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New Results on the Oscillation of Solutions of Third-Order Differential Equations with Multiple Delays

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Abstract: This study aims to examine the oscillatory behavior of third-order differential equations involving various delays within the context of functional differential equations of the neutral type. The oscillation criteria for the solutions of our equation have been obtained in this study to extend and supplement existing findings in the literature. In this study, a technique that relies on repeatedly improving monotonic properties was used in order to exclude positive solutions to the studied equation. Negative solutions are excluded based on the symmetry between the positive and negative solutions. Our results are important because they become sharper when applied to a Euler-type equation as compared to previous studies of the same equation. The significance of the findings was illustrated through the application of these findings to specific cases of the investigated equation.

Keywords: delay differential equation; neutral; oscillation; third order; multiple delays

MSC: 34C10; 34K11

1. Introduction

Delay differential equations (DDEs), a subclass of functional differential equations, take into account the system's reliance on the past to produce predictions for the future that are more precise and effective. One of the most important roles that the concept of delay in systems is thought to play is modeling the length of time needed to complete certain unseen activities. The predator–prey model demonstrates a delay when the birth rate of the predator takes into account both present and past numbers of predators and prey. With the rapid development of communication technologies, transmitting measured signals to a remote control center has become much simpler. However, the biggest obstacle for engineers is the time that it takes for the signal to reach the controller after a measurement has been taken. In order to minimize the possibility of experimental instability and potential harm, this lag must be considered during the planning phase. Modeling such phenomena, as well as others, requires the use of DDEs (see [1–5]).

Neutral delay differential equations (NDDEs) are encountered in several kinds of phenomena, such as electric transmission line problems, which are utilized for interconnecting switching circuits in high-speed computers, the study of vibrating masses connected to elastic bars, the solution of variational problems involving time delays or in the theory of automatic control, and neuro-mechanical systems where inertia is a significant factor (see [6–10]). The reader is directed to consult the references [11–15] for comprehensive insights into the methodologies, techniques, and findings relating to the investigation of oscillatory behavior in third-order NDDEs. Furthermore, the aforementioned studies [16–20] primarily center their attention on the examination of DDEs with odd orders.

This study focuses on third-order NDDEs expressed in linear form with several delays

$$\left(\kappa_2(\ell)\left(\kappa_1(\ell)z'(\ell)\right)'\right)' + \sum_{i=1}^{\Omega} q_i(\ell)y(\tau_i(\ell)) = 0, \tag{1}$$

where $\ell \geq \ell_0$, $z(\ell) := y(\ell) + p(\ell)y(\sigma(\ell))$, and Ω is a positive natural number. We suppose throughout this paper that the following hypotheses are fulfilled:

(A$_1$) $\kappa_1 \in C^2([\ell_0, \infty), (0, \infty))$, $\kappa_2 \in C([\ell_0, \infty), (0, \infty))$ and

$$\int_{\ell_0}^{\infty} \frac{1}{\kappa_1(\rho)}\mathrm{d}\rho = \int_{\ell_0}^{\infty} \frac{1}{\kappa_2(\rho)}\mathrm{d}\rho = \infty; \tag{2}$$

(A$_2$) $p, q_i \in C([\ell_0, \infty), [0, \infty))$, $q_i(\ell) \geq 0$, $q_i(\ell)$ does not vanish identically for each $i = 1, 2, \ldots, \Omega$ and $0 \leq p(\ell) \leq p_0 < 1$;

(A$_3$) $\tau_i, \sigma \in C([\ell_0, \infty), \mathbb{R})$, $\tau_i(\ell) \leq \ell$, $\sigma(\ell) \leq \ell$, $\lim_{\ell \to \infty} \sigma(\ell) = \infty$, and $\lim_{\ell \to \infty} \tau_i(\ell) = \infty$, for each $i = 1, 2, \ldots, \Omega$.

A function $y \in C^3([\ell_y, \infty), \mathbb{R})$, $\ell_y \geqslant \ell_0$, is said to be a solution of (1), which has the property z, $\kappa_1 z'$, and $\kappa_2(\kappa_1 z')'$ belong to $C^1[\ell_y, \infty)$ and satisfies (1) on $[\ell_y, \infty)$. Furthermore, we consider only solutions y of (1) that satisfy

$$\sup\{|y(\ell)| : \ell \geqslant L\} > 0, \text{ for all } L \geq \ell_y.$$

If a solution y is neither eventually positive nor eventually negative, then it is said to be oscillatory. Otherwise, it is said to be non-oscillatory. The equation itself is termed oscillatory if all of its solutions oscillate.

The previous studies on the oscillatory characteristics of neutral differential equations with odd orders primarily concentrated on establishing a suitable criterion for verifying whether the solutions exhibit oscillatory behavior or approach zero, as referenced in [21–24]. In the following, we provide some background details regarding the study of various classes of neutral differential equations.

In 2010, Baculíková and Džurina [11,25] investigated the asymptotic properties of the third-order NDDE

$$\left(\kappa(\ell)\left(z''(\ell)\right)^{\gamma}\right)' + q(\ell)f(y(\tau(\ell))) = 0. \tag{3}$$

They obtained conditions that test the convergence of all non-oscillatory solutions to zero. In [11], they used comparisons with first-order equations, while in [25], they obtained Hille and Nehari criteria. Thandapani and Li [26] found some fulfilling conditions that confirm that every solution of (3) either converges to zero or is oscillatory by using the Riccati transformation. In [27], Baculíková and Džurina examined the oscillation of the NDDE

$$\left(\kappa(\ell)(z(\ell))''\right)' + q(\ell)y(\tau(\ell)) = 0.$$

They obtained results based on the comparison theorems, which allowed them to reduce the problem of the oscillation in a third-order equation coupled to a first-order equation.

As an improvement over and completion of previous studies, Džurina et al. [28] established conditions to ensure that all solutions of linear NDDE

$$\left(\kappa_2(\ell)\left(\kappa_1(\ell)z'(\ell)\right)'\right)' + q(\ell)y(\tau(\ell)) = 0,$$

using a comparison with first-order delay equations. Moaaz et al. [29] investigated the oscillatory behavior of the NDDE

$$\left(\kappa(\ell)\left(z''(\ell)\right)^{\alpha}\right)' + q(\ell)y^{\alpha}(\tau(\ell)) = 0.$$

Utilizing the iterative technique, they established criteria of an iterative nature and found a criterion for the nonexistence of the so-called Kneser solutions.

Recently, Jadlovská et al. [30] studied the oscillation of the NDDE

$$\left(\kappa_2(\ell)(\kappa_1(\ell)y(\ell))''\right)' + q(\ell)y(\tau(\ell)) = 0.$$

Their results tested the convergence of all non-oscillatory solutions to zero. Their results are also sharp when applied to the Euler-type DDE, and they improved all previous results with regard to the criterion that tests the convergence of all non-oscillatory solutions to zero.

Our paper investigates the oscillatory properties of a third-order NDDE with multiple delays. The main motivation of this study is to extend the results of [28] to equations with multiple delays with respect to the convergence of non-oscillatory solutions to zero. Moreover, we create standards that guarantee the oscillation of all solutions of the studied equation by establishing a standard that excludes so-called Kneser solutions. Applying our results to a particular case of the considered equation supported the findings.

2. Preliminary Results

For convenience, we define the following:

$$\tau(\ell) := \min\{\tau_i(\ell); \ i = 1, 2, \ldots, \Omega\}, \ q_i(\ell) := \min\{q_i(\ell), q_i(\sigma(\ell)\}, $$

$$\widetilde{\tau}(\ell) := \max\{\tau_i(\ell); \ i = 1, 2, \ldots, \Omega\},$$

$$\mathcal{J}_0 z = z, \ \mathcal{J}_1 z = \kappa_1 z', \ \mathcal{J}_2 z = \kappa_2(\kappa_1 z')', \ \mathcal{J}_3 z = \left(\kappa_2(\kappa_1 z')'\right)',$$

$$M_1(\ell) := \int_{\ell_0}^{\ell} \frac{\mathrm{d}\rho}{\kappa_1(\rho)}, M_2(\ell) := \int_{\ell_0}^{\ell} \frac{\mathrm{d}\rho}{\kappa_2(\rho)},$$

and

$$M_{12}(\ell) := \int_{\ell_0}^{\ell} \frac{M_2(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho.$$

Lemma 1. *Ref. [28] Lemma 1—suppose that there is a constant $l > 0$ such that*

$$\lim_{\ell \longrightarrow \infty} \frac{\kappa_1(\ell)M_1(\ell)}{\kappa_2(\ell)M_2(\ell)} = l. \tag{4}$$

Then,

$$M_{12}(\ell) \geq \frac{\epsilon}{1+l} M_1(\ell)M_2(\ell), \tag{5}$$

eventually for all $\epsilon \in (0,1)$.

To proceed with proving our results we need to define the following limits:

$$\liminf_{\ell \to \infty} \frac{M_{12}(\ell)}{M_{12}(\tau(\ell))} := \lambda_*,$$

$$\liminf_{\ell \to \infty} \kappa_2(\ell)M_2(\ell)M_{12}(\tau(\ell)) \sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell)) := \beta_*,$$

and

$$\liminf_{\ell \to \infty} \frac{M_2^{\beta_*}(\ell)}{M_{12}(\ell)} \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta_*}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho := k_*, \text{ for } \beta_* \in (0,1).$$

3. Main Results

In this section, we provide sufficient conditions to ensure the oscillation of all solutions of the studied equation. For the following results, we assume that $\lambda_*, \beta_*, k_* \in (0, \infty)$.

Lemma 2. *Suppose that y is a positive solution of* (1). *Then,* $\mathcal{J}_3 z(\ell) \leq 0$, *and there are only two categories:*

$$\text{Class (1)} \quad : \quad z > 0,\ \mathcal{J}_1 z > 0,\ \mathcal{J}_2 z > 0,$$
$$\text{Class (2)} \quad : \quad z > 0,\ \mathcal{J}_1 z < 0,\ \mathcal{J}_2 z > 0.$$

Proof. The proof is straightforward; hence, we omit the details. □

Notation 1. *By* $x \in \Im_1$ *we mean that solution x with corresponding function has class (1) properties, while by* $x \in \Im_2$ *we mean that the solution x with corresponding function has class (2) properties.*

3.1. Class (1)

In this section, we present some characteristics of solutions that belong to class $\Im_1$. We also obtain criteria that rule out the existence of solutions with class $\Im_1$ properties.

Remark 1. *From the definition of* M_{12}, β_*, *and* k_*, *the following can be concluded:*

(R1) For any $\beta \in (0, \beta_*)$, *there is a* $\ell_\beta \geq \ell_0$ *such that*

$$\kappa_2(\ell) M_2(\ell) M_{12}(\tau(\ell)) \sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell))) \geq \beta, \text{ for all } \ell \geq \ell_\beta. \tag{6}$$

Lemma 3. *Suppose that* $y \in \Im_1$ *and* $\beta_* > 0$. *Then, eventually,*

(a) the functions $\mathcal{J}_2 z(\ell)$, $\mathcal{J}_1 z(\ell)/M_2(\ell)$, *and* $z(\ell)/M_{12}(\ell)$ *converge to zero;*
(b) $\mathcal{J}_1 z/M_2$ is decreasing;
(c) z/M_{12} is decreasing.

Proof. Suppose that $y \in \Im_1$. Since $z(\ell) \geq y(\ell)$ and $z'(\ell) > 0$, we have $y(\ell) \geq (1 - p(\ell))z(\ell)$, and so $y(\tau_i(\ell)) \geq (1 - p(\tau_i(\ell)))z(\tau_i(\ell))$. Thus, (1) becomes

$$
\begin{aligned}
\mathcal{J}_3 z(\ell) &= -\sum_{i=1}^{\Omega} q_i(\ell) y(\tau_i(\ell)) \\
&\leq -\sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell)))z(\tau_i(\ell)) \\
&\leq -z(\tau(\ell)) \sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell))).
\end{aligned}
\tag{7}
$$

(a) Using the facts $\mathcal{J}_2 z(\ell) > 0$ and $\mathcal{J}_3 z(\ell) \leq 0$, it is obvious that $\mathcal{J}_2 z(\ell) \to \varrho_0$ as $\ell \to \infty$. Assume the contrary that $\varrho_0 > 0$. Hence, it follows that $\mathcal{J}_2 z(\ell) \geq \varrho_0 > 0$. Therefore,

$$\mathcal{J}_1 z(\ell) \geq \int_{\ell_1}^{\ell} \frac{1}{\kappa_2(\rho)} \mathcal{J}_2 z(\rho) \mathrm{d}\rho, \tag{8}$$

and so

$$
\begin{aligned}
z(\ell) &\geq \int_{\ell_1}^{\ell} \frac{1}{\kappa_1(\rho)} \mathcal{J}_1 z(\rho) \mathrm{d}\rho \\
&\geq \int_{\ell_1}^{\ell} \frac{1}{\kappa_1(\rho)} \left(\int_{\ell_1}^{\rho} \frac{1}{\kappa_2(u)} \mathcal{J}_2 z(u) \mathrm{d}u \right) \mathrm{d}\rho \\
&\geq \varrho_0 \int_{\ell_1}^{\ell} \frac{1}{\kappa_1(\rho)} \left(\int_{\ell_1}^{\rho} \frac{1}{\kappa_2(u)} \mathrm{d}u \right) \mathrm{d}\rho \\
&= \varrho_0 \int_{\ell_1}^{\ell} \frac{M_2(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho > \delta \varrho_0 M_{12}(\ell).
\end{aligned}
\tag{9}
$$

for all $\delta \in (0,1)$. Hence, (7) reduces to

$$\mathcal{J}_3 z(\ell) \geq \varrho_0 M_{12}(\tau(\ell)) \sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell))).$$

Integrating this inequality from $\ell_1 \geq \ell$ and using (6), we obtain for $\ell_2 = \max\{\ell_1, \ell_\beta\}$,

$$\begin{aligned}
\mathcal{J}_2 z(\ell) &\geq \varrho_0 \int_{\ell_1}^{\ell} M_{12}(\tau(\rho)) \sum_{i=1}^{\Omega} q_i(\rho)(1 - p(\tau_i(\rho))) \mathrm{d}\rho \\
&\geq \beta \varrho_0 \int_{\ell_1}^{\ell} \frac{1}{\kappa_2(\rho) M_2(\rho)} \mathrm{d}\rho \\
&= \beta \varrho_0 \ln \frac{M_2(\ell)}{M_2(\ell_1)}.
\end{aligned}$$

Then, $\lim_{\ell \to \infty} \mathcal{J}_2 z(\ell) = \infty$, a contradiction. Consequently, $\xi = 0$. The rest of the properties in (a) are proven directly by using L'Hopital's principle.

(b) Using (8) and the fact that $\mathcal{J}_3 z(\ell) \leq 0$, we arrive at

$$\mathcal{J}_1 z(\ell) \geq \mathcal{J}_2 z(\ell) \int_{\ell_1}^{\ell} \frac{1}{\kappa_2(\rho)} \mathrm{d}\rho = M_2(\ell) \mathcal{J}_2 z(\ell).$$

and thus,

$$\left(\frac{\mathcal{J}_1 z}{M_2}\right)' = \frac{\mathcal{J}_2 z M_2 - \mathcal{J}_1 z}{M_2^2 \kappa_2} < 0.$$

(c) Since $\mathcal{J}_1 z / M_2$ tend to zero and the function is decreasing, we find

$$\begin{aligned}
z(\ell) &\geq \int_{\ell_1}^{\ell} \frac{M_2(\rho)}{\kappa_1(\rho)} \frac{\mathcal{J}_1 z(\rho)}{M_2(\rho)} \mathrm{d}\rho \geq \frac{\mathcal{J}_1 z(\ell)}{M_2(\ell)} \int_{\ell_1}^{\ell} \frac{M_2(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho \\
&> \frac{\mathcal{J}_1 z(\ell)}{M_2(\ell)} M_{12}(\ell).
\end{aligned}$$

Therefore,

$$\left(\frac{z}{M_{12}}\right)' = \frac{\mathcal{J}_1 z M_{12} - z M_2}{M_{12}^2 \kappa_1} < 0.$$

We have reached the end of the proof. $\quad\square$

Remark 2. *From the definition of M_{12}, β_*, and k_*, the following can be concluded:*

(R2) Assume that $\beta_* \in (0,1)$, we can conclude that $k_* \geq 1$. For any $k \in (1,\infty)$, there is $\ell_k \geq \ell_0$ such that

$$\frac{M_2^\beta(\ell)}{M_{12}(\ell)} \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta}(\rho)}{\kappa_1(\rho)} d\rho \geq k, \ \ell \geq \ell_k. \tag{10}$$

Lemma 4. *Suppose that $y \in \Im_1$ and $\beta_* > 0$. Then, eventually,*

(C_{01}) $\mathcal{J}_1 z / M_2^{1-\beta_*}(\ell)$ is decreasing;
(C_{02}) $\mathcal{J}_1 z(\ell) / M_2^{1-\beta_*}(\ell)$ converges to zero;
(C_{03}) $z > k(M_{12}/M_2)\mathcal{J}_1 z$, and $z / M_{12}^{1/k}$ is decreasing.

Proof. Assume that $y \in \Im_1$. From Lemma 3, we have that (a), (b), and (c) hold.

(C_{01}) We define

$$\omega(\ell) := \mathcal{J}_1 z(\ell) - M_2(\ell) \mathcal{J}_2 z(\ell). \tag{11}$$

Thus, from (b), $\omega(\ell) > 0$ for all $\ell \geq \ell_1$. Hence, we obtain

$$\omega'(\ell) = (\mathcal{J}_1 z(\ell) - M_2(\ell) \mathcal{J}_2 z(\ell))' = -M_2(\ell) \mathcal{J}_3 z(\ell),$$

which with (1) gives

$$\omega'(\ell) = M_2(\ell) \sum_{i=1}^{\Omega} q_i(\ell) y(\tau_i(\ell)). \tag{12}$$

Since $z(\ell) \geq y(\ell)$ and $z'(\ell) > 0$, we have $y(\ell) \geq (1 - p(\ell)) z(\ell)$. Thus,

$$\begin{aligned} \omega'(\ell) &\geq M_2(\ell) \sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell))) z(\tau_i(\ell)) \\ &\geq z(\tau(\ell)) M_2(\ell) \sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell))). \end{aligned}$$

Using (6) and (c), we obtain

$$\omega'(\ell) \geq \beta \frac{z(\tau(\ell))}{\kappa_2(\ell) M_{12}(\tau(\ell))} \geq \beta \frac{z(\ell)}{\kappa_2(\ell) M_{12}(\ell)} \geq \beta \frac{\mathcal{J}_1 z(\ell)}{\kappa_2(\ell) M_2(\ell)}.$$

for $\ell \geq \ell_2 \geq \ell_1$. Integrating this inequality from ℓ_2 to ℓ, we have

$$\begin{aligned} \omega(\ell) &\geq \beta \int_{\ell_2}^{\ell} \frac{\mathcal{J}_1 z(\rho)}{\kappa_2(\rho) M_2(\rho)} \mathrm{d}\rho \geq \beta \frac{\mathcal{J}_1 z(\ell)}{M_2(\ell)} \int_{\ell_2}^{\ell} \frac{1}{\kappa_2(\rho)} \mathrm{d}\rho \\ &> \beta \mathcal{J}_1 z(\ell). \end{aligned} \tag{13}$$

From the definition of ω, we obtain $(1 - \beta)\mathcal{J}_1 z(\ell) > M_2(\ell) \mathcal{J}_2 z(\ell)$, and so

$$\left(\frac{\mathcal{J}_1 z}{M_2^{1-\beta}} \right)' = \frac{\mathcal{J}_2 z M_2 - (1 - \beta) \mathcal{J}_1 z M_2}{M_3^{2-\beta} \kappa_2} < 0, \ \ell \geq \ell_3. \tag{14}$$

(C_{02}) From (14), we have $\beta < 1$. From (13), we find that

$$\begin{aligned} \omega(\ell) &\geq \beta \int_{\ell_2}^{\ell} \frac{\mathcal{J}_1 z(\rho)}{\kappa_2(\ell) M_2(\rho)} \mathrm{d}\rho \\ &\geq \beta \frac{\mathcal{J}_1 z(\ell)}{M_2^{1-\beta}(\ell)} \int_{\ell_2}^{\ell} \frac{1}{\kappa_2(\ell) M_2^{\beta}(\rho)} \mathrm{d}\rho \\ &\geq \frac{\beta}{1-\beta} \frac{\mathcal{J}_1 z(\ell)}{M_2^{1-\beta}(\ell)} \left(M_2^{1-\beta}(\ell) - M_2^{1-\beta}(\ell_2) \right) \\ &\geq \frac{\epsilon_1 \beta}{1-\beta} \mathcal{J}_1 z(\ell) \end{aligned}$$

for all $\epsilon_1 \in (0, 1)$. Then, eventually,

$$\omega(\ell) \geq (\beta_* + c_2) \mathcal{J}_1 z(\ell),$$

for $c_2 > 0$, by choosing $\beta \in (\beta_*/(1 + \beta_*), \beta_*)$. Hence,

$$(1 - \beta_*) \mathcal{J}_1 z(\ell) > (1 - \beta_* - c_2) \mathcal{J}_1 z(\ell) > M_2(\ell) \mathcal{J}_2 z(\ell). \tag{15}$$

Thus, $\mathcal{J}_1 z / M_2^{1-\beta_* - c_2}$ is decreasing. Now, if we assume $\lim_{\ell \to \infty} \mathcal{J}_1 y / M_2^{1-\beta_*} > 0$, we find

$$\frac{\mathcal{J}_1 z(\ell)}{M_2^{1-\beta_* - c_2}(\ell)} = \frac{\mathcal{J}_1 z(\ell)}{M_2^{1-\beta_*}(\ell)} M_2^{c_2}(\ell) \longrightarrow \infty \text{ as } \ell \longrightarrow \infty, \tag{16}$$

a contradiction. Therefore, $\lim_{\ell \to \infty} \mathcal{J}_1 y / M_2^{1-\beta_*} = 0$.

(C_{03}) From the previous facts we can conclude that

$$
\begin{aligned}
z(\ell) &= z(\ell_3) + \int_{\ell_3}^{\ell} \frac{M_2^{1-\beta_*}(\rho)}{\kappa_1(\rho)} \frac{\mathcal{J}_1 z(\rho)}{M_2^{1-\beta_*}(\rho)} \mathrm{d}\rho \\
&\geq z(\ell_3) + \frac{\mathcal{J}_1 z(\ell)}{M_2^{1-\beta_*}(\ell)} \int_{\ell_3}^{\ell} \frac{M_2^{1-\beta_*}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho \\
&= z(\ell_3) + \frac{\mathcal{J}_1 z(\ell)}{M_2^{1-\beta_*}(\ell)} \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta_*}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho - \frac{\mathcal{J}_1 z(\ell)}{M_2^{1-\beta_*}(\ell)} \int_{\ell_0}^{\ell_3} \frac{M_2^{1-\beta_*}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho \\
&> \frac{\mathcal{J}_1 z(\ell)}{M_2^{1-\beta_*}(\ell)} \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta_*}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho \\
&\geq k \frac{M_{12}(\ell)}{M_2(\ell)} \mathcal{J}_1 z(\ell), \quad \ell \geq \ell_5 \geq \ell_3.
\end{aligned}
$$

Therefore, eventually,

$$
\left(\frac{z}{M_{12}^{1/k}} \right)' < 0.
$$

We have reached the end of the proof. $\square$

Lemma 5. *If $\beta_* \geq 1$, then $\Im_1 = \varnothing$.*

Proof. Assume that $y \in \Im_1$. Since $\mathcal{J}_1 z / M_2^{1-\beta_*}(\ell)$ is decreasing and $\mathcal{J}_2 z > 0$, we obtain $\beta_* < 1$, a contradiction. Thus, $\Im_1 = \varnothing$. $\square$

We can improve the previous properties by defining the following sequences:

$$
\beta_n = \begin{cases} \beta_* & \text{for } n = 0 \\ \dfrac{\beta_0 k_{n-1} \lambda_*^{1-\frac{1}{k_{n-1}}}}{(1-\beta_{n-1})} & \text{for } n = 1, 2, \dots, \end{cases}
$$

and

$$
k_n = \liminf_{\ell \to \infty} \frac{M_2^{\beta_n}(\ell)}{M_{12}(\ell)} \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta_n}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho, \text{ for } n = 0, 1, 2, \dots. \tag{17}
$$

Remark 3. *If $\beta_i < 1$ and $k_i \in [1, \infty)$ for $i = 0, 1, \dots, n$, then β_{n+1} can be clearly determined. In this case, the following inequality holds:*

$$
\beta_1 = \beta_0 \frac{k_0}{1-\beta_0} \lambda_*^{1-\frac{1}{k_0}} > \beta_0.
$$

Thus,

$$
\begin{aligned}
k_1 &= \liminf_{\ell \to \infty} \frac{M_2^{\beta_0}(\ell) \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta_0}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho}{M_{12}(\ell)} = \liminf_{\ell \to \infty} \frac{M_2^{\beta_1}(\ell) \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta_0-(\beta_1-\beta_0)}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho}{M_{12}(\ell)} \\
&\geq \liminf_{\ell \to \infty} \frac{M_2^{\beta_0}(\ell) \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta_0}(\rho)}{\kappa_1(\rho)} \mathrm{d}\rho}{M_{12}(\ell)} = k,
\end{aligned}
$$

Applying induction to n, it is straightforward to demonstrate that

$$\frac{\beta_{n+1}}{\beta_n} = l_n > 1,\tag{18}$$

where

$$\ell_0 \quad : \quad = \frac{k_0 \lambda_*^{1-\frac{1}{k_0}}}{1 - \beta_0},\tag{19}$$

$$\ell_n \quad : \quad = \frac{k_n(1 - \beta_{n-1})\lambda_*^{1-\frac{1}{k_{n-1}}-\frac{1}{k_n}}}{(1 - \beta_n)k_{n-1}}, \, n \in \mathbb{N},$$

with

$$k_n \geq k_{n-1}.$$

Lemma 6. *Ref. [28] Lemma 2—suppose that (4) holds, $\beta_* > 0$, and $\beta_i < 1$, $i = 0, 1, \ldots, n$. We can conclude that*

$$k_n(\ell) \geq \beta_n \frac{l}{1+l} + 1 > 1, \, n \in \mathbb{N}_0.$$

Lemma 7. *Suppose that $y \in \Im_1$ and $\beta_* > 0$. Then, eventually,*

(C_{n1}) $\mathcal{J}_1 z / M_2^{1-\beta_n}$ *is decreasing;*
(C_{n2}) $\lim_{\ell \to \infty} \mathcal{J}_1 z(\ell) / M_2^{1-\beta_n}(\ell)$ *converges to zero;*
(C_{n3}) $z > \varepsilon_n k_n (M_{12}/M_2) \mathcal{J}_1 z$ *and $y / M_{12}^{1/(\varepsilon_n k_n)}$ is decreasing for any $\varepsilon_n \in (0,1)$.*

Proof. Assume that $y \in \Im_1$. We will employ an induction argument on n. For $n = 0$, the conclusion directly follows from Lemma 4 with $\varepsilon_0 = k/k_*$. Next, assuming that (C_{n1})–(C_{n3}) hold when $n \geq 1$ for $\ell \geq \ell_n \geq \ell_1$, we need to demonstrate that these conditions also hold for $n + 1$.

Based on Lemma 4, the proof is exactly similar to the proof of Lemma 5 in [28]; therefore, it was omitted. □

Corollary 1. *If $\beta_i < 1$ for $i = 0, 1, \ldots, n - 1$, and $\beta_n \geq 1$, then $\Im_1 = \emptyset$.*

From the previous results and taking into account (18), the sequence $\{\beta_n\}$ has the limit

$$\lim_{n \to \infty} \beta_n = \beta_\zeta := \frac{\beta_* k_\zeta \lambda_*^{1-1/k_\zeta}}{1 - \beta_\zeta} \in (0,1),\tag{20}$$

where

$$k_\zeta = \liminf_{\ell \to \infty} \frac{M_2^{\beta_\zeta}(\ell)}{M_3(\ell)} \int_{\ell_0}^{\ell} \frac{M_2^{1-\beta_\zeta}(\rho)}{\kappa_1(\rho)} d\rho.$$

Theorem 1. *If (20) does not possess a root on $(0,1)$, then $\Im_1 = \emptyset$.*

Corollary 2. *If*

$$\beta_* > \alpha,\tag{21}$$

then $\Im_1 = \emptyset$, where

$$\alpha := \max\left\{\beta_\zeta(1 - \beta_\zeta)k_\zeta^{-1}\lambda_*^{1/k_\zeta - 1} : 0 < \beta_\zeta < 1\right\}.$$

3.2. Class (2)

Lemma 8. *Assume $y \in \Im_2$ and*

$$\int_{\ell_0}^{\infty} \frac{1}{\kappa_1(s)} \int_s^{\infty} \frac{1}{\kappa_2(\rho)} \int_{\rho}^{\infty} \sum_{i=1}^{\Omega} q_i(u) du d\rho ds = \infty. \tag{22}$$

Then, $\lim_{\ell \to \infty} z(\ell) = 0$.

Proof. Assume that $y \in \Im_2$. Since $z(\ell) > 0$ and $z'(\ell) < 0$, we have $z(\ell) \to \nu_0$ as $\ell \to \infty$, where $\nu_0 \geq 0$. Assume that $\nu_0 > 0$; then, we have for all $\epsilon > 0$, $\nu_0 < z(\ell) < \nu_0 + \epsilon$, eventually. By choosing $0 < \epsilon < \frac{\nu_0(1-p_0)}{p_0}$, it is easy to verify that

$$\begin{aligned} y(\ell) &= z(\ell) - p(\ell)y(\sigma(\ell)) > \nu_0 - p_0 z(\sigma(\ell)) \\ &> \nu_0 - p_0(\nu_0 + \epsilon) > h(\nu_0 + \epsilon) > hz(\ell), \end{aligned}$$

where $h = \frac{\nu_0 - p_0(\nu_0 + \epsilon)}{\nu_0 + \epsilon} > 0$. Then, (1) becomes

$$\begin{aligned} \mathcal{J}_3 z(\ell) &= -\sum_{i=1}^{\Omega} q_i(\ell) y(\tau_i(\ell)) \\ &\leq -h \sum_{i=1}^{\Omega} q(\ell_i) z(\tau(\ell)) \\ &\leq -h\nu_0 \sum_{i=1}^{\Omega} q(\ell_i). \end{aligned}$$

Integrating from ℓ to ∞, we obtain

$$\mathcal{J}_2 z(\ell) \geq \nu_0 h \int_{\ell}^{\infty} \sum_{i=1}^{\Omega} q_i(\rho) h\nu_0 d\rho,$$

In other words,

$$(\mathcal{J}_1 z(\ell))' \geq \frac{\nu_0 h}{\kappa_2(\ell)} \int_{\ell}^{\infty} \sum_{i=1}^{\Omega} q_i(\rho) d\rho. \tag{23}$$

Integration (23) from ℓ to ∞ gives

$$-z'(\ell) \geq \frac{\nu_0 h}{\kappa_1(\ell)} \int_{\ell}^{\infty} \frac{1}{\kappa_2(u)} \int_{u}^{\infty} \sum_{i=1}^{\Omega} q_i(\rho) d\rho du,$$

and hence,

$$z(\ell) \leq z(\ell_2) - \nu_0 h \int_{\ell_2}^{\ell} \frac{1}{\kappa_1(x)} \int_{x}^{\infty} \frac{1}{\kappa_2(u)} \int_{u}^{\infty} \sum_{i=1}^{\Omega} q_i(\rho) d\rho du dx \to -\infty \text{ as } \ell \to \infty,$$

which contradicts the positivity of z. Then, the proof of this lemma is complete. □

In the following theorem, we establish certain conditions that guarantee the absence of Kneser solutions, which are solutions whose corresponding function satisfies the properties in class (2). In the following, we need the conditions

$$\tau_i(\sigma(\ell)) = \sigma(\tau_i(\ell)), \text{ and } \sigma'(\ell) \geq \sigma_0 > 0.$$

Theorem 2. *Suppose that there is a function $\zeta(\ell) \in C([\ell_0, \infty), (0, \infty))$ satisfying $\tilde{\tau}(\ell) < \zeta(\ell)$ and $\sigma^{-1}(\zeta(\ell)) < \ell$. If the DDE*

$$w'(\ell) + \frac{\sigma_0}{\sigma_0 + p_0} K(\zeta(\ell), \tilde{\tau}(\ell)) w\left(\sigma^{-1}(\zeta(\ell))\right) \sum_{i=1}^{\Omega} \tilde{q}_i(\ell) = 0, \tag{24}$$

is oscillatory, then $\Im_2 = \varnothing$, where

$$K(\varsigma, \varrho) := \int_{\varrho}^{\varsigma} \frac{1}{\kappa_1(s)} \int_{s}^{\varsigma} \frac{1}{\kappa_2(u)} du ds.$$

Proof. Assume that $y \in \Im_2$. This implies that

$$z > 0, \ \mathcal{J}_1 z < 0, \text{ and } \mathcal{J}_2 z > 0. \tag{25}$$

From (1), we see that

$$\begin{aligned}
0 &\geq \frac{p_0}{\sigma'(\ell)} \left(\kappa_2(\sigma(\ell))\left(\kappa_1(\sigma(\ell))z'(\sigma(\ell))\right)'\right)' + p_0 \sum_{i=1}^{\Omega} q_i(\sigma(\ell))x(\tau_i(\sigma(\ell))) \\
&\geq \frac{p_0}{\sigma_0} \mathcal{J}_3 z(\sigma(\ell)) + p_0 \sum_{i=1}^{\Omega} q_i(\sigma(\ell))x(\tau_i(\sigma(\ell))) \\
&= \frac{p_0}{\sigma_0} \mathcal{J}_3 z(\sigma(\ell)) + p_0 \sum_{i=1}^{\Omega} q_i(\sigma(\ell))x(\sigma(\tau_i(\ell))).
\end{aligned} \tag{26}$$

Combining (1) and (26), we obtain

$$\begin{aligned}
0 &\geq \mathcal{J}_3 z(\ell) + \frac{p_0}{\sigma_0} \mathcal{J}_3 z(\sigma(\ell)) \\
&\quad + \sum_{i=1}^{m} q_i(\ell)x(\tau_i(\ell)) + p_0 \sum_{i=1}^{\Omega} q_i(\sigma(\ell))x(\sigma(\tau_i(\ell))) \\
&\geq \mathcal{J}_3 z(\ell) + \frac{p_0}{\sigma_0} \mathcal{J}_3 z(\sigma(\ell)) + \sum_{i=1}^{\Omega} \tilde{q}_i(t)[x(\tau_i(\ell)) + p_0 x(\sigma(\tau_i(\ell)))].
\end{aligned} \tag{27}$$

From definition of z, we have

$$z(\tau_i(\ell)) = x(\tau_i(\ell)) + p(\tau_i(\ell))x(\sigma(\tau_i(\ell))) \leq x(\tau_i(\ell)) + p_0 x(\sigma(\tau_i(\ell))).$$

By using the latter inequality in (27), we obtain

$$0 \geq \mathcal{J}_3 z(\ell) + \frac{p_0}{\sigma_0} \mathcal{J}_3 z(\sigma(\ell)) + \sum_{i=1}^{\Omega} \tilde{q}_i(t)z(\tau_i(\ell)).$$

Since z is decreasing, then

$$0 \geq \mathcal{J}_3 z(\ell) + \frac{p_0}{\sigma_0} \mathcal{J}_3 z(\sigma(\ell)) + z(\tilde{\tau}(\ell)) \sum_{i=1}^{\Omega} \tilde{q}_i(\ell)$$

That is

$$0 \geq \left(\mathcal{J}_2 z(\ell) + \frac{p_0}{\tau_0} \mathcal{J}_2 z(\sigma(\ell))\right)' + z(\tilde{\tau}(\ell)) \sum_{i=1}^{\Omega} \tilde{q}_i(\ell). \tag{28}$$

On the other hand, it follows from the monotonicity of $\mathcal{J}_2 z$ that

$$
\begin{aligned}
-\mathcal{J}_1 z(\varrho) &\geq \mathcal{J}_1 z(\varsigma) - \mathcal{J}_1 z(\varrho) = \int_\varrho^\varsigma (\mathcal{J}_1 z(s))' \mathrm{d}s = \int_\varrho^\varsigma \frac{\mathcal{J}_2 z(s)}{\kappa_2(s)} \mathrm{d}s \\
&\geq \mathcal{J}_2 z(\varsigma) \int_\varrho^\varsigma \frac{1}{\kappa_2(s)} \mathrm{d}s
\end{aligned}
\tag{29}
$$

Integrating (29) from ϱ to ς, we have

$$
z(\varrho) \geq \mathcal{J}_2 z(\varsigma) \int_\varrho^\varsigma \frac{1}{\kappa_1(s)} \int_s^\varsigma \frac{1}{\kappa_2(u)} \mathrm{d}u \mathrm{d}s.
\tag{30}
$$

Thus, we have

$$
z(\widetilde{\tau}(\ell)) \geq \mathcal{J}_2 z(\zeta(\ell)) K(\zeta(\ell), \widetilde{\tau}(\ell)),
$$

which, by virtue of (28), yields that

$$
\left(\mathcal{J}_2 z(\ell) + \frac{p_0}{\sigma_0} \mathcal{J}_2 z(\sigma(\ell)) \right)' + K(\zeta(\ell), \widetilde{\tau}(\ell)) \mathcal{J}_2 z(\zeta(\ell)) \sum_{i=1}^\Omega \widetilde{q}_i(\ell) \leq 0.
\tag{31}
$$

Now, set

$$
w(t) = \mathcal{J}_2 z(\ell) + \frac{p_0}{\sigma_0} \mathcal{J}_2 z(\sigma(\ell)) > 0.
$$

From the fact that $\mathcal{J}_2 z$ is non-increasing, we have

$$
w(t) \leq \mathcal{J}_2 z(\sigma(\ell)) \left(1 + \frac{p_0}{\sigma_0} \right),
$$

or equivalently,

$$
\mathcal{J}_2 z(\zeta(\ell)) \geq \frac{\sigma_0}{\sigma_0 + p_0} w\left(\sigma^{-1}(\zeta(\ell)) \right).
\tag{32}
$$

Using (32) in (31), we see that w is a positive solution of the differential inequality

$$
w'(\ell) + \frac{\sigma_0}{\sigma_0 + p_0} K(\zeta(\ell), \widetilde{\tau}(\ell)) w\left(\sigma^{-1}(\zeta(\ell)) \right) \sum_{i=1}^\Omega \widetilde{q}_i(\ell) \leq 0.
$$

In view of [31] Theorem 1, we have that (24) also has a positive solution, a contradiction. Thus, the proof is complete. □

Corollary 3. *Suppose that there is a function $\zeta(\ell) \in C([\ell_0, \infty), (0, \infty))$ satisfying $\widetilde{\tau}(\ell) < \zeta(\ell)$ and $\sigma^{-1}(\zeta(\ell)) < \ell$. If*

$$
\liminf_{t \to \infty} \int_{\sigma^{-1}(\zeta(t))}^t K(\zeta(s), \tau(s)) \sum_{i=1}^\Omega \widetilde{q}_i(s) \mathrm{d}s > \frac{\sigma_0 + p_0}{\sigma_0 \mathrm{e}},
\tag{33}
$$

then $\Im_2 = \varnothing$.

Proof. The results in [32] guarantee the oscillation of Equation (24) under condition (33). □

3.3. Oscillatory Theorems and Examples

We obtain the criteria in the following theorems by directly combining the results in the previous two subsections. Assuming that the solution is positive means that it belongs to one of two categories: $\Im_2$ or $\Im_2$. Therefore, when it is confirmed that categories $\Im_2$ and $\Im_2$ are empty, this means that there are no positive solutions, and accordingly, all solutions are oscillatory (this is based on the principle of symmetry between positive and negative solutions).

Theorem 3. *Suppose $\beta_* > 1$, and (22) holds. Then, every solution of (1) either converges to zero or is oscillatory.*

Theorem 4. *Suppose (21) and (22) hold. Then, every solution of (1) either converges to zero or is oscillatory.*

Theorem 5. *Suppose $\beta_* > 1$, and (33) holds. Then, every solution of (1) is oscillatory.*

Theorem 6. *Suppose (21) and (33) hold. Then, every solution of (1) is oscillatory.*

The following example demonstrates the significance of the results obtained.

Example 1. *Consider*

$$(y(\ell) + p_0 y(\sigma_0 \ell))''' + \sum_{i=1}^{\Omega} \frac{q_0}{\ell^3} y(\tau_i \ell) = 0, \tag{34}$$

where $0 \leq p_0 < 1$, and $\tau_i, \sigma_0 \in (0,1)$. Clearly:
$\kappa_1(\ell) = \kappa_2(\ell) = 1$, $\sigma(\ell) = \rho_0 \ell$, $\tau(\ell) = \tau_0 \ell = \min\{\tau_i \ell, \ i = 1, 2, \ldots, \Omega\}$, $p(\ell) = p_0$, $q_i(\ell) = q_0 / \ell^3$, *and*

$$M_1(\ell) \sim \ell, \ M_2(\ell) \sim \ell, \ M_{12}(\ell) \sim \ell^2 / 2.$$

Then, we can compute the value of β_* as follows:

$$\begin{aligned}
\beta_* &= \liminf_{\ell \to \infty} M_2(\ell) \kappa_2(\ell) M_{12}(\tau(\ell)) \sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell))) \\
&= \liminf_{\ell \to \infty} \ell \frac{\tau_0^2 \ell^2}{2} \frac{\Omega q_0}{\ell^3} (1 - p_0) \\
&= \frac{1}{2} \Omega \tau_0^2 (1 - p_0) q_0,
\end{aligned}$$

For $\beta_* \geq 1$, we have

$$q_0 > \frac{2}{\Omega \tau^2 (1 - p_0)}.$$

Moreover

$$\int_{\ell_0}^{\infty} \frac{1}{\kappa_1(v)} \int_{\ell}^{\infty} \frac{1}{\kappa_2(u)} \int_{\rho}^{\infty} \varphi(\rho) \mathrm{d}\rho \mathrm{d}u \mathrm{d}v = \int_{1}^{\infty} \int_{\ell}^{\infty} \int_{\rho}^{\infty} \frac{\Omega q_0}{\rho^3} (1 - p_0) \mathrm{d}\rho \mathrm{d}u \mathrm{d}v = \infty.$$

Thus, the assumption of Theorem 3 is satisfied, and then, every solution of (34) either converges to zero or is oscillatory.

Example 2. *Consider the third-order neutral delay differential equation*

$$\left(e^{-\ell} (y(\ell) + p_0 y(\sigma(\ell))'')' \right)' + \sum_{i=1}^{\Omega} q_0 y(\tau_i \ell) = 0, \quad \ell > 1. \tag{35}$$

where $q_0 > 0$. It is easy to verify that

$\kappa_1(\ell) = 1, \kappa_2(\ell) = e^{-\ell}, \sigma(\ell) = \sigma_0 \ell, \tau(\ell) = \tau_0 \ell = \min\{\tau_i \ell, \ i = 1, 2, \ldots, \Omega\}, p(\ell) = p_0$, $q_i(\ell) = q_0$, and

$$M_1(\ell) \sim \ell, \quad M_2(\ell) \sim e^{\ell}, \quad M_{12}(\ell) \sim e^{\ell}.$$

Then

$$\lambda_* = \liminf_{t \to \infty} \frac{M_{12}(\ell)}{M_{12}(\tau(\ell))} = \liminf_{\ell \to \infty} e^{(1-\tau)\ell} = \infty,$$

and

$$\beta_* = \liminf_{\ell \to \infty} M_2(\ell) \kappa_2(\ell) M_{12}(\tau(\ell)) \sum_{i=1}^{\Omega} q_i(\ell)(1 - p(\tau_i(\ell))) = \liminf_{\ell \to \infty} e^{\tau \ell} \Omega q_0 (1 - p_0) > 0, \tag{36}$$

where $0 \le p_0 < 1$. Moreover,

$$\int_{\ell_0}^{\infty} \frac{1}{\kappa_2(u)} \int_u^{\infty} \sum_{i=1}^{\Omega} q_i(\rho) \mathrm{d}\rho \mathrm{d}u = \int_1^{\infty} e^u \int_u^{\infty} \Omega q_0 \mathrm{d}\rho \mathrm{d}u = \int_1^{\infty} \Omega q_0 (e^u - e) \mathrm{d}u = \infty.$$

Hence, if (36) holds, all assumptions of Theorem 3 are satisfied, and then every solution of (35) either converges to zero or is oscillatory.

Remark 4. *Consider the differential equation*

$$\left(y(\ell) + 0.5 y\left(\frac{\ell}{6} \right) \right)''' + \frac{q_0}{\ell^3} (x(0.5\ell) + x(0.6\ell) + x(0.7\ell)) = 0, \tag{37}$$

where $\Omega = 3$, $p_0 = 0.5$, $\sigma(\ell) = \ell/6$ and $\tau(\ell) = \min\{0.5\ell, 0.6\ell, 0.7\ell\} = 0.5\ell$. Then, every solution of (37) either converges to zero or is oscillatory.

$$q_0 > \frac{2}{3(0.5)^2 \left(1 - \frac{1}{2}\right)} = 5.3. \tag{38}$$

4. Conclusions

This study focuses on the oscillatory characteristics of solutions to the third-order neutral equation with several delays. Although there have been numerous studies related to this subject, we have discovered enough evidence in these studies to assert that any non-oscillatory solution will lead to zero. In this study, we introduce new standards which guarantee that all solutions to Equation (1) are oscillatory. Our results expand and improve upon those found in the literature [28]. For a certain type of general third-order delay differential equation, we propose new oscillation criteria in the event that the functions κ_i are of the same kind, using an iterative technique in Theorems 5 and 6. In a particular case, a single condition ensures that Equation (1) oscillates. It is noteworthy that our criteria are relevant even when $\tau(\ell) = \ell$, as they do not require $\tau(\ell)$ to be a non-decreasing function.

In future research endeavors within this particular domain, we are enthusiastic about the potential to expand our investigation to include quasi-linear third-order neutral differential equations of the form:

$$\left(\kappa_2(\ell) \left(\left(\kappa_1(\ell) z'(\ell) \right)^{\alpha} \right)' \right)' + \sum_{i=1}^{\Omega} q_i(\ell) y^{\alpha}(\tau_i(\ell)) = 0.$$

Author Contributions: Conceptualization N.O., O.M., G.A. and E.M.E.; Methodology, N.O., O.M., G.A. and E.M.E.; Investigation, N.O., O.M., G.A.; Writing– original draft, N.O.; Writing – review and editing, O.M., G.A. and E.M.E. All authors have read and agreed to the published version of the manuscript.

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