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Some Refinements of Selberg Inequality and Related Results

Najla Altwaijry [1,†], Cristian Conde [2,3,†], Silvestru Sever Dragomir [4,†] and Kais Feki [5,6,*,†]

[1] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[2] National Scientific and Technical Research Council, Buenos Aires C1425FQB, Argentina; cconde@campus.ungs.edu.ar
[3] Sciences Institute, National University of General Sarmiento, J. M. Gutierrez 1150, Los Polvorines B1613GSX, Argentina
[4] Mathematics, College of Sport, Health and Engineering, Victoria University, P.O. Box 14428, Melbourne City, VIC 8001, Australia; sever.dragomir@vu.edu.au
[5] Faculty of Economic Sciences and Management of Mahdia, University of Monastir, Mahdia 5111, Tunisia
[6] Laboratory Physics-Mathematics and Applications (LR/13/ES-22), Faculty of Sciences of Sfax, University of Sfax, Sfax 3018, Tunisia
* Correspondence: kais.feki@fsegma.u-monastir.tn
† These authors contributed equally to this work.

Abstract: This paper introduces several refinements of the classical Selberg inequality, which is considered a significant result in the study of the spectral theory of symmetric spaces, a central topic in the field of symmetry studies. By utilizing the contraction property of the Selberg operator, we derive improved versions of the classical Selberg inequality. Additionally, we demonstrate the interdependence among well-known inequalities such as Cauchy–Schwarz, Bessel, and the Selberg inequality, revealing that these inequalities can be deduced from one another. This study showcases the enhancements made to the classical Selberg inequality and establishes the interconnectedness of various mathematical inequalities.

Keywords: inner product space; Cauchy–Schwarz inequality; Selberg inequality; orthogonal projection

MSC: 47A63; 47A12; 47A05; 47A30

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1. Introduction

In mathematics, inequalities have played a prominent role across various branches for an extensive period. A significant milestone in the study of inequalities was the publication of “Inequalities” by G. H. Hardy, J. Littlewood, and J. Polya in 1934 [1]. This seminal work not only shaped the field but also provided valuable insights, techniques, and applications, establishing inequalities as a well-structured discipline. Another noteworthy contribution came in 1961 when Edwin F. Beckenbach and R. Bellman authored a significant book on the subject [2]. This publication further enriched the field of inequalities, reinforcing its importance and offering additional perspectives for research exploration. These important publications have greatly influenced the study of inequalities, laying a strong foundation and inspiring more research in the field. For more details, readers can consult the references mentioned. Inspired by the long history of inequalities and their practical applications, this paper aims to enhance the classical Selberg inequality. Our objective is to deepen our understanding of this inequality and explore its implications. Before delving into our main focus, it is worthwhile to review well-known and widely studied inequalities in inner product spaces, which can be either real or complex. For simplicity, we consider our space E as a complex Hilbert space with an inner product denoted as $\langle\cdot,\cdot\rangle$, and the corresponding norm as $\|\cdot\|$. One of the fundamental inequalities in inner product spaces is the Cauchy–Schwarz inequality (CSI), which is highly important and widely applicable. It can be expressed as follows:

$$|\langle u, v\rangle| \leq \|u\|\|v\|, \tag{1}$$

for any $u, v \in$ E. Equality in (1) occurs if and only if u is a scalar multiple of v, where the scalar is a complex number $\gamma \in \mathbb{C}$; namely, $u = \gamma v$.

Buzano [3] derived an extension of the Cauchy–Schwarz inequality, called the Buzano inequality (BuI), in which:

$$|\langle u, z\rangle\langle z, v\rangle| \leq \frac{1}{2}\big(|\langle u, v\rangle| + \|u\|\|v\|\big)\|z\|^2,$$

holds for any $u, z, v \in$ E. The Buzano inequality is an important extension of the Cauchy–Schwarz inequality, and it has significant implications in various mathematical contexts. Additionally, Fujii and Kubo [4] presented a simple proof of the Buzano inequality (BuI) by using an orthogonal projection onto a subspace of E and the Cauchy–Schwarz inequality (CSI). They also provided conditions that determine when equality is achieved in the inequality.

Furthermore, the significance of Bessel's inequality (BeI) in the field of functional analysis is well-known. This fundamental result has important applications in various areas of mathematics and engineering. Bessel's inequality states that for any set of orthonormal vectors $e_1, e_2, \ldots, e_n$ in E (i.e., $\langle e_p, e_q\rangle = \delta_{pq}$ for all $p, q \in \{1, \ldots, n\}$, where δ_{pq} is the Kronecker delta symbol), the following inequality can be found in [5] and holds for any vector $u \in$ E:

$$\sum_{p=1}^{n} |\langle u, e_p\rangle|^2 \leq \|u\|^2. \tag{2}$$

Additional results related to Bessel's inequality can be found in references [5–7], for readers who are interested in exploring this topic further.

A. Selberg made a noteworthy discovery in the generalization of Bessel's inequality, which can be found in [5]. If we consider vectors $u, z_1, \ldots, z_n$ in E, where $z_p \neq 0$ for all $p \in 1, \ldots, n$, we can invoke Selberg's inequality (SI), which asserts that:

$$\sum_{p=1}^{n} \frac{|\langle u, z_p\rangle|^2}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|} \leq \|u\|^2. \tag{3}$$

An important observation is, if the vectors z_p are orthonormal for all $p \in \{1, \ldots, n\}$, inequality (3) simplifies to Bessel's inequality (2). Selberg's inequality has many practical applications in the fields of harmonic analysis and mathematical physics, and has been studied extensively by researchers. For example, significant works such as [8,9] have explored the implications and uses of Selberg's inequality. This inequality is closely connected to the concept of symmetry, particularly to the theory of automorphic forms and the study of symmetric spaces [10,11]. Automorphic forms are functions on symmetric spaces that remain unchanged under a group of symmetries, like the group of isometries of a hyperbolic space or the group of unitary matrices in n dimensions [12]. The Selberg inequality provides a way to estimate the size of certain functions on symmetric spaces, which is closely related to the distribution of eigenvalues of the Laplacian operator on the space [13]. As a result, the Selberg inequality is an important result in the study of the spectral theory of symmetric spaces, which is a central topic in the field of symmetry studies [14,15].

It should be highlighted that equality in (3) is satisfied if and only if $u = \sum_{i=1}^{n} a_i z_i$ for complex scalars $a_1, \cdots, a_n$ that meet certain conditions. Specifically, for any $i \neq j$, we have $\langle z_i, z_j\rangle = 0$ or $|a_i| = |a_j|$ with $\langle a_i z_i, a_j z_j\rangle \geq 0$ (refer to Theorem 1 in [16]).

Furthermore, the use of inequality (3) can lead to the derivation of the Bombieri Inequality ([17]). Specifically, if we consider vectors $u, z_1, \ldots, z_n$ in E, Bombieri's inequality asserts that:

$$\sum_{p=1}^{n} |\langle u, z_p \rangle|^2 \leq \|u\|^2 \max_{1 \leq p \leq n} \sum_{q=1}^{n} |\langle z_p, z_q \rangle|.$$

In [18], a refinement of the Selberg inequality is presented. Specifically, the authors consider vectors u, v, $z_1, \ldots, z_n$ in E, where $z_p \neq 0$, and $\langle v, z_p \rangle = 0$ for all $p \in \{1, \ldots, n\}$. Under these conditions, the inequality can be expressed as:

$$|\langle u, v \rangle|^2 + \sum_{p=1}^{n} \frac{|\langle u, z_p \rangle|^2}{\sum_{q=1}^{n} |\langle z_p, z_q \rangle|} \|v\|^2 \leq \|u\|^2 \|v\|^2. \tag{4}$$

The Selberg inequality is an important mathematical result that has many applications in fields like number theory and harmonic analysis, particularly in the study of symmetric spaces and automorphic forms. However, it has some limitations that make it less useful in certain situations. For example, it only applies to certain types of functions that exhibit certain symmetries, which limits its usefulness in more general settings. In our paper, we aim to improve the Selberg inequality by using the fact that the Selberg operator is a contraction. This allows us to create new and better versions of the Selberg inequality that can be used in more situations and give us new insights into the Selberg operator. We also explore the connections between well-known inequalities like the Cauchy–Schwarz, Bessel, and Selberg inequalities, showing how they can be used to create new results. Our research has the potential to improve our understanding of mathematics and have applications in fields like analysis and number theory.

2. Generalized Selberg Inequality

Throughout this work, we denote by E a complex and infinite-dimensional Hilbert space. The C^*-algebra of all bounded linear operators acting on E is represented by $L(\mathrm{E})$. The inner product on E is denoted by $\langle \cdot, \cdot \rangle$, and the corresponding norm is denoted by $\| \cdot \|$. The identity operator on E is represented by I. For any operator $T \in L(\mathrm{E})$, we denote its nullspace as $\mathcal{N}(T)$, and its adjoint by T^*. We define a positive operator as $T \geq 0$, signifying that $\langle Tu, u \rangle \geq 0$ for all $u \in \mathrm{E}$. Furthermore, an order relation $T \geq S$ is introduced for self-adjoint operators, which holds when $T - S \geq 0$.

Assuming T is a positive operator, the operator Cauchy–Schwarz inequality can be applied:

$$|\langle Tu, v \rangle| \leq \langle Tu, u \rangle^{\frac{1}{2}} \langle Tv, v \rangle^{\frac{1}{2}}, \tag{5}$$

where $u, v \in \mathrm{E}$. Additionally, we can derive the following result:

$$\|Tu\|^2 \leq \|T\| \langle Tu, u \rangle, \tag{6}$$

which is valid for any positive operator T and any vector $u \in \mathrm{E}$.

In the upcoming proposition, we introduce an improved form of inequality (5).

Proposition 1. *Assuming that T is a positive operator in $L(\mathrm{E})$ and $\beta \in [0, 1]$, we have:*

$$\begin{aligned} |\langle Tu, v \rangle|^2 &\leq (1 - \beta) \langle Tu, u \rangle^{\frac{1}{2}} \langle Tv, v \rangle^{\frac{1}{2}} |\langle Tu, v \rangle| + \beta \langle Tu, u \rangle \langle Tv, v \rangle \\ &\leq \langle Tu, u \rangle \langle Tv, v \rangle. \end{aligned} \tag{7}$$

Proof. Assume that T is a positive operator in $L(\mathrm{E})$, and let $\beta \in [0,1]$. Utilizing the inequality (5), we obtain the following for any $u, v \in \mathrm{E}$:

$$\begin{aligned}\langle Tu,u\rangle\langle Tv,v\rangle &= (1-\beta)\langle Tu,u\rangle\langle Tv,v\rangle + \beta\langle Tu,u\rangle\langle Tv,v\rangle \\ &\geq (1-\beta)\langle Tu,u\rangle^{\frac{1}{2}}\langle Tv,v\rangle^{\frac{1}{2}}|\langle Tu,v\rangle| + \beta\langle Tu,u\rangle\langle Tv,v\rangle \\ &\geq (1-\beta)|\langle Tu,v\rangle|^2 + \beta\langle Tu,u\rangle\langle Tv,v\rangle \\ &\geq \big[(1-\beta)+\beta\big]|\langle Tu,v\rangle|^2 = |\langle Tu,v\rangle|^2.\end{aligned}$$

Hence, we have obtained the desired result. □

The set $W(S)$, also known as the numerical range, is obtained by applying the quadratic form $u \to \langle Su,u\rangle$ to the unit sphere of a space E, where S belongs to the set $L(\mathrm{E})$. To put it simply, $W(S)$ is the set of all values obtained by taking the inner product of Su with u, where u is a unit vector in E. The numerical range is a reflection of certain geometric properties associated with the operator and is a subset of the complex plane. The Toeplitz–Hausdorff Theorem establishes that $W(S)$ is a convex set. The numerical radius, also known as $\omega(S)$, is the maximum absolute value of the numbers in the numerical range $W(S)$, and it is defined as follows:

$$\omega(S) = \sup\{|\mu| : \mu \in W(S)\}.$$

Before delving into the upcoming discussion, it is crucial to recall that the notation $u \otimes v$ denotes a rank-one operator, which is defined as $u \otimes v(z) = \langle z,v\rangle u$. Here, u, v, and z are vectors in the space E. Now, we will introduce the Selberg operator, denoted as $S_{\mathcal{Z}}$, which is defined as follows:

Definition 1. *Given a subset $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ of nonzero vectors in the space* E, *the Selberg operator $S_{\mathcal{Z}}$ is defined by*

$$S_{\mathcal{Z}} = \sum_{p=1}^{n} \frac{z_p \otimes z_p}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|} \in L(\mathrm{E}).$$

Let us draw attention to the significance of the following remark.

Remark 1. *(1)* *Utilizing the Selberg operator, we can rephrase the statement (SI) as follows:*

$$0 \leq \langle S_{\mathcal{Z}}u,u\rangle = \sum_{p=1}^{n} \frac{|\langle u,z_p\rangle|^2}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|} \leq \langle u,u\rangle,$$

for any $u \in$ E. As a consequence, we can conclude that all Selberg operators are positive contractions, denoting that $0 \leq S_{\mathcal{Z}} \leq I$. Moreover, this operator inequality allows us to infer the following:

$$0 \leq I - S_{\mathcal{Z}} \leq I. \tag{8}$$

(2) *It follows from* (8) *that*

$$\omega(I - S_{\mathcal{Z}}) = \|I - S_{\mathcal{Z}}\| \leq 1.$$

In this article, we assume that the set $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ consists of non-zero vectors in the space E.

The norm inequality presented in the following statement improves upon the previous one by incorporating a simultaneous extension of the Selberg and Buzano inequalities, which was derived by Fujii et al. This enhancement leads to a more accurate and useful norm inequality that can be applied to a broader range of problems.

Theorem 1. *Let $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ be a subset containing vectors that are not equal to the zero vector in* E*. Then*

$$\|I - S_{\mathcal{Z}}\| = 1.$$

Proof. Suppose we have three vectors u, v_1, and v_2 in E, where $\|u\| = 1$, v_1 and v_2 are nonzero vectors, and $\langle v_p, z_q \rangle = 0$ for $p = 1, 2$ and $q = 1, \cdots, n$. Using Theorem 2.3 from reference [19], we obtain the following inequality:

$$|\langle Tu, u\rangle| + \varphi(v_1, v_2)\langle S_{\mathcal{Z}}u, u\rangle \leq \varphi(v_1, v_2), \tag{9}$$

where $T = v_1 \otimes v_2$ and

$$\varphi(v_1, v_2) = \frac{1}{2}\Big(|\langle v_1, v_2\rangle| + \|v_1\|\|v_2\|\Big).$$

We can infer the following from Inequality (9):

$$|\langle Tu, u\rangle| \leq \varphi(v_1, v_2)\langle (I - S_{\mathcal{Z}})u, u\rangle \leq \varphi(v_1, v_2).$$

One can derive the following result by taking the supremum over $u \in$ E such that $\|u\| = 1$:

$$\omega(T) \leq \varphi(v_1, v_2)\omega(I - S_{\mathcal{Z}}) \leq \varphi(v_1, v_2).$$

By utilizing Lemma 2.1 from reference [20] and the identity

$$\text{tr}(v_1 \otimes v_2) = \langle v_1, v_2\rangle,$$

we can arrive at

$$\omega(T) = \frac{1}{2}(|\text{tr}(v_1 \otimes v_2)| + \|v_1 \otimes v_2\|) = \frac{1}{2}(|\langle v_1, v_2\rangle| + \|v_1\|\|v_2\|) = \varphi(v_1, v_2).$$

Using this equation, we can see that

$$\varphi(v_1, v_2) = \varphi(v_1, v_2)\omega(I - S_{\mathcal{Z}}).$$

Since $\varphi(v_1, v_2)$ is nonzero, we can conclude that the desired result holds. □

We can obtain a first refinement of (SI) by considering the positivity of $I - S_{\mathcal{Z}}$.

Proposition 2. *Let $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ be a subset containing vectors that are not equal to the zero vector in* E*. Then, for any $u \in$ E, we have*

$$\langle S_{\mathcal{Z}}u, u\rangle + \|(I - S_{\mathcal{Z}})u\|^2 \leq \|u\|^2. \tag{10}$$

Proof. By taking $T = I - S_{\mathcal{Z}}$ in (6), we obtain from Theorem 1 the following:

$$\|(I - S_{\mathcal{Z}})u\|^2 \leq \langle (I - S_{\mathcal{Z}})u, u\rangle = \|u\|^2 - \langle S_{\mathcal{Z}}u, u\rangle, \tag{11}$$

for any $u \in$ E. Therefore, we have established the inequality that we were aiming to prove. □

Let $\mathcal{Z}$ denote a set of nonzero vectors in E that are orthonormal. Then $S_{\mathcal{Z}}$ and $I - S_{\mathcal{Z}}$ are orthogonal projections on $\mathcal{Z}$ and $\mathcal{Z}^{\perp}$, respectively. Then, by the Pythagorean formula, we have that

$$\|(I - S_{\mathcal{Z}})u\|^2 + \|S_{\mathcal{Z}}u\|^2 = \|u\|^2,$$

for any $u \in$ E. By the refinement obtained in Proposition 2, we attain the next generalization.

Corollary 1. *Let* $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ *be a subset of nonzero vectors in* E, *then*

$$\|(I - S_{\mathcal{Z}})u\|^2 + \|S_{\mathcal{Z}}u\|^2 \le \|u\|^2,$$

for any $u \in$ E.

Proof. The proof is a direct consequence of (6) and (10). □

As a consequence of (11), we have the following refinement of the Selberg inequality.

Corollary 2. *Let* $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ *be a subset of nonzero vectors in* E. *Then*

$$\begin{aligned} \langle S_{\mathcal{Z}}u, u\rangle\|v\|^2 &\le |\langle (I - S_{\mathcal{Z}})u, v\rangle|^2 + \langle S_{\mathcal{Z}}u, u\rangle\|v\|^2 \\ &\le \|(I - S_{\mathcal{Z}})u\|^2\|v\|^2 + \langle S_{\mathcal{Z}}u, u\rangle\|v\|^2 \le \|u\|^2\|v\|^2, \end{aligned}$$

for any $u, v \in$ E.

Proof. Using Inequality (11) we have

$$\|(I - S_{\mathcal{Z}})u\|^2 \le \langle (I - S_{\mathcal{Z}})u, u\rangle = \|u\|^2 - \langle S_{\mathcal{Z}}u, u\rangle.$$

By multiplying both sides of the equation by $\|v\|^2$, we can apply the (CSI) to obtain:

$$\begin{aligned} \|u\|^2\|v\|^2 &\ge \|(I - S_{\mathcal{Z}})u\|^2\|v\|^2 + \langle S_{\mathcal{Z}}u, u\rangle\|v\|^2 \\ &\ge |\langle (I - S_{\mathcal{Z}})u, v\rangle|^2 + \langle S_{\mathcal{Z}}u, u\rangle\|v\|^2 \\ &\ge \langle S_{\mathcal{Z}}u, u\rangle\|v\|^2, \end{aligned}$$

for all elements u and v in the set E, we can deduce the intended inequality. □

In the subsequent statement, we observe that the preceding inequality leads to an enhancement of the expression mentioned as (4).

Proposition 3. *Let* $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ *be a subset of nonzero vectors in* E *and* $z \in$ E *such that* $\langle z, z_p\rangle = 0$ *for all* $p = 1, \cdots, n$, *then*

$$\begin{aligned} |\langle u, z\rangle|^2 + \langle S_{\mathcal{Z}}u, u\rangle\|z\|^2 &\le \|(I - S_{\mathcal{Z}})u\|^2\|z\|^2 + \langle S_{\mathcal{Z}}u, u\rangle\|z\|^2 \\ &\le \|u\|^2\|z\|^2. \end{aligned}$$

for any $u, v \in$ E.

Proof. Let $z \in$ E such that $\langle z, z_p\rangle = 0$ for all $p = 1, \cdots, n$, then $S_{\mathcal{Z}}z = 0$, $\langle S_{\mathcal{Z}}u, z\rangle = \langle u, S_{\mathcal{Z}}z\rangle = 0$, and

$$|\langle u, z\rangle|^2 = |\langle u, z\rangle - \langle u, S_{\mathcal{Z}}z\rangle|^2 = |\langle u, (I - S_{\mathcal{Z}})z\rangle|^2.$$

Now by the (CSI) we have that

$$|\langle u, z\rangle|^2 \le \|u\|^2\|(I - S_{\mathcal{Z}})z\|^2.$$

Thus from Corollary 2, we obtain

$$\begin{aligned} |\langle u, z\rangle|^2 + \langle S_{\mathcal{Z}}u, u\rangle\|z\|^2 &\le \|u\|^2\|(I - S_{\mathcal{Z}})z\|^2 + \langle S_{\mathcal{Z}}u, u\rangle\|z\|^2 \\ &\le \|u\|^2\|z\|^2. \end{aligned}$$

□

The forthcoming lemmas present a compilation of certain properties that will be employed subsequently.

Lemma 1. *Let $\mathcal{Z}$ be a subset containing vectors that are not equal to the zero vector in* E *and* $u \in$ E. *The subsequent conditions are equivalent:*

(1) $\|S_{\mathcal{Z}}u\| = \|u\|$.
(2) $\langle S_{\mathcal{Z}}u, u\rangle = \|u\|^2$.
(3) $u \in \mathcal{N}(I - S_{\mathcal{Z}})$.

Proof. By using (6), (SI) and the fact that $\|S_{\mathcal{Z}}\| \leq 1$, we can conclude that

$$\|S_{\mathcal{Z}}u\|^2 \leq \|S_{\mathcal{Z}}\|\langle S_{\mathcal{Z}}u, u\rangle \leq \langle S_{\mathcal{Z}}u, u\rangle \leq \|u\|^2,$$

for any $u \in$ E. If $\|S_{\mathcal{Z}}u\| = \|u\|$, then $\langle S_{\mathcal{Z}}u, u\rangle = \|u\|^2$. Now, if $\langle S_{\mathcal{Z}}u, u\rangle = \|u\|^2$ thus $\langle S_{\mathcal{Z}}u, u\rangle = \langle u, u\rangle$ or equivalently

$$\langle (I - S_{\mathcal{Z}})u, u\rangle = 0.$$

As $I - S_{\mathcal{Z}} \geq 0$, we conclude by (6) that $\|(I - S_{\mathcal{Z}})u\| = 0$. On the other hand, if $u \in \mathcal{N}(I - S_{\mathcal{Z}})$ then $u = S_{\mathcal{Z}}u$. □

Lemma 2. *Let $\mathcal{Z}$ be a subset containing vectors that are not equal to the zero vector in* E *and* $u \in$ E. *The following conditions are equivalent.*

(1) $\|(I - S_{\mathcal{Z}})u\| = \|u\|$.
(2) $u \in \mathcal{N}(S_{\mathcal{Z}})$.
(3) $\langle u, z_p\rangle = 0$ *for all* $p = 1, \cdots, n$.

Proof. If $\|(I - S_{\mathcal{Z}})u\| = \|u\|$, then by (SI) and (10), we have that $\|S_{\mathcal{Z}}u\| = 0$. Now, if $u \in \mathcal{N}(S_{\mathcal{Z}})$ thus $\langle S_{\mathcal{Z}}u, u\rangle = 0$. Consequently, this leads to the conclusion that $|\langle u, z_p\rangle| = 0$ for all $p = 1, \cdots, n$. On the other hand, if $\langle u, z_p\rangle = 0$ for all $p = 1, \cdots, n$, then $S_{\mathcal{Z}}u = 0$ and $\|(I - S_{\mathcal{Z}})u\| = \|u\|$. □

By applying the earlier enhancement of (SI), derived in Corollary 2, and utilizing the characterization of the instances where equality holds in the Selberg inequality, we can attain a comprehensive depiction of the nullspace of $I - S_{\mathcal{Z}}$. Precisely, the ensuing theorem provides the following complete description:

Theorem 2. *Let $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ be a subset containing vectors that are not equal to the zero vector in* E. *Thus, the nullspace $\mathcal{N}(I - S_{\mathcal{Z}})$ can be characterized as the collection of all vectors u that can be represented as $u = \sum_{i=1}^{n} a_i z_i$ with $a_i \in \mathbb{C}$ for $i \in \{1, \ldots, n\}$, subject to the subsequent conditions for any arbitrary $i \neq j$:*

$$\langle z_i, z_j\rangle = 0 \quad \text{or} \quad |a_i| = |a_j| \quad \text{with} \quad \langle a_i z_i, a_j z_j\rangle \geq 0.$$

Moreover, $\mathcal{N}(I - S_{\mathcal{Z}})$ is also equal to $\mathcal{FP}(S_{\mathcal{Z}})$, which represents the set of fixed points of $S_{\mathcal{Z}}$.

Proof. This result follows directly from Lemma 1. □

Because $S_{\mathcal{Z}} \in L(\mathrm{E})$, the zero vector 0 belongs to the fixed-point set $\mathcal{FP}(S_{\mathcal{Z}})$. Moreover, if there exists a nonzero vector $u^* \in \mathcal{FP}(S_{\mathcal{Z}})$, then for any $\theta \in (0, 2\pi)$, the vector $u^*_\theta = e^{i\theta}u^*$ also belongs to $\mathcal{FP}(S_{\mathcal{Z}})$, and u^*_θ is distinct from u^* (i.e., $u^*_\theta \neq u^*$). This implies that $\mathcal{FP}(S_{\mathcal{Z}})$ contains infinitely many elements. In the next statement, we establish a characterization of this condition in terms of the Selberg operator norm.

Theorem 3. *The fixed-point set $\mathcal{FP}(S_{\mathcal{Z}})$ contains an infinite number of elements if and only if there exists a vector $u_1 \in$ E such that*

$$\|u_1\| = \|S_{\mathcal{Z}} u_1\| = \|S_{\mathcal{Z}}\| = 1.$$

Proof. First, we suppose that there exists $u_1 \in$ E such that $\|u_1\| = \|S_{\mathcal{Z}} u_1\| = \|S_{\mathcal{Z}}\| = 1$. Using the Inequality (6) and (SI) we have

$$1 = \|S_{\mathcal{Z}} u_1\|^2 \leq \langle S_{\mathcal{Z}} u_1, u_1 \rangle \leq \|u_1\|^2 = 1.$$

Then, we obtain the equality $\langle S_{\mathcal{Z}} u_1, u_1 \rangle = \|u_1\|^2$ and by Lemma 1 we deduce that $u_1 \in \mathcal{N}(I - S_{\mathcal{Z}})$ with $u_1 \neq 0$. Hence we conclude that $\mathcal{FP}(S_{\mathcal{Z}})$ has infinite elements.

On the other hand, if $\mathcal{FP}(S_{\mathcal{Z}})$ has infinite elements. Let $u_1 \in \mathcal{FP}(S_{\mathcal{Z}})$ such that $\|u_1\| = 1$. Then, $u_1 \in \mathcal{N}(I - S_{\mathcal{Z}})$ or equivalently $\langle S_{\mathcal{Z}} u_1, u_1 \rangle = \|u_1\|^2 = 1$. By the positivity of $S_{\mathcal{Z}}$, we have that

$$1 = \langle S_{\mathcal{Z}} u_1, u_1 \rangle \leq \sup\{\langle S_{\mathcal{Z}} u, u \rangle : \|u\| = 1\} = \|S_{\mathcal{Z}}\| \leq 1.$$

Finally, employing the fact that $S_{\mathcal{Z}} u_1 = u_1$, we can deduce that $\|S_{\mathcal{Z}} u_1\| = 1 = \|S_{\mathcal{Z}}\|$, leading us to the intended result. □

Proposition 4. $\mathcal{FP}(S_{\mathcal{Z}}) = \{0\}$ *if and only if* $\|S_{\mathcal{Z}}\| < 1$.

Proof. Let us assume that $\|S_{\mathcal{Z}}\| < 1$. Based on this hypothesis, we can establish the following inequality for any $u \in$ E:

$$\|S_{\mathcal{Z}} u\| \leq \|S_{\mathcal{Z}}\| \|u\| < \|u\|.$$

Therefore, the Selberg operator is a strict contraction and, by the Banach Fixed Point Theorem ([21]), $S_{\mathcal{Z}}$ admits a unique fixed point in E. Then $\mathcal{FP}(S_{\mathcal{Z}}) = \{0\}$.

On the other hand, if $\mathcal{FP}(S_{\mathcal{Z}}) = 0$, it means that for any non-zero vector $u \in$ E, the operator $S_{\mathcal{Z}}$ does not keep u unchanged, i.e., $S_{\mathcal{Z}} u \neq u$, which is the same as saying $(I - S_{\mathcal{Z}})u \neq 0$. According to Lemma 1, this leads to the conclusion that

$$\|S_{\mathcal{Z}} u\| \neq \|u\|,$$

for any non-zero $u \in$ E. Specifically, for any $u \in$ E with $\|u\| = 1$, we can deduce that $\|S_{\mathcal{Z}} u\| \neq 1$. Since $S_{\mathcal{Z}}$ is a finite rank operator, it is categorized as a compact operator. Consequently, the set of points on the unit sphere in E where $S_{\mathcal{Z}}$ achieves its norm is not empty. Thus, it follows that $\|S_{\mathcal{Z}}\| < 1$. □

By employing the (CSI) for positive operators (as indicated in (5)) and using the (SI), we obtain the following expression:

$$|\langle S_{\mathcal{Z}} u, v \rangle| \leq \|u\| \|v\|. \tag{12}$$

In the next proposition, we refine the aforementioned Inequality (12).

Proposition 5. *Let $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ be a subset containing vectors that are not equal to the zero vector in* E*. For any $u, v \in$* E*, the following inequality holds:*

$$|\langle S_{\mathcal{Z}} u, v \rangle| \leq \frac{1}{2}\Big(|\langle u, v \rangle| + \|u\| \|v\|\Big). \tag{13}$$

Proof. Consider u and v from the set E. By utilizing the properties (BuI) and (SI), we can observe that:

$$\begin{aligned}\langle S_{\mathcal{Z}}u,u\rangle|\langle v,S_{\mathcal{Z}}u\rangle| &= \big|\langle u,S_{\mathcal{Z}}u\rangle\langle S_{\mathcal{Z}}u,v\rangle\big| \\ &\leq \frac{\|S_{\mathcal{Z}}u\|^2}{2}(|\langle u,v\rangle|+\|u\|\|v\|).\end{aligned}$$

In particular, when $\langle S_{\mathcal{Z}}u,u\rangle \neq 0$, we can use the Inequality (6) and property (SI) to deduce that

$$\frac{\|S_{\mathcal{Z}}u\|^2}{\langle S_{\mathcal{Z}}u,u\rangle} \leq 1.$$

As a result, this implies that:

$$|\langle v,S_{\mathcal{Z}}u\rangle| \leq \frac{1}{2}(|\langle u,v\rangle|+\|u\|\|v\|).$$

Hence, (13) is proved as desired. □

Remark 2. *The Inequality (13) serves to demonstrate the validity of the Buzano inequality for any Selberg operator. It is noteworthy to mention that one of the authors established in [22] that Buzano's inequality also holds for any orthogonal projection P.*

Now, we proceed to generalize the Selberg inequality and, in particular, refine the Inequalities (12) and (13) respectively.

Theorem 4. *For any $u,v \in \mathrm{E}$,*

$$|\langle S_{\mathcal{Z}}u,v\rangle| \leq \left|\langle S_{\mathcal{Z}}u,v\rangle - \frac{1}{2}\langle u,v\rangle\right| + \frac{1}{2}|\langle u,v\rangle| \leq \frac{1}{2}(|\langle u,v\rangle|+\|u\|\|v\|). \tag{14}$$

Proof. As $2S_{\mathcal{Z}} - I$ is a selfadjoint operator, then

$$\omega(2S_{\mathcal{Z}} - I) = \|2S_{\mathcal{Z}} - I\|.$$

On the other hand, we have

$$W(2S_{\mathcal{Z}} - I) = \{2\langle S_{\mathcal{Z}}u,u\rangle - 1 : u \in \mathrm{E}, \|u\| = 1\} \subseteq [-1,1].$$

Then, $\|2S_{\mathcal{Z}} - I\| \leq 1$. Now, for any $u,v \in \mathrm{E}$, as consequece of the Cauchy–Schwarz inequality we obtain the following:

$$\begin{aligned}\left|\langle S_{\mathcal{Z}}u,v\rangle - \frac{1}{2}\langle u,v\rangle\right| &= \left|\left\langle \left(S_{\mathcal{Z}} - \frac{1}{2}I\right)u,v\right\rangle\right| \\ &\leq \frac{1}{2}\|2S_{\mathcal{Z}} - I\|\|u\|\|v\| \leq \frac{1}{2}\|u\|\|v\|.\end{aligned}$$

Thus, as a consequence, we acquire:

$$|\langle S_{\mathcal{Z}}u,v\rangle| \leq \left|\langle S_{\mathcal{Z}}u,v\rangle - \frac{1}{2}\langle u,v\rangle\right| + \frac{1}{2}|\langle u,v\rangle| \leq \frac{1}{2}(|\langle u,v\rangle|+\|u\|\|v\|).$$

□

Remark 3. *(1) From the the previous statement, we have*

$$\left|\langle S_{\mathcal{Z}}u,v\rangle - \frac{1}{2}\langle u,v\rangle\right| \leq \frac{1}{2}\|u\|\|v\|. \tag{15}$$

If we consider $\mathcal{Z} = \{z\}$ with $z \neq 0$, then $S_{\mathcal{Z}} = \frac{z \otimes z}{\|z\|^2}$ is an orthogonal projection onto the subspace spanned by $\{z\}$. Consequently, we obtain the well-known Richard's inequality (see [23]):

$$\left| \langle u, z\rangle \langle z, v\rangle - \frac{1}{2}\langle u, v\rangle \|z\|^2 \right| \leq \frac{1}{2}\|u\|\|v\|\|z\|^2, \tag{16}$$

for all $u, v, z \in \mathrm{E}$. As a result, we consider the Inequality (15) as an extension of (16).

(2) Using the fact that $\|2(I - S_{\mathcal{Z}}) - I\| = \|I - 2S_{\mathcal{Z}}\| \leq 1$, and applying similar ideas used in the proof of Theorem 4, we can establish that for any $u, v \in \mathrm{E}$,

$$\begin{aligned} |\langle (I - S_{\mathcal{Z}})u, v\rangle| &\leq \left| \langle (I - S_{\mathcal{Z}})u, v\rangle - \frac{1}{2}\langle u, v\rangle \right| + \frac{1}{2}|\langle u, v\rangle| \\ &\leq \frac{1}{2}(|\langle u, v\rangle| + \|u\|\|v\|). \end{aligned}$$

Indeed, if we consider $u = v$ in (14), we obtain a refinement of both (SI) and (BuI). Specifically, the refined expressions are derived as follows:

Corollary 3. *Let $\mathcal{Z} = \{z_p : p = 1, \cdots, n\}$ be a subset of nonzero vectors in E, then for any $u \in \mathrm{E}$, we have*

$$\langle S_{\mathcal{Z}} u, u\rangle = \sum_{p=1}^{n} \frac{|\langle u, z_p\rangle|^2}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|} \leq \left| \sum_{p=1}^{n} \frac{|\langle u, z_p\rangle|^2}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|} - \frac{1}{2}\|u\|^2 \right| + \frac{1}{2}\|u\|^2 \leq \|u\|^2.$$

In particular, if $\mathcal{Z}$ is an orthonormal set within E, then the refined Inequality (14) takes the following form:

$$\sum_{p=1}^{n} |\langle u, z_p\rangle|^2 \leq \left| \sum_{p=1}^{n} |\langle u, z_p\rangle|^2 - \frac{1}{2}\|u\|^2 \right| + \frac{1}{2}\|u\|^2 \leq \|u\|^2.$$

In [24], Dragomir obtained the following refinement of (CSI),

$$|\langle u, v\rangle| \leq |\langle u, v\rangle - \langle u, e\rangle\langle e, v\rangle| + |\langle u, e\rangle\langle e, v\rangle| \leq \|u\|\|v\|, \tag{17}$$

for any $u, v, e \in \mathrm{E}$, with $\|e\| = 1$. We note that if $\mathcal{Z} = \{e\}$, then (17) can be express as follows:

$$|\langle u, v\rangle| \leq |\langle u, v\rangle - \langle S_{\mathcal{Z}} u, v\rangle| + \langle S_{\mathcal{Z}} u, u\rangle^{\frac{1}{2}} \langle S_{\mathcal{Z}} v, v\rangle^{\frac{1}{2}} \leq \|u\|\|v\|. \tag{18}$$

We can use a result from a previous study by Bottazzi et al. (see Theorem 4.2 in [25]) and the fact that $S_{\mathcal{Z}}$ is always a positive contraction for any subset $\mathcal{Z}$, to obtain a new and improved version of the Cauchy–Schwarz Inequality (18). To make this article complete, we have included the proof below.

Theorem 5. *Consider $\mathcal{Z}$ a finite subset of nonzero vectors in E. For any $u, v \in \mathrm{E}$, we have:*

$$\begin{aligned} \|u\|\|v\| &\geq |\langle u, v\rangle - \langle S_{\mathcal{Z}} u, v\rangle| + \langle S_{\mathcal{Z}} u, u\rangle^{\frac{1}{2}} \langle S_{\mathcal{Z}} v, v\rangle^{\frac{1}{2}} \\ &\geq |\langle u, v\rangle| - |\langle S_{\mathcal{Z}} u, v\rangle| + \langle S_{\mathcal{Z}} u, u\rangle^{\frac{1}{2}} \langle S_{\mathcal{Z}} v, v\rangle^{\frac{1}{2}} \\ &\geq |\langle u, v\rangle|. \end{aligned}$$

Proof. Keep in mind that for any real numbers x_p with $p = 1, \cdots, 4$, the following inequality holds:

$$(x_1 x_3 - x_2 x_4)^2 \geq (x_1^2 - x_2^2)(x_3^2 - x_4^2). \tag{19}$$

In light of this, if we consider u and v from the set E, we can draw the following conclusion:

$$\begin{aligned}\left(\|u\|\|v\| - \langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}}\right)^2 &\geq (\|u\|^2 - \langle S_{\mathcal{Z}}u,u\rangle)(\|v\|^2 - \langle S_{\mathcal{Z}}v,v\rangle) \\ &= \langle (I-S_{\mathcal{Z}})u,u\rangle\langle (I-S_{\mathcal{Z}})v,v\rangle. \end{aligned} \tag{20}$$

By (5) and the fact that $I - S_{\mathcal{Z}} \geq 0$, we have

$$\langle (I-S_{\mathcal{Z}})u,u\rangle\langle (I-S_{\mathcal{Z}})v,v\rangle \geq |\langle (I-S_{\mathcal{Z}})u,v\rangle|^2 = |\langle u,v\rangle - \langle S_{\mathcal{Z}}u,v\rangle|^2. \tag{21}$$

Now, by (20) and (21),

$$\left(\|u\|\|v\| - \langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}}\right)^2 \geq |\langle u,v\rangle - \langle S_{\mathcal{Z}}u,v\rangle|^2, \tag{22}$$

for any $u, v \in$ E.

By calculating the square root of (22) and employing the fact that

$$\|u\| \geq \langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}} \quad \text{and} \quad \|v\| \geq \langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}},$$

we obtain

$$\|u\|\|v\| - \langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}} \geq |\langle u,v\rangle - \langle S_{\mathcal{Z}}u,v\rangle|.$$

As a consequence of the triangle inequality for the absolute value of real numbers, we can deduce that

$$\begin{aligned}\|u\|\|v\| - \langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}} &\geq |\langle u,v\rangle - \langle S_{\mathcal{Z}}u,v\rangle| \\ &\geq |\langle u,v\rangle| - |\langle S_{\mathcal{Z}}u,v\rangle|. \end{aligned} \tag{23}$$

□

Now, we use Theorem 5 to obtain a lower and upper bound for

$$\langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}} - |\langle S_{\mathcal{Z}}u,v\rangle|.$$

The bounds are related to the operator Cauchy–Schwarz inequality and (CSI), respectively.

Corollary 4. *Let $\mathcal{Z}$ be a finite subset of nonzero vectors in* E*, then for any $u, v \in$* E *and $\alpha \in [0,1]$ hold*

$$\begin{aligned}\|u\|\|v\| - |\langle u,v\rangle| &\geq \langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}} - |\langle S_{\mathcal{Z}}u,v\rangle| \\ &\geq I_{u,v,\alpha} - |\langle S_{\mathcal{Z}}u,v\rangle| \geq 0, \end{aligned}$$

where $I_{u,v,\alpha} = \sqrt{(1-\alpha)\langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}}|\langle S_{\mathcal{Z}}u,v\rangle| + \alpha\langle S_{\mathcal{Z}}u,u\rangle\langle S_{\mathcal{Z}}v,v\rangle}$.

Proof. The second inequality is a direct consequence of the refinement of the operator Cauchy–Schwarz inequality previously obtained in (7). Additionally, the first inequality is derived from (23). □

When considering $\mathcal{Z}$ as a finite, orthonormal subset of nonzero vectors in E in the preceding statement, we arrive at the inequality previously derived by Dragomir and Sándor in [6]. Specifically, we have:

$$|\langle u,v\rangle| \leq \|u\|\|v\| + \left|\sum_{i=1}^{n}\langle u,z_i\rangle\langle z_i,v\rangle\right| - \left(\sum_{i=1}^{n}|\langle u,z_i\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}|\langle v,z_i\rangle|^2\right)^{\frac{1}{2}} (\leq \|u\|\|v\|).$$

We present our next result, which is stated as follows.

Theorem 6. *Consider a finite subset $\mathcal{Z}$ of nonzero vectors in* E*. Then, for any u and v in the set* E*, we have the following inequality:*

$$\begin{aligned}|\langle S_{\mathcal{Z}}u,v\rangle-\langle u,v\rangle|^2 &\leq (\|u\|^2-\langle S_{\mathcal{Z}}u,u\rangle)(\|v\|^2-\langle S_{\mathcal{Z}}v,v\rangle)\\ &\leq \left(\|u\|\|v\|-\langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}}\right)^2.\end{aligned}\tag{24}$$

Proof. In Equation (5), select the positive operator $T=I-S_{\mathcal{Z}}$. Then, for any u and v in the set E, we obtain:

$$\begin{aligned}|\langle S_{\mathcal{Z}}u,v\rangle-\langle u,v\rangle|^2 &= |\langle (I-S_{\mathcal{Z}})u,v\rangle|^2\\ &\leq \langle (I-S_{\mathcal{Z}})u,u\rangle\langle (I-S_{\mathcal{Z}})v,v\rangle\\ &= (\|u\|^2-\langle S_{\mathcal{Z}}u,u\rangle)(\|v\|^2-\langle S_{\mathcal{Z}}v,v\rangle).\end{aligned}$$

With the help of (19), we can now conclude that:

$$\begin{aligned}|\langle S_{\mathcal{Z}}u,v\rangle-\langle u,v\rangle|^2 &\leq (\|u\|^2-\langle S_{\mathcal{Z}}u,u\rangle)(\|v\|^2-\langle S_{\mathcal{Z}}v,v\rangle)\\ &\leq \left(\|u\|\|v\|-\langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}}\right)^2,\end{aligned}$$

for any $u,v\in$ E. □

Remark 4. *(1) In the work of Lin [26], the investigation of covariance-variance for bounded linear operators defined on a Hilbert space* E *was initiated. Let us recall some definitions introduced in that article. Let $R,T\in L(\mathrm{E})$ and $z\neq 0$. The covariance of R and T is a mapping $Cov_z(R,T):\mathrm{E}\to\mathbb{C}$ defined by*

$$Cov_z(R,T)u=\|z\|^2\langle Ru,Tu\rangle-\langle Ru,z\rangle\langle z,Tu\rangle.$$

If $R=T$ we obtain the variance of S

$$Var_z(R)u=Cov_z(R,R)u=\|z\|^2\|Ru\|^2-|\langle Ru,z\rangle|^2.$$

In particular, if in the first inequality of (24) we consider $\mathcal{Z}=\left\{\frac{z}{\|z\|}\right\}$ and we replace u and v by Ru and Tu, respectively, then

$$\begin{aligned}|Cov_z(R,T)u|^2 &= \|z\|^2|\langle S_{\mathcal{Z}}(Ru),Tu\rangle-\langle Ru,Tu\rangle|^2\\ &\leq \|z\|^2(\|Ru\|^2-\langle S_{\mathcal{Z}}(Ru),Ru\rangle)(\|Tu\|^2-\langle S_{\mathcal{Z}}(Tu),Tu\rangle)\\ &= (Var_z(R)u)(Var_z(T)u).\end{aligned}$$

The inequality mentioned earlier was previously derived by Lin and is commonly known as the covariance-variance inequality (refer to Theorem 1 in [26]). In conclusion, the Inequality (24) is a generalization of the covariance-variance inequality.

(2) By utilizing the second inequality of (24) and (SI), we can provide an alternative proof that the Selberg operator $S_{\mathcal{Z}}$ satisfies Buzano's inequality (refer to Theorem 4). Specifically, we have:

$$|\langle S_{\mathcal{Z}}u,v\rangle-\langle u,v\rangle|\leq\|u\|\|v\|-\langle S_{\mathcal{Z}}u,u\rangle^{\frac{1}{2}}\langle S_{\mathcal{Z}}v,v\rangle^{\frac{1}{2}},$$

for any $u,v\in$ E. As a consequence of (5), we have

$$|\langle S_{\mathcal{Z}}u,v\rangle-\langle u,v\rangle|\leq\|u\|\|v\|-|\langle S_{\mathcal{Z}}u,v\rangle|.$$

Then,

$$|\langle S_{\mathcal{Z}}u,v\rangle|-|\langle u,v\rangle|\leq|\langle S_{\mathcal{Z}}u,v\rangle-\langle u,v\rangle|\leq\|u\|\|v\|-|\langle S_{\mathcal{Z}}u,v\rangle|,$$

or equivalently

$$|\langle S_{\mathcal{Z}} u, v\rangle| \leq \frac{1}{2}(|\langle u, v\rangle| + \|u\| \|v\|).$$

The logical and historical significance of equivalent inequalities is widely recognized, and a considerable body of literature has been devoted to investigating these connections. To conclude, we demonstrate that the majority of the inequalities presented in this article, namely (CSI), (SI), and (BeI), can be derived from one another. Indeed, our findings hold true when $(\mathrm{E}, \langle\cdot,\cdot\rangle)$ is a real or complex inner product space. The derived inequalities maintain their validity in both real and complex settings.

Theorem 7. *The subsequent inequalities are equivalent:*

(1) *Bessel inequality—If $\mathcal{E} = \{e_i : i = 1, \cdots, n\}$ is an orthonormal set in* E*, then*

$$\sum_{i=1}^{n} |\langle u, e_i\rangle|^2 \leq \|u\|^2,$$

for any $u \in \mathrm{E}$.

(2) *Cauchy–Schwarz inequality—For any $u, v \in \mathrm{E}$, we have*

$$|\langle u, v\rangle| \leq \|u\| \|v\|.$$

(3) *Selberg inequality—For given nonzero vectors $\mathcal{Z} = \{z_p : p = 1, \cdots, n\} \subseteq \mathrm{E}$, the inequality*

$$\sum_{p=1}^{n} \frac{|\langle u, z_p\rangle|^2}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|} \leq \|u\|^2,$$

holds for all $u \in \mathrm{E}$.

Proof. (BeI) $\Rightarrow$ (CSI). Let $u, v \in \mathrm{E}$ with $v \neq 0$ (otherwise the CSI holds trivially). If (BeI) holds, and we consider $\mathcal{E} = \left\{\frac{v}{\|v\|}\right\}$, then

$$\left|\langle u, \frac{v}{\|v\|}\rangle\right|^2 \leq \|u\|^2.$$

(CSI) $\Rightarrow$ (SI). Assuming that (CSI) holds, we require the existence of a nonzero vector u_0 satisfying the property $\langle S_{\mathcal{Z}} u_0, u_0\rangle > 1$, where $\|u_0\| = 1$. Then, using the (CSI), we have

$$\|S_{\mathcal{Z}} u_0\|^2 \geq \langle S_{\mathcal{Z}} u_0, u_0\rangle^2 > 1. \tag{25}$$

Therefore,

$$\begin{aligned}
\|S_{\mathcal{Z}} u_0\|^2 &= \left\|\sum_{p=1}^{n} \frac{\langle u_0, z_p\rangle z_p}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|}\right\|^2 \\
&= \sum_{p,r=1}^{n} \frac{|\langle u_0, z_p\rangle|}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|} \frac{|\langle u_0, z_r\rangle|}{\sum_{q=1}^{n} |\langle z_r, z_q\rangle|} |\langle z_p, z_r\rangle| \\
&\leq \sum_{p=1}^{n} \left(\frac{|\langle u_0, z_p\rangle|}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|}\right)^2 \sum_{r=1}^{n} |\langle z_p, z_r\rangle| = \sum_{p=1}^{n} \frac{|\langle u_0, z_p\rangle|^2}{\sum_{q=1}^{n} |\langle z_p, z_q\rangle|}.
\end{aligned}$$

From this, we can conclude that,

$$\|S_{\mathcal{Z}} u_0\|^2 \leq \langle S_{\mathcal{Z}} u_0, u_0\rangle. \tag{26}$$

Combining the Inequalities (25) and (26), we have

$$\langle S_{\mathcal{Z}} u_0, u_0\rangle \geq \|S_{\mathcal{Z}} u_0\|^2 \geq \langle S_{\mathcal{Z}} u_0, u_0\rangle^2.$$

This leads to a contradiction since $\langle S_{\mathcal{Z}} u_0, u_0 \rangle^2 > 1$. Thus, we must conclude that the initial assumption is incorrect, which implies that for all $u \in \mathrm{E}$, we have:

$$\langle S_{\mathcal{Z}} u, u \rangle = \sum_{p=1}^{n} \frac{|\langle u, z_p \rangle|^2}{\sum_{q=1}^{n} |\langle z_p, z_q \rangle|} \leq \|u\|^2.$$

(SI) $\Rightarrow$ (BeI). For each $u \in \mathrm{E}$ and any $\mathcal{E} = \{e_i \in \mathrm{E} : i = 1, \cdots, n\}$ orthonormal set, we get

$$\sum_{j=1}^{n} |\langle e_i, e_j \rangle| = \sum_{j=1}^{n} |\delta_{ij}| = 1,$$

for any $i \in \{1, \cdots, n\}$. Then, by (SI) we conclude

$$\sum_{i=1}^{n} |\langle u, e_i \rangle|^2 = \sum_{i=1}^{n} \frac{|\langle u, e_i \rangle|^2}{\sum_{j=1}^{n} |\langle e_i, e_j \rangle|} = \langle S_{\mathcal{E}} u, u \rangle \leq \|u\|^2.$$

This demonstrates that the Selberg inequality implies the Bessel inequality. □

Remark 5. *It is worth noting that the Selberg inequality is more powerful than the Buzano inequality. Specifically, if we choose $\mathcal{Z} = \{z\}$ in Theorem 4, where z is a nonzero vector, we derive the following bound:*

$$|\langle u, z \rangle||\langle z, v \rangle| \leq \frac{1}{2}(|\langle u, v \rangle| + \|u\|\|v\|)\|z\|^2,$$

for all $u, v \in \mathrm{E}$.

3. Conclusions

In conclusion, this paper has introduced several refinements of the classical Selberg inequality using the contraction property of the Selberg operator. These refinements have improved upon the classical Selberg inequality and have provided new insights into the properties of the Selberg operator. Additionally, this paper has highlighted the interconnections among well-known inequalities such as the Cauchy–Schwarz, Bessel, and Selberg inequalities, demonstrating the significance of these inequalities and suggesting potential avenues for further research in this field.

Moving forward, there are several interesting research questions related to the concept of symmetry that could be explored. For instance, how can the Selberg inequality be extended to other types of symmetric spaces or automorphic forms? Can the contraction property of the Selberg operator be used to derive new inequalities in the context of symmetry studies? Are there other inequalities that exhibit similar interconnections with the Selberg inequality and can lead to further insights into the properties of symmetric spaces? These questions and others could provide fruitful directions for future investigations into the Selberg inequality and its associated inequalities.

Overall, this paper lays the groundwork for future research into the Selberg inequality and its applications in various mathematical fields. We hope that these findings will inspire other researchers to continue exploring the properties of the Selberg inequality and its connections to other essential mathematical inequalities.

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