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Cycle Embedding in Enhanced Hypercubes with Faulty Vertices

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Abstract: The enhanced hypercube is a well-known variant of the hypercube and can be constructed from a hypercube by adding an edge to every pair of vertices with complementary addresses. Let F_v denote the set of faulty vertices in an n-dimensional enhanced hypercube $Q_{n,k}$ $(1 \leq k \leq n-1)$. In this paper, we conclude that if $n \geq 2$, then every fault-free edge of $Q_{n,k} - F_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$, and if n (≥ 2) and k have the different parity, then every fault-free edge of $Q_{n,k} - F_v$ lies on a fault-free cycle of every possible odd length from $n-k+4$ to $2^n - 2|F_v| - 1$, where $|F_v| \leq n-2$.

Keywords: enhanced hypercube; cycle embedding; faulty vertices; interconnection network

1. Introduction

The well-known hypercue has several excellent properties, such as recursive structure, regularity, symmetry, small diameter, low degree, and logarithmic diameter [1]. One variant of the hypercube that has been the focus of a great deal of research is the enhanced hypercube $Q_{n,k}$ [2,3], which can be obtained from the well-known n-dimensional hypercube Q_n by adding each edge from the vertex $x_1x_2 \ldots x_{k-1}x_k \ldots x_n$ to the vertex $x_1x_2 \ldots x_{k-1}\bar{x}_k \ldots \bar{x}_n$. The n-dimensional enhanced hypercube $Q_{n,k}$ $(1 \leq k \leq n-1)$ is proposed to improve the efficiency of the hypercube structure Q_n, as it possesses many attractive properties that are superior to that of the hypercube [4–11]. Moreover, the folded hypercube FQ_n is the special case of the enhanced hypercube $Q_{n,k}$ when $k = 1$ [12–21].

In computer network topology design, one of the central issues in evaluating a network is to study the network embedding problem. The embedding of one guest graph G_1 into another host graph G_2 is a one-to-one mapping m from the vertex set of G_1 to the vertex set G_2 [1]. Recently, the multiprocessor system is becoming prevalent and significant. Using the fault-tolerant embedding properties to evaluate the reliability of a parallel computing system is a significant issue. Therefore, many research fields and topics focus on the reliability analysis problems regarding the fault-tolerant embedding of distributed networks [19].

The concept of ISTs was first introduced by Itai and Rodeh [22]. At a later time, a large number of researchers were attracted by the problems regarding the reliability of parallel and distributed networks. The construction of ISTs is obtained to receive high levels of fault-tolerant properties and security. To pursue the above goals, one is to design an efficient construction or investigate the fault-tolerant embedding properties. Note that the class of enhanced hypercube is a general case of the folded hypercube. The enhanced hypercubes have attracted much attention, e.g., the diagnosability, embedding, and others. Fault-tolerant cycle embedding with respect to vertex is related to investigating the property of a more cost-effective constructure.

Problems regarding the fault-tolerant embedding for hypercubes and folded hypercubes have been studied in [14,15,23,24]. Let F_v and F_e be the sets of faulty vertices and faulty edges, respectively. Tsai [25] proved that every fault-free edge of $Q_n - F_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$ inclusive, where $|F_v| \leq n-2$. Furthermore, Hsieh and Shen [26] extended the above result to show that every fault-free edge of $Q_n - F_e - F_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$, where $|F_v| + |F_e| \leq n-2$ and $n \geq 3$. Xu et al. [21] showed that every fault-free edge

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of $FQ_n - F_e$ lies on a fault-free cycle of every even length from 4 to 2^n and also lies on a fault-free cycle of every odd length from $n+1$ to $2^n - 1$ if n is even, where $|F_e| \le n-1$. Cheng et al. [13] proved that every fault-free edge of $FQ_n - F_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$ if $n \ge 3$, and also lies on a fault-free cycle of every odd length from $n+1$ to $2^n - 2|F_v| - 1$ if n is even and $n \ge 2$, where $|F_v| \le n-2$. After that, Kuo and Stewart [16] further proved that every fault-free edge of $FQ_n - F_v - F_e$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$ if $n \ge 3$, and also lies on a fault-free cycle of every odd length from $n+1$ to $2^n - 2|F_v| - 1$ if $n \ge 2$ is even, where $|F_v| + |F_e| \le n-2$. Due to the above motivations, in this paper, we consider the faulty enhanced hypercube $Q_{n,k}$ $(1 \le k \le n-1)$ with $|F_v| \le n-2$, where F_v denotes the set of faulty vertices of $Q_{n,k}$ $(1 \le k \le n-1)$, proving that every fault-free edge of $Q_{n,k} - F_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$ if $n \ge 2$, and every fault-free edge of $Q_{n,k} - F_v$ lies on a fault-free cycle of every possible odd length from $n-k+2$ to $2^n - 2|F_v| - 1$ if n (≥ 2) and k have different parity.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic definitions and lemmas used in our discussion. We give the main results related to even cycles and odd cycles embedding in the faulty enhanced hypercube in Sections 3 and 4 respectively. Finally, we conclude this paper in Section 5.

2. Preliminaries

For the graph theoretical terminology and notations not mentioned here, see [27]. A graph $G = (V, E)$ is an ordered pair in which V is a finite set and E is a subset of $\{(u,v)|(u,v)$ is an unordered pair of $V\}$. We call V as the vertex set and E as the edge set. For a set of edges or vertices S in G, the graph $G - S$ is a subgraph of G by deleting all elements in S from G. Two vertices u and v are adjacent if $(u,v) \in E$. A path, represented as $P[v_0, v_m] = \langle v_0, v_1, v_2, \ldots, v_m \rangle$, is a sequence of distinct vertices in which any two consecutive vertices are adjacent. We call v_0 and v_m the end-vertices of the path $P[v_0, v_m]$. A path $P[v_0, v_m]$ forms a cycle if $v_0 = v_m$ and $m \ge 3$. The length of a path P (respectively, a cycle C) is denoted by $l(P)$ (respectively, $l(C)$). Let F_v and F_e be the sets of faulty vertices and faulty edges in G, where $F_v \subseteq V(G)$, $F_e \subseteq E(G)$. A vertex v is fault-free if $v \in F_v$. An edge $e \in E(G)$ is fault-free if the two end-vertices and the edge between them are not faulty, i.e., $e \in F_e$. A path (respectively, a cycle) is fault-free if it contains no faulty edges.

The n-dimensional hypercube, denoted by Q_n, is a graph with 2^n vertices which are labeled as binary strings of length n from $\underbrace{00\ldots0}_{n}$ to $\underbrace{11\ldots1}_{n}$. Two vertices u and v in Q_n are linked by an edge if and only if u and v differ exactly on one bit position. For convenience, we define the vertex $u = x_1x_2\ldots x_{i-1}x_ix_{i+1}\ldots x_n$ and the vertex $u^i = x_1x_2\ldots x_{i-1}\bar{x}_ix_{i+1}\ldots x_n$, where $\bar{x}_i$ is the complement of x_i, i.e., $\bar{x}_i = 1 - x_i$ for some $1 \le i \le n$ and $x_i \in \{0,1\}$. In other words, u and u^i have the different binary strings exactly on the ith position. We call the edge (u, u^i) as an ith dimension edge which is along dimension i. Let E_i be the set of ith dimensional edges. Clearly, $|E(Q_n)| = n \cdot 2^{n-1}$. For a given integer i $(1 \le i \le n)$, partition Q_n along dimension i into two $(n-1)$-dimensional cubes, then Q^{i0}_{n-1} (respectively, Q^{i1}_{n-1}) denotes the subgraph of Q_n induced by $x_1x_2\ldots x_{i-1}0x_{i+1}\ldots x_n$ (respectively, $x_1x_2\ldots x_{i-1}1x_{i+1}\ldots x_n$), where $x_j \in \{0,1\}$, $1 \le j \le n$ $j \ne i$. Obviously, we have Q^{i0}_{n-1} and Q^{i0}_{n-1} being isomorphic to Q_n.

A graph G is bipartite if the vertex set V can be divided into two disjoint partite subsets V_0 and V_1 such that each edge in G connects one end-vertex in V_0 and another in V_1. A bipartite graph $G = (V_0 \cup V_1, E)$ is hyper-Hamiltonian laceable if for any vertex $v \in V_i$, $i = 0, 1$, there exists a Hamiltonian path of $G - \{v\}$ between any two vertices in V_{1-i}.

The distance between u and v denoted by $d_G(u,v)$ is the length of the shortest path between u and v in G. The Hamming distance between two vertices $u = x_1x_2\ldots x_n$ and $v = y_1y_2\ldots y_n$ in Q_n is denoted by $d_H(u,v) = \sum_{i=1}^{n} |x_i - y_i|$, where $x_i \in \{0,1\}$ and $y_i \in \{0,1\}$. The Hamming weight of the vertex $u = x_1x_2\ldots x_n$, denoted by $hw(u)$, is the number of i's such that $x_i = 1$. We can use $hw(u)$ to check the parity of the vertex u, i.e., u is an even vertex (respectively, an odd vertex) if $hw(u)$ is even (respectively, $hw(u)$ is odd). Note that Q_n is a bipartite graph with two disjoint partite subsets $\{u|\ hw(u)$ is odd$\}$ and $\{u|\ hw(u)$ is even$\}$. Clearly, $d_{Q_n}(u,v) = d_H(u,v)$, $E_i = \{(u,u^i)|\ d_H(u,u^i) = 1, i \in \{1,2,\ldots,n\}\}$.

Definition 1. *Ref.* [2] *Enhanced hypercube* $Q_{n,k}$ $(1 \le k \le n-1)$ *is an undirected simple graph. Its vertex set is* $V(Q_{n,k}) = \{x_1x_2\cdots x_n : x_i = 0 \text{ or } 1, 1 \le i \le n\}$*. Its edge set is* $E(Q_{n,k}) = \{(x,y)\}$*; for clarity,* $x = x_1x_2\cdots x_n$*,* $x_i \in \{0,1\}$*, and* y *satisfies one of the following two conditions:* (1) $y = x_1x_2\cdots x_{i-1}\bar{x}_i x_{i+1}\ldots x_n$, $1 \le i \le n$ *or* (2) $y = x_1x_2\cdots x_{k-1}\bar{x}_k\bar{x}_{k+1}\ldots\bar{x}_n$.

One can observe that the enhanced hypercube $Q_{n,k}$ $(1 \le k \le n-1)$ is obtained from the well known hypercube by adding the edges in the set $\{(x_1x_2\cdots x_n, x_1x_2\cdots x_{k-1}\bar{x}_k\bar{x}_{k+1}\cdots\bar{x}_n), \forall k, 1 \le k \le n-1\}$, which is called the set of complementary edges, denoted by $E_c = \{(u,\bar{u}) \in E(Q_{n,k})|\ d_H(u,\bar{u}) = n-k+1, u = x_1x_2\cdots x_n, \bar{u} = x_1x_2\cdots x_{k-1}\bar{x}_k\bar{x}_{k+1}\cdots\bar{x}_n\}$. As mentioned above, $|V(Q_{n,k})| = 2^n$ and $|E(Q_{n,k})| = (n+1)2^{n-1}$. We can define the edge set of $Q_{n,k}$ as $E(Q_{n,k}) = E(Q_n) \cup E_c = \{(u,v)|\ d_H(u,v) = 1, (u,v) \in E(Q_n)\} \cup \{(u,v)|\ d_H(u,v) = n-k+1, (u,v) \in E_c\}$. Note that $Q_{n,k}$ $(1 \le k \le n-1)$ is $(n+1)$-regular, vertex-transitive, but not edge-transitive [3]. For three-dimensional enhanced hypercubes $Q_{3,1}$ and $Q_{3,2}$, see Figure 1.

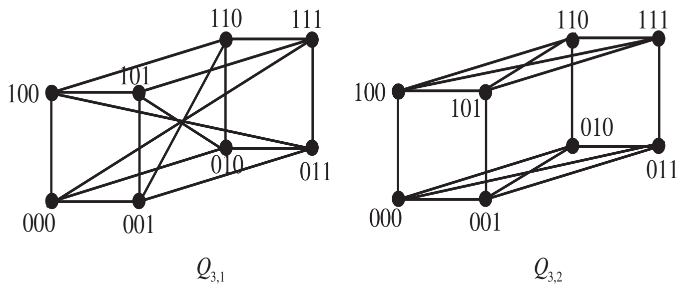

Figure 1. Illustrations of $Q_{3,1}$ and $Q_{3,2}$.

Definition 2. *Ref.* [6] *An i-partition on* $Q_{n,k}$ $(1 \le k \le n-1)$*, where* $1 \le i \le n$*, is a partition of* $Q_{n,k}$ $(1 \le k \le n-1)$ *along dimension i into two* $(n-1)$*-dimensional cubes.*

For $k \le i \le n$, $1 \le k \le n-1$, $Q_{n,k}$ $(1 \le k \le n-1)$ *can be partitioned into two* $(n-1)$*-dimensional hypercubes, we call* Q^{i0}_{n-1} *(respectively,* Q^{i1}_{n-1}*) as the subgraph of* $Q_{n,k}$ *induced by* $x_1x_2\ldots x_k\ldots x_{i-1}0x_{i+1}\ldots x_n$ *(respectively,* $x_1x_2\ldots x_k\ldots x_{i-1}1x_{i+1}\ldots x_n$*). And all the edges in* E_c *are between* Q^{i0}_{n-1} *and* Q^{i1}_{n-1}*, i.e.,* $E(Q_{n,k}) = E(Q^{i0}_{n-1}) \cup E(Q^{i1}_{n-1}) \cup E_c \cup E_i$.

For $1 \le i \le k-1$, $2 \le k \le n-1$, $Q_{n,k}$ $(1 \le k \le n-1)$ *can be partitioned into two* $(n-1)$*-dimensional enhanced hypercubes, we call* $Q^{i0}_{n-1,k-1}$ *(respectively,* $Q^{i1}_{n-1,k-1}$*) as the subgraph of* $Q_{n,k}$ *induced by* $x_1x_2\ldots x_{i-1}0x_{i+1}\ldots x_k\ldots x_n$ *(respectively,* $x_1x_2\ldots x_{i-1}1x_{i+1}\ldots x_k\ldots x_n$*). And we have* $E(Q_{n,k}) = E(Q^{i0}_{n-1,k-1}) \cup E(Q^{i1}_{n-1,k-1}) \cup E_i$.

Lemma 1. *Ref.* [8] *For any positive integers* $i, j \in \{1,2,\ldots,n,c\}$*, an automorphism* σ *of* $Q_{n,k}$ $(1 \le k \le n-1)$ *is denoted as* $\sigma(E_i) = E_j$ *. Moreover, if* $i \in \{k, k+1, k+2, \ldots, n\}$*, it follows that* $Q_{n,k} - E_i = Q^{i0}_{n-1} \cup Q^{i1}_{n-1} \cup E_c$ *is isomorphic to* Q_n *(represented as* $Q_{n,k} - E_i \cong Q_n$*,* $k \le i \le n$*). Particularly, if* $i = c$*,* $Q_{n,k} - E_c \cong Q_n$*. However, if* $i \in \{1,2,\ldots k-1\}$*, it implies that* $Q_{n,k} - E_i = Q^{i0}_{n-1,k-1} \cup Q^{i1}_{n-1,k-1}$ *is a disconnected graph.*

Let F_v and F_e denote the sets of faulty vertices and faulty edges, respectively.

Lemma 2. *Ref.* [25] *Let* Q_n $(n \ge 3)$ *be with* $|F_v| \le n-2$ *, every fault-free edge of* $Q_n - F_v$ *lies on a fault-free cycle, whose length is of every even length from* 4 *to* $2^n - 2|F_v|$ *inclusive.*

Lemma 3. *Ref.* [28] *Assume that u and v are any two distinct vertices in* Q_n $(n \ge 2)$*. Thus, there exists a path of length l joining u and v with* $d_H(u,v) \le l \le 2^n - 1$ *and* $2|(l - d_H(u,v))$.

Lemma 4. *Ref.* [29] *Let Q_n be with $n \geq 2$ and $|F_v| \leq n-2$. Assume that u and v are any two distinct fault-free vertices in Q_n. Thus, $Q_n - F_v$ contains a fault-free path joining u and v, whose length l satisfies that $d_H(u,v) + 2 \leq l \leq 2^n - 2|F_v| - 1$ and $2|(l - d_H(u,v))$.*

Lemma 5. *Ref.* [30] *Let X and Y be any two partite subsets of Q_n $(n \geq 2)$. Assume x and u are two distinct vertices of X, and y and v are two distinct vertices of Y. Then there exist two vertex-disjoint paths P_1 and P_2 such that P_1 connects x and y, P_2 connects u and v. Moreover, $P_1[x,y]$ and $P_2[u,v]$ spanning $V(Q_n)$, i.e., $V(P_1[x,y]) \cup V(P_2[u,v]) = V(Q_n)$.*

Lemma 6. *Ref.* [31] *Let Q_n $(n \geq 3)$ be with $|F_e| \leq n-3$. Then $Q_n - F_e$ is hyper-Hamiltonian-laceable.*

3. Even Cycles Embedding in $Q_{n,k}$ with $|F_v| \leq n-2$

Theorem 1. *Let $Q_{n,k}$ $(1 \leq k \leq n-1)$ be with $|F_v| \leq n-2$, $n \geq 2$. Then every fault-free edge of $Q_{n,k} - F_v$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$.*

Proof. Applying Definition 1, it follows that $E(Q_{n,k}) = E(Q_n) \cup E_c$, and $V(Q_{n,k}) = V(Q_n)$. Let $e \in E(Q_{n,k})$ be an arbitrary fault-free edge. In $Q_{2,1}$, since $|F_v| = 0$, two 4-cycles, $(00,01,11,10,00)$ and $(00,11,10,01,00)$, contain all the edges of $Q_{n,k}$; i.e., every edge of $Q_{2,1}$ lies on a fault-free cycle of length 4. Now we consider the case $n \geq 3$. We have the following two subcases according to the distributions of the fault-free edge e.

Case 1: $e \in E(Q_n)$.

Lemma 2 ensures that the theorem holds.

Case 2: $e \in E_c$.

For any arbitrary edge $e' \in E(Q_n)$, applying Lemma 1, we know that there exists an automorphism $\sigma \in Aut(Q_{n,k})$ such that $\sigma(e') = e$. Since $e = \sigma(e') \in E(\sigma(Q_n))$, and $\sigma(Q_n)$ is isomorphic to Q_n, it implies that $V(\sigma(Q_n)) = V(Q_n) = V(Q_{n,k})$. Note that e is a fault-free edge, so it implies that $e \in E(\sigma(Q_n) - F_v)$. Lemma 2 indicates that $e \in E(\sigma(Q_n) - F_v)$ lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$.

In summary, all cases have been concerned, so the proof is completed. □

4. Odd Cycles Embedding in $Q_{n,k}$ with $|F_v| \leq n-2$

In this section, let F_v denote the set of faulty vertices in $Q_{n,k}$ $(1 \leq k \leq n-1)$. We will prove that every fault-free edge of $Q_{n,k} - F_v$ lies on a fault-free cycle of every possible odd length from $n-k+4$ to $2^n - 2|F_v| - 1$ if $|F_v| \leq n-2$ and $n(\geq 2), k$ have different parity. We first give three useful lemmas as follows.

Lemma 7. *Partition $Q_{n,k}$ $(1 \leq k \leq n-1,\ n \geq 3)$ along dimension n into two $(n-1)$-dimensional hypercubes Q^{n0}_{n-1} and Q^{n1}_{n-1} by Definition 2. Let (u,v) be a jth dimensional edge in $Q_{n,k}$ such that $(u,v) \in E(Q^{n0}_{n-1})$ and $\{\bar{u}, u^n, \bar{v}, v^n\} \subseteq V(Q^{n1}_{n-1})$, where $1 \leq j \leq n-1$. Then, for $1 \leq j \leq k-1, 2 \leq k \leq n-1$, we have $d_H(\bar{u}, v^n) = n-k+1$ and $d_H(u^n, \bar{v}) = n-k+1$; for $k \leq j \leq n-1, 1 \leq k \leq n-1$, we have $d_H(\bar{u}, v^n) = n-k-1$ and $d_H(u^n, \bar{v}) = n-k-1$.*

Proof. Assume (u,v) is an arbitrary jth dimensional edge in $E(Q^{n0}_{n-1})$. Note that $Q_{n,k}$ is partitioned along dimension n. Obviously, $\{\bar{u}, u^n, \bar{v}, v^n\} \subseteq V(Q^{n1}_{n-1})$. We check two possible situations regarding dimensions of the selected edge (u,v).

(i) For $1 \leq j \leq k-1$, $2 \leq k \leq n-1$. We can denote the two vertices in $V(Q^{n0}_{n-1})$ by the binary strings as $u = x_1x_2 \ldots x_{j-1}x_jx_{j+1} \ldots x_{k-1}x_kx_{k+1} \ldots x_{n-1}0$ and $v = x_1x_2 \ldots x_{j-1}\bar{x}_jx_{j+1} \ldots x_{k-1}x_k\ x_{k+1} \ldots x_{n-1}0$. By the definition and connection of the binary strings of two vertices, we have $\bar{u} = x_1x_2 \ldots x_{j-1}x_jx_{j+1} \ldots x_{k-1}\bar{x}_k\bar{x}_{k+1} \ldots \bar{x}_{n-1}1$ and $v^n = x_1x_2 \ldots x_{j-1}\bar{x}_jx_{j+1} \ldots x_{k-1}x_kx_{k+1} \ldots\ x_{n-1}1$. Accordingly, by the calculating of the numbers of the different bits in the binary bits of the two selected vertices, we can conclude that $d_H(\bar{u}, v^n) = (n-k+1) - 1 + 1 = n-k+1$. As an immediate result, we have $d_H(u^n, \bar{v}) = (n-k+1) - 1 + 1 = n-k+1$.

(ii) For $k \leq j \leq n-1$, $1 \leq k \leq n-1$. We can denote the two vertices in $V(Q^{n0}_{n-1})$ by the binary strings as $u = x_1x_2 \ldots x_{k-1}x_kx_{k+1} \ldots x_{j-1}x_jx_{j+1} \ldots x_{n-1}0$ and $v = x_1x_2 \ldots x_{k-1}x_kx_{k+1} \ldots x_{j-1}\bar{x}_j\, x_{j+1} \ldots x_{n-1}0$. By the definition and connection of the binary strings of two vertices, we have $\bar{u} = x_1x_2 \ldots\ x_{k-1}\bar{x}_k\bar{x}_{k+1} \ldots \bar{x}_{j-1}\bar{x}_j\bar{x}_{j+1} \ldots \bar{x}_{n-1}1$ and $v^n = x_1x_2 \ldots x_{k-1}x_kx_{k+1} \ldots x_{j-1}\bar{x}_jx_{j+1} \ldots\ x_{n-1}1$. Similarly as above, it follows that $d_H(\bar{u}, v^n) = (n-k+1) - 2 = n-k-1$ and $d_H(u^n, \bar{v}) = (n-k+1) - 2 = n-k-1$.

In summary, for any $u \in V(Q^{n0}_{n-1})$, there exist $n-1$ distinct vertices v's in $V(Q^{n0}_{n-1})$, such that $d_H(u^n, \bar{v}\ ) = n-k-1$ or $d_H(u^n, \bar{v}\ ) = n-k+1$ (respectively, $d_H(\bar{u}, v^n) = n-k-1$ or $d_H(\bar{u}, v^n) = n-k+1$). $\square$

Lemma 8. *Partition $Q_{n,k}$ $(1 \leq k \leq n-1,\ n \geq 4)$ along dimension n into two $(n-1)$-dimensional hypercubes Q^{n0}_{n-1} and Q^{n1}_{n-1} by Definition 2. Assume (u, v) is an ith dimensional edge in Q^{n0}_{n-1}, where $1 \leq i \leq n-1$. Then we can select $n-2$ distinct w's in Q^{n0}_{n-1} such that $w \neq v$, and the edges $(\bar{u}, w^n)$ and $(u^n, \bar{w})$ are both the jth dimension edges in $Q_{n,k}$ and $\{(\bar{u}, w^n), (u^n, \bar{w})\} \subseteq E(Q^{n1}_{n-1})$; i.e., $d_H(\bar{u}, w^n) = 1$ and $d_H(u^n, \bar{w}) = 1$, where $1 \leq i \leq n-1$, $1 \leq j \leq n-1$ and $j \neq i$.*

Proof. Let (u, v) be an arbitrary ith dimensional edge in $E(Q^{n0}_{n-1})$. We distinguish two possible situations regarding dimensions of the selected edge (u, v) as follows:

When $1 \leq i \leq k-1$, $2 \leq k \leq n-1$, we can select $(n-2)$ w's in $V(Q^{n0}_{n-1})$ such that $\{(u^n, \bar{w}), (\bar{u}, w^n)\} \subseteq E(Q^{n1}_{n-1})$ are both the jth dimensional edges, $1 \leq j \leq n-1$, $j \neq i$. We distinguish the following two subcases:

- For $1 \leq i \leq k-1$, we can select $(k-2)$ distinct w's in Q^{n0}_{n-1} such that $(u^n, \bar{w})$ and $(\bar{u}, w^n)$ are both the jth dimensional edges in Q^{n1}_{n-1}, i.e., $1 \leq j \leq k-1$, and $j \neq i$. For clarity, let $u = x_1x_2 \ldots x_i \ldots x_j \ldots x_k \ldots x_{n-1}0$ and $v = x_1x_2 \ldots \bar{x}_i \ldots x_j \ldots x_k \ldots x_{n-1}0$ be two vertices in Q^{n0}_{n-1}. Thus $(u, v) \in E(Q^{n0}_{n-1})$ is an ith dimensional edge. We can select a vertex $w = x_1x_2 \ldots x_i \ldots x_{j-1}\bar{x}_jx_{j+1} \ldots x_{k-1}\bar{x}_k \ldots \bar{x}_{n-1}0$, $1 \leq j \leq k-1$ and $j \neq i$. It implies that $d_H(u, w) = n-k+1$ and $d_H(v, w) = n-k+2$. It follows that the vertices $\{u^n, \bar{w}, w^n, \bar{w}\} \subseteq V(Q^{n1}_{n-1})$ can be denoted as $u^n = x_1 \ldots x_i \ldots x_j \ldots x_k \ldots x_{n-1}1$, $\bar{w} = x_1x_2 \ldots x_i \ldots x_{j-1}\bar{x}_jx_{j+1} \ldots x_k \ldots x_{n-1}1$, $\bar{u} = x_1 \ldots x_i \ldots x_j \ldots x_{k-1}\bar{x}_k \ldots \bar{x}_{n-1}1$, and $w^n = x_1 \ldots x_i \ldots x_{j-1}\bar{x}_jx_{j+1} \ldots x_{k-1}\bar{x}_k \ldots \bar{x}_{n-1}1$. As an immediate result, we have $d_H(u^n, \bar{w}) = 1$ and $d_H(\bar{u}, w^n) = 1$.
- For $1 \leq i \leq k-1$, we can select $(n-k)$ distinct w's in Q^{n0}_{n-1} such that $(u^n, \bar{w})$ and $(\bar{u}, w^n)$ are both the jth dimensional edges in Q^{n1}_{n-1}, i.e., $k \leq j \leq n-1$. For clarity, let $u = x_1x_2 \ldots x_i \ldots x_k \ldots x_j \ldots\ x_{n-1}0$ and $v = x_1x_2 \ldots x_{i-1}\bar{x}_ix_{i+1} \ldots x_k \ldots x_j \ldots x_{n-1}0$ be two vertices in Q^{n0}_{n-1}. Thus, $(u, v) \in E(Q^{n0}_{n-1})$ is an ith dimensional edge. We can select a vertex $w = x_1x_2 \ldots x_i \ldots x_{k-1}\bar{x}_k \ldots \bar{x}_{j-1}x_j\bar{x}_{j+1} \ldots \bar{x}_{n-1}0$, $k \leq j \leq n-1$. It implies that $d_H(u, w) = n-k-1$ and $d_H(v, w) = n-k$. Subsequently, we have $d_H(u^n, \bar{w}) = 1$ and $d_H(\bar{u}, w^n) = 1$.

When $k \leq i \leq n-1$, $1 \leq k \leq n-1$, we can select $(n-2)$ w's in $V(Q^{n0}_{n-1})$ such that $\{(u^n, \bar{w}), (\bar{u}, w^n)\} \subseteq E(Q^{n1}_{n-1})$ are both the jth dimensional edges, $1 \leq j \leq n-1$, $j \neq i$. We distinguish the following two subcases:

- For $k \leq i \leq n-1$, we can select $(k-1)$ distinct w's in Q^{n0}_{n-1} such that $(u^n, \bar{w})$ and $(\bar{u}, w^n)$ are both the jth dimensional edges in Q^{n1}_{n-1}, i.e., $1 \leq j \leq k-1$. For clarity, let $u = x_1x_2 \ldots x_j \ldots x_k \ldots x_i \ldots\ x_{n-1}0$ and $v = x_1x_2 \ldots x_j \ldots x_k \ldots x_{i-1}\bar{x}_ix_{i+1} \ldots x_{n-1}0$ be two vertices in Q^{n0}_{n-1}. Thus $(u, v) \in E(Q^{n0}_{n-1})$ is an ith dimensional edge. We can select a vertex $w = x_1x_2 \ldots x_{j-1}\bar{x}_jx_{j+1} \ldots x_{k-1}\bar{x}_k \ldots\ \bar{x}_i \ldots \bar{x}_{n-1}0$, $1 \leq j \leq k-1$. It implies that $d_H(u, w) = n-k+1$ and $d_H(v, w) = n-k$. It is easy to see that $d_H(u^n, \bar{w}) = 1$ and $d_H(\bar{u}, w^n) = 1$.
- For $k \leq i \leq n-1$, we can select $(n-k-1)$ distinct w's in Q^{n0}_{n-1} such that $(u^n, \bar{w})$ and $(\bar{u}, w^n)$ are both the jth dimensional edges in Q^{n1}_{n-1}, i.e., $k \leq j \leq n-1$, and $j \neq i$. For clar-

ity, let $u = x_1x_2 \dots x_k \dots x_i \dots x_j \dots x_{n-1}0$ and $v = x_1x_2 \dots x_k \dots x_{i-1}\bar{x}_i x_{i+1} \dots x_j \dots x_{n-1}0$ be two vertices in Q^{n0}_{n-1}. Thus, $(u, v) \in E(Q^{n0}_{n-1})$ is an ith dimensional edge. We can select a vertex $w = x_1x_2 \dots x_{k-1}\bar{x}_k \dots \bar{x}_{i-1}\bar{x}_i\bar{x}_{i+1} \dots \bar{x}_{j-1}x_j\, \bar{x}_{j+1} \dots \bar{x}_{n-1}0$ in Q^{n0}_{n-1}, $k \le j \le n-1$. Accordingly, we have $d_H(u, w) = n-k-1$ and $d_H(v, w) = n-k-2$. As a consequence, we have $d_H(u^n, \bar{w}) = 1$ and $d_H(\bar{u}, w^n) = 1$.

In summary, for any ith dimension edge (u, v) in Q^{n0}_{n-1}, there exist $n-2$ distinct w's in Q^{n0}_{n-1} such that $w \ne v$, and $d_H(\bar{u}, w^n) = 1$, $d_H(u^n, \bar{w}) = 1$ simultaneously. □

Lemma 9. *Let $Q_{n,k}$ $(1 \le k \le n-1)$ be with $|F_v| = 1$, where n (≥ 3) and k have the different parity. Let (u, v) be an ith dimensional fault-free edge in $E(Q_{n,k})$. If $i \in \{k, k+1, \dots, n, c\}$ (respectively, $i \in \{1, 2, \dots, k-1\}$), then the edge (u, v) lies on a fault-free cycle of every possible odd length l with $n-k+2 \le l \le 2^n - 1$ (respectively, $n-k+4 \le l \le 2^n - 1$).*

Proof. The proof of this lemma is in Appendix A. □

Theorem 2. *Let $Q_{n,k}$ $(1 \le k \le n-1)$ be with $|F_v| = f_v \le n-2$, where n (≥ 2) and k have the different parity. For an ith dimensional edge (u, v), if $i \in \{k, k+1, \dots, n, c\}$, then the edge lies on a fault-free cycle of every possible odd length l with $n-k+2 \le l \le 2^n - 2f_v - 1$ in $Q_{n,k} - F_v$; if $i \in \{1, 2, \dots, k-1\}$, then the edge lies on a fault-free cycle of every possible odd length l with $n-k+4 \le l \le 2^n - 2f_v - 1$ in $Q_{n,k} - F_v$.*

Proof. The proof of this theorem is by induction on n. It is trivial to check the theorem holds for $Q_{2,1}$ and $Q_{3,2}$. Assume the theorem holds for $3 \le m < n$, where m and k have the different parity. We now would like to show the theorem holds for every $m = n \ge 4$, where m and k have the different parity. Recall that Lemma 9 proved the theorem holds for $|F_v| \le 1$. In the following, we consider $2 \le |F_v| \le n-2$. Let $f = x_1x_2 \dots x_n$, $x_i \in \{0, 1\}$ and $f' = y_1y_2 \dots y_n$, $y_i \in \{0, 1\}$ be two arbitrary distinct faulty vertices in $Q_{n,k}$. Thus, there exists an integer i, $1 \le i \le n$, such that $x_i + y_i = 1$. Applying Definition 2, if we partition $Q_{n,k}$ along dimension i, where $1 \le i \le n$, then we can obtain two $(n-1)$-dimensional cubes, and each of the cubes contains at least one faulty vertex. Let $e = (u, v)$ be an arbitrary fault-free edge in $Q_{n,k}$. We distinguish the following subcases according to the partition of $Q_{n,k}$ (see Table 1).

Table 1. Cases in Theorem 2 for the desired cycle containing the edge e.

Case	The Distribution of e	The Desired Cycle of Length l
Case 1.1	$e \in E(Q^{10}_{n-1,k-1}) \cup E(Q^{11}_{n-1,k-1})$	$n-k+2 \le l \le 2^n - 2f_v - 1$
Case 1.2	$e \in E_1$	$n-k+4 \le l \le 2^n - 2f_v - 1$
Case 2.1	$e \in E(Q^{n0}_{n-1}) \cup E(Q^{n1}_{n-1})$	$n-k+2 \le l \le 2^n - 2f_v - 1$
Case 2.2	$e \in E_n$	$n-k+4 \le l \le 2^n - 2f_v - 1$
Case 2.3	$e \in E_c$	$n-k+4 \le l \le 2^n - 2f_v - 1$

Case 1: $1 \le i \le k-1$, $2 \le k \le n-1$. Without loss of generality, we can assume $i = 1$. Definition 2 ensures that $Q_{n,k}$ is partitioned into two $(n-1)$-dimensional enhanced hypercubes, denoted as $Q^{10}_{n-1,k-1}$ and $Q^{11}_{n-1,k-1}$. Denote $F^0_v = F_v \cap V(Q^{10}_{n-1,k-1})$, $F^1_v = F_v \cap V(Q^{11}_{n-1,k-1})$, $f^0_v = |F^0_v|$, and $f^1_v = |F^1_v|$. By the partition of $Q_{n,k}$, it follows that $1 \le f^0_v \le n-3$ and $1 \le f^1_v \le n-3$. We have two subcases according to the distributions of the fault-free edge e.

- First, $e \in E(Q^{10}_{n-1,k-1}) \cup E(Q^{11}_{n-1,k-1})$. By the symmetric structure of $Q^{10}_{n-1,k-1}$ and $Q^{11}_{n-1,k-1}$, and the distribution of faulty vertices, without loss of generality, we can assume that $e \in E(Q^{10}_{n-1,k-1})$. On one hand, in $Q^{10}_{n-1,k-1}$, $f^0_v \le n-3$, by induction hypothesis, the edge e lies on a fault-free cycle of every possible odd length from $n-k+2$ to$2^{n-1} - 2f^0_v - 1$ in $Q^{10}_{n-1,k-1} - F^0_v$. Let C_0 be a cycle of length $2^{n-1} -$

$2f_v^0 - 1$ in $Q^{10}_{n-1,k-1} - F_v^0$ containing the edge e. Let $(s,t) \neq (u,v)$ denote that $(\{s\} \cap \{u,v\}) \cup (\{t\} \cap \{u,v\}) = \emptyset$. Note that we can select an edge $(s,t) \neq (u,v)$ such that $(s,t) \in E(C_0)$, $(s^1,t^1) \in E(Q^{11}_{n-1,k-1})$ and $\{s^1,t^1\} \cap F_v^1 = \emptyset$. (If not, it implies that $f_v^1 \geq \frac{(2^{n-1}-2f_v^0-1)-3}{2}$. Thus, we have $f_v = f_v^0 + f_v^1 \geq 2^{n-2} - 2 > n-2$ for $n \geq 5$, a contradiction. Specially, for $Q_{4,3}$, Lemma 9 implies that $Q^{10}_{3,2}$ contains a cycle of length 7 when $f_v^0 = 1$ and $f_v^1 = 1$. Thus, it is easy to select the desired edge (s,t) on the cycle.) For clarity, we set $C_1 = \langle s^1, P_1[s^1,t^1], t^1, s^1 \rangle$, and $1 \leq l_1 = l(P_1[s^1,t^1]) \leq 2^{n-1} - 2f_v^1 - 1$. On the other hand, we can construct the desired cycle as $\langle s, P_0[s,t], t, t^1, P_1[t^1,s^1], s^1, s \rangle$, whose length is $l = l_0 + l_1 + 2$, i.e., $2^{n-1} - 2f_v^0 + 1 \leq l \leq 2^n - 2f_v - 1$.

- Now, $e \in E_1$. $C_0 = \langle s, P_0[s,t], t, s \rangle$, and $l_0 = l(P_0[s,t]) = 2^{n-1} - 2f_v^0 - 2$. Recall that $f_v^1 \leq n-3$, Theorem 1 implies that the fault-free edge (s^1,t^1) lies on a fault-free cycle C_1 of every even length from 4 to $2^{n-1} - 2f_v^1$ in $Q^{11}_{n-1,k-1}$. Assume that $u \in V(Q^{10}_{n-1,k-1})$ and $v \in V(Q^{11}_{n-1,k-1})$. Thus, $v = u^1$. Note that u have n neighbors in $Q^{10}_{n-1,k-1}$, i.e., w_j, $j \in \{2,3,4,\ldots,n,c\}$, where $w_c = \bar{u}$, and $w_j = u^j$, $j \in \{2,3,\ldots,n\}$. We can observe that there exist n cycles of length four containing the edge $e = (u,u^1)$ in common, i.e., $\langle u, w_j, w_j^1, u^1, u \rangle$, $j \in \{2,3,\ldots,n,c\}$. Recall that $f_v \leq n-2$. Thus, there exists at least one fault-free pair (w_j, w_j^1), $j \in \{2,3,\ldots,n,c\}$ such that the cycle of length 4 is fault-free. Assume $\langle u, w, w^1, u^1, u \rangle$ forms such a fault-free cycle of length 4 containing the edge $e = (u,u^1)$. Obviously, $d_H(u^1,w^1) = 1$. On one hand, by induction hypothesis, (u,w) lies on a fault-free cycle C_0 of every odd length from $n-k+2$ to $2^{n-1} - 2f_v^0 - 1$. For clarity, $C_0 = \langle u, P_0[u,w], w, u \rangle$. Therefore the desired cycle of every odd length from $n-k+4$ to $2^{n-1} - 2f_v^0 + 1$ can be constructed as $\langle u, P_0[u,w], w, w^1, u^1, u \rangle$. On the other hand, we can construct the desired cycle of every odd length from $2^{n-1} - 2f_v^0 + 3$ to $2^n - 2f_v - 1$. Let C_0 be a fault-free cycle of length $2^{n-1} - 2f_v^0 - 1$ in $Q^{10}_{n-1,k-1}$, which contains the edge (u,w). Denote $l_0 = l(P_0[u,w]) = 2^{n-1} - 2f_v^0 - 2$. Applying Theorem 1, (u^1,w^1) lies on a cycle C_1 of every even length from 4 to $2^{n-1} - 2f_v^1$. For clarity, $C_1 = \langle u^1, P_1[u^1,w^1], w^1, u^1 \rangle$, and $3 \leq l_1 = l(P_1[u^1,w^1]) \leq 2^{n-1} - 2f_v^1 - 1$. Subsequently, merging the two paths $P_0[u,w]$ and $P_1[w^1,u^1]$ as well as the two fault-free edges (u,u^1) and (w,w^1), the desired cycle can be constructed as $\langle u, P_0[u,w], w, w^1, P_1[w^1,u^1], u^1, u \rangle$, and the cycle is of every odd length from $2^{n-1} - 2f_v^0 + 3$ to $2^n - 2f_v - 1$.

Case 2: $k \leq i \leq n$, $1 \leq k \leq n-1$. Without loss of generality, we can select $i = n$. Definition 2 ensures that $Q_{n,k}$ can be partitioned into two $(n-1)$-dimensional hypercubes, denoted as Q^{n0}_{n-1} and Q^{n1}_{n-1}. Denote $F_v^0 = F_v \cap V(Q^{n0}_{n-1})$, $F_v^1 = F_v \cap V(Q^{n1}_{n-1})$, $|F_v^0| = f_v^0$ and $|F_v^1| = f_v^1$. It follows that $1 \leq f_v^0 \leq n-3$ and $1 \leq f_v^1 \leq n-3$. We have three subcases according to the distributions of the fault-free edge e.

- First, $e \in E(Q^{n0}_{n-1}) \cup E(Q^{n1}_{n-1})$. Without loss of generality, we can assume that $e = (u,v) \in E(Q^{n0}_{n-1})$.
  (i) $1 \leq f_v^0 \leq n-4$ and $2 \leq f_v^1 \leq n-3$. Lemma 8 ensures that there exist $(n-2)$ distinct vertices w_j's such that $w_j \neq v$, and $d_H(u^n, \bar{w}_j) = 1$ or $d_H(\bar{u}, w_j^n) = 1$. Assuming the vertex u as a faulty vertex temporarily, since $1 \leq f_v^0 \leq n-4$, we obtain $|F_v \cap Q^{n0}_{n-1}) \cup \{u\}| \leq n-3$ for $n \geq 4$. Lemma 4 implies that $Q^{n0}_{n-1} - F_v^0 - \{u\}$ contains a fault-free path $P_0[v,w_j]$ joining v and w_j. Merging the path $P_0[v,w_j]$ and the edges (u,u^n), $(w_j,\bar{w}_j)$ and $(u^n,\bar{w}_j)$ (respectively, $(u,\bar{u})$, (w_j,w_j^n) and $(\bar{u},w_j^n)$), there exist $2(n-2)$ cycles C_j's containing the edge (u,v), i.e., $C_j = \langle u, v, P_0[v,w_j], w_j, \bar{w}_j, u^n, u \rangle$, or $C_j = \langle u, v, P_0[v,w_j], w_j, w_j^n, \bar{u}, u \rangle$. Obviously, the cycles C_j's have the common vertices in the path $P_0[v,w_j]$. We can observe that there are at most $2f_v^0$ cycles C_j's with faulty vertices in $P_0[v,w_j]$. Since $f_v = f_v^0 + f_v^1 \leq n-2$, we have $2(n-2) \geq 2(f_v^0 + f_v^1)$. It implies there are at least $2f_v^1$ cycles C_j's with fault-free vertices in Q^{n0}_{n-1}. Since $f_v^0 \leq f_v^1$, there

exist at least f_v^1 fault-free cycles C_j's in $Q_{n,k}$. Without loss of generality, we can assume that $\langle u, v, P_0[v,w], w, \bar{w}, u^n, u\rangle$ is such a fault-free cycle and $d_H(u^n, \bar{w}) = 1$. Lemma 8 indicates that $d_H(v,w) = n-k-2, n-k,$ or $n-k+2$. Obviously, $n-k \leq l_0 = l(P_0[v,w]) \leq 2^{n-1} - 2(f_v^0+1) - 1$. In Q_{n-1}^{n1}, $f_v^1 \leq n-3$ and $d_H(u^n, \bar{w}) = 1$, applying Lemma 4, there exists a fault-free path $P_1[u^n, \bar{w}]$ of every odd length l_1 joining u^n and $\bar{w}$, where $1 \leq l_1 \leq 2^{n-1} - 2f_v^1 - 1$. As an immediate result, $\langle u, v, P_0[v,w], w, \bar{w}, P_1[\bar{w}, u^n], u^n, u\rangle$ forms the desired cycle of every possible odd length $l = l_0 + l_1 + 3$ in $Q_{n,k} - F_v$. Since $n-k \leq l_0 \leq 2^{n-1} - 2f_v^0 - 3$ and $1 \leq l_1 \leq 2^{n-1} - 2f_v^1 - 1$, we can obtain $n-k+4 \leq l \leq 2^n - 2f_v - 1$ (see Figure 2a). (In Figures 2–5, we use white vertices and black vertices to distinguish the different parity of the vertices, and we use gray vertices to denote the vertices with unknown parity.)

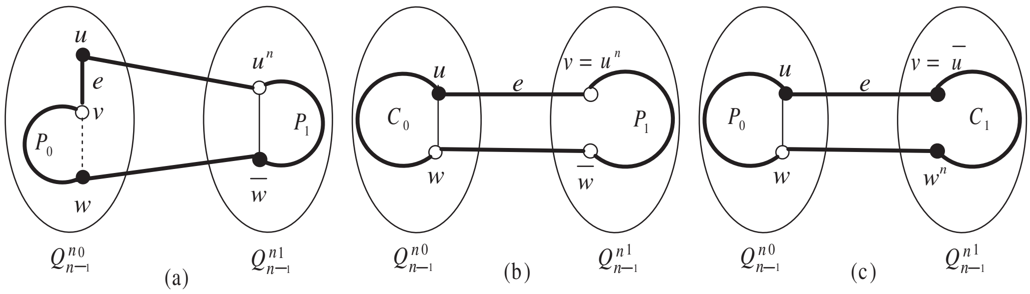

Figure 2. Illustrations of (**a**) Case 2, (**b**) Case 2, and (**c**) Case 2 in the proof of Theorem 2 .

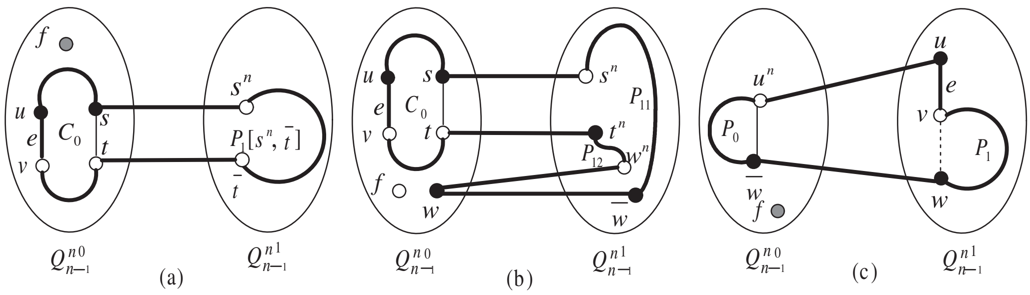

Figure 3. Illustrations of (**a**) Subcase 1.1, (**b**) Subcase 1.2, and (**c**) Subcase 2.1 in the proof of Lemma 9 .

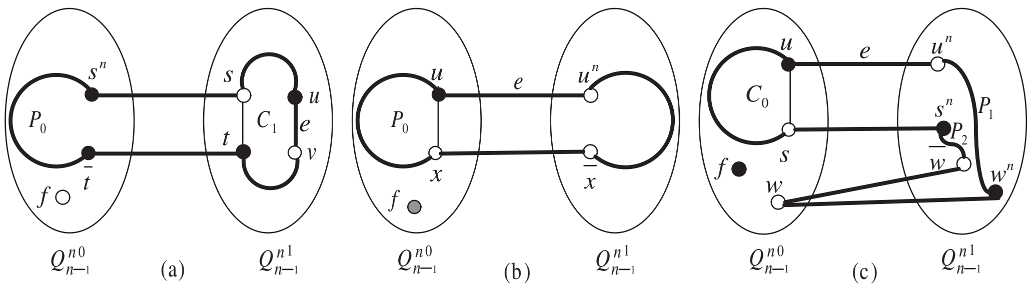

Figure 4. Illustrations of (**a**) Subcase 2.2, (**b**) Subcase 3.1, and (**c**) Subcase 3.2.1 in the proof of Lemma 9.

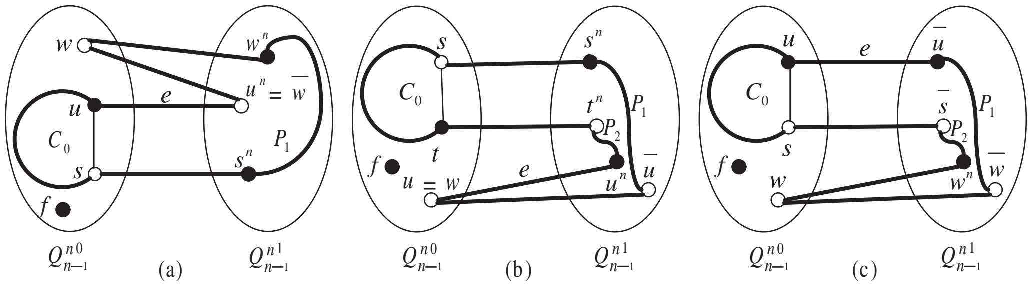

Figure 5. Illustrations of (**a**) Subcase 3.2.1, (**b**) Subcase 3.2.2, and (**c**) Subcase 4.2.1 in the proof of Lemma 9.

(ii) $f_v^0 = n-3$ and $f_v^1 = 1$. Specially, Lemmas 3 and 7 imply that we can construct the cycles $\langle u, v, v^n, P_1[v^n, \bar{u}], \bar{u}, u\rangle$ and $\langle u, v, \bar{v}, P_1[\bar{v}, u^n], u^n, u\rangle$, whose length is $n-k+2$ or $n-k+4$. Since $f_v^1 = 1$, it follows that there exists at least one of the above fault-free cycle of odd length $n-k+2$ or $n-k+4$ containing the edge (u,v). Generally, we construct the desired cycle of every odd length from $n-k+6$ to $2^n - 2f_v - 1$. Since $f_v^0 = n-3$, applying Lemma 2, (u,v) lies on a fault-free cycle C_0 of length l_0' in Q_{n-1}^{n0}, where $4 \le l_0' \le 2^{n-1} - 2f_v^0$. Assume that (s,t) is an edge on C_0, $(s,t) \ne (u,v)$. For clarity, $C_0 = \langle s, P_0[s,t], t, s\rangle$, $3 \le l_0 = l(P_0[s,t]) \le 2^{n-1} - 2f_v^0 - 1$. Since $f_v^1 = 1$, we have $\{s^n, \bar{t}\} \cap F_v^1 = \emptyset$ or $\{\bar{s}, t^n\} \cap F_v^1 = \emptyset$. It implies that we can assume s^n and $\bar{t}$ are both fault-free vertices in $V(Q_{n-1}^{n1})$. Note that $d_H(s^n, \bar{t}) = n-k-1$ or $d_H(s^n, \bar{t}) = n-k+1$. Lemma 4 ensures that $Q_{n-1}^{n1} - F_v^1$ contains a fault-free path $P_1[s^n, \bar{t}]$ of every even length l_1, where $n-k+1 \le l_1 \le 2^{n-1} - 2f_v^1 - 2$. Accordingly, merging the two paths $P_0[s,t]$ and $P_1[s^n, \bar{t}]$ as well as the two edges (s, s^n) and $(t, \bar{t})$, we can construct the desired cycle as $\langle s, P_0[s,t], t, \bar{t}, P_1[\bar{t}, s^n], s^n, s\rangle$, whose length is $n-k+6 \le l = l_0 + l_1 + 2 \le 2^n - 2f_v - 1$.

- Next, $e \in E_n$. Obviously, $v = u^n$. Applying Lemma 7, we can select $n-1$ distinct vertices w_j adjacent to u in Q_{n-1}^{n0} such that $d_H(u^n, \bar{w}_j) = n-k-1$ or $d_H(u^n, \bar{w}_j) = n-k+1$, and $w_j = u^j$ for $j \in \{1, 2, \ldots, n-1\}$. Since $f_v^1 \le n-3$, Lemma 4 implies that Q_{n-1}^{n1} contains a path $P_1[u^n, \bar{w}_j]$ joining u^n and $\bar{w}_j$. Subsequently, there exist $n-1$ cycles C_j's denoted as $C_j = \langle u, w_j, \bar{w}_j, P_1[\bar{w}_j, u^n], u^n, u\rangle$, where $j \in \{1, 2, \ldots, n-1\}$. Note that $f_v \le n-2$. It implies that there exists at least one fault-free cycle C_j. Assume $\langle u, w, \bar{w}, P_1[\bar{w}, u^n], u^n, u\rangle$ is such a fault-free cycle. For clarity, $l_1 = l(P_1[u^n, \bar{w}])$, it follows that $n-k+1 \le l_1 \le 2^{n-1} - 2f_v^0 - 2$. Recall that $f_v^0 \le n-3$, Lemma 4 ensures that Q_{n-1}^{n0} contains a fault-free path $P_0[u,w]$ of every odd length l_0 joining u and v, where $1 \le l_0 \le 2^{n-1} - 2f_v^1 - 1$. Accordingly, the desired cycle containing the edge e can be constructed as $\langle u, P_0[u,w], w, \bar{w}, P_1[\bar{w}, u^n], u^n, u\rangle$, whose length is $l = l_0 + l_1 + 2$. As a result, $n-k+4 \le l \le 2^n - 2f_v - 1$ (see Figure 2b).
- Finally, $e \in E_c$. Note that E_c is the set of complementary edges between Q_{n-1}^{n0} and Q_{n-1}^{n1}. It follows that $v = \bar{u}$. Lemma 7 implies that Q_{n-1}^{n0} contains $n-1$ distinct vertices w_j's, which are adjacent to u and satisfy $d_H(\bar{u}, w_j^n) = n-k-1$ or $d_H(\bar{u}, w_j^n) = n-k+1$, and $w_j = u^j$ for $j \in \{1, 2, \ldots, n-1\}$. As mentioned above, there exist $n-1$ cycles C_j's denoted as $C_j = \langle u, w_j, w_j^n, P_1[w_j^n, \bar{u}], \bar{u}, u\rangle$, where $j \in \{1, 2, \ldots, n-1\}$. Since $f_v \le n-2$, there exists at least one fault-free cycle C_j. Assume $\langle u, w, w^n, P_1[w^n, \bar{u}], \bar{u}, u\rangle$ is such a fault-free cycle in $Q_{n,k}$. Lemma 4 indicates that there exists a fault-free path $P_1[\bar{u}, w^n]$ of every even length l_1 in Q_{n-1}^{n1}, and there also exists a fault-free path $P_0[u,w]$ of every odd length l_0 in Q_{n-1}^{n0}, where $n-k+1 \le l_1 \le 2^{n-1} - 2f_v^1 - 2$ and $1 \le l_0 \le 2^{n-1} - 2f_v^0 - 1$. It is easy to see that $\langle u, P_0[u,w], w, w^n, P_1[w^n, \bar{u}], \bar{u}, u\rangle$ forms the desired odd cycle, whose length is $l = l_0 + l_1 + 2$. It follows that $n-k+4 \le l \le 2^n - 2f_v - 1$ (see Figure 2c).

In summary, all cases have been concerned, so the proof is completed. □

5. Concluding Remarks

Let F_v be the set of faulty vertices in $Q_{n,k}$ $(1 \leq k \leq n-1)$. In this paper, we consider the faulty enhanced hypercube $Q_{n,k}$ $(1 \leq k \leq n-1)$ with $|F_v| \leq n-2$ faulty vertices. For a fault-free edge (u,v) of $Q_{n,k} - F_v$, we show that it lies on a fault-free cycle of every even length from 4 to $2^n - 2|F_v|$, where $n \geq 2$; moreover, it lies on a fault-free cycle of every possible odd length from $n-k+4$ to $2^n - 2|F_v| - 1$ in $Q_{n,k} - F_v$, where n (≥ 2) and k have different parity.

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Appendix A. The proof of Lemma 9

Proof. Since n (≥ 3) and k have the different parity, we first need to check the lemma holds for $Q_{3,2}$. Since $Q_{3,2}$ is vertex-transitive [3], we can assume the vertex 000 is faulty. For $i \in \{2,3,c\}$, we can find the cycles of length $n-k+2 = 3$ containing all the ith dimensional fault-free edges of $Q_{3,2} - \{000\}$, i.e., $(010,001,011,010)$, $(110,101,111,110)$, $(100,101,110,100)$, $(100,101,111,100)$. For $i \in \{1,2,3,c\}$, we can find the cycles of length $n-k+4 = 5$ containing all the ith dimensional fault-free edges of $Q_{3,2} - \{000\}$, i.e., $(010,110,101,111,011,010)$, $(010,001,101,100,110,010)$, $(001,011,111,100,101,001)$, $(110,111,101,001,010,110)$. For $i \in \{1,2,3,c\}$, we can find the cycles of length 7 containing all the ith dimensional fault-free edges of $Q_{3,2} - \{000\}$, i.e., $(010,001,101,110,100,111,011,010)$, $(010,001,011,111,101,100,110,010)$. It implies that for each ith dimensional edge in $Q_{3,2}$, if $i=1$, it lies on a cycle of length 5 and 7; if $i \in \{2,3,c\}$, it lies on a cycle of length 3, 5 and 7.

Now we consider the cases that $n \geq 4$. Definition 2 ensures $Q_{n,k}$ can be partitioned along dimension n into two $(n-1)$-dimensional hypercubes, denoted as Q^{n0}_{n-1} and Q^{n1}_{n-1}. Let f be the faulty vertex in $Q_{n,k}$, i.e., $F_v = \{f\}$. Recall that $Q_{n,k}$ is vertex-transitive [3]. Without loss of generality, we may assume that $f \in V(Q^{n0}_{n-1})$. Choose an ith dimensional arbitrary fault-free edge $e = (u,v)$ in $Q_{n,k}$, $i \in \{1,2,\ldots,n,c\}$. Let l be the length of the desired fault-free cycle. We have four subcases according to the distributions of the fault-free edge e (see Table A1).

Table A1. Cases in Lemma 9 for the desired cycles containing the edge e.

Case	The Distribution of e	The Length l of the Desired Cycle
Case 1	$e \in E(Q^0_{n-1})$	$n-k+2 \leq l \leq 2^n - 1$
Case 2	$e \in E(Q^1_{n-1})$	$n-k+2 \leq l \leq 2^n - 1$
Case 3	$e \in E_n$	$n-k+2 \leq l \leq 2^n - 1$
Case 4	$e \in E_c$	$n-k+2 \leq l \leq 2^n - 1$

Case 1: $e \in E(Q^{n0}_{n-1})$. Obviously, $i \in \{1,2,\ldots,n-1\}$. We distinguish two subcases according to the length l.

Subcase 1.1: $n-k+2 \leq l \leq 2^n - 3$ (respectively, $n-k+4 \leq l \leq 2^n - 3$), for the ith dimensional edge (u,v), where $i \in \{k,k+1,\ldots,n-1\}$ (respectively, $i \in \{1,2,\ldots,k-1\}$).

- Since $(u,v) \in E(Q^{n0}_{n-1})$, it follows that $\{u^n, \bar{v}\} \subseteq V(Q^{n1}_{n-1})$. When $i \in \{k,k+1,\ldots,n-1\}$ and $k = n-1$, we have $d_H(u^n, \bar{v}) = 0$. Thus, $\langle u,v,\bar{v},u \rangle$ forms the desired cycle of length $n-k-1=3$. Lemma 2 implies that (u,v) lies on a fault-free cycle C_0 in Q^{n0}_{n-1} of every even length from 4 to $2^{n-1}-2$. Select $(s,t) \neq (u,v)$ in C_0. For clarity, $C_0 = \langle s, P_0[s,t], t, s\rangle$, where $l_0 = l(P_0[s,t])$ satisfies $3 \leq l_0 \leq 2^{n-1}-3$. Since $k=n-1$, we have $d_H(s^n, \bar{t}) = n-k-1 = 0$. Thus, $\langle s, P_0[s,t], t, \bar{t}, s$ forms the desired cycle of length $l = l_0+2$, i.e., $n-k+4 = 5 \leq l \leq 2^{n-1}-1$. Let C_0 be a cycle of length $2^{n-1}-2$. We can select three distinct edges (s_1,t_1), (s_2,t_2) and (s_3,t_3) which satisfies that $(s_i,t_i) \neq (u,v)$ and $|\{s_i,t_i\} \cap \{s_j,t_j\}| \leq 1$, where $i \in \{1,2,3\}$ and $j \in \{1,2,3\}$. Since $k=n-1$, we have $s_i^n = \bar{t}_i, i \in \{1,2,3\}$. Thus, $\langle s_1, P_0[s_1,t_3], t_3, t_3^n, s_3, P_1[s_3,t_2], t_2, t_2^n, s_2, P_1[s_2,t_1], t_1, t_1^n, s_1\rangle$ forms the desired cycle of length $(2^{n-1}-2)-3+6 = 2^{n-1}+1$.

- Now we consider the case $k \neq n-1$. Specially, Lemmas 3 and 7 ensure that Q^{n1}_{n-1} contains a fault-free path $P_1[u^n, \bar{v}\,]$ of length l_1 with $n-k-1 \leq l_1 \leq 2^{n-1}-2$ (respectively, $n-k+1 \leq l_1 \leq 2^{n-1}-2$) when (u,v) is an ith dimensional edge, where $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1,2,\ldots,k-1\}$). Then $\langle u, u^n, P_1[u^n, \bar{v}], \bar{v}, v, u\rangle$ forms the desired cycle containing the edge (u,v) of length $l = l_1+3$ with $n-k+2 \leq l \leq 2^{n-1}+1$ (respectively, $n-k+4 \leq l \leq 2^{n-1}+1$) when (u,v) is the ith dimensional edge, where $k \neq n-1$ and $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1,2,\ldots,k-1\}$).
- Generally, applying Lemma 2, $Q^{n0}_{n-1} - \{f\}$ contains a fault-free cycle C_0 of every even length from 4 to $2^{n-1}-2$ containing the edge $e = (u,v)$. Select an edge $(s,t) \neq (u,v)$ in C_0. For clarity, $C_0 = \langle s, P_0[s,t], t, s\rangle$, where $l_0 = l(P_0[s,t])$ satisfies $3 \leq l_0 \leq 2^{n-1}-3$. Lemma 3 indicates that Q^{n1}_{n-1} contains a fault-free path $P_1[s^n, \bar{t}]$ of even length l_1 with $l_1 = 2^{n-1}-2$. Consequently, $\langle s, P_0[s,t], t, \bar{t}, P_1[s^n, \bar{t}\,], s^n, s\rangle$ forms the desired fault-free cycle of length $l = l_0 + l_1 + 2$, i.e., $2^{n-1}+3 \leq l \leq 2^n - 3$ (see Figure 3a).

Subcase 1.2: $l = 2^n - 1$.

Applying Lemma 2, the edge $e = (u,v)$ lies on a fault-free cycle C_0 of length $2^{n-1}-2$. Note that $|V(Q^{n0}_{n-1})| = 2^{n-1}$ and $|V(Q^{n0}_{n-1})| - |V(C_0)| - |\{f\}| = 1$. It implies that there must exist a fault-free vertex $w \in V(Q^{n0}_{n-1}) - V(C_0) - \{f\}$. Note that w and f have different parity. Since $l(C_0) - 3 = 2^{n-1} - 5 \geq 3$ for $n \geq 4$, we can select an edge $(s,t) \neq e$ on the cycle C_0, such that $(s^n, t^n) \in E(Q^{n1}_{n-1})$ and $\{s^n, t^n\} \cap \{\bar{w}\} = \emptyset$ (or we can select an edge $(s,t) \neq e$ on the cycle C_0, such that $(\bar{s}, \bar{t}\,) \in E(Q^{n1}_{n-1})$ and $\{\bar{s}, \bar{t}\,\} \cap \{w^n\} = \emptyset$). Without loss of generality, we can assume the first situation holds. For clarity, $C_0 = \langle s, P_0[s,t], t, s\rangle$ and $l_0 = l(P_0[s,t]) = 2^{n-1}-3$. Without loss of generality, we can assume $d_H(s,w)$ is even. Since $d_H(s,t)$ is odd, it follows that $d_H(s^n, \bar{w})$ is odd, and $d_H(t^n, w^n)$ is odd. Lemma 5 indicates that there exist two vertex-disjoint paths $P_1[s^n, \bar{w}]$ and $P_2[t^n, w^n]$ spanning $V(Q^{n1}_{n-1})$, that is, $V(P_1[s^n, \bar{w}]) \cup V(P_2[t^n, w^n]) = V(Q^{n1}_{n-1})$. For clarity, $l(P_1[s^n, \bar{w}]) + l(P_2[t^n, w^n]) = l_1 + l_2 = 2^{n-1}-2$. So the desired cycle containing the edge e can be constructed as $\langle s, P_0[s,t], t, t^n, P_2[t^n, w^n], w^n, w, \bar{w}, P_1[\bar{w}, s^n], s^n, s\rangle$, whose length is $l_0 + l_1 + l_2 + 4 = (2^{n-1}-3) + (2^{n-1}-2) + 4 = 2^n - 1$ in $Q_{n,k} - F_v$ (see Figure 3b).

Case 2: $e \in E(Q^{n1}_{n-1})$. Obviously, $i \in \{1,2,\ldots,n-1\}$. We distinguish two subcases according to the length l.

Subcase 2.1: $n-k+2 \leq l \leq 2^n-3$. (respectively, $n-k+4 \leq l \leq 2^n-3$), for the ith dimensional edge, where $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1,2,\ldots,k-1\}$).

- First, we consider the case $k = n-1$. Lemma 7 indicates that $d_H(u^n, \bar{v}) = d_H(\bar{u}, v^n) = 0$. Since $|F_v| = 1$, with out loss of generality, we can assume $\{u^n, \bar{v}\} \cap F_v = \emptyset$. Obviously, we have $d_H(u^n, \bar{v}) = 0$ (respectively, $d_H(u^n, \bar{v}) = n-k+1 = 2$) when $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1,2,\ldots,k-1\}$). For clarity, $\langle u^n, u, v, u^n\rangle$ (respectively, $\langle u, u^n, v^n, \bar{v}, v, u\rangle$) forms the desired cycle of length $n-k+2 = 3$ (respectively, $n-k+4 = 5$) when $i \in \{k, k+1, \ldots, n-1\}$

(respectively, $i \in \{1, 2, \ldots, k-1\}$). Lemma 2 indicates that (u, v) lies on a fault-free cycle C_1 of length l'_1, where $4 \le l'_1 \le 2^{n-1}$. We can select an edge $(s, t) \neq (u, v)$ on C_1 such that (s, t) is an jth dimensional edge, where $j \in \{1, 2, \ldots, n-1\}$. Note that $1 \le l(P_1[s, t]) \le 2^{n-1} - 1$ and $d_H(s^n, \bar{t}) = n - k - 1 = 0$ (respectively, $d_H(s^n, \bar{t}) = n - k + 1 = 2$) when $j \in \{k, k+1, \ldots, n-1\}$ (respectively, $j \in \{1, 2, \ldots, k-1\}$). Thus, $\langle s^n, s, P_1[s, t], t, \bar{t}\rangle$ (respectively, $\langle s^n, t^n, \bar{t}, t, P_1[t, s], s, s^n\rangle$) forms the desired cycle, whose length is $l = l(P_1[s, t]) + 2$, i.e., $n - k + 2 \le l \le 2^{n-1} + 1$ (respectively, $l = l(P_1[s, t]) + 4$, i.e., $n - k + 4 \le l \le 2^{n-1} + 3$).

- Now we construct the desired cycle of every odd length from $2^{n-1} + 3$ to $2^n - 1$. Let C_1 be a cycle of length 2^{n-1} and contains the edge (u, v). If every edge $(s, t) \neq (u, v)$ on the cycle C_1 are the ith dimensional edge, $i \in \{k, k+1, \ldots, n-1\}$, then we can replace the edge (s, t) by the path $\langle s^n, s, t, s^n\rangle$. Thus, the desired cycle is of length $l = 2(l_1 - 1) + 1 = 2l_1 - 1 = 2^{n-1} - 1$. If there exists an edge $(s, t) \neq (u, v)$ is an ith dimensional edge, $i \in \{1, 2, \ldots, k-1\}$, then we have $d_H(s^n, \bar{t}) = n - k + 1 \neq 0$ and is even. Lemma 2 implies that $(s^n, \bar{t})$ lies on a cycle of length $l_0 = l(P_0[s^n, \bar{t}]) = 2^{n-1} - 2$. Thus, $\langle s^n, P_0[s^n, \bar{t}], \bar{t}, t, P_1[t, s], s, s^n\rangle$ forms the desired cycle of length $l = l_0 + l_1 + 2$, i.e., $2^{n-1} + 3 \le l \le 2^n - 1$.
- Finally, we consider $k \neq n - 1$, since n and k have different parity, we have $1 \le k \le n - 3$. Note that $n \ge 4$ and Q^{n1}_{n-1} is fault-free. Let (u, v) be an ith dimensional edge in Q^{n1}_{n-1}, where $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1, 2, \ldots, k-1\}$). Lemma 8 implies that there exist $n - k - 1 \ge 2$ distinct vertices w_j's in $V(Q^{n1}_{n-1})$ such that $w_j \neq v$, $(u^n, \bar{w}_j)$ is a fault-free jth dimensional edge in $E(Q^{n0}_{n-1})$, where $j \in \{k, k+1, \ldots, n-1\}$. Note that $d_H(v, w_j) = n - k - 2$ (respectively, $d_H(v, w_j) = n - k$) when $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1, 2, \ldots, k-1\}$). Since $|F_v| = 1$, for the ith dimensional edge (u, v), $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1, 2, \ldots, k-1\}$), we can choose such a vertex $w \in V(Q^{n1}_{n-1})$ satisfies $d_H(u^n, \bar{w}) = 1$ and $\{u^n, \bar{w}\} \cap \{f\} = \emptyset$. It follows thatthe edge $(u^n, \bar{w})$ is a jth dimensional edge in $Q_{n,k}$, where $j \in \{k, k+1, \ldots, n-1, c\}$. Obviously, Lemma 3 ensures that Q^{n1}_{n-1} contains a fault-free path $P_1[v, w]$ joining v and w, whose length is $n - k - 2$ (respectively, $n - k$) when $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1, 2, \ldots, k-1\}$). Specially, by Lemma 8, if the fault-free path $P_1[v, w]$ is of length $n - k - 2$ (respectively, $n - k$), then it does not contain the vertex u. Assuming u as a faulty vertex temporarily, we can conclude that $|(F_v \cap V(Q^{n1}_{n-1})) \cup \{u\}| = 1$. Lemma 4 implies that there exists a fault-free path $P_1[v, w]$ of every possible odd length l'_1 joining v and w in $Q^{n1}_{n-1} - \{u\}$, where $n - k \le l'_1 \le 2^{n-1} - 3$ (respectively, $n - k + 2 \le l'_1 \le 2^{n-1} - 3$). Consequently, $n - k - 2 \le l_1 = l(P_1[v, w]) \le 2^{n-1} - 3$ (respectively, $n - k \le l_1 = l(P_1[v, w]) \le 2^{n-1} - 3$). Since $d_H(u^n, \bar{w}) = 1$, by Lemma 4, there exists a fault-free path $P_0[u^n, \bar{w}]$ of every odd length l'_0 joining u^n and $\bar{w}$ in $Q^{n0}_{n-1} - \{f\}$, where $3 \le l'_0 \le 2^{n-1} - 3$. Note that $(u^n, \bar{w})$ is a fault-free edge in Q^{n0}_{n-1}. Thus, $1 \le l_0 = l(P_0[u^n, \bar{w}]) \le 2^{n-1} - 3$. As a result, for the ith dimensional edge (u, v), $i \in \{k, k+1, \ldots, n-1\}$ (respectively, $i \in \{1, 2, \ldots, k-1\}$), $\langle u^n, P_0[u^n, \bar{w}], \bar{w}, w, P_1[w, v], v, u, u^n\rangle$ forms the desired cycle of every possible odd length $l = l_0 + l_1 + 3$ in $Q_{n,k} - F^*_v$, i.e., $n - k + 2 \le l \le 2^n - 3$ (respectively, $n - k + 4 \le l \le 2^n - 3$)(see Figure 3c).

Subcase 2.2: $l = 2^n - 1$.

Lemma 2 ensures that there exists a cycle C_1 containing e of length 2^{n-1} in Q^{n1}_{n-1}. Since $l(C_1) - 3 = 2^{n-1} - 3 \ge 5$ for $n \ge 4$, we can select an edge $(s, t) \in E(C_1)$ such that $(s, t) \neq e$, $\{s^n, \bar{s}, t^n, \bar{t}\} \cap \{f\} = \emptyset$. For clarity, $C_1 = \langle s, P_1[s, t], t, s\rangle$, $l_1 = l(P_1[s, t]) = 2^{n-1} - 1$. Note that $d_H(\bar{t}, s^n)$ is even, $d_H(\bar{s}, t^n)$ is even and $d_H(\bar{s}, s^n)$ is odd. Without loss of generality, we can assume $d_H(s^n, f)$ is odd and $d_H(\bar{t}, f)$ is odd. Lemma 6 indicates that $Q^{n0}_{n-1} - \{f\}$ contains a fault-free Hamiltonian path $P_0[s^n, \bar{t}\,]$ joining s^n and $\bar{t}$ with length $l_0 = 2^{n-1} - 2$. As a consequence, $\langle s^n, P_0[s^n, \bar{t}\,], \bar{t}, t, P_1[t, s], s, s^n\rangle$

forms the desired cycle containing the edge e of odd length $l = l_0 + l_1 + 2 = 2^n - 1$ in $Q_{n,k} - F_v$ (see Figure 4a).

Case 3: $e \in E_n$. Obviously, $i = n$. Assume that $u \in V(Q^{n0}_{n-1})$ and $v \in V(Q^{n1}_{n-1})$. Obviously, $v = u^n$. We distinguish two subcases according to the length l.

Subcase 3.1: $n - k + 2 \leq l \leq 2^n - 3$.

Note that (u, v) is a nth dimensional edge between Q^{n0}_{n-1} and Q^{n1}_{n-1}. Specially, if $k = n - 1$, we can find that $d_H(u^n, \bar{u}) = n - k$. Lemma 3 indicates that there exists a fault-free path $P_1[u^n, \bar{u}]$ of length $n - k$ in Q^{n1}_{n-1}. Thus, $\langle u, u^n, P_1[u^n, \bar{u}], \bar{u}, u\rangle$ forms the desired cycle of length $n - k + 2$. Lemma 7 ensures that there exist $n - 1$ distinct vertices x_j's adjacent to u in $V(Q^{n0}_{n-1})$ for $n \geq 4$, such that $d_H(u^n, \bar{x}_j) = n - k - 1$ or $d_H(u^n, \bar{x}_j) = n - k + 1$. Generally, since $|F_v| = 1$, we can select a fault-free vertex $x \in V(Q^{n0}_{n-1})$ such that (u, x) is a jth dimensional edge in Q^{n0}_{n-1}, where $j \in \{k, k+1, \ldots, n-1\}$ and $k \neq n - 1$. Lemma 7 indicates that $d_H(u^n, \bar{x}) = n - k - 1$. Lemma 4 indicates that Q^{n0}_{n-1} contains a fault-free path $P_0[u, x]$ of every odd length from 3 to $2^{n-1} - 3$. For convenience, we denote $l_0 = l(P_0[u, x])$, $1 \leq l_0 \leq 2^{n-1} - 3$. In Q^{n1}_{n-1}, Lemma 3 ensures that Q^{n1}_{n-1} contains a path $P_1[u^n, \bar{x}]$ of every even length l_1 joining u^n and $\bar{x}$, where $n - k - 1 \leq l_1 \leq 2^{n-1} - 2$. Accordingly, the desired cycle can be constructed as $\langle u, P_0[u, x], x, \bar{x}, P_1[\bar{x}, u^n], u^n, u\rangle$ with every possible odd length $l = l_0 + l_1 + 2$ in $Q_{n,k} - F_v$, $n - k + 2 \leq l \leq 2^n - 3$ (see Figure 4b).

Subcase 3.2: $l = 2^n - 1$.

By Lemma 2, Q^{n0}_{n-1} contains a fault-free cycle C_0 of length $2^{n-1} - 2$. Note that $|V(Q^{n0}_{n-1})| = 2^{n-1}$ and $|V(Q^{n0}_{n-1})| - |V(C_0)| - |\{f\}| = 1$. It implies that there must exist a fault-free vertex $w \in V(Q^{n0}_{n-1}) - V(C_0) - \{f\}$. According to the distribution of the node $u \in V(Q^{n0}_{n-1})$, we consider the following subcases:

Subcase 3.2.1 $u \neq w$, i.e., $u \in V(C_0)$.

Since the number of vertices that adjacent to the vertex u in C_0 is 2, there must exist such a vertex $s \in C_0$ adjacent to the vertex u such that $s^n \neq \bar{w}$. Hence, the cycle $C_0 \in Q^{n0}_{n-1}$ can be represented as $C_0 = \langle u, P_0[u, s], s, u\rangle$. Therefore, $l_0 = l(P_0[u, s]) = 2^{n-1} - 3$. Considering the relationship between u^n and $\bar{w}$, we distinguish the following subcases:

- First, we consider the case that $u^n \neq \bar{w}$. One can observe that we may assume $d_H(u, w)$ is odd and $d_H(s, w)$ is even. It implies that $d_H(u^n, w^n)$ is odd and $d_H(s^n, \bar{w})$ is odd. Applying Lemma 5, there exist two vertex-disjoint paths $P_1[u^n, w^n]$ and $P_2[s^n, \bar{w}]$ spanning $V(Q^{n1}_{n-1})$, whose length totally is $2^{n-1} - 2$. For clarity, $l(P_1[u^n, w^n]) + l(P_2[s^n, \bar{w}]) = l_1 + l_2 = 2^{n-1} - 2$. Consequently, $\langle u, P_0[u, s], s, s^n, P_2[s^n, \bar{w}], \bar{w}, w, w^n, P_1[w^n, u^n], u^n, u\rangle$ forms the desired cycle with length $l = l_0 + l_1 + l_2 + 4$. Since $l_0 = 2^{n-1} - 3$, $l_1 + l_2 = 2^{n-1} - 2$, it follows that $l = 2^n - 1$ (see Figure 4c).
- Now, we consider the case that $u^n = \bar{w}$. Note that $d_H(w^n, s^n)$ is even. Applying Lemma 6, $Q^{n1}_{n-1} - \{u^n\}$ contains a fault-free Hamiltonian path $P_1[w^n, s^n]$ joining w^n and s^n with length $l_1 = 2^{n-1} - 2$. As a result, $\langle u, P_0[u, s], s, s^n, P_1[s^n, w^n], w^n, w, u^n, u\rangle$ forms the desired cycle containing the edge e, whose length is $l = l_0 + l_1 + 4 = 2^n - 1$ (see Figure 5a).

Subcase 3.2.2: $u = w$, i.e., $u \in V(Q^{n0}_{n-1}) - V(C_0) - \{f\}$.

Recall that $l(C_0) = 2^{n-1} - 2 \geq 6$ for $n \geq 4$. It follows that we can select an edge $(s, t) \in E(C_0)$ such that $\{s^n, t^n\} \cap \{\bar{u}\} = \emptyset$. For convenience, we may assume that $d_H(u, s)$ is even and $d_H(u, t)$ is odd. It implies that $d_H(u^n, t^n)$ is odd and $d_H(\bar{u}, s^n)$ is odd. Lemma 5 indicates that Q^{n1}_{n-1} contain two vertex-disjoint paths $P_1[s^n, \bar{u}]$ and $P_2[t^n, u^n]$ with total length $2^{n-1} - 2$, that is, $l(P_1[s^n, \bar{u}]) +$

$l(P_2[t^n, u^n]) = l_1 + l_2 = 2^{n-1} - 2$. For clarity, $l(P_0[s,t]) = l_0 = 2^{n-1} - 3$. As an immediate result, $\langle s, P_0[s,t], t, t^n, P_2[t^n, u^n], u^n, u, \bar{u}, P_1[\bar{u}, s^n], s^n, s\rangle$ forms the desired cycle of length l with $l = l_0 + l_1 + l_2 + 4 = 2^n - 1$ (see Figure 5b).

Case 4: $e \in E_c$. Obviously, $i = c$. Note that E_c is the set of complementary edges between Q^{n0}_{n-1} and Q^{n1}_{n-1}. Without loss of generality, we can assume $u \in V(Q^{n0}_{n-1})$ and $v \in V(Q^{n1}_{n-1})$. Obviously, $v = \bar{u}$. We distinguish two subcases according to the length l.

Subcase 4.1: $n - k + 2 \leq l \leq 2^n - 3$.

Specially, when $k = n - 1$, one can observe that $d_H(\bar{u}, u^n) = n - k$. Lemma 3 indicates that Q^{n1}_{n-1} contains a fault-free path $P_1[\bar{u}, u^n]$ of length $n - k$. Thus, $\langle u, \bar{u}, P_1[\bar{u}, u^n], u^n, u\rangle$ forms the desired cycle of length $n - k + 2$. In Q^{n0}_{n-1}, Lemma 7 ensures that there exists $n - 1$ distinct vertices w_j's adjacent to the vertex u in $V(Q^{n0}_{n-1})$ such that $d_H(\bar{u}, w^n_j) = n - k - 1$ or $d_H(\bar{u}, w^n_j) = n - k + 1$. Generally, since $|F_v| = 1$, we can select a fault-free vertex $w \in V(Q^{n0}_{n-1})$ such that (u, w) is a jth dimensional edge in Q^{n0}_{n-1} when $j \in \{k, k+1, \ldots, n-1\}$ and $k \neq n - 1$. Lemma 7 indicates that $d_H(\bar{u}, w^n) = n - k - 1$. Applying Lemma 4, $Q^{n0}_{n-1} - \{f\}$ contains a fault-free cycle $P_0[u, w]$ of every odd length from 3 to $2^{n-1} - 3$ joining u and w. For clarity, $l_0 = l(P_0[u, w])$, $1 \leq l_0 \leq 2^{n-1} - 3$. In Q^{n1}_{n-1}, Lemma 3 indicates that there exists a fault-free path $P_1[\bar{u}, w^n]$ of every even length l_1 joining $\bar{u}$ and w^n, where $n - k - 1 \leq l_1 \leq 2^{n-1} - 2$. Consequently, $\langle u, P_0[u, w], w, w^n, P_1[w^n, \bar{u}], \bar{u}, u\rangle$ forms the desired cycle of length l with $l = l_0 + l_1 + 2$ in $Q_{n,k} - F_v$. Since $1 \leq l_0 \leq 2^{n-1} - 3$, $n - k - 1 \leq l_1 \leq 2^{n-1} - 2$, it follows that $n - k + 2 \leq l \leq 2^n - 3$.

Subcase 4.2: $l = 2^n - 1$.

Applying Lemma 2, Q^{n0}_{n-1} contains a fault-free cycle C_0 of length $2^{n-1} - 2$. Note that $|V(Q^{n0}_{n-1})| - |V(C_0)| - |\{f\}| = 1$. It implies that there must exist a fault-free vertex $w \in V(Q^{n0}_{n-1}) - V(C_0) - \{f\}$. Considering the distribution of the vertex $u \in V(Q^{n0}_{n-1})$, we distinguish the following subcases:

Subcase 4.2.1: $u \neq w$, i.e., $u \in V(C_0)$.

Since the number of vertices that adjacent to the vertex u in C_0 is 2, there must exist such a vertex $s \in C_0$ adjacent to the vertex u such that $\bar{s} \neq w^n$. For clarity, $C_0 = \langle u, P_0[u, s], s, u\rangle$, $l_0 = l(P_0[u, s]) = 2^{n-1} - 3$. According to the relationship between $\bar{u}$ and w^n, we distinguish the following subcases:

- First, we consider the case that $\bar{u} \neq w^n$. Note that we can assume $d_H(\bar{u}, \bar{w})$ is odd and $d_H(\bar{s}, w^n)$ is odd. Lemma 5 ensures that there exist two vertex-disjoint paths $P_1[\bar{u}, \bar{w}]$ and $P_2[\bar{s}, w^n]$ spanning $V(Q^{n1}_{n-1})$, that is, $l(P_1[\bar{u}, \bar{w}]) + l(P_2[\bar{s}, w^n]) = l_1 + l_2 = 2^{n-1} - 2$. Accordingly, $\langle u, P_0[u, s], s, \bar{s}, P_2[\bar{s}, w^n], w^n, w, \bar{w}, P_1[\bar{w}, \bar{u}], \bar{u}, u\rangle$ forms the desired cycle of length $l = l_0 + l_1 + l_2 + 4 = 2^n - 1$ in $Q_{n,k} - F_v$ (see Figure 5c).
- Now, we consider the case that $\bar{u} = w^n$. Note that $d_H(\bar{w}, \bar{s}\ )$ is even, $d_H(\bar{u}, \bar{w})$ is odd and $|F_v \cup \{\bar{u}\}| = 1$. Lemma 6 indicates that $Q^{n1}_{n-1} - \{\bar{u}\}$ contains a fault-free Hamiltonian path $P_1[\bar{w}, \bar{s}]$ joining $\bar{w}$ and $\bar{s}$ with length $l_1 = 2^{n-1} - 2$. As a result, $\langle u, P_0[u, s], s, \bar{s}, P_1[\bar{s}, \bar{w}], \bar{w}, w, w^n, u\rangle$ forms the desired cycle with length is $l = l_0 + l_1 + 4 = 2^n - 1$.

Subcase 4.2.2: $u = w$, i.e., $u \in V(Q^{n0}_{n-1}) - V(C_0) - \{f\}$.

This proof is similar to that in Case 3.2.2.

In summary, all cases have been concerned, so the proof is completed. □

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