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Generalized χ and η Cross-Helicities in Non-Ideal Magnetohydrodynamics

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Abstract: We study the generalized χ and η cross-helicities for non-ideal non-barotropic magnetohydrodynamics (MHD). χ and η, the additional label translation symmetry group, are used to generalize cross-helicity in ideal flows. Both new helicities are additional topological invariants of ideal MHD. To study there behavior in non-ideal MHD, we calculate the time derivative of both helicities using non-ideal MHD equations in which viscosity, finite resistivity, and heat conduction are taken into account. Physical variables are divided into ideal and non-ideal quantities separately during the mathematical analysis for simplification. The analytical results indicate that χ and η cross-helicities are not strict constants of motion in non-ideal MHD and show a rate of dissipation that is comparable to the dissipation of other topological constants of motion.

Keywords: MHD; topological constants of motion; non-ideal flows

1. Introduction

Topological constants of motion are useful for different physical structures, and there are such constants in MHD. Most importantly, magnetic helicity [1–3] and cross-helicity have long been studied in relation to the controlled nuclear fusion problem and astrophysical scenarios. In the absence of dissipation, cross-helicity is conserved. Previous works [4–6] have concentrated on the deep relations between topological invariants and continuous symmetries of ideal MHD. MHD connects electromagnetism with fluid dynamics of very conductive flows to elucidate the dynamics of conducting fluids such as plasmas. Considering the fact that ideal MHD does not precisely describe real plasmas was the paramount motivation for the present work. Major natural processes are missing in the ideal depiction, which include heating due to finite electrical resistivity, conduction of heat, and heating due to friction and viscosity. Viscous processes are significant on the dissipation scale for plasma turbulence in solar wind and in other turbulent plasmas. Magnetic diffusivity (due to finite electrical conductivity) is responsible, among other things, for the magnetic reconnection phenomena. Thermal conductivity is also a significant process that one needs to understand in real plasmas. It affects the perturbations of physical variables causing them to spread through the fluid. These significant attributes of all three non-ideal processes are the stimulus for this current analysis.

Cross-helicity is defined as [7–9]:

$$H_C = \int \vec{B} \cdot \vec{v} \, d^3x, \tag{1}$$

in which the volume integral is calculated over the entire flow regime. Here, H_C is constant in time for barotropic or incompressible MHD (but it is not conserved for non-barotropic MHD) and can be given a topological meaning in terms of the knots between magnetic and flow field lines. This correlation is of great importance in the case of Alfvén waves.

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However, it has a relation with magnetosonic waves in the compressible case [10]. A generalization for non-barotropic MHD of this quantity was given by [11,12]. This is analogous to the generalization of barotropic fluid dynamics topological invariants such as helicity to non-barotropic flows as suggested by [13]. Generalized cross-helicity, which is invariant in non-barotropic MHD, was studied by [14] in multi-symplectic MHD. Potential vorticity conservation for non-barotropic MHD was suggested by the same authors [15].

More recently non-barotropic cross-helicity was further generalized using additional label translation continuous symmetry sub-groups (χ and η translations) [16], which resulted in additional topological invariants: the χ and η cross-helicities. The quantities χ and η are also denoted 'Euler potentials', 'Clebsch variables', and 'flux representation functions' [17].

The notion of metage as a label for fluid elements specifying there whereabouts along a vortex line in ideal fluids was introduced by Lynden-Bell and Katz [18]. A translation symmetry sub-group of this label was found to be connected to the conservation of Moffat's [19] helicity by Yahalom [6] using a Lagrangian variational principle and the theorem of Noether. The metage notion was generalized by Yahalom and Lynden-Bell [4] for barotropic MHD, but not for labeling flow elements along vortex lines but rather as a label for fluid elements along magnetic field lines, which are co-moving with the flow in the case of ideal MHD (vortex lines are not co-moving in MHD). Yahalom and Lynden-Bell [4] demonstrated that the translation symmetry of magnetic metage is connected to Woltjer's [7,8] conservation of cross-helicity for barotropic MHD. Later the concept of metage was generalized to non-barotropic MHD in which magnetic field lines lie on entropy surfaces [20]. This was further generalized by removing the entropy condition on magnetic field lines [21]. In those early papers, the metage translation symmetry group was used to obtain a Noether current for the generalized non-barotropic cross-helicity using a Lagrangian action.

Cross-helicities are expected to play a significant role in MHD related to global magnetic-field generation, turbulence suppression, etc. It provides a linkage measure of the vortex tubes of the velocity field with the flux tubes of the magnetic field. Cross-helicity plays a major role with respect to the operation of turbulent dynamo [22]. Cross-helicity density conservation for barotropic MHD turbulence theory is significant [23–25]. Plasma velocity and magnetic field empirical data, which were recorded in the Voyager 2 mission, is important for the study of solar wind turbulence in low velocity solar wind [26] and can be used to characterize its cross-helicity. Verma [27] studied MHD turbulence in detail. He has closely examined the Alfvénic MHD turbulence of zero and non-zero helicities. Energy fluxes of MHD turbulence measure transfers of energy between velocity and magnetic fields [28,29].

Magnetic helicity is also one of the important topological invariants in fluid dynamics. Faraco and Lindberg [30] demonstrated the conservation of magnetic helicity in turbulent flows. However, flux tubes diffuse through one another on resistive time durations; thus, eventually, magnetic helicity dissipates [31]. Candelaresi and Del Sordo [32] studied the role of magnetic helicity in plasmas stabilization by performing a series of experiments and numerical simulations. Further, magnetic helicity's significant role in determining the structures, dynamics, and heating of the solar corona was studied by Knizhnik et al. [33].

The χ and η cross-helicities derive their name from the label translation symmetry sub-group to which their associated Noether currents are connected. In addition, they are expected to be as important as their predecessors: the magnetic and generalized cross-helicities, for the stability and dynamics of MHD [16]. Thus, it is of paramount importance to study the effects of non-ideal processes on their development, which is the major innovation of this paper. Here, we would like to mention a recent work in which we studied non-ideal processes and their effect on the conservation of magnetic and generalized cross-helicities [34], although, of course, there is no overlap with the current work as χ and η cross-helicities are completely different quantities.

There are three sections in the current paper. Section 2 describes the basic quantities and equations in non-ideal MHD. The calculations for the time derivative of χ helicity are outlined in Section 3, and Section 4 deals with the mathematics of η helicity.

2. Standard Formulation of Non-Ideal Non-Barotropic MHD

The standard set of equations solved for non-ideal non-barotropic MHD is given below (here, we use the EMU system of units):

$$\frac{\partial \vec{v}_n}{\partial t} = -(\vec{v}_n \cdot \vec{\nabla})\vec{v}_n - \frac{\vec{\nabla} p_n}{\rho_n} + \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} - \vec{\nabla}\phi + \frac{1}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k}, \tag{2}$$

$$\frac{\partial \rho_n}{\partial t} + \vec{\nabla} \cdot (\rho_n \vec{v}_n) = 0, \tag{3}$$

$$\vec{\nabla} \cdot \vec{B}_n = 0, \tag{4}$$

$$\frac{\partial \vec{B}_n}{\partial t} = \vec{\nabla} \times (\vec{v}_n \times \vec{B}_n) + \frac{\eta_v}{4\pi}\nabla^2 \vec{B}_n, \tag{5}$$

$$\rho_n T_n \frac{d s_n}{dt} = \sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \eta_v J_n^2 + \vec{\nabla} \cdot (k \vec{\nabla} T_n), \tag{6}$$

where $\frac{\partial}{\partial t}$ is the partial temporal derivative; $\vec{\nabla}$ takes its standard meaning in vector calculus; we use the sub-script $_n$ to describe non-ideal processes. Thus, $\vec{v}$ is the ideal velocity and $\vec{v}_n$ is the velocity for a non-ideal fluid, etc.; ρ_n is the density and p_n is the pressure, which, through the equation of state, depends on the density and entropy s_n (the non-barotropic case); T_n is the temperature and ϕ is a potential.The stress tensor is defined as:

$$\sigma'_{ik} = \mu_v\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_{nl}}{\partial x_l}\right), \tag{7}$$

where μ_v is a coefficient of kinematic viscosity (not to be confused with the metage μ defined in [16]). Notice that we take the coefficient of second viscosity (or volume viscosity) to be zero for the sake of simplicity. The entropy Equation (6) depends on the heat conduction coefficient k. According to classical kinetic theory, viscosity arises from collisions between particles. The justification for those equations and the conditions under which they apply can be found in standard books on MHD (see, for example, [9,35–37]).

The current density $\vec{J}_n$ and the magnetic field $\vec{b}_n$ of MHD are related by Ampere's law:

$$\vec{\nabla} \times \vec{B}_n = 4\pi \vec{J}_n, \tag{8}$$

where the displacement current is neglected. Equation (5) depends on the non-ideal magnetic diffusivity η_v (not to be confused with the label η introduced in [16]). In the limit of ideal flows, all the non-ideal coefficients μ_v, η_v, k tend to zero and the ideal MHD equations are recovered. Moreover, we prime the difference between ideal and non-ideal quantities, for example, $\vec{v'} = \vec{v}_n - \vec{v}$. It is clear that in the ideal limit, all primed quantities tend to zero, for example, $\lim_{(\mu_v,\eta_v,k)\to 0} \vec{v'} = 0$.

3. Direct Derivation of the Constancy of Non-Barotropic χ Cross-Helicity

We introduce the abstract 'magnetic fields' as follows [16]:

$$\vec{B}_\chi = \vec{\nabla}\mu \times \vec{\nabla}\eta, \tag{9}$$

Non-barotropic χ cross-helicity is given by:

$$H_{CNB\chi} = \int \vec{v_{nt}} \cdot \vec{B}_\chi d^3x, \tag{10}$$

where the topological non-ideal velocity field is defined as $\vec{v_{nt}} = \vec{v_n} - \sigma_n \vec{\nabla} s_n$ [16], and σ_n is an auxiliary variable, which depends on the Lagrangian time integral of the temperature, i.e.,

$$\frac{d\sigma_n}{dt} = T_n. \tag{11}$$

Please refer to [20] for a detailed justification for the relation of non-barotropic cross-helicity.

Taking the temporal derivative of the non-barotropic χ cross-helicity:

$$\frac{dH_{CNB\chi}}{dt} = \int d^3x\, (\vec{v}_{nt} \cdot \frac{\partial \vec{B}_\chi}{\partial t} + \vec{B}_\chi \cdot \frac{\partial \vec{v}_{nt}}{\partial t}), \tag{12}$$

Now, after identifying the value of the first term of RHS, we take the time derivative of Equation (9):

$$\frac{\partial \vec{B}_\chi}{\partial t} = \vec{\nabla}(\frac{\partial \mu}{\partial t}) \times \vec{\nabla}\eta + \vec{\nabla}\mu \times \vec{\nabla}(\frac{\partial \eta}{\partial t}), \tag{13}$$

Notice that both the labels are co-moving and conserved under an ideal material derivative [16]; thus,

$$\frac{\partial \mu}{\partial t} + (\vec{v} \cdot \vec{\nabla})\mu = 0. \tag{14}$$

Similarly,

$$\frac{\partial \eta}{\partial t} + (\vec{v} \cdot \vec{\nabla})\eta = 0. \tag{15}$$

Therefore,

$$\frac{\partial \vec{B}_\chi}{\partial t} = \vec{\nabla}[(-\vec{v} \cdot \vec{\nabla}\mu)] \times \vec{\nabla}\eta + \vec{\nabla}\mu \times \vec{\nabla}[(-\vec{v} \cdot \vec{\nabla}\eta)], \tag{16}$$

Using the vector identity:

$$\vec{\nabla} \times (\psi\vec{a}) = \vec{\nabla}\psi \times \vec{a} + \psi\vec{\nabla} \times \vec{a}, \tag{17}$$

Equation (16) takes the form:

$$\frac{\partial \vec{B}_\chi}{\partial t} = [\vec{\nabla} \times \{\vec{\nabla}\mu(\vec{v} \cdot \vec{\nabla}\eta) - \vec{\nabla}\eta(\vec{v} \cdot \vec{\nabla}\mu)\}], \tag{18}$$

Now, with the help of the identity:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}), \tag{19}$$

we obtain:

$$\frac{\partial \vec{B}_\chi}{\partial t} = [\vec{\nabla} \times \{\vec{v} \times (\vec{\nabla}\mu \times \vec{\nabla}\eta)\}]. \tag{20}$$

Substituting $\vec{B}_\chi$ defined in Equation (9), we obtain:

$$\frac{\partial \vec{B}_\chi}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}_\chi). \tag{21}$$

Next, we add $\vec{v}_{nt}$ to both sides:

$$\vec{v}_{nt} \cdot \frac{\partial \vec{B}_\chi}{\partial t} = \vec{v}_{nt} \cdot \vec{\nabla} \times (\vec{v} \times \vec{B}_\chi), \tag{22}$$

and using a well-known vector identity, we obtain:

$$\vec{v}_{nt} \cdot \frac{\partial \vec{B}_\chi}{\partial t} = \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\chi) \times \vec{v}_{nt}\} + (\vec{v} \times \vec{B}_\chi) \cdot \vec{\omega}_{nt}, \tag{23}$$

where we define the topological vorticity of the non-ideal flow field as:

$$\vec{\omega}_{nt} \equiv \vec{\nabla} \times \vec{v}_{nt}. \tag{24}$$

Next, we calculate the second term on the RHS of Equation (12):

$$\partial_t \vec{v}_{nt} \cdot \vec{B}_\chi = \vec{B}_\chi \cdot \partial_t(\vec{v}_n - \sigma_n \vec{\nabla} s_n) = \vec{B}_\chi \cdot (\partial_t \vec{v}_n - \partial_t \sigma_n \vec{\nabla} s_n - \sigma_n \vec{\nabla} \partial_t s_n). \tag{25}$$

Now, we simplify the right hand side of Equation (25) in three steps. The first term is calculated with the help of Equation (2):

$$\frac{\partial \vec{v}_n}{\partial\, t} = (\vec{v}_n \times \vec{\omega}_n) + \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} - \vec{\nabla}(\frac{v_n^2}{2}) - \vec{\nabla} w_n + T_n \vec{\nabla} s_n - \vec{\nabla}\phi + \frac{1}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k}, \tag{26}$$

in which the non-ideal vorticity is:

$$\vec{\omega}_n \equiv \vec{\nabla} \times \vec{v}_n, \tag{27}$$

and we use the thermo-dynamical identity:

$$dw_n = d\varepsilon_n + d(\frac{p_n}{\rho_n}) = T_n ds_n + \frac{1}{\rho_n} dp_n \Rightarrow \vec{\nabla} w_n = T_n \vec{\nabla} s_n + \frac{1}{\rho_n}\vec{\nabla} p_n. \tag{28}$$

Thus,

$$\vec{B}_\chi \cdot \frac{\partial \vec{v}_n}{\partial\, t} = \vec{B}_\chi \cdot \left[(\vec{v}_n \times \vec{\omega}_n) - \vec{\nabla}(\frac{v_n^2}{2} + w_n) + T_n \vec{\nabla} s_n - \vec{\nabla}\phi\right] + \frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} + \vec{B}_\chi \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n}. \tag{29}$$

In the second term, we use Equation (11) to obtain:

$$-\frac{\partial \sigma_n}{\partial t}\vec{\nabla} s_n = (\vec{v}_n \cdot \vec{\nabla}\sigma_n - T_n)\vec{\nabla} s_n. \tag{30}$$

In the third term, we use the equation for the rate of change of entropy in non-ideal MHD, also known as the heat equation (Equation (6)):

$$-\sigma_n \vec{\nabla}\frac{\partial s_n}{\partial t} = \sigma_n \vec{\nabla}[\vec{v}_n \cdot \vec{\nabla} s_n - \frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} - \frac{\eta_v}{\rho_n T_n} J_n^2 - \frac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla} T_n)]. \tag{31}$$

Combining Equations (29)–(31), we obtain:

$$\begin{aligned}
\vec{B}_\chi \cdot \tfrac{\partial \vec{v}_{nt}}{\partial\, t} &= \vec{B}_\chi \cdot [(\vec{v}_n \times \vec{\omega}_{nt}) + \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \tfrac{v_n^2}{2} - w_n - \phi\} \\
&- \sigma_n \vec{\nabla}\{\tfrac{1}{\rho_n T_n}\sigma'_{ik}\tfrac{\partial v_{ni}}{\partial x_k} + \tfrac{\eta_v}{\rho_n T_n} J_n^2 + \tfrac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla} T_n)\}] \\
&+ \tfrac{B_{\chi i}}{\rho_n}\tfrac{\partial \sigma'_{ik}}{\partial x_k} + \vec{B}_\chi \cdot \tfrac{\vec{J}_n \times \vec{B}_n}{\rho_n},
\end{aligned} \tag{32}$$

in which the current density is given by

$$\vec{J}_n = \frac{\vec{\nabla} \times \vec{B}_n}{4\pi} \Rightarrow \vec{\nabla} \cdot \vec{J}_n = 0. \tag{33}$$

Now,

$$\vec{B}_\chi \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} = \frac{1}{\rho_n}\vec{J}_n \cdot \vec{B}_n \times \vec{B}_\chi, \tag{34}$$

$$\vec{B}_n \times \vec{B}_\chi = \vec{B}_n \times (\vec{\nabla}\mu \times \vec{\nabla}\eta) = \vec{\nabla}\mu(\vec{B}_n \cdot \vec{\nabla}\eta) - \vec{\nabla}\eta(\vec{B}_n \cdot \vec{\nabla}\mu), \tag{35}$$

$$\vec{B}_n \times \vec{B}_\chi = \vec{\nabla}\mu(\vec{B} \cdot \vec{\nabla}\eta) + \vec{\nabla}\mu(\vec{B}' \cdot \vec{\nabla}\eta) - \vec{\nabla}\eta(\vec{B} \cdot \vec{\nabla}\mu) - \vec{\nabla}\eta(\vec{B}' \cdot \vec{\nabla}\mu), \tag{36}$$

however, using Equations (15) and (17) of [16]: $\vec{B} \cdot \vec{\nabla}\eta = 0$ and $\vec{B} \cdot \vec{\nabla}\mu = \rho$. And thus:

$$\vec{B}_n \times \vec{B}_\chi = \vec{\nabla}\mu(\vec{B}' \cdot \vec{\nabla}\eta) - \vec{\nabla}\eta(\vec{B}' \cdot \vec{\nabla}\mu) - \rho\vec{\nabla}\eta, \tag{37}$$

Therefore,

$$\vec{B}_n \times \vec{B}_\chi = \vec{B}' \times (\vec{\nabla}\mu \times \vec{\nabla}\eta) - \rho\vec{\nabla}\eta = \vec{B}' \times \vec{B}_\chi - \rho\vec{\nabla}\eta, \tag{38}$$

It thus follows that:

$$\vec{B}_\chi \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} = -\frac{\rho}{\rho_n}(\vec{J}_n \cdot \vec{\nabla}\eta) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\chi)}{\rho_n} = -\frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n\eta) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\chi)}{\rho_n}. \tag{39}$$

Now, Equation (32) can be written as:

$$\begin{aligned} & \vec{B}_\chi \cdot \tfrac{\partial \vec{v}_{nt}}{\partial t} = \vec{B}_\chi \cdot \Big[(\vec{v}_n \times \vec{\omega}_{nt}) + \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla}s_n) - \tfrac{v_n^2}{2} - w_n - \phi\} \\ - \quad & \sigma_n\vec{\nabla}\{\tfrac{1}{\rho_n T_n}\sigma'_{ik}\tfrac{\partial v_{ni}}{\partial x_k} + \tfrac{\eta_v}{\rho_n T_n}J_n^2 + \tfrac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla}T_n)\}\Big] + \tfrac{B_{\chi i}}{\rho_n}\tfrac{\partial \sigma'_{ik}}{\partial x_k} \\ - \quad & \tfrac{\rho}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n\eta) + \tfrac{\vec{J}_n\cdot(\vec{B}'\times\vec{B}_\chi)}{\rho_n}. \end{aligned} \tag{40}$$

Combining Equations (23) and (40) and taking into account that:

$$\vec{B}_\chi \cdot (\vec{v}_n \times \vec{\omega}_t) = -(\vec{v}_n \times \vec{B}_\chi) \cdot \vec{\omega}_t, \tag{41}$$

We obtain:

$$\begin{aligned} & \vec{v}_{nt} \cdot \tfrac{\partial \vec{B}_\chi}{\partial t} + \vec{B}_\chi \cdot \tfrac{\partial \vec{v}_{nt}}{\partial t} = \\ & \vec{\nabla}\cdot\{(\vec{v}\times\vec{B}_\chi)\times\vec{v}_{nt}\} + \vec{B}_\chi\cdot\vec{\nabla}\{\sigma_n(\vec{v}_n\cdot\vec{\nabla}s_n) - \tfrac{v_n^2}{2} - w_n - \phi\} \\ + \quad & \tfrac{B_{\chi i}}{\rho_n}\tfrac{\partial\sigma'_{ik}}{\partial x_k} - \vec{B}_\chi\cdot[\sigma_n\vec{\nabla}\{\tfrac{1}{\rho_n T_n}\sigma'_{ik}\tfrac{\partial v_{ni}}{\partial x_k} + \tfrac{\eta_v}{\rho_n T_n}J_n^2 + \tfrac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla}T_n)\}] \\ - \quad & \tfrac{\rho}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n\eta) + \tfrac{\vec{J}_n\cdot(\vec{B}'\times\vec{B}_\chi)}{\rho_n} + \vec{\omega}_{nt}\cdot(\vec{B}_\chi\times\vec{v}'). \end{aligned} \tag{42}$$

Now, substituting Equation (42) into Equation (12), we obtain:

$$\begin{aligned} & \tfrac{dH_{CNB\chi}}{dt} = \int \vec{\nabla}\cdot\{(\vec{v}\times\vec{B}_\chi)\times\vec{v}_{nt}\} + \vec{B}_\chi\cdot\vec{\nabla}\{\sigma_n(\vec{v}_n\cdot\vec{\nabla}s_n) - \tfrac{v_n^2}{2} - w_n - \phi\}d^3x \\ + \quad & \int[\tfrac{B_{\chi i}}{\rho_n}\tfrac{\partial\sigma'_{ik}}{\partial x_k} - \vec{B}_\chi\cdot[\sigma_n\vec{\nabla}\{\tfrac{1}{\rho_n T_n}\sigma'_{ik}\tfrac{\partial v_{ni}}{\partial x_k} + \tfrac{\eta_v}{\rho_n T_n}J_n^2 + \tfrac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla}T_n)\}]]d^3x \\ - \quad & \int[\tfrac{\rho}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n\eta) - \tfrac{\vec{J}_n\cdot(\vec{B}'\times\vec{B}_\chi)}{\rho_n} - \vec{\omega}_{nt}\cdot(\vec{B}_\chi\times\vec{v}')]d^3x. \end{aligned} \tag{43}$$

Using Gauss's divergence theorem, we can write part of this integral as a surface integral:

$$\begin{aligned} & \tfrac{dH_{CNB\chi}}{dt} = \oint[(\vec{v}\times\vec{B}_\chi)\times\vec{v}_{nt} + \vec{B}_\chi\{\sigma_n(\vec{v}_n\cdot\vec{\nabla}s_n) - \tfrac{v_n^2}{2} - w_n - \phi\} - \vec{J}_n\eta]\cdot d\vec{S} \\ + \quad & \int[\tfrac{B_{\chi i}}{\rho_n}\tfrac{\partial\sigma'_{ik}}{\partial x_k} - \vec{B}_\chi\cdot[\sigma_n\vec{\nabla}\{\tfrac{1}{\rho_n T_n}\sigma'_{ik}\tfrac{\partial v_{ni}}{\partial x_k} + \tfrac{\eta_v}{\rho_n T_n}J_n^2 + \tfrac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla}T_n)\}]]d^3x \\ + \quad & \int[\tfrac{\rho'}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n\eta) + \tfrac{\vec{J}_n\cdot(\vec{B}'\times\vec{B}_\chi)}{\rho_n} + \vec{\omega}_{nt}\cdot(\vec{B}_\chi\times\vec{v}')]d^3x. \end{aligned} \tag{44}$$

Here, the surface integral encapsulates the volume for which the χ non-barotropic cross-helicity is calculated and a cut in the case that η is multiple-valued. If the surface integral vanishes:

$$\frac{dH_{CNB\chi}}{dt} = \int [\frac{B_{\chi i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\chi \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla}\cdot(k\vec{\nabla}T_n)\}]]d^3x$$
$$+ \int [\frac{\rho'}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n\eta) + \frac{\vec{J}_n\cdot(\vec{B}'\times\vec{B}_\chi)}{\rho_n} + \vec{\omega}_{nt}\cdot(\vec{B}_\chi \times \vec{v}')]d^3x. \tag{45}$$

If viscosity and heat conduction are less important, we can write a reduced expression:

$$\begin{aligned} \frac{dH_{CNB\chi}}{dt} &= -\int \sigma_n \vec{B}_\chi \cdot \vec{\nabla}[\frac{\eta_v J_n^2}{\rho_n T_n}]d^3x \\ &+ \int [\frac{\rho'}{\rho_n}\vec{\nabla}\cdot(\vec{J}_n\eta) + \frac{\vec{J}_n\cdot(\vec{B}'\times\vec{B}_\chi)}{\rho_n} + \vec{\omega}_{nt}\cdot(\vec{B}_\chi\times\vec{v}')]d^3x. \end{aligned} \tag{46}$$

To conclude, we notice that is not possible to partition the time derivative of the non-barotropic χ cross-helicity in accordance with the different non-ideal processes: viscosity, finite conductivity, and heat conductivity. Each of these processes may contribute to the primed quantities directly or indirectly. In the limit that all non-ideal coefficients tend to zero, it is easy to see that the non-barotropic χ cross-helicity is indeed conserved:

$$\frac{dH_{CNB\chi}}{dt} = 0. \tag{47}$$

This is as expected for an ideal flow.

4. Direct Derivation of the Constancy of Non-Barotropic η Cross-Helicity

We introduce the abstract 'magnetic field' as follows [16]:

$$\vec{B}_\eta = \vec{\nabla}\chi \times \vec{\nabla}\mu. \tag{48}$$

Non-barotropic η cross-helicity is given by:

$$H_{CNB\eta} = \int \vec{v_{nt}} \cdot \vec{B}_\eta d^3x. \tag{49}$$

Taking the temporal derivative of the non-barotropic η cross-helicity

$$\frac{dH_{CNB\eta}}{dt} = \int d^3x\,(\vec{v}_{nt}\cdot\frac{\partial \vec{B}_\eta}{\partial t} + \vec{B}_\eta \cdot \frac{\partial \vec{v}_{nt}}{\partial t}), \tag{50}$$

$$\frac{\partial \vec{B}_\eta}{\partial t} = \vec{\nabla}(\frac{\partial \chi}{\partial t}) \times \vec{\nabla}\mu + \vec{\nabla}\chi \times \vec{\nabla}(\frac{\partial \mu}{\partial t}). \tag{51}$$

Again, both the labels are co-moving and conserved under an ideal material derivative [16]:

$$\frac{\partial \mu}{\partial t} + (\vec{v}\cdot\vec{\nabla})\mu = 0, \qquad \frac{\partial \chi}{\partial t} + (\vec{v}\cdot\vec{\nabla})\chi = 0. \tag{52}$$

Therefore,

$$\frac{\partial \vec{B}_\eta}{\partial t} = \vec{\nabla}[(-\vec{v}\cdot\vec{\nabla}\chi)] \times \vec{\nabla}\mu + \vec{\nabla}\chi \times \vec{\nabla}[(-\vec{v}\cdot\vec{\nabla}\mu)]. \tag{53}$$

Using the vector identity:

$$\vec{\nabla}\times(\psi\vec{\nabla}\mu) = \vec{\nabla}\psi \times \vec{\nabla}\mu, \tag{54}$$

Equation (53) takes the form.

$$\frac{\partial \vec{B}_\eta}{\partial t} = \vec{\nabla}\times\{\vec{\nabla}\chi(\vec{v}\cdot\vec{\nabla}\mu) - \vec{\nabla}\mu(\vec{v}\cdot\vec{\nabla}\chi)\}. \tag{55}$$

Now, with the help of the identity:

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}), \tag{56}$$

we obtain:

$$\frac{\partial\vec{B}_\eta}{\partial t}=\vec{\nabla}\times\{\vec{v}\times(\vec{\nabla}\chi\times\vec{\nabla}\mu)\}=\vec{\nabla}\times(\vec{v}\times\vec{B}_\eta). \tag{57}$$

Thus,

$$\vec{v}_{nt}\cdot\frac{\partial\vec{B}_\eta}{\partial t}=\vec{v}_{nt}\cdot[\vec{\nabla}\times(\vec{v}\times\vec{B}_\eta)]=\vec{\nabla}\cdot\{(\vec{v}\times\ \vec{B}_\eta)\times\vec{v}_{nt}\}+(\vec{v}\times\ \vec{B}_\eta)\cdot\vec{\omega}_{nt}. \tag{58}$$

Next, we calculate the second term:

$$\partial_t\vec{v}_{nt}\cdot\vec{B}_\eta=\vec{B}_\eta\cdot\partial_t(\vec{v}_n-\sigma_n\vec{\nabla}s_n)=\vec{B}_\eta\cdot(\partial_t\vec{v}_n-\partial_t\sigma_n\vec{\nabla}s_n-\sigma_n\vec{\nabla}\partial_t s_n). \tag{59}$$

Now, we simplify the right hand side of Equation (59) in three steps. The first term is calculated with the help of momentum Equation (26) for the non-ideal case. We multiply both sides of Equation (26) by $\vec{B}_\eta$

$$\vec{B}_\eta\cdot\frac{\partial\vec{v}_n}{\partial\ t}=\vec{B}_\eta\cdot\left[(\vec{v}_n\times\vec{\omega}_n)+\frac{\vec{J}_n\times\ \vec{B}_n}{\rho_n}-\vec{\nabla}(\frac{v_n^2}{2})-\vec{\nabla}w_n+T_n\vec{\nabla}s_n-\vec{\nabla}\phi+\frac{1}{\rho_n}\frac{\partial\sigma'_{ik}}{\partial x_k}\right], \tag{60}$$

In addition, arranging the terms on the right hand side of the equation, we thus obtain:

$$\vec{B}_\eta\cdot\frac{\partial\vec{v}_n}{\partial\ t}=\vec{B}_\eta\cdot\left[(\vec{v}_n\times\vec{\omega}_n)-\vec{\nabla}(\frac{v_n^2}{2}+w_n)+T_n\vec{\nabla}s_n-\vec{\nabla}\phi\right]+\frac{B_{\eta i}}{\rho_n}\frac{\partial\sigma'_{ik}}{\partial x_k}+\vec{B}_\eta\cdot\frac{\vec{J}_n\times\vec{B}_n}{\rho_n}. \tag{61}$$

For the second and third terms, we use Equations (30) and (31) to obtain:

$$\begin{aligned}\vec{B}_\eta\cdot\tfrac{\partial\vec{v}_{nt}}{\partial\ t} &= \vec{B}_\eta\cdot[(\vec{v}_n\times\vec{\omega}_{nt})+\vec{\nabla}\{\sigma_n(\vec{v}_n\cdot\vec{\nabla}s_n)-\tfrac{v_n^2}{2}-w_n-\phi\}\\ &- \sigma_n\vec{\nabla}\{\tfrac{1}{\rho_nT_n}\sigma'_{ik}\tfrac{\partial v_{ni}}{\partial x_k}+\tfrac{\eta_v}{\rho_nT_n}J_n^2+\tfrac{1}{\rho_nT_n}\vec{\nabla}\cdot(k\vec{\nabla}T_n)\}]\\ &+ \tfrac{B_{\eta i}}{\rho_n}\tfrac{\partial\sigma'_{ik}}{\partial x_k}+\vec{B}_\eta\cdot\tfrac{\vec{J}_n\times\vec{B}_n}{\rho_n}.\end{aligned} \tag{62}$$

Now,

$$\vec{B}_\eta\cdot\frac{\vec{J}_n\times\vec{B}_n}{\rho_n}=\frac{1}{\rho_n}\vec{J}_n\cdot\vec{B}_n\times\vec{B}_\eta. \tag{63}$$

And we have:

$$\begin{aligned}\vec{B}_n\times\vec{B}_\eta &= \vec{B}_n\times(\vec{\nabla}\chi\times\vec{\nabla}\mu)=\vec{\nabla}\chi(\vec{B}_n\cdot\vec{\nabla}\mu)-\vec{\nabla}\mu(\vec{B}_n\cdot\vec{\nabla}\chi)\\ &= \vec{\nabla}\chi(\vec{B}\cdot\vec{\nabla}\mu)+\vec{\nabla}\chi(\vec{B}'\cdot\vec{\nabla}\mu)-\vec{\nabla}\mu(\vec{B}\cdot\vec{\nabla}\chi)-\vec{\nabla}\mu(\vec{B}'\cdot\vec{\nabla}\chi),\end{aligned} \tag{64}$$

However, using Equations (15) and (17) of [16], which dictate:

$$\vec{B}\cdot\vec{\nabla}\chi=0,\qquad \vec{B}\cdot\vec{\nabla}\mu=\rho, \tag{65}$$

it follows that:

$$\begin{aligned}\vec{B}_n\times\vec{B}_\eta &= \vec{\nabla}\chi(\vec{B}'\cdot\vec{\nabla}\mu)-\vec{\nabla}\mu(\vec{B}'\cdot\vec{\nabla}\chi)+\rho\vec{\nabla}\chi=\vec{B}'\times(\vec{\nabla}\chi\times\vec{\nabla}\mu)+\rho\vec{\nabla}\chi\\ &= \vec{B}'\times\vec{B}_\eta+\rho\vec{\nabla}\chi.\end{aligned} \tag{66}$$

Inserting Equations (66) into (63) leads to the following identity:

$$\vec{B}_\eta \cdot \frac{\vec{J}_n \times \vec{B}_n}{\rho_n} = \frac{\rho}{\rho_n}(\vec{J}_n \cdot \vec{\nabla}\chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} = \frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n}. \tag{67}$$

Therefore, Equation (62) becomes:

$$\begin{aligned}\vec{B}_\eta \cdot \frac{\partial \vec{v}_{nt}}{\partial t} &= \vec{B}_\eta \cdot [(\vec{v}_n \times \vec{\omega}_{nt}) + \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} \\ &- \sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla} T_n)\}] \\ &+ \frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} + \frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n}.\end{aligned} \tag{68}$$

Combining Equations (58) and (68), we obtain:

$$\begin{aligned}& \vec{v}_{nt} \cdot \frac{\partial \vec{B}_\eta}{\partial t} + \vec{B}_\eta \cdot \frac{\partial \vec{v}_{nt}}{\partial t} = \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\eta) \times \vec{v}_{nt}\} \\ + \; & \vec{B}_\eta \cdot \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} \\ + \; & \frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\eta \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla} T_n)\}] \\ + \; & \frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\eta \times \vec{v}').\end{aligned} \tag{69}$$

Now, substituting Equation (69) into Equation (50), we obtain:

$$\begin{aligned}& \frac{dH_{CNB\eta}}{dt} \\ = \; & \int \vec{\nabla} \cdot \{(\vec{v} \times \vec{B}_\eta) \times \vec{v}_{nt}\} + \vec{B}_\eta \cdot \vec{\nabla}\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} d^3x \\ + \; & \int [\frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\eta \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla} T_n)\}]] d^3x \\ + \; & \int [\frac{\rho}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\eta \times \vec{v}')] d^3x.\end{aligned} \tag{70}$$

Using Gauss' divergence theorem, we obtain:

$$\begin{aligned}& \frac{dH_{CNB\eta}}{dt} \\ = \; & \oint [(\vec{v} \times \vec{B}_\eta) \times \vec{v}_{nt} + \vec{B}_\eta\{\sigma_n(\vec{v}_n \cdot \vec{\nabla} s_n) - \frac{v_n^2}{2} - w_n - \phi\} + \vec{J}_n \chi] \cdot d\vec{S} \\ + \; & \int [\frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\eta \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla} T_n)\}]] d^3x \\ + \; & \int [-\frac{\rho'}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\eta \times \vec{v}')] d^3x.\end{aligned} \tag{71}$$

Here, the surface integral encapsulates the volume for which the χ non-barotropic cross-helicity is calculated (χ is usually single-valued so no cut is required). If the surface integral vanishes:

$$\begin{aligned}& \frac{dH_{CNB\eta}}{dt} \\ = \; & \int [\frac{B_{\eta i}}{\rho_n}\frac{\partial \sigma'_{ik}}{\partial x_k} - \vec{B}_\eta \cdot [\sigma_n \vec{\nabla}\{\frac{1}{\rho_n T_n}\sigma'_{ik}\frac{\partial v_{ni}}{\partial x_k} + \frac{\eta_v}{\rho_n T_n}J_n^2 + \frac{1}{\rho_n T_n}\vec{\nabla} \cdot (k\vec{\nabla} T_n)\}]] d^3x \\ + \; & \int [-\frac{\rho'}{\rho_n}\vec{\nabla} \cdot (\vec{J}_n \chi) + \frac{\vec{J}_n \cdot (\vec{B}' \times \vec{B}_\eta)}{\rho_n} + \vec{\omega}_{nt} \cdot (\vec{B}_\eta \times \vec{v}')] d^3x.\end{aligned} \tag{72}$$

To conclude, we notice that is not possible to partition the time derivative of the non-barotropic η cross-helicity in accordance with the different non-ideal processes: viscosity, finite conductivity, and heat conductivity. Each of these processes may contribute to the primed quantities directly or indirectly. In the limit that all non-ideal coefficients tend to zero, it is easy to see that the non-barotropic η cross-helicity is indeed conserved:

$$\frac{dH_{CNB\eta}}{dt} = 0. \quad (73)$$

This is as expected for an ideal flow.

5. Conclusions

In this paper, we study the constancy of two new topological invariants: the generalized χ and η cross-helicities, when fluid flow is non-ideal, but the non-ideal processes are less significant. We show that the helicities are not conserved and their time derivatives depend on dissipative processes present in the system. Non-zero cross-helicity has important consequences for transport and [38] exhibits a close relation with momentum transfer and cross-helicity. Ref. [39] discussed its significance in MHD turbulence using high-resolution direct numerical simulations. Cross-helicity resembles the correlation between velocity and magnetic field fluctuation and measures the importance of Alfvén waves in global fluctuation. In addition, [40] predicted the generation of turbulence in interstellar media caused by non-linear interactions among shear Alfvén waves. Many studies have shown that cross-helicity is correlated to turbulence self-production.

Cascade processes in MHD turbulence were studied by [41] in detail and it was concluded that "cross helicity blocks the spectral energy transfer in MHD turbulence and results in energy accumulation in the system. This accumulation proceeds until the vortex intensification compensates the decreasing efficiency of nonlinear interactions". Ref. [42] studied the relation between the magnitude of cross-helicity and the rate of decay of isotropic MHD turbulence and showed that an initial non-zero cross-helicity causes imbalanced MHD turbulence.

The coupled action of helical particle flow (the kinetic α effect) and differential rotation is a significant dynamo cause in solar and galactic magnetic fields [43]. The dynamics of magnetic helicity on the small scale is important for non-linear dynamo saturation in which turbulent magnetic helicity fluxes cause a lack of catastrophic quenching due to the α effect. Convective zones in galactic discs, the Sun, and stars that resemble the Sun generate turbulent magnetic helicity fluxes. Using the mean-field method and the τ approximation, Kleeorin and Rogachevskii [43] derived turbulent magnetic helicity fluxes with the help of the Coulomb gauge for the case of density-stratified turbulence. Turbulent magnetic helicity fluxes are composed non-gradient and gradient parts. Non-gradient magnetic helicity flux is proportional to a non-linear effective velocity (which is null if no density stratification exists) multiplied by small-scale magnetic helicity. Gradient contributions depict turbulent magnetic diffusion of the small-scale magnetic helicity. In addition, the turbulent magnetic helicity fluxes contain source terms proportional to the kinetic α effect or its gradients, and also contributions caused by the large-scale shear (solar differential rotation). Kleeorin and Rogachevskii [43] demonstrated that the turbulent magnetic helicity fluxes due to the kinetic α effect and its radial derivative in combination with the non-linear magnetic diffusion of the small-scale magnetic helicity are dominant in the solar convective zone. It is, thus, plausible that the χ and η cross-helicities may also play an important role in regions where convection is more important than non-ideal processes, such as friction, magnetic diffusion, and heat conduction.

The technological (fusion) and astrophysical importance of cross-helicity suggest that the newly discovered χ and η cross-helicities may also play a pivotal rule. The reason for this is two-fold. First, as we have shown above, all processes that change the quantities are slow dissipative processes; thus, any fast processes conserve the χ and η cross-helicities.

Second, given that those quantities are approximately constant, they serve as lower bounds to other quantities such as "energy":

$$|H_{CNB\chi}| = \left| \int \vec{B}_\chi \cdot \vec{v}_t d^3x \right| \leq \frac{1}{2} \int \left(\vec{B}_\chi{}^2 + \vec{v}_t{}^2 \right) d^3x, \tag{74}$$

$$|H_{CNB\chi}| = \left| \int \vec{B}_\chi \cdot \vec{v}_t d^3x \right| \leq \sqrt{\int \vec{v}_t{}^2 d^3x} \sqrt{\int \vec{B}_\chi{}^2 d^3x}, \tag{75}$$

$$|H_{CNB\eta}| = \left| \int \vec{B}_\eta \cdot \vec{v}_t d^3x \right| \leq \frac{1}{2} \int \left(\vec{B}_\eta{}^2 + \vec{v}_t{}^2 \right) d^3x, \tag{76}$$

$$|H_{CNB\eta}| = \left| \int \vec{B}_\eta \cdot \vec{v}_t d^3x \right| \leq \sqrt{\int \vec{v}_t{}^2 d^3x} \sqrt{\int \vec{B}_\eta{}^2 d^3x}, \tag{77}$$

and, thus, may prevent at least part of the fastest and most dangerous instabilities, which may provide some hope that controlled fusion is indeed feasible.

Another important application is that of testing numerical schemes. The inequalities may be used to check the appropriateness of relevant algorithms by checking that the numerical evolution of a specific numerical scheme indeed satisfies the above inequality.

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