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Minimal Non-C-Perfect Hypergraphs with Circular Symmetry

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Abstract: In this research paper, we study 3-uniform hypergraphs $\mathcal{H} = (X, \mathcal{E})$ with circular symmetry. Two parameters are considered: the largest size $\alpha(\mathcal{H})$ of a set $S \subset X$ not containing any edge $E \in \mathcal{E}$, and the maximum number $\overline{\chi}(\mathcal{H})$ of colors in a vertex coloring of $\mathcal{H}$ such that each $E \in \mathcal{E}$ contains two vertices of the same color. The problem considered here is to characterize those $\mathcal{H}$ in which the equality $\overline{\chi}(\mathcal{H}') = \alpha(\mathcal{H}')$ holds for every induced subhypergraph $\mathcal{H}' = (X', \mathcal{E}')$ of $\mathcal{H}$. A well-known objection against $\overline{\chi}(\mathcal{H}') = \alpha(\mathcal{H}')$ is where $|\cap_{E\in\mathcal{E}'} E| = 1$, termed "monostar". Steps toward a solution to this approach is to investigate the properties of monostar-free structures. All such $\mathcal{H}$ are completely identified up to 16 vertices, with the aid of a computer. Most of them can be shown to satisfy $\overline{\chi}(\mathcal{H}) = \alpha(\mathcal{H})$, and the few exceptions contain one or both of two specific induced subhypergraphs $\mathcal{H}_5^{\circlearrowright}$, $\mathcal{H}_6^{\circlearrowright}$ on five and six vertices, respectively, both with $\overline{\chi} = 2$ and $\alpha = 3$. Furthermore, a general conjecture is raised for hypergraphs of prime orders.

Keywords: hypergraph; coloring; circular symmetry; independent set; upper chromatic number; C-perfect hypergraph

MSC: 05-08; 05C15; 05C17; 05C65; 05C69; 05C75

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1. Introduction

Comparison between structural invariants is a fundamental issue in the investigation of discrete mathematical objects. A celebrated area of this nature is the theory of perfect graphs defined in terms of clique number and chromatic number, via the induced-hereditary equation $\chi = \omega$. Initiated in the early 1960s by Claude Berge, it has lead to many deep structural results. Trotignon [1] sketched the proof of the characterization theorem ("Strong Perfect Graph Theorem") due to Chudnovsky, Robertson, Seymour, and Thomas [2], discussed further developments, and at the end of the introduction, also recommended several earlier surveys on the topic.

In the present study, we consider a kind of hypergraph perfectness, which Vitaly Voloshin introduced in the 1990s. His original definitions dealt with their highly influential model of *mixed hypergraphs* [3]; here, we consider only hypergraphs, and more restrictively 3-uniform ones. A *hypergraph* is a pair $\mathcal{H} = (X, \mathcal{E})$, where X is the *vertex set* and the *edge set* $\mathcal{E}$ is a set system over X. In the current context it can be assumed without loss of generality that no multiple edges occur. As a further term, $\mathcal{H}$ is called *r-uniform* if $|E| = r$ holds for all $E \in \mathcal{E}$.

1.1. Symmetry Assumption

Starting with Section 2, it will be assumed throughout this paper that every considered system $\mathcal{H}$ is 3-uniform; moreover, the vertex set admits a cyclic permutation that is an automorphism of $\mathcal{H}$. The corresponding representation will be specified more precisely in Section 2.

1.2. C-Perfectness

A set $S \subset X$ is *independent* if it does not contain any edge $E \in \mathcal{E}$. The largest size of an independent set is termed the *independence number* and is denoted by $\alpha(\mathcal{H})$.x A *C-coloring* of $\mathcal{H}$ is a vertex coloring $\varphi : X \to \mathbb{N}$, such that each $E \in \mathcal{E}$ contains two vertices of the same color. The *upper chromatic number*, denoted by $\overline{\chi}(\mathcal{H})$, is the maximum number of colors in a C-coloring of $\mathcal{H}$. Each color appearing in $\varphi(X)$ defines its *color class* consisting of the vertices assigned to that color. If φ is any C-coloring of $\mathcal{H}$, picking just one vertex from each color class, we obtain an independent set; consequently, the inequality $\overline{\chi}(\mathcal{H}) \leq \alpha(\mathcal{H})$ is universally valid. This simple but important fact was observed in [4]. We refer to [5] for a survey on C-colorings.

An *induced subhypergraph* $\mathcal{H}' = (X', \mathcal{E}')$ of $\mathcal{H} = (X, \mathcal{E})$ is obtained by specifying a subset $X' \subseteq X$ and taking $\mathcal{E}' = \{E \in \mathcal{E} \mid E \subseteq X'\}$. We say that $\mathcal{H} = (X, \mathcal{E})$ is *C-perfect* if:

$$\overline{\chi}(\mathcal{H}') = \alpha(\mathcal{H}')$$

holds for every induced subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$. This notion was introduced in 1995 by Voloshin [4] (in a more general way for the structure class of "mixed hypergraphs", which we will not discuss here), who asked for the characterization of C-perfectness in terms of minimal forbidden induced subhypergraphs. Nevertheless, after nearly three decades, this problem still remains unresolved, and it is so even for the restricted class of 3-uniform hypergraphs.

An easy-to-check class of hypergraphs satisfying $\overline{\chi} < \alpha$ consists of those $\mathcal{H} = (X, \mathcal{E})$, where there is precisely one vertex contained in all edges, i.e., $|\cap_{E \in \mathcal{E}} E| = 1$. Voloshin [4] discovered this class in connection with C-perfectness, and coined the name *monostar* for such hypergraphs. Removing the common vertex of all edges, we obtain an independent set; hence, $\alpha(\mathcal{H}) = |X| - 1$ holds. However, if exactly $|X| - 1$ colors are used on $|X|$ vertices, then just one monochromatic vertex pair arises, and this color assignment would be a C-coloring only if those two vertices were contained in all edges, a contradiction to the defining condition of monostars.

Other constructions of non-C-perfect hypergraphs and further results on C-perfectness can be found in [6–10].

1.3. The 3-Uniform Case

As r grows, the number of minimal non-C-perfect r-uniform hypergraphs is at least exponential in r, as proved by Bujtás and Tuza in [7]. For this reason, here, we only consider $r = 3$ and make no attempt to handle the cases of $r > 3$.

Voloshin [4] listed five minimal non-C-perfect 3-uniform hypergraphs. Four of them are monostars: cutting out the central veretex from their edges, we obtain the graphs K_3 (triangle), $2K_2$ (matching of two edges), P_4 (path of length 3), and C_4 (cycle of length 4). The fifth one is the "cycloid"; setting its vertex set to be $\mathbb{Z}_5$, one edge is $(0, 1, 2)$, and the other edges can be obtained by its rotations: $(1, 2, 3)$, $(2, 3, 4)$, $(3, 4, 0)$, $(4, 0, 1)$ (inside an edge the order of vertices is irrelevant, of course; we use these orders to emphasize rotational symmetry.)

One further minimal non-C-perfect 3-uniform hypergraph was found by Král' [9]. Over the vertex set $\mathbb{Z}_6$, it has the edges $(0, 2, 4)$, $(1, 3, 5)$, and $(0, 1, 3)$, $(1, 2, 4)$, $(2, 3, 5)$, $(3, 4, 0)$, $(4, 5, 1)$, $(5, 0, 2)$. Hence, there is also a rotational symmetry.

With indication of their orders and also their circular symmetry, we use the notation $\mathcal{H}_5^{\circlearrowright}$ for Voloshin's cycloid and $\mathcal{H}_6^{\circlearrowright}$ for Král's construction. (Standard notation for $\mathcal{H}_5^{\circlearrowright}$ are $C_5^{(3)}$ and $C(3, 5)$, while there seems to be no fixed symbol for $\mathcal{H}_6$ in the literature).

1.4. Algorithmic Approach and Structure of the Paper

In this work, we propose an algorithmic approach towards the characterization problem of C-perfect hypergraphs, and derive some theoretical consequences.

The 3-uniform monostars that are minimal under the operation of taking induced subhypergraphs all have at most five vertices. For this reason, it is clear that for a generic input $\mathcal{H}$, it can be deciced in polynomial time (as a function of the number n of vertices) whether or not $\mathcal{H}$ is monostar-free. However, since the number of configurations grows very fast (with a function where n^2 appears in the exponent), a running time of $O(n^5)$ puts a substantial limitation to the checkable range of instances.

For circular systems, we have made the monostar-free test considerably faster; this approach is described in Section 3. To increase the speed, the systems are represented with a certain kind of generating systems described in Section 2. Such improvements admit to extend investigations to a somewhat larger value of n, although combinatorial explosion is impossible to avoid. Nevertheless, the performed tests turned out to be sufficient for the discovery of some infinite classes of monostar-free hypergraphs. On the other hand, most of the obtained structures satisfy the equality $\alpha = \overline{\chi}$, and the others contain $\mathcal{H}_5^{\circlearrowleft}$ or $\mathcal{H}_6^{\circlearrowleft}$ (or both) as an induced subhypergraph.

The goal of the present study is to offer methods that can be applied to approach the characterization problem of C-perfect hypergraphs, or of the minimal non-C-perfect ones, at least in the 3-uniform case. In Section 4, we describe an infinite class of monostar-free hypergraphs. Conjecture 1 below expects that this is the unique cyclic one if the number n of vertices is a prime. In Section 5, we describe a method to prove that $\alpha = \overline{\chi}$ holds when the generator triplets satisfy certain arithmetic conditions. In Section 6, some simple monostar-free sparse hypergraphs are described.

The paper is concluded with Appendix A, where all monostar-free cyclic 3-uniform hypergraphs are listed up to 16 vertices.

1.5. Quantitative Summary

At the end of this introduction, we would like to point out that, somewhat unexpectedly, monostar-free cyclic structures are rather rare. Using the concept of type representation, and generation by combinations of triplet types—see Section 2—we formulate the following open problem.

Conjecture 1. *Let $n \geq 11$ be a prime number, and $\mathcal{H} = (X, \mathcal{E})$ a cyclic, 3-uniform, monostar-free, non-complete hypergraph over $X = \mathbb{Z}_n$. Then, the following properties are valid:*

(i) *The geometric reflection $i \mapsto n - i$ is an automorphism of $\mathcal{H}$, i.e., if $(i, j, k) \in \mathcal{E}$, then also $(n-i, n-j, n-k) \in \mathcal{E}$.*

(ii) *All cyclic, 3-uniform, monostar-free, non-complete hypergraphs over $X = \mathbb{Z}_n$ are isomorphic and can be obtained from $\mathcal{H}$ by multiplying the edges (or equivalently, the generator triplet types) with the powers of a primitive root of n.*

(iii) *As a quantitative form of (ii), there are precisely $\frac{1}{2}(n-1)$ such hypergraphs.*

For $5 \leq n \leq 17$, the number of 3-uniform, monostar-free, non-complete hypergraphs with circular symmetry is exhibited in Table 1. More precisely, for $n = 17$ (marked with * in the table), only the number of symmetric ones—symmetric in the sense of (i) above—has been determined, and a particular case of Conjecture 1 states that no others exist.

A detailed list of monostar-free combinations of generator triplet types is given in Appendix A. From that list, one gets the impression that the number of monostar-free combinations does not grow faster than $O(n^3)$ (or even with the rate of $O(n^2)$) while the number of all combinations is $\exp(\Theta(n^2))$. The following problem of number-theoretic nature arises in a natural way.

Problem 1. *Determine an asymptotically tight upper bound on the number of monostar-free combinations of generator triplets, as a function of the number n of vertices.*

Table 1. Quantitative summary: n = number of vertices; N = number of types ($N = \frac{1}{6}n(n-3) + \epsilon$ with $\epsilon = 1$ for $3 \mid n$ and $\epsilon = 1/3$ for $3 \nmid n$); $2^N - 2$ = number of non-empty non-complete combinations of generator triplet types; MS-free = number of monostar-free non-complete cases (for $n = 17$ marked with * only the symmetric ones were analyzed in full detail).

n	N	$2^N - 2$	MS-Free
5	2	2	2
6	4	14	6
7	5	30	4
8	7	126	6
9	10	1022	7
10	12	4094	18
11	15	32,766	5
12	19	524,286	44
13	22	4,194,302	6
14	26	67,108,862	48
15	31	2,147,483,646	36
16	35	34,359,738,366	68
17	40	1,099,511,627,774	8 *

2. Representations of Circular Systems

We apply several kinds of representation simultaneously, as follows:

Algebraic: The vertex set is identified with $\mathbb{Z}_n$.

Geometric: The vertices are embedded in the Euclidean plane and form a regular n-gon.

Difference triplets: A cyclic 3-tuple is associated with each 3-element subset of $\mathbb{Z}_n$, in the flavor of the geometric view.

Triplet types: A type is associated with each 3-element subset of $\mathbb{Z}_n$ as a canonical representation.

According to the *algebraic* reprsentation, the vertices will be labeled with $0, 1, \ldots, n-1$ throughout this paper, and calculations on edges of hypergraphs are done modulo n.

In the *geometric* view we denote by O_n the *center* of the regular n-gon (of course, this is not a vertex in the hypergraph). Representing as a regular n-gon, with its other benefits, it also makes a distinction between "small" and "large" triplets. The latter are those 3-element vertex sets whose convex hull contains O_n in its interior.

Difference triplets in the study of cyclic edge decompositions of 3-uniform hypergraphs were introduced by Gionfriddo, Milazzo, and Tuza in [11,12] as follows. First, for $i, j \in \mathbb{Z}_n$ the *distance* is defined as:

$$||i-j|| := \min(|i-j|, n-|i-j|)$$

where $|i-j|$ is the absolute value of the (usual) difference between natural numbers i and j. Then, for any $0 \leq a < b < c < n$, the 3-element (unordered) subset $\{a, b, c\} \subset \mathbb{Z}_n$ is associated with the *difference triplet*, that is, the *cyclic sequence:*

$$(||a-b||, ||b-c||, ||c-a||)\,.$$

By definition, $(||b-c||, ||c-a||, ||a-b||)$ and $(||c-a||, ||a-b||, ||b-c||)$ refers to the same difference triplet as $(||a-b||, ||b-c||, ||c-a||)$ does. For example, if $n = 9$ and $(a, b, c) = (1, 5, 7)$, then the three associated difference triplets are $(4, 2, 3)$, $(2, 3, 4)$, $(3, 4, 2)$, because in $\mathbb{Z}_9$, we have $||7-1|| = 3$. On the other hand, the same set $(a, b, c) = (1, 5, 7)$ on $n = 15$ vertices would have the difference triplets $(4, 2, 6)$, $(2, 6, 4)$, $(6, 4, 2)$. Note that all rotations $(a+i, b+i, c+i)$—addition takes modulo n in $\mathbb{Z}_n$, of course—of any 3-element set (a, b, c) have the same difference triplet.

Consider any 3-element set (a,b,c) of vertices. From the three difference triplets associated with it, we take the lexicographically smallest one, say (d_1,d_2,d_3). Then, from the orbit $\{(a+i,b+i,c+i) \mid i \in \mathbb{Z}_n\}$ of all 3-element sets having this difference triplet we use $(0,d_1,c')$, where $c' = d_1+d_2$ if $d_1+d_2 = d_3$, or $c' = n-d_3$ if $d_1+d_2+d_3 = n$. This $(0,d_1,c')$ is called the *triplet type* of (a,b,c), or briefly just the *type* of (a,b,c). In the above example, the set $(a,b,c) = (1,5,7)$ has type $(0,2,5)$ if $n = 9$ and $(0,2,11)$ if $n = 15$. Representation with triplet types was applied previously by Keszler and Tuza [13], concerning cycle decompositions of complete hypergraphs (i.e., for a problem very far from the present one). A slight difference is that here, we normalize the starting element of types to 0, while that in [13] was taken as 1.

Triplet types can be arranged in a lexicographic order, which is useful from the algorithmic point of view. Except for the last part, the blocks of this list have the same general form:

- $(0,1,2),(0,1,3),\dots,(0,1,n-2)$;
- $(0,2,4),(0,2,5),\dots,(0,2,n-3)$;

  $\vdots$
- $(0,i,2i),(0,i,2i+1),\dots,(0,i,n-i-1)$;

  $\vdots$
- $(0,\lfloor n/3\rfloor-1,2\lfloor n/3\rfloor-2),(0,\lfloor n/3\rfloor-1,2\lfloor n/3\rfloor-1),\dots,(0,\lfloor n/3\rfloor-1,n-\lfloor n/3\rfloor)$;
  and the last part depends on n (mod 3), as
- $(0,t,2t)$ if $n = 3t$ or $n = 3t+1$;
- $(0,t,2t),(0,t,2t+1)$ if $n = 3t+2$.

Symmetric vs. Reflected Triplets

It is worth making a distinction between symmetric and non-symmetric cases.

- *Symmetric difference triplets and triplet types:*
  A difference triplet is *symmetric* if it contains two (or three) equal distances. The corresponding symmetric triplet types are those having a symmetric difference triplet.
- *Reflected triplets:*
  If all the three differences are distinct in a difference triplet (d_1,d_2,d_3), then its *reflected pair* is (d_1,d_3,d_2). Analogously, if $(0,a,b)$ is a non-symmetric triplet type, then its reflected pair is $(0,a,n+a-b)$.

On n vertices, there are $\frac{n(n-3)}{6}+1$ types if n is a multiple of 3, and $\frac{(n-1)(n-2)}{6}$ types otherwise [13]. Moreover, for every n, the number of symmetric types is the same as the number of possible distances other than $n/2$, that is $\lfloor\frac{n-1}{2}\rfloor$. In the range $1 \le i \le n/3$, those are the starting types $(0,i,2i)$ of the blocks in the lexicographic listing. For $n/3 < i < n/2$, the symmetric type is $(0,n-2i,n-i)$. Viewing this from another point, and assuming $i \le n/3$, if $i \equiv n-1$ (mod 2) then the ith block has its unique symmetric type $(0,i,2i)$, while $i \equiv n$ (mod 2), admits two symmetric types: $(0,i,2i)$ and $(0,i,(n-i)/2$. In particular, if $n = 3t+2$, then in the last block both types $(0,t,2t),(0,t,2t+1)$ are symmetric.

The number of reflected pairs is more complicated to express. In Lemma 2.1 of [12] the case of $3 \nmid n$ is considered; then, the number of reflected pairs is $\frac{(n-1)(n-5)}{12}$ for n odd and $\frac{(n-2)(n-4)}{12}$ for n even. To complete the picture here, we observe:

- If $n = 6t+3r$ with $r \in \{0,1\}$, then the number of reflected pairs is $\frac{n^2-6n-3r}{12}+1$.

For a proof, let us recall that the number of types is $\frac{n^2-3n}{6}+1$ for every n divisible by 3. Among those, we have $\frac{n-2}{2}$ symmetric types if n is even, and $\frac{n-1}{2}$ if n is odd. A formula covering both cases is $\frac{n+r}{2}-1$. It follows that the number of non-symmetric triplet types is $\frac{n^2-6n-3r}{6}+2$. The number of reflected pairs is half of that.

3. Algorithmic Considerations

For the generating process of hypergraphs, the triplet types introduced in the preceding section are used. The combinations $\mathcal{T}$ of triplet types are systematically inspected for the presence of an induced monostar, and those containing one are abandoned without further inspection. Some applied principles are as follows.

- The type $(0, n/3, 2n/3)$ generates $n/3$ edges of the form $(i, i+n/3, i+2n/3)$ $(i = 0, 1, \dots, n/3-1)$ if $3 \mid n$; any other type $(0, a, b)$ generates exactly n edges of the form $(i, i+a, i+b)$ $(i = 0, 1, \dots, n-1)$ for every n, addition is taken in $\mathbb{Z}_n$. Hence, the number of edges generated by $\mathcal{T}$ is $n \cdot |\mathcal{T}|$ if $(0, n/3, 2n/3) \notin \mathcal{T}$, and $n \cdot |\mathcal{T}| - 2n/3$ otherwise.
- If there is a monostar in $\mathcal{H}$, then also one centered at 0 is present. Two types of monostars can occur:

  M2: It contains two edges $(0, a, b)$, $(0, c, d)$, where $0, a, b, c, d$ are five distinct vertices, and none of the 3-element sets, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, occurs as an edge in $\mathcal{H}$ (cutting out 0 from the two edges, a *2-matching* is obtained).

  C3: It contains three edges $(0, a, b)$, $(0, a, c)$, $(0, b, c)$, and the set $\{a, b, c\}$ is not an edge in $\mathcal{H}$. (Cutting out 0 from the three edges, a *3-cycle* is obtained.)
- A graph $G = G(\mathcal{T})$ with the vertex set $\mathbb{Z}_n \setminus \{0\}$ can be defined, which has $3 \cdot |\mathcal{T}|$ edges if $(0, n/3, 2n/3) \notin \mathcal{T}$, and $3 \cdot |\mathcal{T}| - 2$ edges otherwise. The edges ij of G arise from the edges $(0, i, j)$ of $\mathcal{H}$.
- Having G at hand, the presence of monostar types M2 and C3 can be tested. For the former, pairs of disjoint edges ($2K_2$ subgraphs) are checked whether the set of their four vertices induces no edges of $\mathcal{H}$. For the latter, triangles (K_3 subgraphs) of G are checked whether their three vertices form a non-edge in $\mathcal{H}$.

It is worth noting that there are three monostars on five vertices (obtained by extending the edges of the graphs $2K_2$, P_4, and C_4 with a common new vertex), but we are still able to simplify the monostar test to just one related M2 type. The reason is that $2K_2$, P_4, C_4 all contain disjoint edges.

One may also speculate whether the test should deal with M2 or C3 first, if we intend to decrease the running time. Experience shows that in this direction, the density of $\mathcal{H}$ is relevant. Namely, in sparse $\mathcal{H}$, an algorithm taking type M2 first is definitely faster, and starting with type C3, is preferable only if $\mathcal{T}$ contains most of the triplet types.

Verification of Computer Codes

During the years, three independent installations of algorithms were prepared, taking a pair (n, g) as input (n = number of vertices, g = number of generating triplet types) and printing the generators of all monostar-free systems with the given parameters as part of the output. Not all cases reported here have been run with all codes, but many of them were, and the three codes mutually verified each other with their unanimous outputs. The code applied for the generation of symmetric systems on 17 vertices was a modified version of the fastest general code. In addition to the generating combinations, one of the codes also determined α and $\overline{\chi}$ for all monostar-free systems within a certain (naturally smaller) range of n, and its outputs agree with the theoretical results that we develop in this paper.

4. A Dense Infinite Class

Assume that $n \geq 5$ is odd. Let $\mathcal{F}_n^*$ denote the hypergraph of order n generated by the triplet types.

- $(0, a, b)$ and $(0, a, n+a-b)$ such that $b < n/2$ and $0 < a \leq b/2$.

In the geometric representation, these generators are precisely the "small" triplets, which do not contain the center of the regular n-gon in their interior.

The following assertion shows that $\{\mathcal{F}_n^* \mid n \geq 7,\ n \text{ odd}\}$ is an infinite family of non-C-perfect—but not minimal non-C-perfect—hypergraphs containing no monostars (the smallest case $\mathcal{F}_5^*$ is not C-perfect either, but it is $\mathcal{H}_5^{\circlearrowright}$ itself, and hence, it is minimal).

Theorem 1. *For every odd $n \geq 5$ the hypergraph $\mathcal{F}_n^*$ has the following properties:*

1. *$\mathcal{F}_n^*$ is monostar-free;*
2. *$\mathcal{H}_5^{\circlearrowright}$ is an induced subhypergraph of $\mathcal{F}_n^*$;*
3. *The number of generator triplets is $\frac{(n-1)(n-3)}{8}$.*

Proof. To prove 1 assume for a contradiction that $\mathcal{F}_n^*$ contains a monostar, say M. Because of circular symmetry, we may assume that the center of M is vertex 0.

Let K be the convex hull of the vertices of $M \setminus \{0\}$ in the geometric representation. The geometric center O_n must be in the interior of K, otherwise all vertex triples of $M \setminus \{0\}$ are edges in $\mathcal{F}_n^*$ and the proof is done. Thus, assume that O_n is in K.

If K is a triangle (i.e., M is of type C3), consider the line that passes through 0 and O_n. This line intersects two sides of K, say ab and ac, where $a, b, c \in \mathbb{Z}_n$ are vertices of the regular n-gon. Assume that ac is farther from 0 than ab. Then, the triangle $0ac$ contains O_n in its interior, implying the contradiction that $(0, a, c)$ is not among the generator triplets of $\mathcal{F}_n^*$.

If K is a quadrangle (i.e., M is of type M2), let a, b, c, d be its four vertices. We now consider the line L passing through a and O_n. Two of the other three vertices, say b and c, are on the same side of L. However, then the convex hull of $\{a, b, c\}$ does not contain O_n, implying the contradiction that (a, b, c) is an edge of $\mathcal{F}_n^*$. This completes the proof of 1.

If $n = 2k + 1$, then in the five-cycle $(0, 1, k, k + 1, 2k)$, the edge $(2k, 0, 1)$ is from the orbit of $(0, 1, 2)$, and the other four edges are from the orbits of $(0, 1, k)$ and $(0, 1, k + 1)$. The other five 3-element sets contained in $\{0, 1, k, k + 1, 2k\}$ are non-edges because their convex hull contains the center O_n. Hence, an induced $\mathcal{H}_5^{\circlearrowright}$ is found.

Finally, to enumerate the generator triplet types, let us write $n = 2k + 1$, so that the formula gets the simpler form $\frac{(n-1)(n-3)}{8} = \binom{k}{2}$. It is easily checked that for small values of n this is the correct number of generator triplets. Indeed, if $n = 5 = 2 \cdot 2 + 1$, then $\binom{2}{2} = 1$, and the only type not containing O_5 is $(0, 1, 2)$; and if $n = 7 = 2 \cdot 3 + 1$, then $\binom{3}{2} = 3$, and we have three types avoiding O_7, namely $(0, 1, 2)$, $(0, 1, 3)$, $(0, 1, 5)$.

For larger orders we apply induction from n to $n + 2$, that means a step from k to $k + 1$. The new generating types for $\mathcal{F}_{n+2}^*$ are then the triplets $(0, i, k + 1)$ and their reflections, where $1 \leq i \leq (k + 1)/2$. We distinguish between odd and even values of k.

Consider first $k = 2t - 1$. Then, the symmetric triplet $(0, t, 2t)$ becomes one of the generators, and the other new ones are $(0, 1, 2t), (0, 2, 2t), \ldots, (0, t - 1, 2t)$ with their reflected pairs. Hence, the number of new generator triplets is equal to $2t - 1 = k$ and we have $\binom{k}{2} + k = \binom{k+1}{2}$. On the other hand, if $k = 2t$, then by increasing it to $2t + 1$ no new symmetric triplet becomes a generator: the new members are $(0, 1, 2t + 1), (0, 2, 2t + 1), \ldots, (0, t, 2t + 1)$ and their reflections. Hence, we have $2t = k$ new generators, and induction validates the assertion. □

Corollary 1. *If $n \geq 7$, then $\mathcal{F}_n^*$ is neither C-perfect, nor minimal non-C-perfect.*

Remark 1. *It is interesting that the analogues of $\mathcal{F}_n^*$ for even n do not provide examples with properties similar to those expressed in Theorem 1, as the following two examples show.*

If we keep the condition $b < n/2$ in the definition, then we can take, e.g., the generating triplets $(0, 1, 2)$ and $(0, 1, n/2 - 1)$. A rotation of the latter yields $(0, n/2 + 1, n/2 + 2)$. All the four 3-element subsets of $\{1, 2, n/2 + 1, n/2 + 2\}$ contain the center O_n on a boundary line, and hence, none of them is an edge of the hypergraph, and therefore, the 5-tuple $\{0, 1, 2, n/2 + 1, n/2 + 2\}$ induces a monostar of type M2.

If we modify the condition to $b \leq n/2$ in the definition, then the three edges $(n-1,0,1)$, $(0,1,n/2)$, $(n/2,n-1,0)$ are present in the hypergraph but $(1,n/2,n-1)$ is not an edge; therefore, a monostar of type C3 occurs.

5. Distances Dividing the Number of Vertices

Recall that $\overline{\chi}(\mathcal{H}) \leq \alpha(\mathcal{H})$ is a universally valid inequality for all hypergraphs $\mathcal{H}$. In this section, we present a method suitable for proving the equality $\overline{\chi} = \alpha$ in many cases. It is applicable when there is an integer d that is a common divisor of n and at least one difference in the difference triplets of all generators. In such situations the color assignment:

$$\varphi : i \mapsto (i \bmod d)$$

is a C-coloring with d colors, and consequently, the lower bound $\overline{\chi} \geq d$ is valid. Thus, for proving $\overline{\chi} = \alpha$, it suffices to verify the upper bound $\alpha \leq d$. In the next four subsections, we illustrate the ideas of such proofs with explicit examples for $n/d = 2,3,4,5$. That means hypergraphs whose upper chromatic number and independence number are equal to $n/2$, $n/3$, $n/4$, or $n/5$.

5.1. Method to Prove $\alpha = \overline{\chi} = n/2$

Consider $n = 16$ with six generator types. One of the monostar-free structures has the following generators:

$$(0,1,8)\ (0,1,9)\ (0,2,8)\ (0,2,10)\ (0,3,8)\ (0,4,8)\,.$$

This is system #1 of $n = 16$, $g = 6$ in Appendix A.

We know that $\overline{\chi} \geq 8$ holds. Let S be a largest independent set in $\mathcal{H}$. If S contains at most one vertex from each pair $\{i, i+8\}$, then we are done. Otherwise consider the auxiliary directed graph H whose vertex set is $\mathbb{Z}_8$, and for $1 \leq j \leq 4$, an edge $\overrightarrow{i, i+j}$ is oriented from each $i \in \mathbb{Z}_8$ to $i+j$ if at least one of the triplets $(0,j,8)$ and $(0,8-j,16-j)$ is among the generators. In our example, the edges are $\overrightarrow{i, i\pm 1}$, $\overrightarrow{i, i\pm 2}$, $\overrightarrow{i, i+3}$, and $\overrightarrow{i, i+4}$, derived from the correspondig values of j as $1, 7 = 8-1, 2, 6 = 8-2, 3$, and 4, respectively.

In general, the next step is to cover the vertex set $V(H)$ of H with vertex-disjoint directed cycles. In our case, each of the distances $1,2,3,4,6,7$ provides a possibility. More explicitly, each of the distances $1,3,7$ determines a directed Hamiltonian cycle; 2 and 6 decompose $V(H)$ into cycles of length 4; and distance 4 yields four cycles of length 2.

The point is that if both $i, i+8 \in S$, then both $i+j, i+j+8 \notin S$ because the 3-element sets $\{i, i+j, i+8\}$ and $\{i, i+8, i+j+8\}$ in $\mathbb{Z}_{16}$ are in the orbit of $(0,j,8)$ if $j \leq 4$ and of $(0, 8-j, 16-j)$ if $5 \leq j \leq 7$. For each $i \in \mathbb{Z}_8$, define the weight $w(i) = |S \cap \{i, i+8\}|$. Then, $\alpha(\mathcal{H}) = |S| = \sum_{i \in \mathbb{Z}_8} w(i)$. By what has been said, if $w(i) = 2$ then $w(i+j) = 0$; hence, the average weight is 1 on such vertex pairs $(i, i+j)$. This fact is valid in each cycle of the vertex partition of $\mathcal{H}$. Since each of the other vertices has weight 0 or 1, it follows that also the overall average weight is at most 1, and thus, $\alpha(\mathcal{H}) \leq |\mathbb{Z}_8| = 8$.

5.2. Method to Prove $\alpha = \overline{\chi} = n/3$

Let us begin this discussion with $n = 15$ and ten generator types, for which a different idea can be presented. One of the monostar-free structures has the following generators:

$$(0,1,5)\ (0,1,6)\ (0,1,10)\ (0,1,11)\ (0,2,5)\ (0,2,10)\ (0,3,8)\ (0,4,9)\ (0,4,10)\ (0,5,10)\,.$$

This is system #1 of $n = 15$, $g = 10$ in Appendix A.

We know that $\overline{\chi} \geq 5$ holds. Let S be a largest independent set in $\mathcal{H}$. If S contains at most one vertex from each of the five 3-element sets $\{i, i+5, i+10\}$, then we are done. Otherwise, relabeling by rotation if necessary, we may assume without loss of generality that $0, 5 \in S$. Then, using the generator triplets, we observe the following implications:

- $1 \notin S$ by $(0,1,5)$;

- $14 \notin S$ by $(0,1,6)$;
- $6 \notin S$ by $(0,1,10)$;
- $4 \notin S$ by $(0,1,11)$;
- $2 \notin S$ by $(0,2,5)$;
- $7 \notin S$ by $(0,2,10)$;
- $12 \notin S$ by $(0,3,8)$;
- $11 \notin S$ by $(0,4,9)$;
- $9 \notin S$ by $(0,4,10)$;
- $10 \notin S$ by $(0,5,10)$.

Thus, beside 0 and 5, only some of 3, 8, 13 can occur in S. However, those three vertices form an edge, so at least one of them is missing. This would imply $|S| < 5$, and the proof is done.

The combination of the two approaches is also suitable for handling the very sparse cases. We illustrate this with system #1 of $n = 15$, $g = 4$ from Appendix A, having the following generators:

$$(0,1,5)\ (0,1,10)\ (0,4,9)\ (0,5,10)\,.$$

Additionally, here, the proof of $\alpha \le 5$ is done if a largest independent set S contains at most one vertex from each of the five 3-element sets $\{i, i+5, i+10\}$. If this is not the case, assume without loss of generality that $0, 5 \in S$. Then, by the same reason as above:

- $1 \notin S$ by $(0,1,5)$;
- $6 \notin S$ by $(0,1,10)$;
- $11 \notin S$ by $(0,4,9)$;
- $10 \notin S$ by $(0,5,10)$.

Hence, we encounter a situation analogous to the one described in the proof concerning $n/2$ ($d = 2$). Namely, if S contains two elements from an edge $\{i, i+5, i+10\}$, then it does not contain any from the next edge $\{i+1, i+6, i+11\}$. Thus, we can define an auxiliary graph over the vertex set $\mathbb{Z}_5$ (or more generally over some $\mathbb{Z}_m$ when another n is considered), introduce the weights $w(i)$ as above, and infer that the average weight of vertices is at most 1. In our present example, this argument yields $\alpha(\mathcal{H}) \le |\mathbb{Z}_5| = 5$.

5.3. Method to Prove $\alpha = \overline{\chi} = n/4$

Here, we take system #1 of $n = 16$, $g = 7$ for illustration. It has the generator types:

$$(0,1,4)\ (0,1,8)\ (0,1,12)\ (0,3,7)\ (0,3,11)\ (0,4,8)\ (0,4,9)\,.$$

We now consider the four 4-element sets $\{i, i+4, i+8, i+12\}$. If a largest independent set S contains at most one vertex from each of them, then $\alpha \le 4$ follows. Otherwise, two cases have to be investigated.

If $0, 4 \in S$, then we have the implications:

- $1 \notin S$ by $(0,1,4)$;
- $5 \notin S$ by $(0,1,12)$;
- $13 \notin S$ by $(0,3,7)$;
- $8, 12 \notin S$ by $(0,4,8)$;
- $9 \notin S$ by $(0,4,9)$.

Similarly, if $0, 8 \in S$, then:

- $1, 9 \notin S$ by $(0,1,8)$;
- $5, 13 \notin S$ by $(0,3,11)$;
- $4, 12 \notin S$ by $(0,4,8)$.

From this, we conclude that, in either case, $\{i, i+4, i+8, i+12\} \cup \{i+1, i+5, i+9, i+13\}$ can contain at most two vertices from S. Thus, the vertices of the corresponding auxiliary graph on $\mathbb{Z}_4$ will get average weight at most 1, and the required upper bound $\alpha(\mathcal{H}) \le |\mathbb{Z}_4| = 4$ is obtained.

5.4. Method to Prove $\alpha = \overline{\chi} = n/5$

In this final example for the methods based on common divisors of difference triplets, we consider system #1 of $n = 15$, $g = 12$. Then, the generator types are:

$$(0,1,3)\ (0,1,6)\ (0,1,9)\ (0,1,12)\ (0,2,5)\ (0,2,8)\ (0,2,11)$$
$$(0,3,6)\ (0,3,7)\ (0,3,9)\ (0,3,10)\ (0,4,9)$$

Now, the common divisor is $d = 3$; therefore, we consider the three 5-element sets $\{i, i+3, i+6, i+9, i+12\}$. Making them monochromatic—assigning the color (i mod 3) to each vertex—is a C-coloring method; hence, $\overline{\chi} \geq 3$. Note further that any three vertices from $\{i, i+3, i+6, i+9, i+12\}$ form an edge in the hypergraph, due to the generator types $(0,3,6)$ and $(0,3,9)$. For this reason, in order to prove $\alpha \leq 3$, we need to analyze two cases for a largest independent set S.

If $0, 3 \in S$, then:

- $1 \notin S$ by $(0,1,3)$;
- $4 \notin S$ by $(0,1,12)$;
- $13 \notin S$ by $(0,2,5)$;
- $6, 12 \notin S$ by $(0,3,6)$;
- $7 \notin S$ by $(0,3,7)$;
- $9 \notin S$ by $(0,3,9)$;
- $10 \notin S$ by $(0,3,10)$.

If $0, 6 \in S$, then:

- $1 \notin S$ by $(0,1,6)$;
- $7 \notin S$ by $(0,1,9)$;
- $13 \notin S$ by $(0,2,8)$;
- $4 \notin S$ by $(0,2,11)$;
- $3 \notin S$ by $(0,3,6)$;
- $9, 12 \notin S$ by $(0,3,9)$;
- $10 \notin S$ by $(0,4,9)$.

Thus, no matter if 3 or 6 is in S together with 0, only five vertices remain non-excluded, namely $2, 5, 8, 11, 14$. However, this is just the case $i = 2$ of $\{i, i+3, i+6, i+9, i+12\}$, and hence, no more than two of its vertices may occur in S. We verify—using only the assumption $0 \in S$, and hence, in a unified way for both subcases—that only one of those five vertices is possible:

- $14 \in S$ excludes all of $2, 5, 8, 11$, due to the types $(0,1,3)$, $(0,1,6)$, $(0,1,9)$, $(0,1,12)$;
- $2 \in S$ excludes $5, 8, 11$, due to the types $(0,2,5)$, $(0,2,8)$, $(0,2,11)$;
- $5 \in S$ excludes 8 and 11, due to $(0,3,10)$ and $(0,4,9)$;
- $8 \in S$ excludes 11, due to $(0,3,7)$.

6. Some Very Sparse Infinite Classes

In this section, we identify nomostar-free systems that have only one or two generator types. In both cases, a substantial difference occurs between odd and even numbers of vertices.

6.1. Systems Generated by One Type

For just one generator, the following standard types can be identified:

- (1A)—if n is a multiple of 3, then one type: $(0, n/3, 2n/3)$;
- (1B)—if n is a multiple of 5, then two types: $(0, n/5, 2n/5)$ and $(0, n/5, 3n/5)$;
- (1C)—if $n = 2k$ is even, then $k-1$ types: $(0, i, n/2)$ and $(0, i, n/2 + i)$ where $1 \leq i \leq n/4$.

We note that if k is even, i.e., if n is a multiple of 4, then the 3-element sets $(0, n/4, n/2)$ and $(0, n/4, 3n/4)$ are of the same type; hence, the number $k-1$ is the correct value for the number of types containing the vertex pair $\{0, n/2\}$ for all even orders $n = 2k$.

Proposition 1. *The above types satisfy the following coloring properties:*

- *A hypergraph $\mathcal{H}$ of type* (1A) *for any order n divisible by* 3 *consists of $n/3$ disjoint edges, and has $\overline{\chi}(\mathcal{H}) = \alpha(\mathcal{H}) = 2n/3$.*
- *A hypergraph $\mathcal{H}$ of type* (1B) *for any order n divisible by* 5 *consists of $n/5$ vertex-disjoint copies of $\mathcal{H}_5^{\circlearrowright}$, and has $\overline{\chi}(\mathcal{H}) = 2n/5$ and $\alpha(\mathcal{H}) = 3n/5$.*
- *A hypergraph $\mathcal{H}$ of type* (1C) *for any even order n has $\overline{\chi}(\mathcal{H}) = \alpha(\mathcal{H}) = n/2$.*

It turns out that the above types describe all possibilities of monostar-free systems generated by just one type.

Proposition 2. *A single triplet generates a monostar-free hypergraph if and only if it is of type* (1A) *or* (1B) *or* (1C)*. As a function of n, the number of such systems is equal to:*

- 0*, if $n \equiv 1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}$;*
- 1*, if $n \equiv 3, 9, 21, 27 \pmod{30}$;*
- 2*, if $n \equiv 5, 25 \pmod{30}$;*
- 3*, if $n \equiv 15 \pmod{30}$;*
- *$n/2 - 1$, if $n \equiv 2, 4, 8, 14, 16, 22, 26, 28 \pmod{30}$;*
- *$n/2$, if $n \equiv 6, 12, 18, 24 \pmod{30}$;*
- *$n/2 + 1$, if $n \equiv 10, 20 \pmod{30}$;*
- *$n/2 + 2$, if $n \equiv 0 \pmod{30}$.*

In particular, for odd n, it is periodic modulo 30*, while for even n, the function becomes periodic modulo* 30 *after the subtraction of $n/2$.*

6.2. Systems Generated by Two Types

We begin with Table 2 that exhibits the number of odd systems of small orders with two generator types.

Table 2. Number of monostar-free systems of odd orders $n < 40$ with two generator types.

n	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
$\#_n$	1	1	•	•	•	1	•	•	1	•	1	•	•	•	•	2	•	•

As a comparison, the number of monostar-free systems with one generator is determined in terms of divisibility by 3 and 5, whereas the corresponding number for two generators depends on divisibility by 5 and 7. Namely, for odd n, the latter is as follows:

- 2, if $35 \mid n$;
- 1, if $5 \mid n$ and $7 \nmid n$, or $7 \mid n$ and $5 \nmid n$;
- 0, if $\gcd(n, 35) = 1$.

There is a complete analogy between the two sparsest classes in the sense that each odd system consists of vertex-disjoint copies of the unique system of order 3, 5, or 7. Of course, such constructions exist for the even values $n = 10t$, $n = 14t$, and $n = 70t$ as well. For this reason, we try to make Table 3 more transparent by including two further data points beside the number of monostar-free systems: in the third line for D', we subtract $(n/2 - 2)$, and in the fourth line, D' is decreased further by 1 or 2 to get D'' if n is divisible by 5 or 7 or both.

Let us define:

$$f(n) = \begin{cases} 0 & \textit{if } n \equiv 4, 8 \pmod{12}, \\ 1 & \textit{if } n \equiv 0, 2, 10 \pmod{12}, \\ 2 & \textit{if } n \equiv 6 \pmod{12} \end{cases}$$

Now, we can state Proposition 3.

Table 3. Number of monostar-free systems of even orders $n \leq 100$ with two generator types.

n	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
$\#_n$	3	2	5	5	7	6	9	9	10	11	12	13	16	14	16	17
D'	2	0	2	1	2	0	2	1	1	1	1	1	3	0	1	1
D''	2	0	1	1	1	0	2	0	1	1	1	0	2	0	1	1
n	38	40	42	44	46	48	50	52	54	56	58	60	62	64	66	68
$\#_n$	18	19	22	20	22	23	25	24	27	27	28	30	30	30	33	32
D'	1	1	3	0	1	1	2	0	2	1	1	2	1	0	2	0
D''	1	0	2	0	1	1	1	0	2	0	1	1	1	0	2	0
n	70	72	74	76	78	80	82	84	86	88	90	92	94	96	98	100
$\#_n$	36	35	36	36	39	39	40	42	42	42	46	44	46	47	49	49
D'	3	1	1	0	2	1	1	2	1	0	3	0	1	1	2	1
D''	1	1	1	0	2	0	1	1	1	0	2	0	1	1	1	0

Proposition 3. *The number of monostar-free hypergraphs generated by two triplet types, as a function of n, is equal to:*

- 0, *if* $\gcd(n, 70) = 1$;
- 1, *if* $\gcd(n, 70) = 5$ *or* 7;
- 2, *if* $\gcd(n, 70) = 35$;
- $n/2 - 2 + f(n)$, *if* $\gcd(n, 70) = 2$;
- $n/2 - 1 + f(n)$, *if* $\gcd(n, 70) = 10$ *or* 14;
- $n/2 + f(n)$, *if* $n \equiv 0 \pmod{70}$.

In particular, for odd n, it is periodic modulo 70, *and for even n, the function becomes periodic modulo* 420 *after the subtraction of $n/2$.*

6.3. Few Further Generators

The situation for more generators, $g \geq 3$, is increasingly difficult to analyze. Nevertheless, we have conducted many further computational experiments.

6.3.1. $g = 3$

For three generators, we have run the computer code for all $n \leq 110$. The output demonstrates that the number of monostar-free hypergraphs of odd order, generated by three triplet types, is as follows:

- 0, if $\gcd(n, 14) = 1$;
- 3, if $\gcd(n, 14) = 7$.

Moreover, in the latter case, the hypergraph consists of $n/7$ copies of any one of the constructions on seven vertices (in fact, the three constructions are isomorphic, as shown in Appendix A).

For even orders, the situation is much more complicated. The number of systems is exhibited in Table 4 for $n \leq 100$; here, D' is the increase above $(n/2 - 3)$.

Unfortunately, these data are insufficient for us to guess a general formula for the number of systems as a function of n. However, at least one fact can be observed in the currently inspected range:

- For every even n, there are at least $n/2 - 3$ monostar-free systems generated by three triplet types, and in case of equality, either $n = 8$ or $n \equiv 6 \pmod{12}$ holds.

Further tests proved that $n/2 - 3$ is the correct number also for $n = 102$ and $n = 114$. However, $n = 126$ breaks the rule with its 65 systems instead of 60. Here, three of the excess 5 originate from the systems of order 7, similarly to the case of general odd n.

Table 4. Number of monostar-free systems of even orders $n \leq 100$ with three generator types.

n	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36
$\#_n$	0	1	4	4	11	7	6	12	10	11	12	23	12	19	16	22
D'	0	0	2	1	7	2	0	5	2	2	2	12	0	6	2	7
n	38	40	42	44	46	48	50	52	54	56	58	60	62	64	66	68
$\#_n$	18	25	23	30	22	29	24	36	24	42	28	40	30	43	30	48
D'	2	8	5	11	2	8	2	13	0	17	2	13	2	14	0	17
n	70	72	74	76	78	80	82	84	86	88	90	92	94	96	98	100
$\#_n$	39	47	36	54	36	55	40	63	42	61	42	66	46	65	53	72
D'	7	14	2	19	0	18	2	24	2	20	0	23	2	20	7	25

6.3.2. $g = 4$

Six vertices admit exactly four triplet types; hence, those types generate all the $\binom{6}{3}$ triplets, and of course, this system is monostar-free. Corresponding data up to $n = 51$ have been computed, as shown in Table 5. A general formula is unclear; however, the following remarkable facts can be observed:

- In the inspected range, if $n = 6k + 3$, then the number of monostar-free hypergraphs generated by four triplet types is equal to $2k$.
- For $n > 8$ the number of systems is 0 if and only if n either is a prime or all of its prime factors are 5 and/or 7.

Table 5. Number of monostar-free systems of orders $7 \leq n \leq 51$ with four generator types.

n	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\#_n$	0	0	2	1	0	7	0	12	4	8	0	21	0	5	6
n	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
$\#_n$	10	0	19	0	12	8	18	0	26	0	16	10	16	0	41
n	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
$\#_n$	0	18	12	13	0	49	0	20	14	22	0	51	0	21	16

For the first case, $n = 6k + 3$, let us use the notation $t = 2k + 1$, i.e., $n = 3t$. Then, the general form of the $2k$ systems mentioned above is as follows:

$$
\begin{array}{rllll}
1. & (0,1,t) & (0,1,2t) & (0,t-1,2t-1) & (0,t,2t) \\
2. & (0,1,t+1) & (0,1,2t+1) & (0,t-1,2t) & (0,t,2t) \\
3. & (0,2,t) & (0,2,2t) & (0,t-2,2t-2) & (0,t,2t) \\
4. & (0,2,t+2) & (0,2,2t+2) & (0,t-2,2t) & (0,t,2t) \\
\vdots & & & & \\
2k-1. & (0,k,t) & (0,k,2t) & (0,t-k,2t-k) & (0,t,2t) \\
2k. & (0,k,t+k) & (0,k,2t+k) & (0,t-k,2t) & (0,t,2t)
\end{array}
$$

Using the methods of Section 5, we prove:

Proposition 4. *All hypergraphs $\mathcal{H}$ generated in this way satisfy the equality $\alpha(\mathcal{H}) = \overline{\chi}(\mathcal{H}) = n/3$.*

Proof. Observe that each generator triplet contains at least one difference equal to $t = n/3$ in its difference triplet (d_1, d_2, d_3), implying $\overline{\chi}(\mathcal{H}) \geq n/3$ by the C-coloring $i \mapsto (i \bmod t)$.

Therefore, it remains to be proved that the upper bound $\alpha(\mathcal{H}) \leq t$ is valid. Let S be a largest independent set in $\mathcal{H}$. If S contains at most one vertex from each of the 3-element sets $\{i, i+t, i+2t\}$, then the proof is done. Otherwise, assume that $\{0, t\} \subset S$, and consider the hypergraph numbered $2i-1$ or $2i$ above. We handle the two cases separately.

Case of $2i-1$: Generators $(0, i, t)$, $(0, i, 2t)$, $(0, t-i, 2t-i)$, $(0, t, 2t)$.

We then have the following implications:

- $i \notin S$, due to $(0, i, t)$;
- $i+t \notin S$, due to $(0, i, 2t)$;
- $i+2t \notin S$, due to $(0, t-i, 2t-i)$;
- $2t \notin S$, due to $(0, t, 2t)$.

As a consequence, for each $j = 0, 1, \ldots, t-1$, if a set $\{j, j+t, j+2t\}$ contains two vertices from S, then its ith successor $\{i+j, i+j+t, i+j+2t\}$ is disjoint from S. Now we can consider the auxiliary directed graph H over $\mathbb{Z}_t$ whose edges are $\overrightarrow{j, i+j}$. This H consists of vertex-disjoint directed cycles, and in the same way as in Section 4, we obtain by weighting its vertices with $0, 1, 2$ that the upper bound $\alpha(\mathcal{H}) \leq |\mathbb{Z}_t| = t$ is valid.

Case of $2i$: Generators $(0, i, t+i)$, $(0, i, 2t+i)$, $(0, t-i, 2t)$, $(0, t, 2t)$.

Now we have the following implications:

- $3t-i \notin S$, due to $(0, i, t+i)$;
- $t-i \notin S$, due to $(0, i, 2t+i)$;
- $2t-i \notin S$, due to $(0, t-i, 2t)$;
- $2t \notin S$, due to $(0, t, 2t)$.

In this case, for each $j = 0, 1, \ldots, t-1$, if a set $\{j, j+t, j+2t\}$ contains two vertices from S, then its ith predecessor $\{-i+j, -i+j+t, -i+j+2t\}$ is disjoint from S. Thus, the upper bound $\alpha(\mathcal{H}) \leq |\mathbb{Z}_t| = t$ follows in the same way as above, with the only difference that now we go around $\mathbb{Z}_{3t}$ in the other direction. $\square$

6.3.3. $5 \leq g \leq 9$

As the number of generators gets larger, running time becomes an issue because the number of triplet types grows essentially with $n^2/6$. Furthermore, as the orbits of types grow with n, monostar test gets slower to some extent. Nevertheless, for every g other than the smallest ones, we performed calulations at least until the number of combinations exceeded one billion. More explicitly, we considered $g = 5$ for $n \leq 35$; $g = 6$ for $n \leq 27$; $g = 7$ for $n \leq 22$; $g = 8$ for $n \leq 20$; and $g = 9$ for $n \leq 18$.

For odd n, the findings are fairly simple to summarize:

- If $g = 5$ and $n \leq 35$ is odd with $7 \nmid n$, then no monostar-free systems exist.
- If $g = 5$ and $n \leq 35$ is odd with $7 \mid n$, then exactly one monostar-free system exists, and it is the vertex-disjoint union of $n/7$ copies of the complete hypergraph whose edges are all the $\binom{7}{3}$ vertex triples on 7 vertices.
- If $g = 6$ and $n \leq 27$ is odd with $9 \nmid n$, then no monostar-free systems exist.
- If $g = 6$ and $n \leq 27$ is odd with $9 \mid n$ (i.e., $n = 9$ or $n = 27$), then exactly three monostar-free systems exist, and those with $n = 27$ consist of three vertex-disjoint copies of a construction on 9 vertices.
- If $g = 7$ and $n \leq 22$ is odd, then a monostar-free system exists if and only if $3 \mid n$.
- If $g = 8$ and $n \leq 20$ is odd, then no monostar-free systems exist.
- If $g = 9$ and $n \leq 18$ is odd, then no monostar-free systems exist.

As a supplement to the last two cases for even n, the following holds:

- If $g = 8$ and $n \leq 20$ is even, then the number of monostar-free systems is 2 if $n = 10$; 8 if $n = 12$; 11 if $n = 18$; 10 if $n = 20$; and no such systems exist for any other values of $n \leq 20$.
- If $g = 9$ and $n \leq 18$ is even, then the number of monostar-free systems is 2 if $n = 12$ or $n = 16$; it is 6 if $n = 18$; and no such systems exist for any other values of $n \leq 18$.

For $g = 5$ and $g = 6$ (n even) and for $g = 7$ (all $n \leq 22$), the results are shown in Tables 6–8.

Table 6. Number of monostar-free systems of even orders $n \leq 34$ with five generator types.

n	8	10	12	14	16	18	20	22	24	26	28	30	32	34
$\#_n$	0	0	1	7	12	32	16	32	13	12	19	12	24	16

Table 7. Number of monostar-free systems of even orders $n \leq 26$ with six generator types.

n	8	10	12	14	16	18	20	22	24	26
$\#_n$	0	0	2	1	6	27	32	80	34	64

Table 8. Number of monostar-free systems of orders $n \leq 22$ with seven generator types.

n	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$\#_n$	1	1	0	0	5	0	0	4	4	0	13	0	28	6	80

6.3.4. $n \leq 17$

We considered all possible numbers of generator triplets for all $n \leq 17$. More precisely, an exhaustive search was executed up to $n = 16$. For $n = 17$, the total number of combinations is 2^{40}; it would have required rather too much running time. On the other hand, we observed that for all smaller prime orders n, up to isomorphism, there is a unique monostar-free system, and its generators form a *symmetric* combination. By this we mean that either both members of a reflected pair occur among the generators or none of them is present. Motivated by this fact, we made an exhaustive search on the restricted class of symmetric combinations. On 17 vertices, there are 8 symmetric triplet types and 16 reflected pairs; in this way, the search space was reduced to 2^{24} combinations. It turned out that, disregarding the trivial family of all 3-element sets (the complete 3-uniform hypergraph of order 17), the monostar-free combination is unique up to isomorphism. Furthermore, we performed an exhaustive search on 17 vertices for at most 12 and at least 28 generators—nearly 18.3 billion combinations in all—and no monostar-free combinations occurred, in accordance with Conjecture 1. All these results are given in Table 9.

Table 9. The number of monostar-free combinations generated by g triplet types on $n \leq 17$ vertices; • = there are none; x = no symmetric systems exist.

g \ n	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	2	3	•	3	1	6	•	6	•	6	3	7	•
2		1	3	1	2	•	5	•	5	•	7	1	6	•
3			•	3	1	•	4	•	4	•	11	•	7	•
4			1	•	•	2	1	•	7	•	12	4	8	•
5				1	•	•	•	•	1	•	7	•	12	•
6					•	3	•	•	2	•	1	•	6	•
7					1	1	•	•	5	•	•	4	4	•
8						•	2	•	8	•	•	•	•	•
9						•	•	•	2	•	•	•	2	•
10						1	•	5	1	•	•	4	8	•
11							•	•	2	•	•	•	3	•
12							1	•	•	•	•	2	•	•
13								•	1	•	•	11	2	x
14								•	•	•	1	•	•	x
15								1	•	6	•	•	2	x
16									•	•	•	•	•	x
17									•	•	•	•	•	x
18									•	•	3	•	•	x
19									1	•	•	•	1	x
20										•	•	•	•	x
21										•	•	4	•	x
22										1	•	3	•	x
23											•	•	•	x
24											•	•	•	x
25											•	•	•	x
26											1	•	•	x
27												•	•	x
28												•	•	8
29												•	•	•
30												•	•	•
31												1	•	•
32													•	•
33													•	•
34													•	•
35													1	•
36														•
37														•
38														•
39														•
40														1

7. Discussion

In this paper, we proposed a computer-aided approach to the long-standing unsolved characterization problem of C-perfect hypergraphs. This class is defined in terms of a hereditary equality between two basic parameters of set systems. Substructures called monostars are excluded by theory; we applied this fact to design a fast algorithm suitable to list all relevant small configurations. Those structures turned out to be so unexpectedly rare that they can be investigated one by one, using theoretical tools.

The two relevant combinatorial invariants—upper chromatic number and independence number—would still be rather tedious to determine. We offer a method to handle them on the few remaining constructions that are of interest in the context of C-perfect 3-uniform cyclic hypergraphs.

There are several directions for further research on the problem. As a particular case, it would be desirable to complete the analysis on 17 vertices; with the aid of a computer, additional principles for the algorithm may be needed. A more ambitious plan would be to prove Conjecture 1 in general. Furthermore, it would be interesting to see further infinite classes of monostar-free hypergraphs with circular symmetry, and to find concise descriptions of them for relatively small numbers g of generator triplets. Furthermore, of course, the ultimate goal of this track of research is to solve the characterization problem of C-perfect 3-uniform hypergraphs.

It is our hope that the methods proposed in this paper will prove to be useful towards the solution of the characterization problem of C-perfect hypergraphs. Furthermore, beyond that, some elements of the representation applied here may help in various other studies of systems with circular symmetry as well.

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Appendix A

Below, the complete list of monostar-free systems is given for $5 \leq n \leq 16$. This has been determined by an exhaustive search. The trivial case, where the system is generated by all triplet types, is not listed because it is the complete 3-uniform hypergraph which clearly is unique and C-perfect for each n, both α and $\overline{\chi}$ being equal to 2. That would be the situation also for $n = 4$, where only one type exists, namely $(0, 1, 2)$.

For each number n of vertices we provide the following data:

- The number N of triplet types, which is equal to $\frac{1}{6}n(n-3)+1$ if n is divisible by 3, and $\frac{1}{6}(n-1)(n-2)$ otherwise;
- The number of those values g ($1 \leq g \leq N$) for which no combination of exactly g distinct triplet types can generate a monostar-free hypergraph;
- For all the other values of g, the number $\#_g$ of combinations of g distinct triplet types that generate a monostar-free hypergraph;
- All combinations, one by one, that generate monostar-free hypergraphs, in increasing order of g;
- Isomorphisms established by multiplication (augmentation) between the listed hypergraphs;
- The lexicographically smallest copy of $\mathcal{H}_5^{\circlearrowright}$ and/or $\mathcal{H}_6^{\circlearrowright}$ if there is one.

In order to facilitate checking of isomorphism and especially verifying that no isomorphisms exist beyond the claimed ones, at the very end for the most complicated case $n = 16$, we provide a table of triplet type transformations when multiplication by odd numbers is applied (Table A1).

As an example, if $n = 16$, then the type $(0,3,12)$ multiplied by 13 is $(0,39,156) \equiv (0,7,12)$; the three difference triplets of the latter are $(7,5,4)$, $(5,4,7)$, $(4,7,5)$. Among these three, the lexicographically smallest one is $(4,7,5)$; hence, the type of this triplet is $(0,4,11)$.

Appendix A.1. $n = 5$

Number of types: 2

Groups not generating any monostar-free systems: none

Number of monostar-free systems, $g \to \#_g$**:**
$1 \to 2$

Detailed list:
$g = 1$

1. $(0,1,2)$
2. $(0,1,3)$

Isomorphism: 1 $\xrightarrow{\times 3}$ **2**
Copy of $\mathcal{H}_5^{\circlearrowright}$ **in 1:** $(0,1,2,3,4)$

Appendix A.2. $n = 6$

Number of types: 4

Groups not generating any monostar-free systems: $g = 3$

Number of monostar-free systems, $g \to \#_g$**:**
$1 \to 3 \quad 2 \to 3$

Detailed list:
$g = 1$

1. $(0,1,3)$
2. $(0,1,4)$
3. $(0,2,4)$

Isomorphism: 1 $\xrightarrow{\times 5}$ **2**

$g = 2$

1. $(0,1,3)$ $(0,1,4)$
2. $(0,1,3)$ $(0,2,4)$
3. $(0,1,4)$ $(0,2,4)$

Isomorphism: 2 $\xrightarrow{\times 5}$ **3**
Copy of $\mathcal{H}_6^{\circlearrowright}$ **in 1:** $(0,1,2,3,4,5)$

Appendix A.3. $n = 7$

Number of types: 5

Groups not generating any monostar-free systems: $g = 1,\ 4$

Number of monostar-free systems, $g \to \#_g$**:**
$2 \to 1 \quad 3 \to 3$

Detailed list:
$g = 2$

1. $(0,1,3)$ $(0,1,5)$

Copy of $\mathcal{H}_6^{\circlearrowright}$ **in 2:** $(0,1,6,3,2,4)$

$g = 3$

1. $(0,1,2)$ $(0,1,3)$ $(0,1,5)$
2. $(0,1,3)$ $(0,1,4)$ $(0,1,5)$
3. $(0,1,3)$ $(0,1,5)$ $(0,2,4)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{3} \xrightarrow{\times 2} \mathbf{2}$
Copy of $\mathcal{H}_5^{\circlearrowleft}$ **in 1:** $(0,1,2,4,5)$

Appendix A.4. $n = 8$

Number of types: 7

Groups not generating any monostar-free systems: $g = 4,\ 5,\ 6$

Number of monostar-free systems, $g \to \#_g$**:**
$1 \to 3 \quad 2 \to 2 \quad 3 \to 1$

Detailed list:

$g = 1$

1. $(0,1,4)$
2. $(0,1,5)$
3. $(0,2,4)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2}$

$g = 2$

1. $(0,1,4)$ $(0,2,4)$
2. $(0,1,5)$ $(0,2,4)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 7} \mathbf{2}$

$g = 3$

1. $(0,1,4)$ $(0,1,5)$ $(0,2,4)$

Appendix A.5. $n = 9$

Number of types: 10

Groups not generating any monostar-free systems: $g = 2,\ 3,\ 5,\ 8,\ 9$

Number of monostar-free systems, $g \to \#_g$**:**
$1 \to 1 \quad 4 \to 2 \quad 6 \to 3 \quad 7 \to 1$

Detailed list:

$g = 1$

1. $(0,3,6)$

$g = 4$

1. $(0,1,3)$ $(0,1,6)$ $(0,2,5)$ $(0,3,6)$
2. $(0,1,4)$ $(0,1,7)$ $(0,2,6)$ $(0,3,6)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{2}$

$g = 6$

1. $(0,1,2)$ $(0,1,3)$ $(0,1,4)$ $(0,1,6)$ $(0,1,7)$ $(0,2,4)$
2. $(0,1,2)$ $(0,1,4)$ $(0,1,5)$ $(0,1,6)$ $(0,2,5)$ $(0,2,6)$
3. $(0,1,3)$ $(0,1,5)$ $(0,1,7)$ $(0,2,4)$ $(0,2,5)$ $(0,2,6)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{3} \xrightarrow{\times 2} \mathbf{2}$

Copy of $\mathcal{H}_5^{\circlearrowleft}$ **in 1:** $(0,1,2,5,6)$

$g = 7$

1. $(0,1,3)$ $(0,1,4)$ $(0,1,6)$ $(0,1,7)$ $(0,2,5)$ $(0,2,6)$ $(0,3,6)$

Appendix A.6. $n = 10$

Number of types: 12

Groups not generating any monostar-free systems: $g = 5,\ 6,\ 7,\ 9,\ 10,\ 11$

Number of monostar-free systems, $g \to \#_g$ **:**
$1 \to 6 \quad 2 \to 5 \quad 3 \to 4 \quad 4 \to 1 \quad 8 \to 2$

Detailed list:

$g = 1$

1. $(0,1,5)$
2. $(0,1,6)$
3. $(0,2,4)$
4. $(0,2,5)$
5. $(0,2,6)$
6. $(0,2,7)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{6} \xrightarrow{\times 3} \mathbf{2} \xrightarrow{\times 3} \mathbf{4}\,\mathbf{;}\ \ \mathbf{3} \xrightarrow{\times 3} \mathbf{5}$

Copy of $\mathcal{H}_5^{\circlearrowleft}$ **in 3:** $(0,2,4,6,8)$

$g = 2$

1. $(0,1,5)$ $(0,2,5)$
2. $(0,1,5)$ $(0,2,7)$
3. $(0,1,6)$ $(0,2,5)$
4. $(0,1,6)$ $(0,2,7)$
5. $(0,2,4)$ $(0,2,6)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2} \xrightarrow{\times 3} \mathbf{4} \xrightarrow{\times 3} \mathbf{3}$

$g = 3$

1. $(0,1,5)$ $(0,1,6)$ $(0,2,5)$
2. $(0,1,5)$ $(0,1,6)$ $(0,2,7)$
3. $(0,1,5)$ $(0,2,5)$ $(0,2,7)$
4. $(0,1,6)$ $(0,2,5)$ $(0,2,7)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{3} \xrightarrow{\times 3} \mathbf{2} \xrightarrow{\times 3} \mathbf{4}$

$g = 4$

1. $(0,1,5)$ $(0,1,6)$ $(0,2,5)$ $(0,2,7)$

$g = 8$

1. $(0,1,2)$ $(0,1,4)$ $(0,1,5)$ $(0,1,6)$ $(0,1,7)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$
2. $(0,1,3)$ $(0,1,5)$ $(0,1,6)$ $(0,1,8)$ $(0,2,4)$ $(0,2,5)$ $(0,2,7)$ $(0,3,6)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2}$
Copy of $\mathcal{H}_5^{\circlearrowleft}$ **in 1:** $(0,1,2,3,4)$

Appendix A.7. $n = 11$

Number of types: 15

Groups not generating any monostar-free systems: all $g \leq 14$ except $g = 10$

Number of monostar-free systems, $g \to \#_g$ **:**
$10 \to 5$

Detailed list:

$g = 10$

1. $(0,1,2)$ $(0,1,3)$ $(0,1,4)$ $(0,1,5)$ $(0,1,7)$ $(0,1,8)$ $(0,1,9)$ $(0,2,4)$ $(0,2,5)$ $(0,2,8)$
2. $(0,1,2)$ $(0,1,4)$ $(0,1,5)$ $(0,1,6)$ $(0,1,7)$ $(0,1,8)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$ $(0,2,8)$
3. $(0,1,3)$ $(0,1,4)$ $(0,1,5)$ $(0,1,7)$ $(0,1,8)$ $(0,1,9)$ $(0,2,6)$ $(0,2,7)$ $(0,3,6)$ $(0,3,7)$
4. $(0,1,3)$ $(0,1,4)$ $(0,1,6)$ $(0,1,8)$ $(0,1,9)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$ $(0,2,8)$ $(0,3,6)$
5. $(0,1,3)$ $(0,1,5)$ $(0,1,7)$ $(0,1,9)$ $(0,2,4)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$ $(0,2,8)$ $(0,3,7)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{5} \xrightarrow{\times 2} \mathbf{3} \xrightarrow{\times 2} \mathbf{4} \xrightarrow{\times 2} \mathbf{2}$
Copy of $\mathcal{H}_5^{\circlearrowleft}$ in 1: $(0,1,2,6,7)$

Appendix A.8. $n = 12$

Number of types: 19

Groups not generating any monostar-free systems: $g = 12$ and $14 \leq g \leq 18$

Number of monostar-free systems, $g \to \#_g$:
$1 \to 6$ $2 \to 5$ $3 \to 4$ $4 \to 7$ $5 \to 1$ $6 \to 2$ $7 \to 5$ $8 \to 8$ $9 \to 2$ $10 \to 1$ $11 \to 2$ $13 \to 1$

Detailed list:

$g = 1$

1. $(0,1,6)$
2. $(0,1,7)$
3. $(0,2,6)$
4. $(0,2,8)$
5. $(0,3,6)$
6. $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 5} \mathbf{2}\,;$ $\mathbf{3} \xrightarrow{\times 5} \mathbf{4}$

$g = 2$

1. $(0,1,6)$ $(0,2,6)$
2. $(0,1,7)$ $(0,2,8)$
3. $(0,2,6)$ $(0,2,8)$
4. $(0,2,6)$ $(0,4,8)$
5. $(0,2,8)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 5} \mathbf{2}\,;$ $\mathbf{4} \xrightarrow{\times 5} \mathbf{5}$

Copy of $\mathcal{H}_6^{\circlearrowleft}$ in 4: $(0,2,4,6,8,10)$

$g = 3$

1. $(0,1,6)$ $(0,2,6)$ $(0,3,6)$
2. $(0,1,6)$ $(0,2,8)$ $(0,3,6)$
3. $(0,1,7)$ $(0,2,6)$ $(0,3,6)$
4. $(0,1,7)$ $(0,2,8)$ $(0,3,6)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 5} \mathbf{4}\,;$ $\mathbf{2} \xrightarrow{\times 5} \mathbf{3}$

$g = 4$

1. $(0,1,4)$ $(0,1,8)$ $(0,3,7)$ $(0,4,8)$
2. $(0,1,5)$ $(0,1,9)$ $(0,3,8)$ $(0,4,8)$
3. $(0,1,6)$ $(0,1,7)$ $(0,2,6)$ $(0,3,6)$

4. $(0,1,6)$ $(0,1,7)$ $(0,2,8)$ $(0,3,6)$
5. $(0,1,6)$ $(0,2,6)$ $(0,2,8)$ $(0,3,6)$
6. $(0,1,7)$ $(0,2,6)$ $(0,2,8)$ $(0,3,6)$
7. $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 7} \mathbf{2}\,;\ \ \mathbf{3} \xrightarrow{\times 5} \mathbf{4}\,;\ \ \mathbf{5} \xrightarrow{\times 5} \mathbf{6}$

$g = 5$

1. $(0,1,6)$ $(0,1,7)$ $(0,2,6)$ $(0,2,8)$ $(0,3,6)$

$g = 6$

1. $(0,1,6)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,3,6)$ $(0,4,8)$
2. $(0,1,7)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,3,6)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 5} \mathbf{2}$

$g = 7$

1. $(0,1,3)$ $(0,1,6)$ $(0,1,9)$ $(0,2,5)$ $(0,2,8)$ $(0,3,6)$ $(0,3,7)$
2. $(0,1,4)$ $(0,1,7)$ $(0,1,10)$ $(0,2,6)$ $(0,2,9)$ $(0,3,6)$ $(0,3,8)$
3. $(0,1,4)$ $(0,1,8)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,3,7)$ $(0,4,8)$
4. $(0,1,5)$ $(0,1,9)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,3,8)$ $(0,4,8)$
5. $(0,1,6)$ $(0,1,7)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,3,6)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 5} \mathbf{2}\,;\ \ \mathbf{3} \xrightarrow{\times 7} \mathbf{4}$

$g = 8$

1. $(0,1,3)$ $(0,1,5)$ $(0,1,6)$ $(0,1,9)$ $(0,2,5)$ $(0,2,8)$ $(0,3,6)$ $(0,3,7)$
2. $(0,1,3)$ $(0,1,6)$ $(0,1,8)$ $(0,1,9)$ $(0,2,5)$ $(0,2,8)$ $(0,3,6)$ $(0,3,7)$
3. $(0,1,3)$ $(0,1,6)$ $(0,1,9)$ $(0,2,5)$ $(0,2,8)$ $(0,3,6)$ $(0,3,7)$ $(0,4,8)$
4. $(0,1,4)$ $(0,1,5)$ $(0,1,7)$ $(0,1,10)$ $(0,2,6)$ $(0,2,9)$ $(0,3,6)$ $(0,3,8)$
5. $(0,1,4)$ $(0,1,5)$ $(0,1,8)$ $(0,1,9)$ $(0,2,6)$ $(0,3,7)$ $(0,3,8)$ $(0,4,8)$
6. $(0,1,4)$ $(0,1,5)$ $(0,1,8)$ $(0,1,9)$ $(0,2,8)$ $(0,3,7)$ $(0,3,8)$ $(0,4,8)$
7. $(0,1,4)$ $(0,1,7)$ $(0,1,8)$ $(0,1,10)$ $(0,2,6)$ $(0,2,9)$ $(0,3,6)$ $(0,3,8)$
8. $(0,1,4)$ $(0,1,7)$ $(0,1,10)$ $(0,2,6)$ $(0,2,9)$ $(0,3,6)$ $(0,3,8)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 5} \mathbf{4} \xrightarrow{\times 7} \mathbf{7} \xrightarrow{\times 5} \mathbf{2}\,;\ \ \mathbf{3} \xrightarrow{\times 5} \mathbf{8}\,;\ \ \mathbf{5} \xrightarrow{\times 5} \mathbf{6}$

Copy of $\mathcal{H}_5^{\circlearrowright}$ in 1: $(0,1,5,4,9)$ — **Copy of $\mathcal{H}_6^{\circlearrowright}$ in 1:** $(0,1,8,6,7,2)$

Copy of $\mathcal{H}_6^{\circlearrowright}$ in 3: $(0,1,8,9,4,5)$

Copy of $\mathcal{H}_6^{\circlearrowright}$ in 5: $(0,2,4,6,8,10)$

$g = 9$

1. $(0,1,3)$ $(0,1,6)$ $(0,1,7)$ $(0,1,9)$ $(0,2,5)$ $(0,2,6)$ $(0,2,8)$ $(0,3,6)$ $(0,3,7)$
2. $(0,1,4)$ $(0,1,6)$ $(0,1,7)$ $(0,1,10)$ $(0,2,6)$ $(0,2,8)$ $(0,2,9)$ $(0,3,6)$ $(0,3,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 5} \mathbf{2}$

$g = 10$

1. $(0,1,4)$ $(0,1,5)$ $(0,1,8)$ $(0,1,9)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,3,7)$ $(0,3,8)$ $(0,4,8)$

$g = 11$

1. $(0,1,3)$ $(0,1,6)$ $(0,1,7)$ $(0,1,9)$ $(0,2,4)$ $(0,2,5)$ $(0,2,6)$ $(0,2,8)$ $(0,3,6)$ $(0,3,7)$ $(0,4,8)$
2. $(0,1,4)$ $(0,1,6)$ $(0,1,7)$ $(0,1,10)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,9)$ $(0,3,6)$ $(0,3,8)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 5} \mathbf{2}$

Copy of $\mathcal{H}_6^{\circlearrowleft}$ **in 1:** $(0,1,2,3,4,5)$

$g = 13$

1. $(0,1,3)$ $(0,1,4)$ $(0,1,6)$ $(0,1,7)$ $(0,1,9)$ $(0,1,10)$ $(0,2,5)$ $(0,2,6)$ $(0,2,8)$ $(0,2,9)$ $(0,3,6)$ $(0,3,7)$ $(0,3,8)$

Appendix A.9. $n = 13$

Number of types: 22

Groups not generating any monostar-free systems: all $g \leq 21$ except $g = 15$

Number of monostar-free systems, $g \to \#_g$**:**
$15 \to 6$

Detailed list:
$g = 15$

1. $(0,1,2)$ $(0,1,3)$ $(0,1,4)$ $(0,1,5)$ $(0,1,6)$ $(0,1,8)$ $(0,1,9)$ $(0,1,10)$ $(0,1,11)$ $(0,2,4)$ $(0,2,5)$ $(0,2,6)$ $(0,2,9)$ $(0,2,10)$ $(0,3,6)$
2. $(0,1,2)$ $(0,1,3)$ $(0,1,5)$ $(0,1,6)$ $(0,1,7)$ $(0,1,8)$ $(0,1,9)$ $(0,1,11)$ $(0,2,6)$ $(0,2,7)$ $(0,2,8)$ $(0,2,9)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$
3. $(0,1,2)$ $(0,1,4)$ $(0,1,5)$ $(0,1,6)$ $(0,1,8)$ $(0,1,9)$ $(0,1,10)$ $(0,2,5)$ $(0,2,6)$ $(0,2,9)$ $(0,2,10)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$ $(0,4,8)$
4. $(0,1,3)$ $(0,1,4)$ $(0,1,6)$ $(0,1,7)$ $(0,1,8)$ $(0,1,10)$ $(0,1,11)$ $(0,2,5)$ $(0,2,6)$ $(0,2,9)$ $(0,2,10)$ $(0,3,6)$ $(0,3,7)$ $(0,3,9)$ $(0,4,8)$
5. $(0,1,3)$ $(0,1,5)$ $(0,1,6)$ $(0,1,8)$ $(0,1,9)$ $(0,1,11)$ $(0,2,4)$ $(0,2,5)$ $(0,2,7)$ $(0,2,8)$ $(0,2,10)$ $(0,3,6)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$
6. $(0,1,3)$ $(0,1,5)$ $(0,1,7)$ $(0,1,9)$ $(0,1,11)$ $(0,2,4)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$ $(0,2,8)$ $(0,2,9)$ $(0,2,10)$ $(0,3,7)$ $(0,3,9)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{6} \xrightarrow{\times 2} \mathbf{3} \xrightarrow{\times 2} \mathbf{5} \xrightarrow{\times 2} \mathbf{4} \xrightarrow{\times 2} \mathbf{2}$
Copy of $\mathcal{H}_5^{\circlearrowleft}$ **in 1:** $(0,1,2,7,8)$

Appendix A.10. $n = 14$

Number of types: 26

Groups not generating any monostar-free systems: all $7 \leq g \leq 25$ except $g = 14,\ 18$

Number of monostar-free systems, $g \to \#_g$**:**
$1 \to 6$ $2 \to 7$ $3 \to 11$ $4 \to 12$ $5 \to 7$ $6 \to 1$ $14 \to 1$ $18 \to 3$

Detailed list:
$g = 1$

1. $(0,1,7)$
2. $(0,1,8)$
3. $(0,2,7)$
4. $(0,2,9)$
5. $(0,3,7)$
6. $(0,3,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{5} \xrightarrow{\times 3} \mathbf{3} \xrightarrow{\times 3} \mathbf{2} \xrightarrow{\times 3} \mathbf{6} \xrightarrow{\times 3} \mathbf{4}$

$g = 2$

1. $(0,1,7)$ $(0,2,7)$
2. $(0,1,7)$ $(0,3,10)$
3. $(0,1,8)$ $(0,2,9)$
4. $(0,1,8)$ $(0,3,7)$

5. $(0,2,6)$ $(0,2,10)$
6. $(0,2,7)$ $(0,3,10)$
7. $(0,2,9)$ $(0,3,7)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{4} \xrightarrow{\times 3} \mathbf{6} \xrightarrow{\times 3} \mathbf{3} \xrightarrow{\times 3} \mathbf{2} \xrightarrow{\times 3} \mathbf{7}$

Copy of $\mathcal{H}_6^{\circlearrowright}$ in 5: $(0,2,12,6,4,8)$

$g = 3$

1. $(0,1,7)$ $(0,2,7)$ $(0,3,7)$
2. $(0,1,7)$ $(0,2,7)$ $(0,3,10)$
3. $(0,1,7)$ $(0,2,9)$ $(0,3,7)$
4. $(0,1,7)$ $(0,2,9)$ $(0,3,10)$
5. $(0,1,8)$ $(0,2,7)$ $(0,3,7)$
6. $(0,1,8)$ $(0,2,7)$ $(0,3,10)$
7. $(0,1,8)$ $(0,2,9)$ $(0,3,7)$
8. $(0,1,8)$ $(0,2,9)$ $(0,3,10)$
9. $(0,2,4)$ $(0,2,6)$ $(0,2,10)$
10. $(0,2,6)$ $(0,2,8)$ $(0,2,10)$
11. $(0,2,6)$ $(0,2,10)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{5} \xrightarrow{\times 3} \mathbf{6} \xrightarrow{\times 3} \mathbf{8} \xrightarrow{\times 3} \mathbf{4} \xrightarrow{\times 3} \mathbf{3}$ **;** $\mathbf{2} \xrightarrow{\times 3} \mathbf{7}$ **;** $\mathbf{9} \xrightarrow{\times 3} \mathbf{10} \xrightarrow{\times 3} \mathbf{11}$

Copy of $\mathcal{H}_5^{\circlearrowright}$ in 9: $(0,2,4,8,10)$

$g = 4$

1. $(0,1,7)$ $(0,1,8)$ $(0,2,7)$ $(0,3,7)$
2. $(0,1,7)$ $(0,1,8)$ $(0,2,7)$ $(0,3,10)$
3. $(0,1,7)$ $(0,1,8)$ $(0,2,9)$ $(0,3,7)$
4. $(0,1,7)$ $(0,1,8)$ $(0,2,9)$ $(0,3,10)$
5. $(0,1,7)$ $(0,2,7)$ $(0,2,9)$ $(0,3,7)$
6. $(0,1,7)$ $(0,2,7)$ $(0,2,9)$ $(0,3,10)$
7. $(0,1,7)$ $(0,2,7)$ $(0,3,7)$ $(0,3,10)$
8. $(0,1,7)$ $(0,2,9)$ $(0,3,7)$ $(0,3,10)$
9. $(0,1,8)$ $(0,2,7)$ $(0,2,9)$ $(0,3,7)$
10. $(0,1,8)$ $(0,2,7)$ $(0,2,9)$ $(0,3,10)$
11. $(0,1,8)$ $(0,2,7)$ $(0,3,7)$ $(0,3,10)$
12. $(0,1,8)$ $(0,2,9)$ $(0,3,7)$ $(0,3,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{11} \xrightarrow{\times 3} \mathbf{10} \xrightarrow{\times 3} \mathbf{4} \xrightarrow{\times 3} \mathbf{8} \xrightarrow{\times 3} \mathbf{5}$ **;** $\mathbf{2} \xrightarrow{\times 3} \mathbf{12} \xrightarrow{\times 3} \mathbf{6} \xrightarrow{\times 3} \mathbf{3} \xrightarrow{\times 3} \mathbf{7} \xrightarrow{\times 3} \mathbf{9}$

$g = 5$

1. $(0,1,7)$ $(0,1,8)$ $(0,2,7)$ $(0,2,9)$ $(0,3,7)$
2. $(0,1,7)$ $(0,1,8)$ $(0,2,7)$ $(0,2,9)$ $(0,3,10)$
3. $(0,1,7)$ $(0,1,8)$ $(0,2,7)$ $(0,3,7)$ $(0,3,10)$
4. $(0,1,7)$ $(0,1,8)$ $(0,2,9)$ $(0,3,7)$ $(0,3,10)$
5. $(0,1,7)$ $(0,2,7)$ $(0,2,9)$ $(0,3,7)$ $(0,3,10)$
6. $(0,1,8)$ $(0,2,7)$ $(0,2,9)$ $(0,3,7)$ $(0,3,10)$
7. $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{3} \xrightarrow{\times 3} \mathbf{6} \xrightarrow{\times 3} \mathbf{2} \xrightarrow{\times 3} \mathbf{4} \xrightarrow{\times 3} \mathbf{5}$

$g = 6$

1. $(0,1,7)$ $(0,1,8)$ $(0,2,7)$ $(0,2,9)$ $(0,3,7)$ $(0,3,10)$

$g = 14$

1. $(0,1,3)$ $(0,1,5)$ $(0,1,7)$ $(0,1,8)$ $(0,1,10)$ $(0,1,12)$ $(0,2,6)$ $(0,2,7)$ $(0,2,9)$ $(0,2,10)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$ $(0,3,10)$

Copy of $\mathcal{H}_6^{\circlearrowright}$ in 1: $(0,1,6,3,2,4)$

$g = 18$

1. $(0,1,2)$ $(0,1,3)$ $(0,1,5)$ $(0,1,6)$ $(0,1,7)$ $(0,1,8)$ $(0,1,9)$ $(0,1,10)$ $(0,1,12)$ $(0,2,6)$ $(0,2,7)$ $(0,2,8)$ $(0,2,9)$ $(0,2,10)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$ $(0,3,10)$
2. $(0,1,3)$ $(0,1,4)$ $(0,1,5)$ $(0,1,7)$ $(0,1,8)$ $(0,1,10)$ $(0,1,11)$ $(0,1,12)$ $(0,2,6)$ $(0,2,7)$ $(0,2,9)$ $(0,2,10)$ $(0,3,6)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$ $(0,3,10)$ $(0,4,8)$
3. $(0,1,3)$ $(0,1,5)$ $(0,1,7)$ $(0,1,8)$ $(0,1,10)$ $(0,1,12)$ $(0,2,4)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$ $(0,2,9)$ $(0,2,10)$ $(0,2,11)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$ $(0,3,10)$ $(0,4,9)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{3} \xrightarrow{\times 3} \mathbf{2}$
Copy of $\mathcal{H}_5^{\circlearrowright}$ in 1: $(0,1,2,4,5)$

Appendix A.11. $n = 15$

Number of types: 31

Groups not generating any monostar-free systems: $g = 3,\ 5,\ 6,\ 8,\ 9,\ 11\,;\ 14 \le g \le 20\,;$ $23 \le g \le 30$

Number of monostar-free systems, $g \to \#_g$:
$1 \to 3$ $2 \to 1$ $4 \to 4$ $7 \to 4$ $10 \to 4$ $12 \to 2$ $13 \to 11$ $21 \to 4$ $22 \to 3$

Detailed list:

$g = 1$

1. $(0,3,6)$
2. $(0,3,9)$
3. $(0,5,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{2}$

Copy of $\mathcal{H}_5^{\circlearrowright}$ in 1: $(0,3,6,9,12)$

$g = 2$

1. $(0,3,6)$ $(0,3,9)$

$g = 4$

1. $(0,1,5)$ $(0,1,10)$ $(0,4,9)$ $(0,5,10)$
2. $(0,1,6)$ $(0,1,11)$ $(0,4,10)$ $(0,5,10)$
3. $(0,2,5)$ $(0,2,10)$ $(0,3,8)$ $(0,5,10)$
4. $(0,2,7)$ $(0,2,12)$ $(0,3,10)$ $(0,5,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{3} \xrightarrow{\times 2} \mathbf{2} \xrightarrow{\times 2} \mathbf{4}$

$g = 7$

1. $(0,1,5)$ $(0,1,10)$ $(0,2,5)$ $(0,2,10)$ $(0,3,8)$ $(0,4,9)$ $(0,5,10)$
2. $(0,1,5)$ $(0,1,10)$ $(0,2,7)$ $(0,2,12)$ $(0,3,10)$ $(0,4,9)$ $(0,5,10)$
3. $(0,1,6)$ $(0,1,11)$ $(0,2,5)$ $(0,2,10)$ $(0,3,8)$ $(0,4,10)$ $(0,5,10)$
4. $(0,1,6)$ $(0,1,11)$ $(0,2,7)$ $(0,2,12)$ $(0,3,10)$ $(0,4,10)$ $(0,5,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{3} \xrightarrow{\times 2} \mathbf{4} \xrightarrow{\times 2} \mathbf{2}$

$g = 10$

1. $(0,1,5)$ $(0,1,6)$ $(0,1,10)$ $(0,1,11)$ $(0,2,5)$ $(0,2,10)$ $(0,3,8)$ $(0,4,9)$ $(0,4,10)$ $(0,5,10)$

2. $(0,1,5)$ $(0,1,6)$ $(0,1,10)$ $(0,1,11)$ $(0,2,7)$ $(0,2,12)$ $(0,3,10)$ $(0,4,9)$ $(0,4,10)$ $(0,5,10)$
3. $(0,1,5)$ $(0,1,10)$ $(0,2,5)$ $(0,2,7)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,3,10)$ $(0,4,9)$ $(0,5,10)$
4. $(0,1,6)$ $(0,1,11)$ $(0,2,5)$ $(0,2,7)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,3,10)$ $(0,4,10)$ $(0,5,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{4} \xrightarrow{\times 2} \mathbf{2} \xrightarrow{\times 2} \mathbf{3}$

$g = 12$

1. $(0,1,3)$ $(0,1,6)$ $(0,1,9)$ $(0,1,12)$ $(0,2,5)$ $(0,2,8)$ $(0,2,11)$ $(0,3,6)$ $(0,3,7)$ $(0,3,9)$ $(0,3,10)$ $(0,4,9)$
2. $(0,1,4)$ $(0,1,7)$ $(0,1,10)$ $(0,1,13)$ $(0,2,6)$ $(0,2,9)$ $(0,2,12)$ $(0,3,6)$ $(0,3,8)$ $(0,3,9)$ $(0,3,11)$ $(0,4,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{2}$

$g = 13$

1. $(0,1,3)$ $(0,1,5)$ $(0,1,6)$ $(0,1,9)$ $(0,1,12)$ $(0,2,5)$ $(0,2,8)$ $(0,2,11)$ $(0,3,6)$ $(0,3,7)$ $(0,3,9)$ $(0,3,10)$ $(0,4,9)$
2. $(0,1,3)$ $(0,1,6)$ $(0,1,9)$ $(0,1,11)$ $(0,1,12)$ $(0,2,5)$ $(0,2,8)$ $(0,2,11)$ $(0,3,6)$ $(0,3,7)$ $(0,3,9)$ $(0,3,10)$ $(0,4,9)$
3. $(0,1,3)$ $(0,1,6)$ $(0,1,9)$ $(0,1,12)$ $(0,2,5)$ $(0,2,7)$ $(0,2,8)$ $(0,2,11)$ $(0,3,6)$ $(0,3,7)$ $(0,3,9)$ $(0,3,10)$ $(0,4,9)$
4. $(0,1,3)$ $(0,1,6)$ $(0,1,9)$ $(0,1,12)$ $(0,2,5)$ $(0,2,8)$ $(0,2,10)$ $(0,2,11)$ $(0,3,6)$ $(0,3,7)$ $(0,3,9)$ $(0,3,10)$ $(0,4,9)$
5. $(0,1,3)$ $(0,1,6)$ $(0,1,9)$ $(0,1,12)$ $(0,2,5)$ $(0,2,8)$ $(0,2,11)$ $(0,3,6)$ $(0,3,7)$ $(0,3,9)$ $(0,3,10)$ $(0,4,9)$ $(0,5,10)$
6. $(0,1,4)$ $(0,1,5)$ $(0,1,7)$ $(0,1,10)$ $(0,1,13)$ $(0,2,6)$ $(0,2,9)$ $(0,2,12)$ $(0,3,6)$ $(0,3,8)$ $(0,3,9)$ $(0,3,11)$ $(0,4,10)$
7. $(0,1,4)$ $(0,1,7)$ $(0,1,10)$ $(0,1,11)$ $(0,1,13)$ $(0,2,6)$ $(0,2,9)$ $(0,2,12)$ $(0,3,6)$ $(0,3,8)$ $(0,3,9)$ $(0,3,11)$ $(0,4,10)$
8. $(0,1,4)$ $(0,1,7)$ $(0,1,10)$ $(0,1,13)$ $(0,2,6)$ $(0,2,7)$ $(0,2,9)$ $(0,2,12)$ $(0,3,6)$ $(0,3,8)$ $(0,3,9)$ $(0,3,11)$ $(0,4,10)$
9. $(0,1,4)$ $(0,1,7)$ $(0,1,10)$ $(0,1,13)$ $(0,2,6)$ $(0,2,9)$ $(0,2,10)$ $(0,2,12)$ $(0,3,6)$ $(0,3,8)$ $(0,3,9)$ $(0,3,11)$ $(0,4,10)$
10. $(0,1,4)$ $(0,1,7)$ $(0,1,10)$ $(0,1,13)$ $(0,2,6)$ $(0,2,9)$ $(0,2,12)$ $(0,3,6)$ $(0,3,8)$ $(0,3,9)$ $(0,3,11)$ $(0,4,10)$ $(0,5,10)$
11. $(0,1,5)$ $(0,1,6)$ $(0,1,10)$ $(0,1,11)$ $(0,2,5)$ $(0,2,7)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,3,10)$ $(0,4,9)$ $(0,4,10)$ $(0,5,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{9} \xrightarrow{\times 2} \mathbf{2} \xrightarrow{\times 2} \mathbf{8}$ **;** $\mathbf{3} \xrightarrow{\times 2} \mathbf{6} \xrightarrow{\times 2} \mathbf{4} \xrightarrow{\times 2} \mathbf{7}$ **;** $\mathbf{5} \xrightarrow{\times 2} \mathbf{10}$

Copy of $\mathcal{H}_5^{\circlearrowright}$ in 1: $(0,1,5,4,9)$ — **Copy of $\mathcal{H}_6^{\circlearrowright}$ in 1:** $(0,1,11,6,10,2)$

Copy of $\mathcal{H}_5^{\circlearrowright}$ in 3: $(0,2,5,12,7)$ — **Copy of $\mathcal{H}_6^{\circlearrowright}$ in 3:** $(0,1,2,3,7,8)$

Copy of $\mathcal{H}_6^{\circlearrowright}$ in 5: $(0,1,5,6,10,11)$

$g = 21$

1. $(0,1,2)$ $(0,1,3)$ $(0,1,4)$ $(0,1,5)$ $(0,1,6)$ $(0,1,7)$ $(0,1,9)$ $(0,1,10)$ $(0,1,11)$ $(0,1,12)$ $(0,1,13)$ $(0,2,4)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$ $(0,2,10)$ $(0,2,11)$ $(0,2,12)$ $(0,3,6)$ $(0,3,7)$ $(0,3,11)$
2. $(0,1,2)$ $(0,1,3)$ $(0,1,5)$ $(0,1,6)$ $(0,1,7)$ $(0,1,8)$ $(0,1,9)$ $(0,1,10)$ $(0,1,11)$ $(0,1,13)$ $(0,2,6)$ $(0,2,7)$ $(0,2,8)$ $(0,2,9)$ $(0,2,10)$ $(0,2,11)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$ $(0,3,10)$ $(0,3,11)$

3. $(0,1,3)$ $(0,1,4)$ $(0,1,5)$ $(0,1,7)$ $(0,1,8)$ $(0,1,9)$ $(0,1,11)$ $(0,1,12)$ $(0,1,13)$ $(0,2,6)$ $(0,2,7)$ $(0,2,10)$ $(0,2,11)$ $(0,3,6)$ $(0,3,7)$ $(0,3,8)$ $(0,3,10)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$ $(0,4,10)$
4. $(0,1,3)$ $(0,1,5)$ $(0,1,7)$ $(0,1,9)$ $(0,1,11)$ $(0,1,13)$ $(0,2,4)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$ $(0,2,8)$ $(0,2,9)$ $(0,2,10)$ $(0,2,11)$ $(0,2,12)$ $(0,3,7)$ $(0,3,9)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$ $(0,4,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{4} \xrightarrow{\times 2} \mathbf{3} \xrightarrow{\times 2} \mathbf{2}$

Copy of $\mathcal{H}_5^{\circlearrowright}$ in 1: $(0,1,2,8,9)$

$g = 22$

1. $(0,1,2)$ $(0,1,4)$ $(0,1,5)$ $(0,1,6)$ $(0,1,7)$ $(0,1,9)$ $(0,1,10)$ $(0,1,11)$ $(0,1,12)$ $(0,2,5)$ $(0,2,6)$ $(0,2,7)$ $(0,2,10)$ $(0,2,11)$ $(0,2,12)$ $(0,3,8)$ $(0,3,9)$ $(0,3,10)$ $(0,4,8)$ $(0,4,9)$ $(0,4,10)$ $(0,5,10)$
2. $(0,1,3)$ $(0,1,4)$ $(0,1,6)$ $(0,1,7)$ $(0,1,9)$ $(0,1,10)$ $(0,1,12)$ $(0,1,13)$ $(0,2,5)$ $(0,2,6)$ $(0,2,8)$ $(0,2,9)$ $(0,2,11)$ $(0,2,12)$ $(0,3,6)$ $(0,3,7)$ $(0,3,8)$ $(0,3,9)$ $(0,3,10)$ $(0,3,11)$ $(0,4,9)$ $(0,4,10)$
3. $(0,1,3)$ $(0,1,5)$ $(0,1,6)$ $(0,1,8)$ $(0,1,10)$ $(0,1,11)$ $(0,1,13)$ $(0,2,4)$ $(0,2,5)$ $(0,2,7)$ $(0,2,8)$ $(0,2,9)$ $(0,2,10)$ $(0,2,12)$ $(0,3,6)$ $(0,3,7)$ $(0,3,8)$ $(0,3,10)$ $(0,3,11)$ $(0,4,9)$ $(0,4,10)$ $(0,5,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 2} \mathbf{3}$
Copy of $\mathcal{H}_5^{\circlearrowright}$ in 1: $(0,1,2,3,4)$

Appendix A.12. $n = 16$

Number of types: 35

Groups not generating any monostar-free systems: $g = 8,\ 12,\ 14,\ 16,\ 17,\ 18;\ 20 \le g \le 34$

Number of monostar-free systems, $g \to \#_g$ **:**
$1 \to 7$ $2 \to 6$ $3 \to 7$ $4 \to 8$ $5 \to 12$ $6 \to 6$ $7 \to 4$ $9 \to 2$ $10 \to 8$ $11 \to 3$ $13 \to 2$ $15 \to 2$ $19 \to 1$

Detailed list:

$g = 1$

1. $(0,1,8)$
2. $(0,1,9)$
3. $(0,2,8)$
4. $(0,2,10)$
5. $(0,3,8)$
6. $(0,3,11)$
7. $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{5} \xrightarrow{\times 5} \mathbf{2} \xrightarrow{\times 3} \mathbf{6}\,;\ \ \mathbf{3} \xrightarrow{\times 3} \mathbf{4}$

$g = 2$

1. $(0,1,8)$ $(0,2,8)$
2. $(0,1,9)$ $(0,2,10)$
3. $(0,2,8)$ $(0,3,11)$
4. $(0,2,8)$ $(0,4,8)$
5. $(0,2,10)$ $(0,3,8)$
6. $(0,2,10)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{5} \xrightarrow{\times 5} \mathbf{2} \xrightarrow{\times 3} \mathbf{3}\,;\ \ \mathbf{4} \xrightarrow{\times 3} \mathbf{6}$

$g = 3$

1. $(0,1,8)$ $(0,2,8)$ $(0,3,8)$
2. $(0,1,8)$ $(0,2,10)$ $(0,3,8)$
3. $(0,1,8)$ $(0,3,11)$ $(0,4,8)$
4. $(0,1,9)$ $(0,2,8)$ $(0,3,11)$
5. $(0,1,9)$ $(0,2,10)$ $(0,3,11)$
6. $(0,1,9)$ $(0,3,8)$ $(0,4,8)$
7. $(0,2,8)$ $(0,2,10)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2} \xrightarrow{\times 5} \mathbf{5} \xrightarrow{\times 3} \mathbf{4}\,\mathbf{;}\ \ \mathbf{3} \xrightarrow{\times 3} \mathbf{6}$

$g = 4$

1. $(0,1,8)$ $(0,2,8)$ $(0,3,8)$ $(0,4,8)$
2. $(0,1,8)$ $(0,2,8)$ $(0,3,11)$ $(0,4,8)$
3. $(0,1,8)$ $(0,2,10)$ $(0,3,8)$ $(0,4,8)$
4. $(0,1,8)$ $(0,2,10)$ $(0,3,11)$ $(0,4,8)$
5. $(0,1,9)$ $(0,2,8)$ $(0,3,8)$ $(0,4,8)$
6. $(0,1,9)$ $(0,2,8)$ $(0,3,11)$ $(0,4,8)$
7. $(0,1,9)$ $(0,2,10)$ $(0,3,8)$ $(0,4,8)$
8. $(0,1,9)$ $(0,2,10)$ $(0,3,11)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{3} \xrightarrow{\times 5} \mathbf{8} \xrightarrow{\times 3} \mathbf{6}\,\mathbf{;}\ \ \mathbf{2} \xrightarrow{\times 3} \mathbf{7}\,\mathbf{;}\ \ \mathbf{4} \xrightarrow{\times 3} \mathbf{5}$

$g = 5$

1. $(0,1,8)$ $(0,1,9)$ $(0,2,8)$ $(0,3,8)$ $(0,4,8)$
2. $(0,1,8)$ $(0,1,9)$ $(0,2,8)$ $(0,3,11)$ $(0,4,8)$
3. $(0,1,8)$ $(0,1,9)$ $(0,2,10)$ $(0,3,8)$ $(0,4,8)$
4. $(0,1,8)$ $(0,1,9)$ $(0,2,10)$ $(0,3,11)$ $(0,4,8)$
5. $(0,1,8)$ $(0,2,8)$ $(0,2,10)$ $(0,3,8)$ $(0,4,8)$
6. $(0,1,8)$ $(0,2,8)$ $(0,2,10)$ $(0,3,11)$ $(0,4,8)$
7. $(0,1,8)$ $(0,2,8)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$
8. $(0,1,8)$ $(0,2,10)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$
9. $(0,1,9)$ $(0,2,8)$ $(0,2,10)$ $(0,3,8)$ $(0,4,8)$
10. $(0,1,9)$ $(0,2,8)$ $(0,2,10)$ $(0,3,11)$ $(0,4,8)$
11. $(0,1,9)$ $(0,2,8)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$
12. $(0,1,9)$ $(0,2,10)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{8} \xrightarrow{\times 5} \mathbf{4} \xrightarrow{\times 3} \mathbf{11}\,\mathbf{;}\ \ \mathbf{2} \xrightarrow{\times 3} \mathbf{12} \xrightarrow{\times 5} \mathbf{3} \xrightarrow{\times 3} \mathbf{7}\,\mathbf{;}\ \ \mathbf{5} \xrightarrow{\times 5} \mathbf{10}\,\mathbf{;}$ $\mathbf{6} \xrightarrow{\times 3} \mathbf{9}$

$g = 6$

1. $(0,1,8)$ $(0,1,9)$ $(0,2,8)$ $(0,2,10)$ $(0,3,8)$ $(0,4,8)$
2. $(0,1,8)$ $(0,1,9)$ $(0,2,8)$ $(0,2,10)$ $(0,3,11)$ $(0,4,8)$
3. $(0,1,8)$ $(0,1,9)$ $(0,2,8)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$
4. $(0,1,8)$ $(0,1,9)$ $(0,2,10)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$
5. $(0,1,8)$ $(0,2,8)$ $(0,2,10)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$
6. $(0,1,9)$ $(0,2,8)$ $(0,2,10)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{5} \xrightarrow{\times 5} \mathbf{2} \xrightarrow{\times 3} \mathbf{6}\,\mathbf{;}\ \ \mathbf{3} \xrightarrow{\times 3} \mathbf{4}$

$g = 7$

1. $(0,1,4)$ $(0,1,8)$ $(0,1,12)$ $(0,3,7)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$
2. $(0,1,5)$ $(0,1,9)$ $(0,1,13)$ $(0,3,8)$ $(0,3,12)$ $(0,4,8)$ $(0,4,11)$
3. $(0,1,8)$ $(0,1,9)$ $(0,2,8)$ $(0,2,10)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$
4. $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,4,8)$ $(0,4,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2}$

$g = 9$

1. $(0,1,4)$ $(0,1,8)$ $(0,1,12)$ $(0,2,8)$ $(0,2,10)$ $(0,3,7)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$
2. $(0,1,5)$ $(0,1,9)$ $(0,1,13)$ $(0,2,8)$ $(0,2,10)$ $(0,3,8)$ $(0,3,12)$ $(0,4,8)$ $(0,4,11)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2}$

$g = 10$

1. $(0,1,4)$ $(0,1,8)$ $(0,1,9)$ $(0,1,12)$ $(0,2,8)$ $(0,3,7)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$
2. $(0,1,4)$ $(0,1,8)$ $(0,1,9)$ $(0,1,12)$ $(0,2,10)$ $(0,3,7)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$
3. $(0,1,5)$ $(0,1,8)$ $(0,1,9)$ $(0,1,13)$ $(0,2,8)$ $(0,3,8)$ $(0,3,11)$ $(0,3,12)$ $(0,4,8)$ $(0,4,11)$
4. $(0,1,5)$ $(0,1,8)$ $(0,1,9)$ $(0,1,13)$ $(0,2,10)$ $(0,3,8)$ $(0,3,11)$ $(0,3,12)$ $(0,4,8)$ $(0,4,11)$
5. $(0,1,8)$ $(0,1,9)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,4,8)$ $(0,4,10)$
6. $(0,1,8)$ $(0,1,9)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,11)$ $(0,4,8)$ $(0,4,10)$
7. $(0,1,8)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$ $(0,4,10)$
8. $(0,1,9)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$ $(0,4,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{4}$; $\mathbf{2} \xrightarrow{\times 3} \mathbf{3}$; $\mathbf{5} \xrightarrow{\times 3} \mathbf{7} \xrightarrow{\times 5} \mathbf{6} \xrightarrow{\times 3} \mathbf{8}$

$g = 11$

1. $(0,1,4)$ $(0,1,8)$ $(0,1,9)$ $(0,1,12)$ $(0,2,8)$ $(0,2,10)$ $(0,3,7)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$
2. $(0,1,5)$ $(0,1,8)$ $(0,1,9)$ $(0,1,13)$ $(0,2,8)$ $(0,2,10)$ $(0,3,8)$ $(0,3,11)$ $(0,3,12)$ $(0,4,8)$ $(0,4,11)$
3. $(0,1,8)$ $(0,1,9)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$ $(0,4,10)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2}$

$g = 13$

1. $(0,1,4)$ $(0,1,8)$ $(0,1,12)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,7)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$ $(0,4,10)$
2. $(0,1,5)$ $(0,1,9)$ $(0,1,13)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,3,12)$ $(0,4,8)$ $(0,4,10)$ $(0,4,11)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2}$

$g = 15$

1. $(0,1,4)$ $(0,1,8)$ $(0,1,9)$ $(0,1,12)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,7)$ $(0,3,8)$ $(0,3,11)$ $(0,4,8)$ $(0,4,9)$ $(0,4,10)$
2. $(0,1,5)$ $(0,1,8)$ $(0,1,9)$ $(0,1,13)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,8)$ $(0,3,11)$ $(0,3,12)$ $(0,4,8)$ $(0,4,10)$ $(0,4,11)$

Isomorphism: $\mathbf{1} \xrightarrow{\times 3} \mathbf{2}$

$g = 19$

1. $(0,1,4)$ $(0,1,5)$ $(0,1,8)$ $(0,1,9)$ $(0,1,12)$ $(0,1,13)$ $(0,2,4)$ $(0,2,6)$ $(0,2,8)$ $(0,2,10)$ $(0,2,12)$ $(0,3,7)$ $(0,3,8)$ $(0,3,11)$ $(0,3,12)$ $(0,4,8)$ $(0,4,9)$ $(0,4,10)$ $(0,4,11)$

In Table A1, each pair $i\ j$ of numbers toghether in a cell means the triplet type $(0, i, j)$. The first column contains the 19 types (out of the 35 existing ones) that occur in monostar-free combinations for $n = 16$. The subsequent columns $3, 5, \ldots, 15$ exhibit the types obtained when the first type in that row is multiplied by the corresponding number.

Table A1. Triplet types and the types of their odd multiples over the vertex set $\mathbb{Z}_{16}$. For simplicity, each type $(0, i, j)$ is abbreviated as $i\ j$.

1	**3**	**5**	**7**	**9**	**11**	**13**	**15**
1 4	3 12	1 12	4 11	4 9	1 5	3 7	1 13
1 5	1 4	4 11	3 7	3 12	4 9	1 13	1 12
1 8	3 8	3 11	1 9	1 8	3 8	3 11	1 9
1 9	3 11	3 8	1 8	1 9	3 11	3 8	1 8
1 12	1 13	4 9	3 12	3 7	4 11	1 4	1 5
1 13	3 7	1 5	4 9	4 11	1 12	3 12	1 4
2 4	4 10	4 10	2 4	2 4	4 10	4 10	2 4
2 6	2 6	2 12	2 12	2 6	2 6	2 12	2 12
2 8	2 10	2 8	2 10	2 8	2 10	2 8	2 10
2 10	2 8	2 10	2 8	2 10	2 8	2 10	2 8
2 12	2 12	2 6	2 6	2 12	2 12	2 6	2 6
3 7	4 11	1 4	1 5	1 12	1 13	4 9	3 12
3 8	1 8	1 9	3 11	3 8	1 8	1 9	3 11
3 11	1 9	1 8	3 8	3 11	1 9	1 8	3 8
3 12	4 9	1 13	1 12	1 5	1 4	4 11	3 7
4 8	4 8	4 8	4 8	4 8	4 8	4 8	4 8
4 9	1 5	3 7	1 13	1 4	3 12	1 12	4 11
4 10	2 4	2 4	4 10	4 10	2 4	2 4	4 10
4 11	1 12	3 12	1 4	1 13	3 7	1 5	4 9

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