D^{(2)}_{10,5},\ D^{(1)}_{12,8} = D^{(2)}_{10,6},$$

$$D^{(1)}_{12,9} = D^{(2)}_{10,7},\ D^{(1)}_{12,10} = D^{(2)}_{10,8},\ D^{(1)}_{12,12} = D^{(2)}_{10,10},\ D^{(1)}_{10,5} = D^{(2)}_{85},\ D^{(1)}_{10,12} = D^{(2)}_{8,10},$$
$$D^{(2)}_{18,1} = D^{(3)}_{12,1},\ D^{(2)}_{18,2} = D^{(3)}_{12,2},\ D^{(2)}_{18,3} = D^{(3)}_{12,3},$$
$$D^{(2)}_{18,6} = D^{(3)}_{12,4},\ D^{(2)}_{18,7} = D^{(3)}_{12,5},\ D^{(2)}_{18,8} = D^{(3)}_{12,6},$$
$$D^{(2)}_{18,16} = D^{(3)}_{12,10},\ D^{(2)}_{18,17} = D^{(3)}_{12,11},\ D^{(2)}_{18,18} = D^{(3)}_{12,12},$$
$$D^{(2)}_{8,16} = D^{(3)}_{6,10},\ D^{(2)}_{8,17} = D^{(3)}_{6,11},\ D^{(2)}_{8,18} = D^{(3)}_{6,12},$$
$$D^{(1)}_{10,1} + D^{(3)}_{61} = D^{(2)}_{81},\ D^{(1)}_{10,2} + D^{(3)}_{62} = D^{(2)}_{82},\ D^{(1)}_{10,3} + D^{(3)}_{63} = D^{(2)}_{83},$$
$$D^{(1)}_{10,8} + D^{(3)}_{64} = D^{(2)}_{86},\ D^{(1)}_{10,9} + D^{(3)}_{65} = D^{(2)}_{87},\ D^{(1)}_{10,10} + D^{(3)}_{66} = D^{(2)}_{88}.$$

$$r\begin{pmatrix} A_1 & C_1 & B_1 & 0 & 0 & 0 \\ 0 & (B_1)_\phi & 0 & (A_2)_\phi & 0 & 0 \\ 0 & 0 & A_2 & -C_2 & B_2 & 0 \\ 0 & 0 & 0 & (B_2)_\phi & 0 & (A_3)_\phi \\ 0 & 0 & 0 & 0 & A_3 & C_3 \\ 0 & 0 & 0 & 0 & 0 & (B_3)_\phi \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 & 0 \\ 0 & A_2 & B_2 \\ 0 & 0 & A_3 \end{pmatrix} + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_1} & D^{(1)} & S_{b_1} & 0 & 0 & 0 \\ 0 & (S_{b_1})_\phi & 0 & (S_{a_2})_\phi & 0 & 0 \\ 0 & 0 & S_{a_2} & -D^{(2)} & S_{b_2} & 0 \\ 0 & 0 & 0 & (S_{b_2})_\phi & 0 & (S_{a_3})_\phi \\ 0 & 0 & 0 & 0 & S_{a_3} & D^{(3)} \\ 0 & 0 & 0 & 0 & 0 & (S_{b_3})_\phi \end{pmatrix}$$
$$= r\begin{pmatrix} S_{a_1} & S_{b_1} & 0 \\ 0 & S_{a_2} & S_{b_2} \\ 0 & 0 & S_{a_3} \end{pmatrix} + r\begin{pmatrix} S_{b_1} & 0 & 0 \\ S_{a_2} & S_{b_2} & 0 \\ 0 & S_{a_3} & S_{b_3} \end{pmatrix}$$

$$\Leftrightarrow D^{(1)}_{i,14} = 0,\ (i = 8,\ldots,14),\ D^{(1)}_{14,j} = 0,\ (j = 3,5,7,10,12),\ D^{(1)}_{i,7} = 0,\ (i = 8,\ldots,10,12),$$
$$D^{(2)}_{i,20} = 0,\ (i = 1,2,3,5,6,7,8,10,16,17,18,20),\ D^{(2)}_{20,j} = 0,\ (j = 3,5,8,10,18,20),$$
$$D^{(2)}_{i,5} = 0,\ (i = 16,17,18),\ D^{(2)}_{i,10} = 0,\ (i = 16,17,18),\ D^{(2)}_{10,18} = 0,$$
$$D^{(3)}_{i,18} = 0,\ (i = 4,5,6,10,11,12,16,17,18),$$
$$D^{(3)}_{i,3} = 0,\ D^{(3)}_{i,6} = 0,\ D^{(3)}_{i,12} = 0,\ (i = 16,17,18),$$
$$D^{(1)}_{12,3} = D^{(2)}_{10,3},\ D^{(1)}_{85} = D^{(2)}_{65},\ D^{(1)}_{95} = D^{(2)}_{75},\ D^{(1)}_{10,5} = D^{(2)}_{85},\ D^{(1)}_{12,5} = D^{(2)}_{10,5},$$
$$D^{(1)}_{12,10} = D^{(2)}_{10,8},\ D^{(1)}_{8,12} = D^{(2)}_{6,10},\ D^{(1)}_{9,12} = D^{(2)}_{7,10},\ D^{(1)}_{10,12} = D^{(2)}_{8,10},\ D^{(1)}_{12,12} = D^{(2)}_{10,10},$$
$$D^{(2)}_{6,18} = D^{(3)}_{4,12},\ D^{(2)}_{7,18} = D^{(3)}_{5,12},\ D^{(2)}_{8,18} = D^{(3)}_{6,12},$$
$$D^{(2)}_{16,3} = D^{(3)}_{10,3},\ D^{(2)}_{17,3} = D^{(3)}_{11,3},\ D^{(2)}_{18,3} = D^{(3)}_{12,3},$$
$$D^{(2)}_{16,8} = D^{(3)}_{10,6},\ D^{(2)}_{17,8} = D^{(3)}_{11,6},\ D^{(2)}_{18,8} = D^{(3)}_{12,6},$$
$$D^{(2)}_{16,18} = D^{(3)}_{10,12},\ D^{(2)}_{17,18} = D^{(3)}_{11,12},\ D^{(2)}_{18,18} = D^{(3)}_{12,12},$$
$$D^{(1)}_{83} + D^{(3)}_{43} = D^{(2)}_{63},\ D^{(1)}_{93} + D^{(3)}_{53} = D^{(2)}_{73},\ D^{(1)}_{10,3} + D^{(3)}_{63} = D^{(2)}_{83},$$

$$D_{8,10}^{(1)} + D_{46}^{(3)} = D_{68}^{(2)},\ D_{9,10}^{(1)} + D_{56}^{(3)} = D_{78}^{(2)},\ D_{10,10}^{(1)} + D_{66}^{(3)} = D_{88}^{(2)}.$$

$$r\begin{pmatrix} A_2 & C_2 & B_2 & 0 & 0 & 0 \\ 0 & (B_2)_\phi & 0 & (A_3)_\phi & 0 & 0 \\ 0 & 0 & A_3 & -C_3 & B_3 & 0 \\ 0 & 0 & 0 & (B_3)_\phi & 0 & (A_4)_\phi \\ 0 & 0 & 0 & 0 & A_4 & C_4 \end{pmatrix} = r\begin{pmatrix} A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \\ 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} B_2 & 0 \\ A_3 & B_3 \\ 0 & A_4 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_2} & D^{(2)} & S_{b_2} & 0 & 0 & 0 \\ 0 & (S_{b_2})_\phi & 0 & (S_{a_3})_\phi & 0 & 0 \\ 0 & 0 & S_{a_3} & -D^{(3)} & S_{b_3} & 0 \\ 0 & 0 & 0 & (S_{b_3})_\phi & 0 & (S_{a_4})_\phi \\ 0 & 0 & 0 & 0 & S_{a_4} & D^{(4)} \end{pmatrix} = r\begin{pmatrix} S_{a_2} & S_{b_2} & 0 \\ 0 & S_{a_3} & S_{b_3} \\ 0 & 0 & S_{a_4} \end{pmatrix} + r\begin{pmatrix} S_{b_2} & 0 \\ S_{a_3} & S_{b_3} \\ 0 & S_{a_4} \end{pmatrix}$$

$$\Leftrightarrow D_{i,20}^{(2)} = 0,\ (i = 16, 18, 20),\ D_{20,j}^{(2)} = 0,\ (j = 1, 3, 5, 6, 8, 10, 11, 13, 15, 16, 18),$$

$$D_{16,5}^{(2)} = 0,\ D_{18,5}^{(2)} = 0,\ D_{16,10}^{(2)} = 0,\ D_{18,10}^{(2)} = 0,\ D_{16,15}^{(2)} = 0,\ D_{18,15}^{(2)} = 0,$$

$$D_{i,18}^{(3)} = 0,\ (i = 10, 12, 16, 18),\ D_{18,j}^{(3)} = 0,\ (i = 1, 3, 4, 6, 7, 9, 10, 12, 16),$$

$$D_{12,16}^{(3)} = 0,\ D_{16,3}^{(3)} = 0,\ D_{16,6}^{(3)} = 0,\ D_{16,9}^{(3)} = 0,\ D_{16,12}^{(3)} = 0,$$

$$D_{i8}^{(4)} = 0,\ (i = 4, 6, 8),\ D_{8j}^{(4)} = 0,\ (j = 1, 2, 3, 4, 6),$$

$$D_{16,3}^{(2)} = D_{10,3}^{(3)},\ D_{16,8}^{(2)} = D_{10,6}^{(3)},\ D_{16,13}^{(2)} = D_{10,9}^{(3)},$$

$$D_{16,18}^{(2)} = D_{10,12}^{(3)},\ D_{18,1}^{(2)} = D_{12,1}^{(3)},\ D_{18,3}^{(2)} = D_{12,3}^{(3)},$$

$$D_{18,6}^{(2)} = D_{12,4}^{(3)},\ D_{18,8}^{(2)} = D_{12,6}^{(3)},\ D_{18,11}^{(2)} = D_{12,7}^{(3)},$$

$$D_{18,13}^{(2)} = D_{12,9}^{(3)},\ D_{18,16}^{(2)} = D_{12,10}^{(3)},\ D_{18,18}^{(2)} = D_{12,12}^{(3)},$$

$$D_{10,16}^{(3)} = D_{46}^{(4)},\ D_{16,1}^{(3)} = D_{61}^{(4)},\ D_{16,4}^{(3)} = D_{62}^{(4)},$$

$$D_{16,7}^{(3)} = D_{63}^{(4)},\ D_{16,10}^{(3)} = D_{64}^{(4)},\ D_{16,16}^{(3)} = D_{66}^{(4)},$$

$$D_{16,1}^{(2)} + D_{41}^{(4)} = D_{10,1}^{(3)},\ D_{16,6}^{(2)} + D_{42}^{(4)} = D_{10,4}^{(3)},$$

$$D_{16,11}^{(2)} + D_{43}^{(4)} = D_{10,7}^{(3)},\ D_{16,16}^{(2)} + D_{44}^{(4)} = D_{10,10}^{(3)}.$$

$$r\begin{pmatrix} A_1 & C_1 & B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (B_1)_\phi & 0 & (A_2)_\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & -C_2 & B_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & (B_2)_\phi & 0 & (A_3)_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & A_3 & C_3 & B_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & (B_3)_\phi & 0 & (A_4)_\phi \\ 0 & 0 & 0 & 0 & 0 & 0 & A_4 & -C_4 \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 & 0 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_3 & B_3 \\ 0 & 0 & 0 & A_4 \end{pmatrix} + r\begin{pmatrix} B_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ 0 & A_3 & B_3 \\ 0 & 0 & A_4 \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} S_{a_1} & D^{(1)} & S_{b_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (S_{b_1})_\phi & 0 & (S_{a_2})_\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & S_{a_2} & -D^{(2)} & S_{b_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & (S_{b_2})_\phi & 0 & (S_{a_3})_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{a_3} & D^{(3)} & S_{b_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & (S_{b_3})_\phi & 0 & (S_{a_4})_\phi \\ 0 & 0 & 0 & 0 & 0 & 0 & S_{a_4} & -D^{(4)} \end{pmatrix}$$

$$= r\begin{pmatrix} S_{a_1} & S_{b_1} & 0 & 0 \\ 0 & S_{a_2} & S_{b_2} & 0 \\ 0 & 0 & S_{a_3} & S_{b_3} \\ 0 & 0 & 0 & S_{a_4} \end{pmatrix} + r\begin{pmatrix} S_{b_1} & 0 & 0 \\ S_{a_2} & S_{b_2} & 0 \\ 0 & S_{a_3} & S_{b_3} \\ 0 & 0 & S_{a_4} \end{pmatrix}$$

$$\Leftrightarrow D^{(1)}_{14,j} = 0,\ (j = 1,3,5,7,8,10,12,14),\ D^{(1)}_{87} = 0,\ D^{(1)}_{8,14} = 0,\ D^{(1)}_{10,7} = 0,$$

$$D^{(1)}_{10,14} = 0,\ D^{(1)}_{12,7} = 0,\ D^{(1)}_{12,14} = 0,$$

$$D^{(2)}_{20,j} = 0,\ (j = 1,3,5,6,8,10,16,18,20),\ D^{(2)}_{10,16} = 0,\ D^{(2)}_{10,18} = 0,\ D^{(2)}_{10,20} = 0,$$

$$D^{(2)}_{6,20} = 0,\ D^{(2)}_{8,20} = 0,\ D^{(2)}_{16,5} = 0,$$

$$D^{(2)}_{16,10} = 0,\ D^{(2)}_{16,20} = 0,\ D^{(2)}_{18,5} = 0,\ D^{(2)}_{18,10} = 0,\ D^{(2)}_{18,20} = 0,$$

$$D^{(3)}_{i,18} = 0,\ (i = 4,6,10,12,16,18),\ D^{(3)}_{18,j} = 0,\ (j = 1,3,4,6,10,12,16,18),$$

$$D^{(3)}_{16,3} = 0,\ D^{(3)}_{16,6} = 0,\ D^{(3)}_{16,12} = 0,\ D^{(3)}_{6,16} = 0,\ D^{(3)}_{12,16} = 0,$$

$$D^{(4)}_{i8} = 0,\ (i = 2,4,6,8),\ D^{(4)}_{8j} = 0,\ (j = 1,2,4,6),$$

$$D^{(1)}_{85} = D^{(2)}_{65},\ D^{(1)}_{8,12} = D^{(2)}_{6,10},\ D^{(1)}_{10,5} = D^{(2)}_{85},\ D^{(1)}_{10,12} = D^{(2)}_{8,10},$$

$$D^{(1)}_{12,1} = D^{(2)}_{10,1},\ D^{(1)}_{12,3} = D^{(2)}_{10,3},\ D^{(1)}_{12,5} = D^{(2)}_{10,5},$$

$$D^{(1)}_{12,8} = D^{(2)}_{10,6},\ D^{(1)}_{12,10} = D^{(2)}_{10,8},\ D^{(1)}_{12,12} = D^{(2)}_{10,10},$$

$$D^{(2)}_{6,18} = D^{(3)}_{4,12},\ D^{(2)}_{8,16} = D^{(3)}_{6,10},\ D^{(2)}_{8,18} = D^{(3)}_{6,12},$$

$$D^{(2)}_{16,3} = D^{(3)}_{10,3},\ D^{(2)}_{16,8} = D^{(3)}_{10,6},\ D^{(2)}_{16,18} = D^{(3)}_{10,12},$$

$$D^{(2)}_{18,1} = D^{(3)}_{12,1},\ D^{(2)}_{18,3} = D^{(3)}_{12,3},\ D^{(2)}_{18,6} = D^{(3)}_{12,4},$$

$$D^{(2)}_{18,8} = D^{(3)}_{12,6},\ D^{(2)}_{18,16} = D^{(3)}_{12,10},\ D^{(2)}_{18,18} = D^{(3)}_{12,12},$$

$$D^{(3)}_{16,1} = D^{(4)}_{61},\ D^{(3)}_{16,4} = D^{(4)}_{62},\ D^{(3)}_{16,10} = D^{(4)}_{64},$$

$$D^{(3)}_{16,16} = D^{(4)}_{66},\ D^{(3)}_{4,16} = D^{(4)}_{26},\ D^{(3)}_{10,16} = D^{(4)}_{46},$$

$$D^{(1)}_{10,1} + D^{(3)}_{61} = D^{(2)}_{81},\ D^{(1)}_{10,3} + D^{(3)}_{63} = D^{(2)}_{83},$$

$$D^{(1)}_{10,8} + D^{(3)}_{64} = D^{(2)}_{86},\ D^{(1)}_{10,10} + D^{(3)}_{66} = D^{(2)}_{88},$$

$$D^{(1)}_{83} + D^{(3)}_{43} = D^{(2)}_{63},\ D^{(1)}_{8,10} + D^{(3)}_{46} = D^{(2)}_{68},$$

$$D^{(2)}_{16,1} + D^{(4)}_{41} = D^{(3)}_{10,1},\ D^{(2)}_{16,6} + D^{(4)}_{42} = D^{(3)}_{10,4},$$

$$D^{(2)}_{16,16} + D^{(4)}_{44} = D^{(3)}_{10,10},\ D^{(2)}_{6,16} + D^{(4)}_{24} = D^{(3)}_{4,10},$$

$$D_{81}^{(1)} + D_{41}^{(3)} = D_{61}^{(2)} + D_{21}^{(4)}, \; D_{88}^{(1)} + D_{44}^{(3)} = D_{66}^{(2)} + D_{22}^{(4)}.$$

□

In Theorem 2, let A_3, B_3, A_4, and C_3, C_4 vanish, then we can obtain the necessary and sufficient conditions for the existence of a ϕ-skew-Hermitian solution in the following equation:

Corollary 1. *Let $C_1 = -(C_1)_\phi \in \mathbb{H}^{p\times p}$, $A_1 \in \mathbb{H}^{p\times l}$, $B_1 \in \mathbb{H}^{p\times n}$, $A_2 \in \mathbb{H}^{q\times n}$, $B_2 \in \mathbb{H}^{q\times k}$, and $C_2 = -(C_2)_\phi \in \mathbb{H}^{q\times q}$ are given. The system*

$$\begin{cases} A_1X_1(A_1)_\phi + B_1X_2(B_1)_\phi = C_1, \\ A_2X_2(A_2)_\phi + B_2X_3(B_2)_\phi = C_2 \end{cases} \quad X_i = -(X_i)_\phi, \tag{33}$$

has a ϕ-skew-Hermitian solution $(X_1, X_2, X_3) \in \mathbb{H}^{l\times l} \times \mathbb{H}^{n\times n} \times \mathbb{H}^{k\times k}$ if and only if the ranks satisfy:

$$r(A_1,\ C_1,\ B_1) = r(A_1,\ B_1),\ \ r(A_2,\ C_2,\ B_2) = r(A_2,\ B_2),$$

$$r\begin{pmatrix} A_1 & C_1 \\ 0 & (B_1)_\phi \end{pmatrix} = r(A_1) + r(B_1),\ \ r\begin{pmatrix} A_2 & C_2 \\ 0 & (B_2)_\phi \end{pmatrix} = r(A_2) + r(B_2),$$

$$r\begin{pmatrix} 0 & (B_1)_\phi & (A_2)_\phi & 0 & 0 \\ B_1 & C_1 & 0 & A_1 & 0 \\ A_2 & 0 & -(C_2) & 0 & B_2 \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 & 0 \\ 0 & A_2 & B_2 \end{pmatrix} + r\begin{pmatrix} B_1 \\ A_2 \end{pmatrix},$$

$$r\begin{pmatrix} 0 & (B_1)_\phi & (A_2)_\phi & 0 \\ B_1 & C_1 & 0 & A_1 \\ A_2 & 0 & -(C_2) & 0 \\ 0 & 0 & (B_2)_\phi & 0 \end{pmatrix} = r\begin{pmatrix} A_1 & B_1 \\ 0 & A_2 \end{pmatrix} + r\begin{pmatrix} B_1 & 0 \\ A_2 & B_2 \end{pmatrix}.$$

Remark 1. *Corollary 1 is the main result of [31].*

4. Conclusions

We investigated some necessary and sufficient conditions for the existence of a ϕ-skew-Hermitian solution to the system (1) by using a simultaneous decomposition for a set of quaternion matrices. Some of the known results can be considered special cases in this paper.

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