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An Independent Cascade Model of Graph Burning

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Abstract: Graph burning was introduced to simulate the spreading of news/information/rumors in social networks. The symmetric undirected graph is considered here. That is, vertex u can spread the information to vertex v, and symmetrically vertex v can also spread information to vertex u. When it is modeled as a graph burning process, a vertex can be set on fire directly or burned by its neighbor. Thus, the task is to find the minimum sequence of vertices chosen as sources of fire to burn the entire graph. This problem has been proved to be NP-hard. In this paper, from a new perspective, we introduce a generalized model called the Independent Cascade Graph Burning model, where a vertex v can be burned by one of its burning neighbors u only if the influence that u gives to v is larger than a given threshold $\beta \geq 0$. We determine the graph burning number with this new Independent Cascade Graph Burning model for several graphs and operation graphs and also discuss its upper and lower bounds.

Keywords: burning number; generalized burning number; Independent Cascade Model; generalized Petersen graphs

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1. Introduction

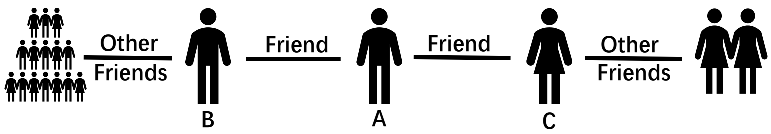
Bonato et al. [1–3] introduced the concept of the graph burning problem as a model for social contagion. Given a finite connected graph G, the burning process on G is a discrete-time process defined as follows.

Initially, all vertices are unburned. One vertex can be set on fire directly or burned by its neighbor. At each step, only one vertex is selected to be set on fire, and simultaneously all those vertices which have caught fire at the last step will burn all their neighbors. The process does not stop until the entire graph is burned. Furthermore, if a vertex is burned, the vertex remains in this state until the end of the process. The vertices selected as the sources of fire are called the burning sequences; the shortest burning sequence is called the optimum burning sequence. The length of the optimum burning sequence is called the burning number $b(G)$. Note that the smaller the burning number is, the faster a contagion (such as news or gossip) spreads in the network. For a given network, finding the optimum burning sequence has important applications in reality.

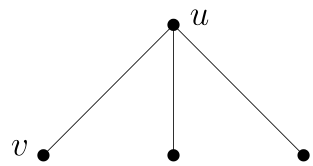
The graph burning problem has been proved to be NP-hard [4], and several approximation algorithms [4–6] and heuristics [7–9] were proposed. Among them, the Burning Farthest-First (BFF) algorithm [6] has a better approximation ratio, which is $3 - \frac{2}{b(G)}$. For heuristics, the Forward-Looking Search Strategy (GFSS) [8] and Component-Based Recursive Heuristic (CBRH) algorithm [7] perform better. In addition to the above studies on algorithms, researchers have paid attention to the bounds of the burning number [1,10,11] and calculated the burning number for some special graphs, such as generalized Petersen graphs [12], theta graphs [13], graph products [14,15], spiders and path forests [16].

The above mentioned graph burning model was used to simulate the spreading of news/information/rumors in social networks. It is supposed that if a person obtains news (or other social information) at time $t-1$, all their neighbors will obtain the news at time t and spread the news at time $t+1$. However, in reality, we find that a person may not accept the news and further spread it to their neighbors if they receive this news from only one of their neighbors. Based on this fact, Li et al. [17] proposed the generalized r-burning process of G where a person will accept the news only if they received the news from more than just their r neighbors, where r is a preset integer that serves as a threshold. This threshold, 'r', defines the minimum number of sources needed before the individual considers the news credible enough for further propagation. They studied the generalized r-burning number for several graphs and operation graphs.

However, in reality, the following situation may happen. For instance, as shown in Figure 1a, if a person, namely A, has two friends, namely B and C. B is popular, while C is not. As a result, the influence of A on these two friends is different. For B, since they have many friends, A will have less influence on B. On the contrary, for C, since C has fewer friends, A is very likely to be an important friend to C. So, A is likely to have more influence on C than on B. Assume that at some step t, A is active/burned. It may happen that at time $t+1$, A can burn C but cannot burn B. However, in the above graph burning models, vertex A's activation ability on its neighbors is not distinguished. Thus, we should introduce another model which can take this difference into account.



(**a**) An example of friends related to A.



(**b**) An example of burning the vertex u.

Figure 1. Two examples of different friends of a person and a burning vertex.

In the information diffusion research field, there is another information diffusion model called the Independent Cascade Model (abbreviated as the IC Model) [18] where whether a vertex can activate its neighbors successfully or not depends on a threshold. Inspired by the IC model, in this paper we propose a new generalized Independent Cascade graph burning model for G, where a burned vertex v can successfully burn its unburned neighbor w only if the influence that v exerts on w is larger than a given threshold β. Note that when $\beta = 0$, it turns to be the traditional graph burning problem. Our task is still to find the minimum sequence of vertices that can be chosen as sources of fire to burn the entire graph. The minimum number of vertices or steps is called the IC burning number $b_\beta(G)$ of a graph G with a given threshold β.

In the following, after presenting some terminology in Section 2, we will discuss the general bounds for $b_\beta(G)$ in Section 3 and discuss the exact values of $b_\beta(G)$ for several special graphs in Section 4.

2. Terminology

All graphs considered in this paper are finite and simple. For notation and terminology not defined here, refer to Bondy and Murty [19].

During the burn process in the IC model, we call a vertex x_i burned outsideif x_i is selected to be set on fire and burned inside if x_i is burned by its neighbor. For a given threshold β, if all vertices of the graph are burned after k time steps, we call the fire source sequence $(x_1, x_2, \ldots, x_k)$ a β-burning sequence of graph G. Clearly, the generalized burning number $b_\beta(G)$ is the length of a minimum burning sequence among all β-burning sequences for graph G. If $b_\beta(G) = k$, we call sequence $(x_1, x_2, \ldots, x_k)$ an optimum β-burning sequence of G.

In reality, the influence a vertex u receives from its neighbor w can be an arbitrary value in $[0, 1]$, only with the restriction that the sum of influences that it receives should not be larger than 1. For simplicity, in this paper, we assume a vertex u receives equal influence from each of its neighbors, which is $\frac{1}{d(u)}$. Note that u's neighbors have equal chance to activate u, but u has a different chance to activate its neighbors. For example, as shown in Figure 1b, the influence from v_1 to u is $\frac{1}{3}$ because u has three friends, while the influence from u to v_1 is 1 because v_1 has only one friend u. Then, whether a vertex can be burned by its neighbor depends heavily on its degree. The more friends a vertex has, the more difficult it is for one of its friends to activate it. For a vertex u, if the number of its neighbors is less than $\frac{1}{\beta}$ (i.e., the influence from each of its neighbors is greater than β), vertex u can be burned by any one of its burned neighbors. See Figure 1b; assume v_1 or v_2 is burned at time $t-1$, then u will be burned by v_1 or v_2 at time t if $\beta = \frac{1}{4} < \frac{1}{3}$ and will not if $\beta = \frac{1}{2} > \frac{1}{3}$. On the contrary, if u is burned at time $t-1$, then v_1 will be burned by u at time t for any β but v_2 will be burned by u at time t only if $\beta \leq \frac{1}{3}$.

For convenience, we assign a parameter $f(u) = \frac{1}{d(u)}$ to each vertex u, representing the influence that it receives from each of its neighbors. For a directed graph, $f(u) = \frac{1}{d^-(u)}$, where $d^-(u)$ is the in-degree of vertex u. Clearly, $b_\beta(G) = b(G)$ while $\beta \leq \frac{1}{\Delta(G)}$, where $\Delta(G)$ is the maximum degree of the graph G.

3. The Bounds for the IC Burning Number of Graphs

If G is a graph and u, v are two vertices of G, and the distance between them is denoted by $d(u, v)$. A shortest path between u and v is denoted by p_{uv}, and the $f_{min}(p_{uv})$ is defined as $min\{f(w) : w \in \ p_{uv}\}$ where $f(w) = \frac{1}{d(w)}$ as described above. The open neighborhood $N(v)$ is the set of vertices at distance one from a vertex v. Notice that $v \notin N(v)$. Meanwhile, the closed neighborhood $N[v]$ is the set of vertices at most one from a vertex v. In other words, $N[v] = N(v) \cup \{v\}$. Given a positive integer k, the k-th closed neighborhood of u is defined to be the set $\{v \in V(G) : d(u, v) \leq k\}$, which is denoted by $N^k(v)$. Similarly, given a positive integer k and fraction β, the k-th closed β-neighborhood of u is defined to be the set $\{v \in V(G) : d(u, v) \leq k,\ f_{min}(p_{uv}) \geq \beta\}$ and is denoted by $N^k_\beta[v]$. Suppose that $(x_1, x_2, \ldots, x_k)$ ($k \geq 3$) is a burning sequence for a given graph G.

The following set equation holds:

$$N^{k-1}_\beta[x_1] \cup N^{k-2}_\beta[x_2] \cup \ldots \cup N^0_\beta[x_k] = V(G) \tag{1}$$

We denote $N^1_\beta[v]$ simply by $N_\beta[v]$.

Observation 1. *Suppose G is a connected graph with n vertices and β_1, β_2 are two fractions with $\beta_1 < \beta_2$. Then, $b_{\beta_1}(G) \leq b_{\beta_2}(G)$.*

From the definition of the IC burning number, the following result can be directly obtained.

Theorem 1. *Suppose G is a connected graph with n vertices and $f(v_i) < \beta$ for $1 \leq i \leq k$, where $0 < \beta \leq 1$. Then, $k \leq b_\beta(G) \leq n$.*

Next, we discuss the extremal cases of the IC burning number.

Theorem 2. *Suppose G is a connected graph with n vertices and $\delta(G)$ is the minimum degree of G. Then, $b_\beta(G) = n$ if and only if $\beta > \frac{1}{\delta(G)}$.*

Proof. Let $\{x_1, x_2, \ldots, x_b\}$ be a β-burning sequence of graph G and v_0 be a vertex with $d(v_0) = \delta(G)$.

First, if $b_\beta(G) = n$, then $\beta > \frac{1}{\delta(G)}$. Otherwise, assume $\beta \leq f(v_0) = \frac{1}{\delta(G)}$. Let the first fire source $x_1 \in N(v_0)$, then v_0 will be burned in the second step. Meanwhile, we can choose another vertex other than v_0 as x_2. Then, we find that there are at least three vertices that will be burned in two steps. Furthermore, thus, there exist at most $n-3$ vertices unburned in the first two steps. So, we obtain $b_\beta(G) \leq 2 + (n-3) = n-1$, which contradicts to $b_\beta(G) = n$. Therefore, $\beta > f(v_0) = \frac{1}{\delta(G)}$.

On the other hand, if $\beta > f(v_0) = \frac{1}{\delta(G)}$, according to Theorem 1, we can easily obtain $b_\beta(G) = n$. □

Theorem 3. *Suppose G is a graph with n vertices and $\Delta(G)$ is the maximum degree. Then, (x_1, x_2) is an optimum β-burning sequence for G if and only if one of the following conditions is met:*

(1) $\Delta(G) = d(x_1) = n-1$, and $f(w) \geq \beta$ for all $\forall w \in N(x_1) \setminus \{x_2\}$.

(2) $\Delta(G) \geq d(x_1) = n-2$, and $f(w) \geq \beta$ for all $\forall w \in N(x_1)$.

Proof. Assume that (x_1, x_2) is an optimum β-burning sequence for G. According to Equation (1), $V(G) = N_\beta[x_1] \cup x_2$, which shows that every vertex in set $V(G) \setminus \{x_1, x_2\}$ is adjacent to x_1, and the influence of its neighbors on it is more than β. Since these vertices can only be burned by x_1 in the second step, there are two possible cases for x_2: (1) If $x_2 \in N(x_1)$, then it implies that $\Delta(G) = d(x_1) = n-1$ and for any $w \in N(x_1) \setminus x_2$, $f(w) \geq \beta$. (2) If $x_2 \notin N(x_1)$, then we must have $\Delta(G) \geq d(x_1) = n-2$ and for any $w \in N(x_1)$, $f(w) \geq \beta$.

Conversely, since G has at least two vertices, then $b_\beta(G) \geq 2$.

If $\Delta(G) = d(x_1) = n-1$, and there is a vertex $u \in N(x_1)$ such that $f(u) < \beta$ while $f(w) \geq \beta$ for any $w \in N(x_1) \setminus \{u\}$, then let $x_2 = u$. If $d(x_1) = n-1$ and $f(w) \geq \beta$ for any $w \in N(x_1)$, then let x_2 be any vertices in $N[x_1]$. If $\Delta(G) \geq d(x_1) = n-2$ and $f(w) \geq \beta$ for any $w \in N(x_1)$, then let $\{x_2\} = V(G) \setminus N(x_1)$. In each case, (x_1, x_2) can burn graph G; thus, it is an optimum β-burning sequence for G. □

Given a graph G, suppose $\varphi = (x_1, x_2, \cdots, x_k)$ is a β-burning sequence. Obviously, if vertex $x_i \in V(G)$ and $f(x_i) < \beta$, then $x_i \in \varphi$, i.e., x_i is a fire source. From the above observation, a bound on the IC burning number of G can be easily concluded.

Theorem 4. *Suppose G is a connected graph with n vertices, and $\pi = (d_1, d_2, \ldots, d_n)$ is a degree sequence such that $d_1 \leq d_2 \leq \ldots \leq d_n$ with $d(v_i) = d_i$. If $\frac{1}{d(v_{i+1})} < \beta \leq \frac{1}{d(v_i)}$, then $b_\beta(G) \leq (n-i) + b_\beta(G')$, and $G' = G - \{v_{i+1}, v_{i+2}, \ldots, v_n\}$.*

Proof. Clearly, we have $f(v_{i+1}) < \beta \leq f(v_i)$, so the influence that each vertex $v_{i+1}, v_{i+2}, \ldots, v_n$ receives from its neighbors is less than β in graph G. As a result, these $n-i$ vertices are firstly chosen as source fires in the β-burning process of graph G and let $G' = G - \{v_{i+1}, v_{i+2}, \ldots, v_n\}$. Then, we directly obtain $b_\beta(G) \leq (n-i) + b_\beta(G')$. □

At the end of this section, we discuss a bound of IC burning number with certain domination numbers. If D is a subset of $V(G)$, D's k-th closed neighborhood is denoted by $N^k(D)$ as defined above. If D satisfies the condition that for every vertex $u \in V(G) \setminus D$, there exists a vertex $v \in D$ and $d(u,v) \leq k$, then D is called a k-step dominating set. Meanwhile, we can define β-way k-step dominating set of G. A k-step dominating set D is called a β-way k-step dominating set of G if for any $u \in V$ either $u \in D$ or there exists $v \in D$ such that $u \in N^k_\beta[v]$. The β-way k-step domination number of G denoted by $\gamma^\beta_k(G)$ is the number of the vertices in a minimum β-way k-step dominating set of G.

From the above definition of β-way k-step dominating set of G, we obtain

Theorem 5. *Suppose G is a connected graph with n vertices. Then, $b_\beta(G) \leq \gamma^\beta_k(G) + k$.*

4. The IC Burning Number of Some Special Graphs

Given a graph G and a vertex v, the eccentricity of v is defined as $max\{d(v,u) : u \in G\}$. The radius is the minimum eccentricity over the vertex set G, which is denoted by rad(G), and the diameter is the maximum eccentricity over the vertex set G, which is denoted by diam(G).

Given two graphs G and H, their Cartesian product is denoted by $G\square H$. In $G\square H$, two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $u_1u_2 \in E(G)$ and $v_1 = v_2$.

We first list several known results for the traditional graph burning number $b(G)$ that will be used later.

Proposition 1 ([1])**.** *Suppose G is a path with n vertices, then* $b(G) = \lceil \sqrt{n} \rceil$.

Proposition 2 ([17])**.** *Suppose G is a complete bipartite graph with* $3 \le m \le n$*, then* $b(G) = 3$.

Proposition 3 ([17])**.** *Suppose G is a Helm graph (See Figure 2 for Helm graph.) with* $n \ge 3$*, then* $b(G) = 3$.

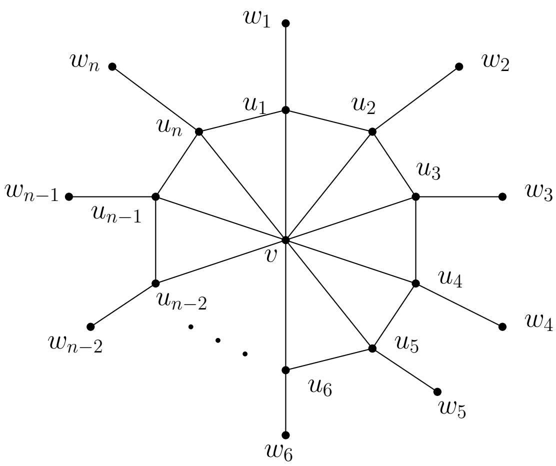


Figure 2. An example of a Helm ($n \ge 6$) graph.

Proposition 4 ([2])**.** *Suppose G is a graph with radius rad(G) and diameter diam(G). Then*

$$\lceil \sqrt{diam(G) + 1} \rceil \le b(G) \le rad(G) + 1$$

Proposition 5 ([14])**.** *Suppose two graphs G and H are connected graphs, then*

$$max\{b(G), b(H)\} \le b(G\square H) \le min\{b(G) + rad(H), b(H) + rad(G)\}$$

For the new generalized graph burning IC model, we present the following results for $b_\beta(G)$.

Theorem 6. *Suppose* P_n *is a path with n vertices and* $0 < \beta \le 1$ *. Then*

$$b_\beta(P_n) = \begin{cases} \lceil \sqrt{n} \rceil & \text{If } 0 < \beta \le \frac{1}{2} \\ \begin{cases} 2 & \text{If } n = 2,3 \\ 3 & \text{If } n = 4 \\ n-2 & \text{If } n > 4 \end{cases} & \text{If } \frac{1}{2} < \beta \le 1 \end{cases}$$

Proof. Suppose $P_n = v_1v_2 \dots v_n$ with $d(v_1) = d(v_n) = 1$ and $d(v_k) = 2$ for $1 < k < n$.

Case 1. $0 < \beta \le \frac{1}{2}$.

Note that $f(v_j) = \frac{1}{d(v_j)} \geq \frac{1}{2}$ for $1 \leq j \leq n$. From Proposition 1, we obtain $b_\beta(P_n) = b(P_n) = \lceil \sqrt{n} \rceil$.

Case 2. $\frac{1}{2} < \beta \leq 1$.

It is easy to obtain $b_\beta(P_2) = b_\beta(P_3) = 2$ and $b_\beta(P_4) = 3$. Here, we consider cases for $n > 4$. We let $x_1 = v_2$, $x_2 = v_{n-1}$, and $x_{i+2} = v_{i+2}$ for $1 \leq i \leq n-4$. Clearly, $(x_1, x_2, \ldots, x_{n-2})$ is a β-burning sequence of P_n and thus $b_\beta(P_n) \leq n-2$. On the other hand, consider $f(v_k) = \frac{1}{d(v_k)} = \frac{1}{2} < \beta$ for $1 < k < n$, so we obtain $b_\beta(P_n) \geq n-2$. Thus, $b_\beta(P_n) = n-2$ while $\beta > \frac{1}{2}$. □

Theorem 7. *Suppose $K_{m,n}$ is a complete bipartite graph with $3 \leq m \leq n$ and $0 < \beta \leq 1$ (See Figure 3). Then*

$$b_\beta(K_{m,n}) = \begin{cases} 3 & \text{If } 0 < \beta \leq \frac{1}{n} \\ m & \text{If } \frac{1}{n} < \beta \leq \frac{1}{m} \\ m+n & \text{If } \frac{1}{m} < \beta \leq 1 \end{cases}$$

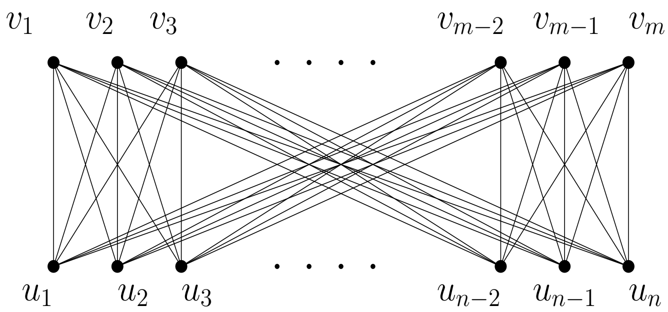


Figure 3. An example of $K_{m,n}$ ($3 \leq m \leq n$).

Proof. Suppose the vertices of $K_{m,n}$ are divided into two parts, A = $\{v_1, v_2, \ldots, v_m\}$ and B = $\{u_1, u_2, \ldots, u_n\}$. Then, $f(v_i) = \frac{1}{n}$ for $i = 1, 2, \ldots, m$ and $f(u_j) = \frac{1}{m}$ for $j = 1, 2, \ldots, n$. We complete the proof by analyzing the following three cases.

Case 1. $0 < \beta \leq \frac{1}{n}$.

In this case, $b_\beta(K_{m,n}) = b(K_{m,n})$. From Proposition 2, we directly obtain $b_\beta(K_{m,n}) = 3$ while $\beta \leq \frac{1}{n}$.

Case 2. $\frac{1}{n} < \beta \leq \frac{1}{m}$.

First, let $x_i = v_i$ for $1 \leq i \leq m$. Obviously, $(x_1, x_2, \ldots, x_m)$ is a β-burning sequence of $K_{m,n}$, which implies $b_\beta(K_{m,n}) \leq m$. On the other hand, because $f(v_i) = \frac{1}{n} < \beta$ for all $v_1, v_2, \ldots, v_m$, according to Theorem 1, we obtain $b_\beta(K_{m,n}) \geq m$. Thus, $b_\beta(K_{m,n}) = m$ while $\frac{1}{n} < \beta \leq \frac{1}{m}$.

Case 3. $\frac{1}{m} < \beta \leq 1$.

According to Theorem 1, it is easy to obtain $n + m \leq b_\beta(K_{m,n}) \leq n + m$. Then, we directly obtain $b_\beta(K_{m,n}) = n + m$ while $\beta > \frac{1}{m}$. □

The Helm graph, which has $2n+1$ vertices, is obtained by adding a pendant edge at each vertex to the cycle of the n-wheel graph; see Figure 2.

Theorem 8. *Suppose G is a Helm graph with $n \geq 3$ and $0 < \beta \leq 1$. Then*

$$b_\beta(G) = \begin{cases} 3 & \text{If } 0 < \beta \leq \frac{1}{4} \\ n+1 & \text{If } \frac{1}{4} < \beta \leq 1 \end{cases}$$

Proof. Let v be the center vertex in the Helm graph and u_iv_i be pendant edges. Meanwhile, $d(w_i) = 1$ for $1 \leq i \leq n$.

Case 1. $0 < \beta \leq \frac{1}{4}$.

Case 1.1. $n = 3$.

For a Helm graph G with $n = 3$, it should be noted that the diameter and radius of G are 3 and 2, respectively. From Proposition 4, we obtain $2 \leq b_\beta(G) \leq 3$. However, note that $\Delta(G) = 4 < 7 - 2$, and according to Theorem 3, it follows that $b_\beta(G) = 3$.

Case 1.2. $n > 3$.

In these cases, the diameter and radius of the Helm graph G are 4 and 2, respectively. If $\beta \leq \frac{1}{n}$, from Proposition 4, we directly obtain $b_\beta(G) = b(G) = 3$. If $\frac{1}{n} < \beta \leq \frac{1}{4}$, we choose $x_1 = v$, $x_2 = w_1$, and $x_3 = w_2$. Obviously, (x_1, x_2, x_3) is a β-burning sequence; thus, $b_\beta(G) \leq 3$. On the other hand, it is clear that $b_\beta(G) \geq b(G) = 3$ from Proposition 3. Thus, we obtain $b_\beta(G) = 3$ when $\frac{1}{n} < \beta \leq \frac{1}{4}$.

In summary, we have $b_\beta(G) = 3$ for $\beta \leq \frac{1}{4}$.

Case 2. $\frac{1}{4} < \beta \leq 1$.

Case 2.1. $n = 3$.

For the case $\frac{1}{4} < \beta \leq \frac{1}{3}$ and $n = 3$, we let $x_1 = u_1$, $x_2 = u_2$, $x_3 = u_3$, and $x_4 = w_3$. Obviously, (x_1, x_2, x_3, x_4) is a β-burning sequence of G, which means $b_\beta(G) \leq 4$. On the other hand, $f(u_i) < \beta$ for $1 \leq i \leq 3$, so we have $b_\beta(G) \geq 3$. Note that in any optimum β-burning sequences, w_i must be burned later than u_i, which means $b_\beta(G) \geq 4$. So, we obtain $b_\beta(G) = 4$ while $\frac{1}{4} < \beta \leq \frac{1}{3}$.

Similarly, for $\beta > \frac{1}{3}$, we let $x_1 = u_1$, $x_2 = u_2$, $x_3 = u_3$, and $x_4 = v$. Clearly, (x_1, x_2, x_3, x_4) is a β-burning sequence of G, which means $b_\beta(G) \leq 4$. On the other hand, $f(u_i) < f(v) < \beta$ for $1 \leq i \leq 3$, which means $b_\beta(G) \geq 4$. We obtain $b_\beta(G) = 4$ while $\beta > \frac{1}{3}$.

So, when $n = 3$, $b_\beta(G) = n + 1 = 4$ for all $\frac{1}{4} < \beta < 1$.

Case 2.2. $n > 3$.

Let $x_1 = u_i$ for $1 \leq i \leq n$ and $x_{n+1} = v$. Obviously, $(x_1, x_2, \cdots, x_{n+1})$ is a β-burning sequence of G, which means $b_\beta(G) \leq n + 1$. Now, we prove $b_\beta(G) \geq n + 1$. In fact, $f(v) \leq f(u_i) < \beta$ for $1 \leq i \leq n$, which implies that there are $n + 1$ vertices that cannot be burned by any of their neighbors. These $n + 1$ vertices must be chosen as the sources fires in any optimum β-burning sequences. So, clearly $b_\beta(G) \geq n + 1$. We directly obtain $b_\beta(G) = n + 1$ for $\beta > \frac{1}{4}$. □

The fan graph is obtained by connecting K_1 to every vertex in the path P_n, where $n \geq 3$, which is denoted by $K_1 + P_n$. The vertex of K_1 is called the center of the fan graph. See Figure 4.

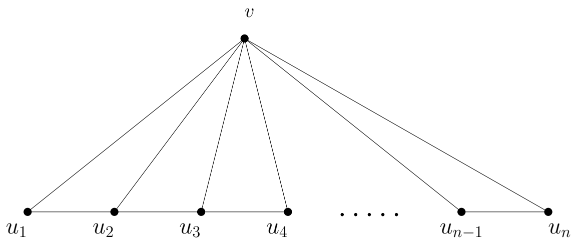


Figure 4. An example of a fan ($n \geq 4$) graph.

Theorem 9. *Suppose G is a fan graph $K_1 + P_n$ with $n \geq 3$ and $0 < \beta \leq 1$. Then*

$$b_\beta(G) = \begin{cases} 2 & \text{If } 0 < \beta \leq \frac{1}{3} \\ n-1 & \text{If } \frac{1}{3} < \beta \leq \frac{1}{2} \\ n+1 & \text{If } \frac{1}{2} < \beta \leq 1 \end{cases}$$

Proof. Let v be the center vertex in the fan graph G and $V(P_n) = \{u_i \mid 1 \leq i \leq n\}$.

Case 1. $0 < \beta \leq \frac{1}{3}$.

Whether $f(v)$ is less than β or not, we choose $x_1 = v$ first and $x_2 = u_1$. Obviously, (x_1, x_2) is the β-burning sequence of G when $0 < \beta \leq \frac{1}{3}$. From the definition, we obtain $b_\beta(G) \leq 2$. On the other hand, for any graph G with more than two vertices, we have $b_\beta(G) \geq 2$. Thus, we have $b_\beta(G) = 2$.

Case 2. $\frac{1}{3} < \beta \leq \frac{1}{2}$.

Let $x_1 = v$, $x_2 = u_{n-1}$ and $x_{i+2} = u_{i+1}$ for $1 \leq i \leq n-3$. Obviously, $(x_1, x_2, \ldots, x_{n-1})$ is a β-burning sequence of G and thus $b_\beta(G) \leq n-1$. On the other hand, considering $f(v) \leq f(u_i) < \beta$ for $2 \leq i \leq n-1$, all vertices u_i $(2 \leq i \leq n-1)$ and v must be chosen as source fires in any optimum β-burning sequences of G. At least $n-1$ steps are thus required to burn all the vertices in G. So, we obtain $b_\beta(G) = n-1$.

Case 3. $\frac{1}{2} < \beta \leq 1$.

It is easy to see that all these $n+1$ vertices should be set on fire directly, so we obtain $b_\beta(G) = n+1$ while $\frac{1}{2} < \beta \leq 1$. □

Given $n \geq 3$ and an integer k such that $1 \leq k \leq n-1$, we define the *generalized Petersen graph* $P(n,k)$ as a graph on $2n$ vertices with vertex set

$$V(P(n,k)) = \{u_i, v_i : i = 1, 2, \ldots, n\}$$

and edge set

$$E(P(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 1, 2, \ldots, n\}.$$

Theorem 10. *Suppose G is a generalized Petersen graph (See Figure 5) with $n \geq 3$, k is an integer with $1 \leq k \leq n-1$, and $0 < \beta \leq 1$. The values of $b_\beta(G)$ for generalized Petersen graphs are listed in the following Table 1.*

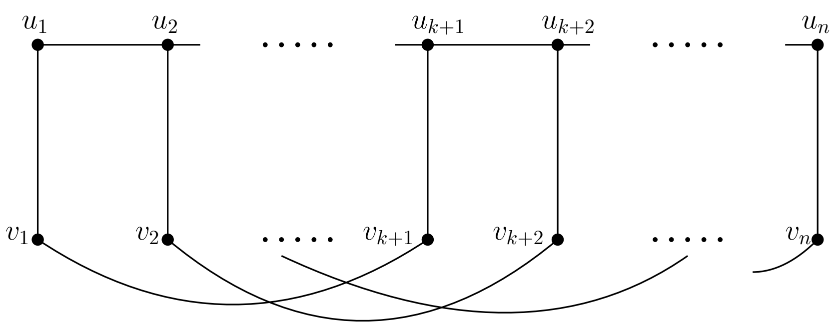


Figure 5. An example of a generalized Petersen ($n \geq 3$) graph.

Proof. Suppose $D_1 = \{u_1, u_2, \ldots, u_n\}$ and $D_2 = \{v_1, v_2, \ldots, v_n\}$ are two partite sets of G. Now, we divide the following four cases to complete the proof.

Case 1. $0 < \beta \leq \frac{1}{3}$.

If $\beta \leq \frac{1}{3}$, $b_\beta(G) = b(G)$, we borrow the results directly from [12]; see Table 1.

Case 2. $\frac{1}{3} < \beta \leq \frac{1}{2}$.

Case 2.1. $n < 2k$.

There exists at least one vertex in D_2 whose degree is 1; see Figure 6, where the solid dots represent that they can be burned by their neighbors, and the hollow dots indicate that they can only be source of fires. For clarity, we converted the graph to another form as shown in Figure 7.

Table 1. $b_\beta(G)$ of generalized Petersen graphs.

β	k, n			$b_\beta(G)$
$0 < \beta \leq \frac{1}{3}$	$k = 1$			$\lceil \sqrt{n} \rceil \leq b_\beta(G) \leq \lceil \sqrt{n} \rceil + 1$ [12]
	$k = 2$			$\lceil \sqrt{\frac{n}{2}} \rceil + 1 \leq b_\beta(G) \leq \lceil \sqrt{\frac{n}{2}} \rceil + 2$ [12]
	otherwise			$\lceil \sqrt{\lfloor \frac{n}{k} \rfloor} \rceil \leq b_\beta(G) \leq \lceil \sqrt{\lfloor \frac{n}{k} \rfloor} \rceil + \lfloor \frac{k}{2} \rfloor + 2$ [12]
$\frac{1}{3} < \beta \leq \frac{1}{2}$	$n < 2k$	$k = n - 1$	$n = 3, k = 2$	3
			otherwise	$n - 1$
		$k \leq n - 2$	$n = 5, k = 3$	4
			otherwise	$n - 2$
	$n = 2k$	$n = 4, k = 2$		3
		otherwise		$n - 2$
	$n > 2k$	$n = 3, k = 1$		3
		otherwise		$2n - 2k - 2$
$\frac{1}{2} < \beta \leq 1$	$n < 2k$			$3n - 2k$
	$n \geq 2k$			$2n$

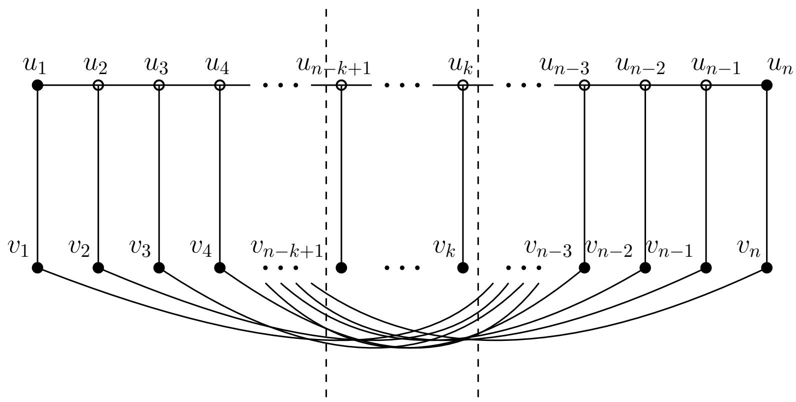


Figure 6. The state of vertices of G in Case 2.1.

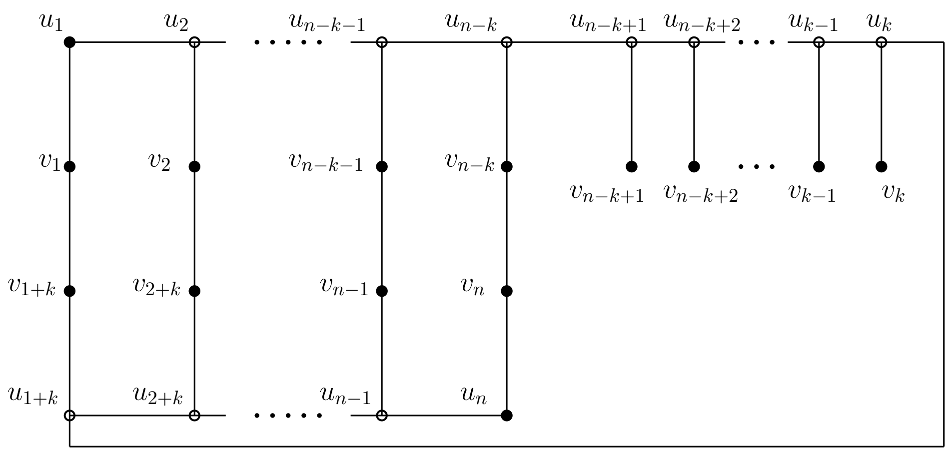


Figure 7. The state of vertices of G in Case 2.1 with another form.

First, we consider the case for $k = n - 1$. As for the case $n = 3$ and $n = 4$, from simple verification, we obtain $b_\beta(G) = 3$ for both $n = 3$ and $n = 4$. Then, we consider the cases for $n \geq 5$. We let $x_1 = u_2$, $x_2 = u_{n-1}$, $x_{i+2} = u_{i+2}$ for $1 \leq i \leq n - 4$ and $x_{n-1} = v_{n-1}$. Obviously, $(x_1, x_2, \ldots, x_{n-1})$ is a β-burning sequence of G. From the definition, we obtain $b_\beta(G) \leq n - 1$. On the other hand, the vertices u_i for $2 \leq i \leq n - 1$ must be chosen as the source fires in any optimum β-burning sequences of G. Suppose that u_i is burned in step t, then v_i will be burned in step $t + 1$. Therefore, no matter how the source fires are chosen, all vertices of G cannot be burned in $n - 2$ steps, which implies that $b_\beta(G) \geq n - 1$. Thus, $b_\beta(G) = n - 1$ while $n \geq 5$.

Next, we consider the case for $k \leq (n - 2)$. In this case, $n < 2k$ and $k \leq n - 2$ implies $n > 4$. See Figure 7.

From simple verification, we obtain $b_\beta(G) = 4$ for $n = 5$. Here, we consider the cases for $n \geq 5$. Let $x_i = u_{i+1}$ for $1 \leq i \leq n - 2$. Obviously, $(x_1, x_2, \ldots, x_{n-2})$ is the β-burning sequence of G. From the definition, we obtain $b_\beta(G) \leq n - 2$. On the other hand, the vertices u_i for $2 \leq i \leq n - 1$ must be chosen as the source fires in any optimum β-burning sequences of G, and we have $b_\beta(G) \geq n - 2$. Thus, we obtain $b_\beta(G) = n - 2$.

Case 2.2. $n = 2k$.

The degree of all the vertices in set D_2 is 2 in this case. We can also convert the graph to another form as shown in Figure 8.

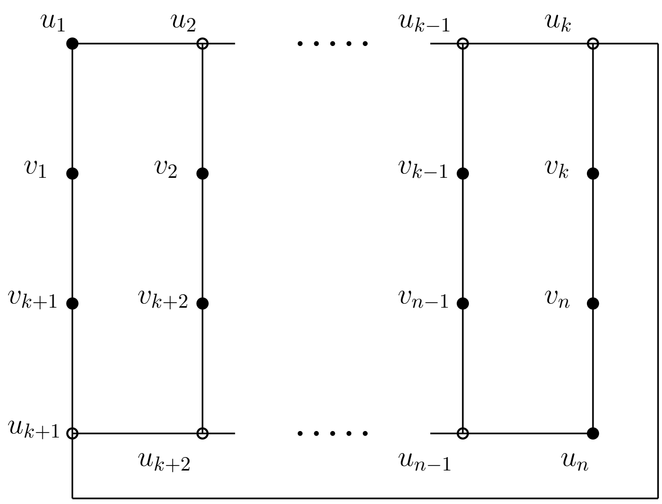


Figure 8. The state of vertices of G in Case 2.2.

First, we consider the case for $k = 2$, which means $n = 4$, and it is not hard to see $b_\beta(G) = 3$. As for the cases when $k \geq 3$, we let $x_1 = u_2$, $x_2 = u_{n-1}$, and $x_{i+2} = u_{i+2}$ for $1 \leq i \leq n - 4$. Clearly, $(x_1, x_2, \ldots, x_{n-2})$ is a β-burning sequence of G. Hence, we obtain $b_\beta(G) \leq n - 2$. On the other hand, all vertices in set D_1 except u_1 and u_n must be chosen as source fires in any optimum β-burning sequences of G. Thus, at least $n - 2$ steps are required to burn all the vertices of G. So, we obtain $b_\beta(G) = n - 2$.

Case 2.3. $n > 2k$.

From simple checking, we obtain $b_\beta(G) = 3$ for $n = 3$ and $k = 1$. As for other cases, see Figure 9. Let $A = \{v_{k+1}, v_{k+2}, \ldots, v_{n-k}\}$, $B_1 = \{v_1, v_2, \ldots, v_k\}$ and $B_2 = \{v_{n-k+1}, \ldots, v_n\}$. Obviously, the degree of each vertex in A is 3, and the degree of each vertex in both B_1 and B_2 is 2. All vertices with degree 3 in this graph must be sources of fires, which means $b_\beta(G) \geq n - 2k + n - 2 = 2n - 2k - 2$. On the other hand, let $x_1 = v_{k+1}$, $x_2 = v_{n-k}$, $x_{i+2} = v_{(k+1)+i}$ for $1 \leq i \leq n - 2k - 2$ and $x_{(n-2k)+j} = u_{j+1}$ for $1 \leq j \leq n - 2$ as source fires. It is easy to check that $(x_1, x_2, \ldots, x_{2n-2k-2})$ is a β-burning sequence of G. Thus, $b_\beta(G) = 2n - 2k - 2$.

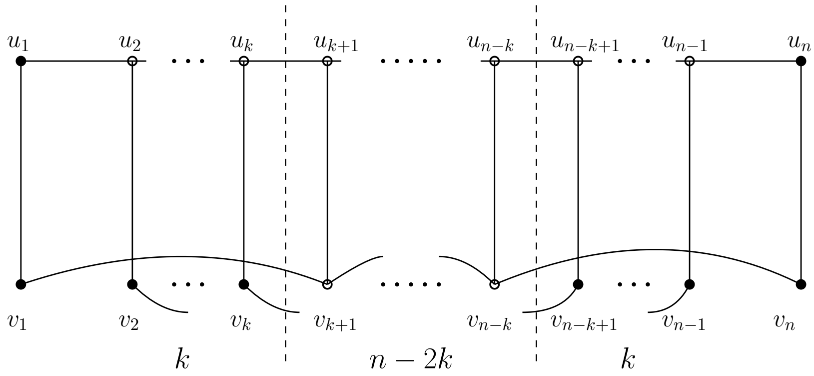


Figure 9. The state of vertices of G in Case 2.3.

Case 3. $\frac{1}{2} < \beta \leq 1$.

Case 3.1. $n < 2k$.

As shown in Figure 6, all vertices except $\{v_{n-k+1}, v_{n-k+2}, \ldots, v_k\}$ with degree 1 cannot be burned by their neighbors; thus, $b_\beta(G) \geq 2n - (2k-n) = 3n - 2k$. Furthermore, also notice that $(u_{n-k+1}, \ldots, u_k, u_1, \ldots, u_{n-k}, u_{k+1}, \ldots, u_n, v_1, \ldots, v_{n-k}, v_{k+1}, \ldots, v_n)$ is a β-burning sequence of G. Thus, $b_\beta(G) = 3n - 2k$ while $n < 2k$.

Case 3.2. $n \geq 2k$.

In this case, all vertices should be the sources of fire, so $b_\beta(G) = 2n$ while $n \geq 2k$. □

Theorem 11. *Suppose P_n and P_m are two paths with $3 \leq m \leq n$ and $0 < \beta \leq 1$. Then*

$$b_\beta(P_n \Box P_m) \begin{cases} \in [max\{\lceil\sqrt{n}\rceil, \lceil\sqrt{m}\rceil\}, min\{\lceil\sqrt{n}\rceil + \lceil\frac{m}{2}\rceil, \lceil\sqrt{m}\rceil + \lceil\frac{n}{2}\rceil\}] & \text{If } 0 < \beta \leq \frac{1}{4} \\ = \begin{cases} 3 & \text{If } n = m = 3 \\ 4 & \text{If } n = 4, m = 3 \text{ or } n = 5, m = 3 \\ (n-2)\cdot(m-2) & otherwise \end{cases} & \text{If } \frac{1}{4} < \beta \leq \frac{1}{3} \\ = mn - 4 & \text{If } \frac{1}{3} < \beta \leq \frac{1}{2} \\ = mn & \text{If } \frac{1}{2} < \beta \leq 1 \end{cases}$$

Proof. Suppose $P_n = v_1, v_2, \ldots, v_n$ and $P_m = u_1, u_2, \ldots, u_m$. There are four cases related to the degree of vertices in graph $P_n \Box P_m$, when $3 \leq m \leq n$. Note that $d[(u_1, v_1)] = d[(u_1, v_n)] = d[(u_m, v_1)] = d[(u_m, v_n)] = 2$, $d[(u_1, v_j)] = d[(u_i, v_1)] = d[(u_m, v_j)] = d[(u_i, v_n)] = 3$ for $2 \leq i \leq m-1$ and $2 \leq j \leq n-1$, and the degree of the vertex is equal to 4 otherwise.

Case 1. $0 < \beta \leq \frac{1}{4}$.

When $\beta \leq \frac{1}{4}$, $b_\beta(G) = b(G)$. Because P_n and P_m are both paths, from Proposition 1, their burning numbers are $\lceil\sqrt{n}\rceil$ and $\lceil\sqrt{m}\rceil$, respectively. Furthermore, it is easy to calculate their radii, $\lceil\frac{n}{2}\rceil$ and $\lceil\frac{m}{2}\rceil$, respectively. Then, we borrow the results in Proposition 5 directly here, which are $max\{\lceil\sqrt{n}\rceil, \lceil\sqrt{m}\rceil\} \leq b_\beta(P_n \Box P_m) \leq min\{\lceil\sqrt{n}\rceil + \lceil\frac{m}{2}\rceil, \lceil\sqrt{m}\rceil + \lceil\frac{n}{2}\rceil\}$ while $0 < \beta \leq \frac{1}{4}$.

Case 2. $\frac{1}{4} < \beta \leq \frac{1}{3}$.

From simple verification, we obtain $b_\beta(P_n \Box P_m) = 3$ for $n = m = 3$ and $b_\beta(P_n \Box P_m) = 4$ for both $n = 4, m = 3$ and $n = 5, m = 3$. Then, we consider other cases.

We claim that $((u_2, v_2), (u_{m-1}, v_{n-1}), (u_2, v_{n-1}), (u_{m-1}, v_2), (u_2, v_3), \ldots, (u_2, v_{m-2}), (u_3, v_2), \ldots, (u_{m-2}, v_{n-1}), (u_{m-1}, v_2), \ldots, (u_{m-1}, v_{n-2}))$ is an optimum β-burning sequence in $P_n \Box P_m$. Clearly, the above sequence is a β-burning sequence of $P_n \Box P_m$, so $b_\beta(P_n \Box P_m) \leq (n-2)\cdot(m-2)$. On the other hand, the number of vertices with influence less than $\frac{1}{4}$ is $(n-2)\cdot(m-2)$ which implies that $b_\beta(P_n \Box P_m) \geq (n-2)\cdot(m-2)$. So, we have $b_\beta(P_n \Box P_m) = (n-2)\cdot(m-2)$.

Case 3. $\frac{1}{3} < \beta \leq \frac{1}{2}$.

As shown in Figure 10, all vertices except $\{(u_1, v_1), (u_1, v_n), (u_m, v_1), (u_m, v_n)\}$ with degree 2 cannot be burned by their neighbor; thus, $b_\beta(P_n \Box P_m) \geq mn - 4$. Furthermore, also notice that $((u_1, v_2), \ldots, (u_1, v_{n-1}), (u_m, v_2), \ldots, (u_m, v_{n-1}), (u_2, v_1), \ldots, (u_{m-1}, v_n))$ is a β-burning sequence of $P_n \square P_m$. Thus, $b_\beta(P_n \square P_m) = mn - 4$ while $\frac{1}{3} < \beta \leq \frac{1}{2}$.

Case 4. $\frac{1}{2} < \beta \leq 1$.

All vertices should be set on fires directly; thus, $b_\beta(P_n \square P_m) = mn$. $\square$

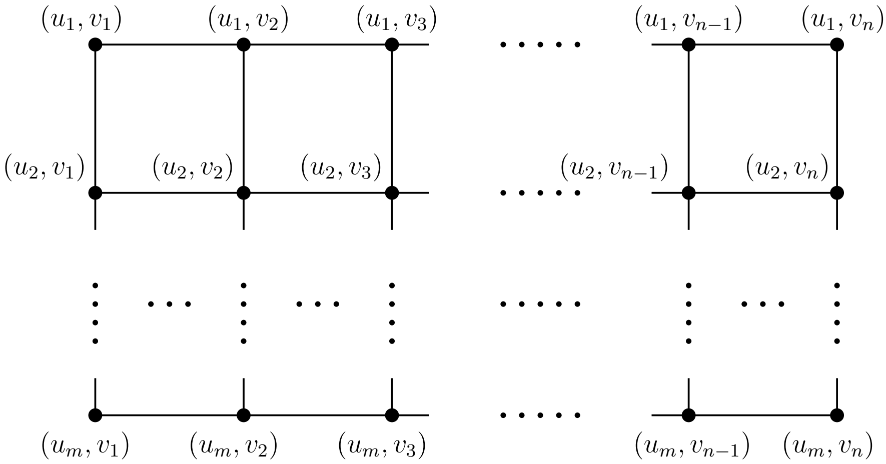


Figure 10. An example of a $P_n \square P_m$ $(3 \leq m \leq n)$ graph.

5. Conclusions and Future Work

In this paper, we introduced a new generalized graph burning model called the Independent Cascade Graph Burning model (abbreviated as the IC Model), which is more realistic than the traditional graph burning model. Regarding the IC burning number, we carried out the following work:

1. The upper and lower bounds of the IC burning numbers of general graphs were discussed.
2. The IC burning numbers of several special graphs were determined.

As for future research, we are interested in related algorithms to calculate IC burning numbers on large networks, and we also intend to calculate the IC burning number of some other special graphs.

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