



Article The Hermitian Solution to a New System of Commutative Quaternion Matrix Equations

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Abstract: This paper considers the Hermitian solutions of a new system of commutative quaternion matrix equations, where we establish both necessary and sufficient conditions for the existence of solutions. Furthermore, we derive an explicit general expression when it is solvable. In addition, we also provide the least squares Hermitian solution in cases where the system of matrix equations is not consistent. To illustrate our main findings, in this paper we present two numerical algorithms and examples.

Keywords: commutative quaternion algebra; matrix equations; Hermitian matrix; least squares solution

MSC: 15A09; 15A24; 15B33; 15B57

1. Introduction

In 1843, Hamilton introduced the concept of real quaternions, which are defined by [1]

$$\mathbb{H} = \{q = q_0 + q_1i + q_2j + q_3k : i^2 = j^2 = k^2 = -1, ijk = -1, q_0, q_1, q_2, q_3 \in \mathbb{R}\},\$$

which is a four-dimensional noncommutative associative algebra over real number field. Quaternions have been used in many areas, such as statistic of quaternion random signals [2], color image processing [3], and face recognition [4]. The non-commutative nature of quaternion multiplication introduces numerous challenges and difficulties when dealing with real quaternions.

A commutative quaternion, which was proposed by Segre [5] in 1892, is in the form of $q = q_0 + q_1i + q_2j + q_3k$, where q_0, q_1, q_2, q_3 belong to the real number field and the imaginary identities *i*, *j*, *k* satisfy $i^2 = k^2 = -1$, $j^2 = 1$, ijk = -1, ij = ji = k, jk = kj = i, ki = ik = -j.

A notable characteristic of a commutative quaternion is its fulfillment of the multiplication commutative rule. The collection of commutative quaternions comprises fourdimensional Clifford algebra, forming a ring. Within this set, we can find noteworthy attributes such as nontrivial idempotents, zero divisors, and nilpotent elements. There are many applications of commutative quaternion algebra in Hopfield neural networks, digital signals, image processing [6–10], and so on. Commutative quaternions have also been extensively researched. Kösal et al. [11] presented complex representations of commutative quaternion matrices and discussed several related properties. In [12], Kösal et al. proposed the real representation of a commutative quaternion matrix and derived explicit expressions for solutions to commutative quaternion matrix equations $X - A\bar{X}B = C$, $X - A\bar{X}B = C$, and $X - A\bar{X}B = C$, which are commonly referred to as Kalman–Yakubovich-conjugate matrix equations. Based on this, Kösal et al. [13] provided a formulation for the general solution to the matrix equation AX = B over the commutative quaternion ring.



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The Hermitian matrix has drawn a significant amount of attentions due to its great importance. In [14], Yu et al. studied Hermitian solutions to the generalizaed quaternion matrix equation $AXB + CX^*D = E$ through the real representation of quaternion matrices. In [15], Yuan et al. also discussed Hermitian solutions to the split quaternion matrix equation AXB + CXD = E by using the complex representation of quaternion matrices. In [16], Kyrchei obtained the determinantal representation formulas of η -(η skew)-Hermitian solutions to the quaternion matrix equations AX = B and $AXA^{\eta *} = B$. In [17], Xu et al. proceeded to delve further into the η Hermitian solutions of the equations, after providing the solvability conditions and expressions for the solutions of the system of equations over the quaternion ring. As a special type of Hermitian solution, research on Hermitian solutions is still in progress. Chen et al. [18] not only investigated the solvability conditions and the general expressions of solution for the matrix equation AXB = Cover dual quaternion algebra but also explored the expression of ϕ Hermitian solutions when they exist. The Sylvester matrix equations are widely utilized in diverse fields. For example, the Sylvester matrix equation $A_1X + XB_1 = C_1$ and the Sylvester-like matrix equation $A_1X + YB_1 = C_1$ have been applied in singular system control [19], perturbation theory [20], sensitivity analysis [21], and control theory [22]. Kyrchei [23] gave the determinantal representation formulas of solutions to the generalized Sylvester quaternion matrix equation $A_1X_1B_1 + A_2X_2B_2 = C$.

Motivated by a sustained interest in Hermitian solutions and the wide applications of commutative quaternion matrix equations, in this paper we aim to explore the solvability conditions and the Hermitian solutions of the following system of commutative quaternion matrix equations,

$$\begin{cases}
A_1 X = C_1, \\
Y B_1 = D_1, \\
A_2 Z = C_2, Z B_2 = D_2, \\
A_3 W = C_3, W B_3 = D_3, A_4 W B_4 = C_4, \\
A_5 X + Y B_5 + A_6 Z B_6 + A_7 W B_7 = C_5,
\end{cases}$$
(1)

where X, Y, Z, W are unknown Hermitian commutative quaternion matrices.

This paper is organized as follows. In Section 2, we review some useful properties and the structures of vec(AXB) over the commutative quaternion algebra when X is a Hermitian commutative quaternion matrix. In Section 3, we derive some practical necessary and sufficient conditions for the existence of Hermitian solutions to the system (1) over \mathbb{H}_c , and the numerical examples are provided in Section 4.

2. Preliminaries

Throughout this paper, let $\mathbb{R}^{m \times n}$, $\mathbb{SR}^{n \times n}$, $\mathbb{ASR}^{n \times n}$, $\mathbb{C}^{m \times n}$, \mathbb{H}_c , \mathbb{H}_c^n , and $\mathbb{H}_c^{m \times n}$ denote the sets of all $m \times n$ real matrices, $n \times n$ real symmetric matrices, $n \times n$ real anti-symmetric matrices, $m \times n$ complex matrices, commutative quaternions, n dimensional commutative quaternion column vectors, and $m \times n$ commutative quaternion matrices, respectively.

The symbol r(A) denotes the rank of A. Let the symbols I, O, A^T, A^{\dagger} stand for the identity matrix, the zero matrix with appropriate size, the transpose of A, and the Moore–Penrose inverse of matrix A, respectively. \overline{A} and A^H denote the conjugate matrix and the conjugate transpose matrix of A, respectively. We call $A \in \mathbb{H}_c^{n \times n}$ a Hermitian matrix if $A^H = A$ and denote it by $A \in \mathbb{HH}_c^{n \times n}$, where $\mathbb{HH}_c^{n \times n}$ is the set of all Hermitian commutative quaternion matrices with a size of $n \times n$.

For any $\overline{A} \in \mathbb{H}_c^{m \times n}$, A can be uniquely expressed as $A = A_0 + A_1 i + A_2 j + A_3 k$, where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m \times n}$. It can also be uniquely expressed as $A = C_1 + C_2 j$, where $C_1 = A_0 + A_1 i, C_2 = A_2 + A_3 i \in \mathbb{C}^{m \times n}$.

Proposition 1 ([11]). *The complex representation matrix for commutative quaternion* $q = d_1 + d_2j$, $d_1 = q_0 + q_1i$, $d_2 = q_2 + q_3i$ is denoted as

$$g(q) = \begin{pmatrix} d_1 & d_2 \\ d_2 & d_1 \end{pmatrix} \in \mathbb{C}^{2 \times 2}.$$

Similarly, for any given $A = A_1 + A_2 j \in \mathbb{H}_c^{m \times n}$, the complex representation matrix of A is

$$G(A) = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix} \in \mathbb{C}^{2m \times 2n}.$$
 (2)

Obviously, G(A) is uniquely determined by A. It is straightforward to confirm that the following statements are valid.

Proposition 2 ([11]). *If* $A, B \in \mathbb{H}_c^{n \times n}$, *then*

- (a) A = B if and only if G(A) = G(B),
- (b) G(A+B) = G(A) + G(B),
- $(c) \quad G(I_n) = I_{2n},$
- (d) G(AB) = G(A)G(B).

Suppose $A = (a_{ij}) \in \mathbb{H}_c^{m \times n}$ and $B = (b_{ij}) \in \mathbb{H}_c^{s \times t}$; the Kronecker product of A and B is defined as $A \otimes B = (a_{ij}B) \in \mathbb{H}_c^{ms \times nt}$. Considering commutative quaternion matrices A, B, C, D, E with appropriate dimensions, along with the real number p, we establish that

$$(pA) \otimes B = A \otimes (pB) = p(A \otimes B),$$

$$(A, B, C) \otimes D = (A \otimes D, B \otimes D, C \otimes D),$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes E = \begin{pmatrix} A \otimes E & B \otimes E \\ C \otimes E & D \otimes E \end{pmatrix}.$$

The vec-operator of $A = (a_{ij}) \in \mathbb{H}_c^{m \times n}$ is defined as

$$\operatorname{vec}(A) = (a_1, a_2, \dots, a_n)^T, a_j = (a_{1j}, a_{2j}, \dots, a_{mj}), j = 1, 2, \dots, n.$$

To investigate the Hermitian solutions of a system of matrix Equation (1) within the framework of the commutative quaternion algebra, we need to review some certain definitions and fundamental properties.

Assume that $A = A_1 + A_2 j \in \mathbb{H}_c^{m \times n}$, $A_1, A_2 \in \mathbb{C}^{m \times n}$, then we have

$$A_1 + A_2 j = A \cong \Phi_A = (A_1, A_2),$$

where the symbol \cong represents an equivalence relation. For a given matrix $A = (a_{ij}) \in \mathbb{C}^{m \times n}$, the corresponding Frobenius norm is defined as follows:

$$||A|| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} ||a_{ij}||^2}, ||a_{ij}||^2 = (\operatorname{Re} a_{ij})^2 + (\operatorname{Im} a_{ij})^2.$$

According to the previously mentioned definition of Frobenius norm for complex matrices, we can define the Frobenius norm for commutative quaternion matrix $A = A_1 + A_2 j \in \mathbb{H}_c^{m \times n}$ as follows:

$$\|\hat{A}\| = \sqrt{\|\operatorname{Re} A_1\|^2 + \|\operatorname{Im} A_1\|^2 + \|\operatorname{Re} A_2\|^2 + \|\operatorname{Im} A_2\|^2},$$

where $\hat{A} = (\operatorname{Re}(A_1) \operatorname{Im}(A_1) \operatorname{Re}(A_2) \operatorname{Im}(A_2))$; then we have

$$||\Phi_A|| = ||\dot{A}|| = ||\operatorname{vec}_{\Phi_A}||.$$

Theorem 1 ([24]). Let $p \in \mathbb{R}$, $A, B \in \mathbb{H}_c^{m \times n}$ and $C \in \mathbb{H}_c^{n \times s}$. Then

- (a) A = B if and only if $\Phi_A = \Phi_B$,
- $(b) \quad \Phi_{A+B} = \Phi_A + \Phi_B, \Phi_{pA} = p\Phi_A,$
- $(c) \quad (AC)^T = C^T A^T,$
- (d) $(AC)^{-1} = C^{-1}A^{-1}$, if the matrices A, C and AC are invertible,
- (e) $\Phi_{AC} = \Phi_A G(C).$

For the purpose of deriving the Hermitian solutions of the system (1), we introduce some relevant definitions and conclusions.

Definition 1 ([15]). For the matrix $A = (a_{ij}) \in \mathbb{H}_c^{n \times n}$, set $a_1 = (a_{11}, \sqrt{2}a_{21}, \dots, \sqrt{2}a_{n1})$, $a_2 = (a_{22}, \sqrt{2}a_{32}, \dots, \sqrt{2}a_{n2}), \dots, a_{n-1} = (a_{(n-1)(n-1)}, \sqrt{2}a_{n(n-1)})$, $a_n = a_{nn}$, and denote by $\operatorname{vec}_S(A)$ the following vector:

$$\operatorname{vec}_{S}(A) = (a_{1}, a_{2}, \dots, a_{n-1}, a_{n})^{T} \in \mathbb{H}_{c}^{(n(n+1))/2}.$$
 (3)

Definition 2 ([15]). For the matrix $B = (b_{ij}) \in \mathbb{H}_c^{n \times n}$, set $b_1 = (b_{21}, b_{31}, \dots, b_{n1}), b_2 = (b_{32}, b_{42}, \dots, b_{n2}), \dots, b_{n-2} = (b_{(n-1)(n-2)}, b_{n(n-2)}), b_{n-1} = b_{n(n-1)}$, and denote by $\operatorname{vec}_A(B)$ the following vector:

$$\operatorname{vec}_{A}(B) = \sqrt{2}(b_{1}, b_{2}, \dots, b_{n-2}, b_{n-1})^{T} \in \mathbb{H}_{c}^{(n(n-1))/2}.$$
 (4)

Proposition 3 ([25]). *Suppose that* $X \in \mathbb{R}^{n \times n}$ *, then* (1)

$$X \in \mathbb{SR}^{n \times n} \iff \operatorname{vec}(X) = K_S \operatorname{vec}_S(X), \tag{5}$$

where the matrix $K_S \in \mathbb{R}^{n^2 \times (n(n+1)/2)}$ is of the following form:

$K_S = \frac{1}{\sqrt{2}}$	$\sqrt{2}e_1$	e_2	• • •	e_{n-1}	e_n	0	0	•••	0	•••	0	0	0)	
	0	e_1	• • •	0	0	$\sqrt{2}e_2$	e_3	• • •	e_n	• • •	0	0	0	
	0	0	• • •	0	0	0	e_2		0	•••	0	0	0	
	÷	÷		÷	÷	÷	÷		÷		:	÷	÷	'
	0	0		e_1	0	0	0	• • •	0		$\sqrt{2}e_{n-1}$	<i>e</i> _n	0	
	\ 0	0	• • •	0	e_1	0	0	• • •	<i>e</i> ₂	• • •	0	e_{n-1}	$\sqrt{2}e_n$	

and e_i is the *i*th column of the identity matrix of order *n*. (2)

$$X \in \mathbb{ASR}^{n \times n} \iff \operatorname{vec}(X) = K_A \operatorname{vec}_A(X), \tag{6}$$

 $\operatorname{vec}_A(X)$ is described as (4) and the matrix $K_A \in \mathbb{R}^{n^2 \times (n(n-1)/2)}$ is of the following form:

$K_A = \frac{1}{\sqrt{2}}$	$\int e_2$	e_3	• • •	e_{n-1}	e_n	0	• • •	0	0	• • •	0)	
	$-e_1$	0	• • •	0	0	e ₃	•••	e_{n-1}	e_n	• • •	0	,
	0	$-e_1$	• • •	0	0	$-e_2$	• • •	0	0	• • •	0	
	:	÷		:	÷	÷		0	0		0	
	0	0		$-e_1$	0	0	• • •	$-e_{2}$	0	• • •	e_n	
	0 /	0	• • •	0	$-e_1$	0	•••	0	$-e_2$	• • •	$-e_{n-1}/$	

where e_i is the column of the identity matrix of order n. It is apparent that $K_S^T K_S = I_{(n(n+1))/2}, K_A^T K_A = I_{(n(n-1))/2}$.

Next, we explore the relationships between the Hermitian commutative quaternion matrices and symmetric matrices, as well as anti-symmetric matrices.

If $X = X_1 + X_2 j \in \mathbb{HH}_c^{n \times n}$, where $X_1, X_2 \in \mathbb{C}^{n \times n}$, we can obtain

$$X \in \mathbb{HH}_{c}^{n \times n} \Leftrightarrow \begin{cases} \operatorname{Re}(X_{1})^{T} = \operatorname{Re}(X_{1}), & \operatorname{Im}(X_{1})^{T} = -\operatorname{Im}(X_{1}), \\ \operatorname{Re}(X_{2})^{T} = -\operatorname{Re}(X_{2}), & \operatorname{Im}(X_{2})^{T} = -\operatorname{Im}(X_{2}). \end{cases}$$

Apparently, $\text{Re}(X_1)$ is symmetric, and $\text{Im}(X_1)$, $\text{Re}(X_2)$, and $\text{Im}(X_2)$ are antisymmetric. By means of Proposition 3, we have the following:

Theorem 2 ([26]). Assume that $X = X_1 + X_2 j \in \mathbb{HH}_c^{n \times n}$, then we obtain

$$\begin{pmatrix} \operatorname{vec}(X_1) \\ \operatorname{vec}(X_2) \end{pmatrix} = M \begin{pmatrix} \operatorname{vec}_S(\operatorname{Re}(X_1)) \\ \operatorname{vec}_A(\operatorname{Im}(X_1)) \\ \operatorname{vec}_A(\operatorname{Re}(X_2)) \\ \operatorname{vec}_A(\operatorname{Im}(X_2)) \end{pmatrix} ,$$
(7)

in which

$$M = \begin{pmatrix} K_S & iK_A & 0 & 0\\ 0 & 0 & K_A & iK_A \end{pmatrix}.$$
 (8)

Theorem 3 ([26]). Suppose that $A = A_1 + A_2 j \in \mathbb{H}_c^{m \times n}$, $B = B_1 + B_2 j \in \mathbb{H}_c^{s \times t}$ and $X = X_1 + X_2 j \in \mathbb{H}_c^{n \times s}$, where $A_1, A_2 \in \mathbb{C}^{m \times n}$, $B_1, B_2 \in \mathbb{C}^{s \times t}$ and $X_1, X_2 \in \mathbb{C}^{n \times s}$. Then

$$\operatorname{vec}(\Phi_{AXB}) = G\left[(B_1^T \otimes A_1 + B_2^T \otimes A_2) + (B_2^T \otimes A_1 + B_1^T \otimes A_2)j \right] \begin{pmatrix} \operatorname{vec}(X_1) \\ \operatorname{vec}(X_2) \end{pmatrix}.$$
(9)

Note that the results of $vec(\Phi_{AXB})$ is very important for calculating the system of commutative quaternion matrix Equation (1). Analogous methods and related conclusions can be found in [15].

By incorporating Theorem 3 with Theorem 2, we can gain the following outcome.

Theorem 4 ([26]). If $A = A_1 + A_2 j \in \mathbb{H}_c^{m \times n}$, $X = X_1 + X_2 j \in \mathbb{H} \mathbb{H}_c^{n \times n}$, and $B = B_1 + B_2 j \in \mathbb{H}_c^{n \times s}$, where $A_i \in \mathbb{C}^{m \times n}$, $X_i \in \mathbb{C}^{n \times n}$, and $B_i \in \mathbb{C}^{n \times s}$ (i = 1, 2). Consequently,

$$\operatorname{vec}(\Phi_{AXB}) = G\left[(B_1^T \otimes A_1 + B_2^T \otimes A_2) + (B_2^T \otimes A_1 + B_1^T \otimes A_2) j \right] M \begin{pmatrix} \operatorname{vec}_S(\operatorname{Re}(X_1)) \\ \operatorname{vec}_A(\operatorname{Im}(X_1)) \\ \operatorname{vec}_A(\operatorname{Re}(X_2)) \\ \operatorname{vec}_A(\operatorname{Im}(X_2)) \end{pmatrix}.$$
(10)

Lemma 1 ([27]). The matrix equation Ax = b, with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, has a solution $x \in \mathbb{R}^n$ if and only if

$$AA^{\dagger}b = b. \tag{11}$$

In this case, it has the general solution

$$x = A^{\dagger}b + (I_n - A^{\dagger}A)y, \tag{12}$$

where $y \in \mathbb{R}^n$ is an arbitrary vector, and it has the unique solution $x = A^{\dagger}b$ for the case when r(A) = n. The solution of the matrix equation Ax = b with the least norm is $x = A^{\dagger}b$.

3. The Hermitian Solution to the System (1)

In accordance with the above discussion, we now focus on solving system (1); for ease of description, we firstly state the following notations.

(13)

Let $A_1 = A_{11} + A_{12}j$, $A_2 = A_{21} + A_{22}j$, $A_3 = A_{31} + A_{32}j \in \mathbb{H}_c^{m \times n}$, $C_1, C_2, C_3 \in \mathbb{H}_c^{m \times n}$, $B_1 = B_{11} + B_{12}j$, $B_2 = B_{21} + B_{22}j$, $B_3 = B_{31} + B_{32}j \in \mathbb{H}_c^{n \times k}$, $D_1, D_2, D_3 \in \mathbb{H}_c^{n \times k}$, $A_4 = A_{41} + A_{42}j \in \mathbb{H}_c^{s \times n}$, $B_4 = B_{41} + B_{42}j \in \mathbb{H}_c^{n \times t}$, $C_4 \in \mathbb{H}_c^{s \in \times t}$, $A_5 = A_{51} + A_{52}j$, $A_6 = A_{61} + A_{62}j$, $A_7 = A_{71} + A_{72}j$, $B_5 = B_{51} + B_{52}j$, $B_6 = B_{61} + B_{62}j$, $B_7 = B_{71} + B_{72}j \in \mathbb{H}_c^{n \times n}$, and $C_5 \in \mathbb{H}_c^{n \times n}$. We set

and

$$E_{1} = \operatorname{Re} E, \quad E_{2} = \operatorname{Im} E, \quad F_{1} = \operatorname{Re} F, \quad F_{2} = \operatorname{Im} F,$$

$$P_{1} = \operatorname{Re} P, \quad P_{2} = \operatorname{Im} P, \quad Q_{1} = \operatorname{Re} Q, \quad Q_{2} = \operatorname{Im} Q,$$

$$U_{1} = (E_{1}, F_{1}, P_{1}, Q_{1}),$$

$$U_{2} = (E_{2}, F_{2}, P_{2}, Q_{2}).$$
(14)

For further study of the structure of Hermitian solution of the system of matrix Equation (1), it is necessary to study the generalized inverse of matrices in the form of column blocks. The following notations are required. Let

$$\begin{split} d &= 6mn + 6kn + 2st + 2n^{2}, \\ H &= \left(I_{8n^{2}-4n} - U_{1}^{\dagger}U_{1}\right)U_{2}^{T}, \\ K &= \left(I_{d} + \left(I_{d} - H^{\dagger}H\right)U_{2}U_{1}^{\dagger}U_{1}^{\dagger T}U_{2}^{T}\left(I_{d} - H^{\dagger}H\right)\right)^{-1}, \\ J &= H^{\dagger} + \left(I_{d} - H^{\dagger}H\right)KU_{2}U_{1}^{\dagger}U_{1}^{\dagger T}\left(I_{8n^{2}-4n} - U_{2}^{T}H^{\dagger}\right), \\ R_{11} &= I_{d} - U_{1}U_{1}^{\dagger} + U_{1}^{\dagger T}U_{2}^{T}K\left(I_{d} - H^{\dagger}H\right)U_{2}U_{1}^{\dagger}, \\ R_{12} &= -U_{1}^{\dagger T}U_{2}^{T}\left(I_{d} - H^{\dagger}H\right)K, \\ R_{22} &= \left(I_{d} - H^{\dagger}H\right)K. \end{split}$$

From the findings [28] presented above, it can be inferred that

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} = \begin{pmatrix} U_1^{\dagger} - J^T U_2 U_1^{\dagger}, J^T \end{pmatrix}, \quad \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = U_1^{\dagger} U_1 + H H^{\dagger}, \quad (15)$$

and

$$I_{2d} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} = \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix}.$$
 (16)

Taking into account the aforementioned results, we then turn our attention to the Hermitian solution of the system (1).

Theorem 5. Let $A_1, A_2, A_3, C_1, C_2, C_3 \in \mathbb{H}_c^{m \times n}, B_1, B_2, B_3, D_1, D_2, D_3 \in \mathbb{H}_c^{n \times k}, A_4 \in \mathbb{H}_c^{s \times n}, B_4 \in \mathbb{H}_c^{n \times t}, C_4 \in \mathbb{H}_c^{s \times t}, A_5, A_6, A_7, B_5, B_6, B_7 \in \mathbb{H}_c^{n \times n}$, and $C_5 \in \mathbb{H}_c^{n \times n}. U_1, U_2$ and ϵ are in the form of (3) and (14), respectively. Then the system of commutative quaternion matrix Equation (1) has a solution $X, Y, Z, W \in \mathbb{H}_c^{n \times n}$ if and only if

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon = \epsilon.$$
 (17)

In this case, the set of Hermitian solutions is as follows:

$$\Lambda = \left\{ (X, Y, Z, W) \middle| \begin{pmatrix} \operatorname{vec}(\check{X}) \\ \operatorname{vec}(\check{Y}) \\ \operatorname{vec}(\check{Z}) \\ \operatorname{vec}(\check{W}) \end{pmatrix} = \mathfrak{W} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon + \mathfrak{W} (I_{8n^2 - 4n} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}) y \right\}, \quad (18)$$

where *y* is an arbitrary vector of appropriate order. Then the system (1) has a unique solution $(X, Y, Z, W) \in \Lambda$ if and only if

$$r\binom{U_1}{U_2} = 8n^2 - 4n. \tag{19}$$

If this condition satisfies, then

$$\Lambda = \left\{ (X, Y, Z, W) | \begin{pmatrix} \operatorname{vec}(\dot{X}) \\ \operatorname{vec}(\dot{Y}) \\ \operatorname{vec}(\dot{Z}) \\ \operatorname{vec}(\dot{W}) \end{pmatrix} = \mathfrak{W} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon \right\}.$$
(20)

Proof. By virtue of Theorems 1 and 4 , we obtain

$$\begin{cases} A_{1}X = C_{1}, \\ YB_{1} = D_{1}, \\ A_{2}Z = C_{2}, ZB_{2} = D_{2}, \\ A_{3}W = C_{3}, WB_{3} = D_{3}, A_{4}WB_{4} = C_{4}, \\ A_{5}X + VB_{5} + A_{6}ZB_{6} + A_{7}WB_{7} = C_{5}, \end{cases}$$

$$\Leftrightarrow \begin{cases} \Phi_{A_{1}X} = \Phi_{C_{1}}, \\ \Phi_{A_{2}Z} = \Phi_{C_{2}}, \Phi_{B_{3}} = \Phi_{D_{2}}, \\ \Phi_{A_{3}X} + \Phi_{YB_{5}} + \Phi_{A_{2}ZB_{6}} = \Phi_{D_{2}}, \\ \Phi_{A_{3}X} + \Phi_{YB_{5}} + \Phi_{A_{2}ZB_{6}} + \Phi_{A_{7}WB_{7}} = \Phi_{C_{4}}, \\ \Phi_{A_{5}X} + \Phi_{YB_{5}} + \Phi_{A_{2}ZB_{6}} + \Phi_{A_{7}WB_{7}} = \Phi_{C_{5}}, \end{cases}$$

$$\Leftrightarrow E \left(\frac{\operatorname{vecs}(\operatorname{Re}(X_{1}))}{\operatorname{vec}_{A}(\operatorname{Im}(X_{1}))} + F \left(\frac{\operatorname{vecs}(\operatorname{Re}(Y_{1}))}{\operatorname{vec}_{A}(\operatorname{Re}(Y_{2}))} \right) + P \left(\frac{\operatorname{vecs}(\operatorname{Re}(Z_{1}))}{\operatorname{vec}_{A}(\operatorname{Re}(Z_{2}))} \right) + Q \left(\frac{\operatorname{vecs}(\operatorname{Re}(W_{1}))}{\operatorname{vec}_{A}(\operatorname{Re}(W_{2}))} \right) \\ \Leftrightarrow (\operatorname{Re} E + i\operatorname{Im} E) \left(\frac{\operatorname{vecs}(\operatorname{Re}(X_{1}))}{\operatorname{vec}_{A}(\operatorname{Im}(X_{2}))} \right) + (\operatorname{Re} F + i\operatorname{Im} F) \left(\frac{\operatorname{vecs}(\operatorname{Re}(Y_{1}))}{\operatorname{vec}_{A}(\operatorname{Im}(Y_{2}))} \right) \\ + (\operatorname{Re} P + i\operatorname{Im} P) \left(\frac{\operatorname{vecs}(\operatorname{Re}(Z_{1}))}{\operatorname{vec}_{A}(\operatorname{Re}(Z_{2}))} \right) + (\operatorname{Re} Q + i\operatorname{Im} Q) \left(\frac{\operatorname{vecs}(\operatorname{Re}(W_{1}))}{\operatorname{vec}_{A}(\operatorname{Im}(W_{2}))} \right) = T_{1} + iT_{2}, \\ \exp \left(\operatorname{Im} E \operatorname{Im} F \operatorname{Im} P \operatorname{Im} Q \right) \left(\frac{\operatorname{vec}(\vec{X})}{\operatorname{vec}(\vec{X})} \right) \\ \exp \left(\frac{U_{1}}{U_{2}} \right) \left(\frac{\operatorname{vec}(\vec{X})}{\operatorname{vec}(\vec{Y})} \right) = \epsilon, \\ \exp \left(\frac{U_{1}}{U_{2}} \right) \left(\frac{\operatorname{vec}(\vec{X})}{\operatorname{vec}(\vec{X})} \right) = \epsilon. \\ \operatorname{Re} E \operatorname{Re} F \operatorname{Re} P \operatorname{Re} Q \operatorname{Im} Q$$

,

By Lemma 2, we conclude that the system (1) has a Hermitian solution $(X, Y, Z, W) \in$ Λ if and only if (17) is satisfied; thus we have

$$\begin{pmatrix} \operatorname{vec}(\vec{X}) \\ \operatorname{vec}(\vec{Y}) \\ \operatorname{vec}(\vec{Z}) \\ \operatorname{vec}(\vec{W}) \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon + \left(I_{8n^2 - 4n} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \right) y.$$

On account of

$$\operatorname{vec}(\acute{X}) = \begin{pmatrix} \operatorname{vec}(\operatorname{Re}(X_1)) \\ \operatorname{vec}(\operatorname{Im}(X_1)) \\ \operatorname{vec}(\operatorname{Re}(X_2)) \\ \operatorname{vec}(\operatorname{Im}(X_2)) \end{pmatrix} = \begin{pmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_A \end{pmatrix} \begin{pmatrix} \operatorname{vec}_S(\operatorname{Re}(X_1)) \\ \operatorname{vec}_A(\operatorname{Im}(X_1)) \\ \operatorname{vec}_A(\operatorname{Re}(X_2)) \\ \operatorname{vec}_A(\operatorname{Im}(X_2)) \end{pmatrix} = \mathcal{W}\operatorname{vec}(\vec{X}),$$

similarly, we can derive $\operatorname{vec}(\acute{Y}) = \mathcal{W}\operatorname{vec}(\acute{Z})$, $\operatorname{vec}(\acute{Z})$, and $\operatorname{vec}(\acute{W}) = \mathcal{W}\operatorname{vec}(\vec{W})$; then we have

$$\begin{pmatrix} \operatorname{vec}(\check{X}) \\ \operatorname{vec}(\check{Y}) \\ \operatorname{vec}(\check{Z}) \\ \operatorname{vec}(\check{W}) \end{pmatrix} = \begin{pmatrix} \mathcal{W} & 0 & 0 & 0 \\ 0 & \mathcal{W} & 0 & 0 \\ 0 & 0 & \mathcal{W} & 0 \\ 0 & 0 & 0 & \mathcal{W} \end{pmatrix} \begin{pmatrix} \operatorname{vec}(\check{X}) \\ \operatorname{vec}(\check{Z}) \\ \operatorname{vec}(\check{W}) \end{pmatrix}$$
$$= \mathfrak{W} \begin{pmatrix} \operatorname{vec}(\check{X}) \\ \operatorname{vec}(\check{Y}) \\ \operatorname{vec}(\check{Z}) \\ \operatorname{vec}(\check{W}) \end{pmatrix}$$
$$= \mathfrak{W} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon + \mathfrak{W} (I_{8n^2 - 4n} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}) y.$$

This means that (18) is true; if (17) holds, the system (1) has a unique solution $(X, Y, Z, W) \in \Lambda$ if and only if

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = I_{8n^2 - 4n}.$$

Thus, by (19) we can obtain (20). \Box

Corollary 1. *The system* (1) *has a solution* $X, Y, Z, W \in \mathbb{HH}_c$ *if and only if*

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} \epsilon = 0.$$
 (21)

Under this circumstance, the set of the Hermitian solution of the system (1) *can be represented as follows:*

$$\Lambda = \left\{ (X, Y, Z, W) \middle| \begin{pmatrix} \operatorname{vec}(\dot{X}) \\ \operatorname{vec}(\dot{Y}) \\ \operatorname{vec}(\dot{Z}) \\ \operatorname{vec}(\dot{W}) \end{pmatrix} = \mathfrak{W} \Big(U_1^{\dagger} - J^T U_2 U_1^{\dagger}, J^T \Big) \epsilon + \mathfrak{W} (I_{8n^2 - 4n} - U_1^{\dagger} U_1 - HH^{\dagger}) y \right\},$$
(22)

in which y is an arbitrary vector of appropriate size. Then, the system (1) *has an unique solution* $(X, Y, Z, W) \in \Lambda$ *when* (21) *and* (19) *are obeyed. In this case,*

$$\Lambda = \left\{ (X, Y, Z, W) | \begin{pmatrix} \operatorname{vec}(\acute{X}) \\ \operatorname{vec}(\acute{Y}) \\ \operatorname{vec}(\acute{Z}) \\ \operatorname{vec}(\acute{W}) \end{pmatrix} = \mathfrak{W} \Big(U_1^{\dagger} - J^T U_2 U_1^{\dagger}, J^T \Big) \epsilon \right\}.$$
(23)

4. Numerical Exemplification

In this section, on the basis of discussions in Sections 2 and 3, we provide Algorithms 1 and 2 for solving the system (1) and present two numerical examples to verify the feasibility of the algorithms.

Algorithm 1 For the system (1)

1: **Input:** the matrix: $A_1, A_2, A_3 \in \mathbb{H}_c^{m \times n}, C_1, C_2, C_3 \in \mathbb{H}_c^{m \times n}, B_1, B_2, B_3 \in \mathbb{H}_c^{n \times k}, D_1, D_2, D_3 \in \mathbb{H}_c^{n \times k}, A_4 \in \mathbb{H}_c^{s \times n}, B_4 \in \mathbb{H}_c^{n \times t}, C_4 \in \mathbb{H}_c^{s \in \times t}, A_5, A_6, A_7, B_5, B_6, B_7 \in \mathbb{H}_c^{n \times n}, C_5 \in \mathbb{H}_c^{n \times n}, K_5 \text{ and } K_A.$ 2: Compute U_1, U_2 and ϵ . 3: If both (17) and (19) hold, then calculate the unique solution $(X, Y, Z, W) \in \Lambda$ by (20). 4: If (17) holds, then calculate $(X, Y, Z, W) \in \Lambda$ according to (18).

5: **Output:**(X, Y, Z, W).

Algorithm 2 For the system (1)

1: **Input**: the matrix: $A_1, A_2, A_3 \in \mathbb{H}_c^{m \times n}, C_1, C_2, C_3 \in \mathbb{H}_c^{m \times n}, B_1, B_2, B_3 \in \mathbb{H}_c^{n \times k}, D_1, D_2, D_3 \in \mathbb{H}_c^{n \times k}, A_4 \in \mathbb{H}_c^{s \times n}, B_4 \in \mathbb{H}_c^{n \times t}, C_4 \in \mathbb{H}_c^{s \in \times t}, A_5, A_6, A_7, B_5, B_6, B_7 \in \mathbb{H}_c^{n \times n}, C_5 \in \mathbb{H}_c^{n \times n}, K_5 \text{ and } K_A.$ 2: Compute $U_1, U_2, H, K, J, R_{11}, R_{12}, R_{22}$ and ϵ . 3: If both (19) and (21) hold, then calculate the unique solution $(X, Y, Z, W) \in \Lambda$ by (23). 4: If (21) holds, then calculate $(X, Y, Z, W) \in \Lambda$ according to (22). Otherwise stop. 5: **Output**: (X, Y, Z, W).

From the previous theoretical analysis, if the system (1) is solvable, then

$$\theta_1 = \left\| \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} \epsilon - \epsilon \right\|, \quad \theta_2 = \left\| \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} \epsilon \right\|$$

and

$$\theta_3 = \left\| I_{2d} - \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}^{\dagger} - \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} \right\|$$

are small enough.

Example 1. Let m = 6, n = 4, k = 5, s = 2, t = 3, and

$$\begin{array}{ll} A_1 = A_{11} + A_{12}j, & A_2 = A_{21} + A_{22}j, & A_3 = A_{31} + A_{32}j, \\ B_1 = B_{11} + B_{12}j, & B_2 = B_{21} + B_{22}j, & B_3 = B_{31} + B_{32}j, \\ A_4 = A_{41} + A_{42}j, & B_4 = B_{41} + B_{42}j, & A_5 = A_{51} + A_{52}j, \\ A_6 = A_{61} + A_{62}j, & A_7 = A_{71} + A_{72}j, & B_5 = B_{51} + B_{52}j, \\ B_6 = B_{61} + B_{62}j, & B_7 = B_{71} + B_{72}j, \\ \dot{X} = \dot{X}_1 + \dot{X}_2j, & \dot{Y} = \dot{Y}_1 + \dot{Y}_2j, \\ \dot{Z} = \dot{Z}_1 + \dot{Z}_2j, & \dot{W} = \dot{W}_1 + \dot{W}_2j, \\ C_1 = A_1\dot{X}, & C_2 = A_2\dot{Z}, & C_3 = A_3\dot{W}, \\ D_1 = \dot{Y}B_1, & D_2 = \dot{Z}B_2, & D_3 = \dot{W}B_3, \\ C_4 = A_4\dot{W}B_4, & C_5 = A_5\dot{X} + \dot{Y}B_5 + A_6\dot{Z}B_6 + A_7\dot{W}B_7. \end{array}$$

where

$$\begin{split} A_{11} &= \begin{pmatrix} I_4 \\ O_{2\times 4} \end{pmatrix}, \quad A_{12} = O_{6\times 4}, \quad A_{21} = O_{6\times 4}, \quad A_{22} = \begin{pmatrix} I_4 \\ O_{2\times 4} \end{pmatrix}, \\ A_{31} &= O_{6\times 4}, \quad A_{32} = \begin{pmatrix} O_{2\times 4} \\ I_4i \end{pmatrix}, \quad A_{41} = \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \end{pmatrix}, \quad A_{42} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ A_{51} &= I_4i, \quad A_{52} = O_{4\times 4}, \quad A_{61} = \begin{pmatrix} I_2 & O_{2\times 2} \\ O_{2\times 2} & O_{2\times 2} \end{pmatrix}, \quad A_{62} = O_{4\times 4}, \\ A_{71} &= \begin{pmatrix} O_{2\times 2} & I_2 \\ I_2 & O_{2\times 2} \end{pmatrix}, \quad A_{72} = \begin{pmatrix} O_{2\times 2} & -I_2i \\ O_{2\times 2} & O_{2\times 2} \end{pmatrix}, \quad B_{11} = O_{4\times 5}, \quad B_{12} = (I_4 & O_{4\times 1}), \\ B_{21} &= (-I_4i & O_{4\times 1}), \quad B_{22} = O_{4\times 5}, \quad B_{31} = O_{4\times 5}, \quad B_{32} = (I_4 & O_{4\times 1}), \\ B_{41} &= \begin{pmatrix} I_3 \\ O_{1\times 3} \end{pmatrix}, \quad B_{42} = O_{4\times 3}, \quad B_{51} = O_{4\times 4}, \quad B_{52} = -I_4i, \\ B_{61} &= I_4, \quad B_{62} = O_{4\times 4}, \quad B_{71} = \begin{pmatrix} O_{2\times 2} & I_2 \\ I_2 & O_{2\times 2} \end{pmatrix}, \quad B_{72} = O_{4\times 4}, \\ C_1 &= \begin{pmatrix} 1 & -1+i & j & 0 \\ -1-i & 2 & i & k \\ -j & -i & 0 & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} j & 0 & 0 & k \\ 0 & j & 0 & -I \\ 0 & 0 & j & 0 \\ -k & 1 & 0 & j \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ C_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & k \\ 0 & -i & 0 & k & 0 \\ 0 & -i & 0 & k & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix}, \\ C_4 &= \begin{pmatrix} -1-k & 2i+j & j \\ -i & 0 & 1-i+j \end{pmatrix}, \\ C_5 &= \begin{pmatrix} 1+i & -i+j+k & i-j & 1+i+j & -2j \\ i-jk & 1-i-j & 0 & i+j \\ 1 & j & -i-j & i-k \end{pmatrix}. \end{split}$$

We take

$$\begin{split} \dot{X}_{1} &= \begin{pmatrix} 1 & -1+i & 0 & 0 \\ -1-i & 2 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \dot{X}_{2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ \end{pmatrix}, \\ \dot{Y}_{1} &= \begin{pmatrix} 0 & i & 1 & 0 \\ -i & 0 & i & 0 \\ 1 & -i & 0 & i \\ 0 & 0 & -i & 1 \end{pmatrix}, \quad \dot{Y}_{2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \dot{Z}_{1} &= \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}, \quad \dot{Z}_{2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \dot{W}_{2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}. \end{split}$$

Let

$$\begin{split} \Phi_{C_1} &= \Phi_{A_1} G(\dot{X}), \\ \Phi_{D_1} &= \Phi_{\dot{Y}} G(B_1), \\ \Phi_{C_2} &= \Phi_{A_2} G(\dot{Z}), \quad \Phi_{D_2} = \Phi_{\dot{Z}} G(B_2), \\ \Phi_{C_3} &= \Phi_{A_3} G(\dot{W}), \quad \Phi_{D_3} = \Phi_{\dot{W}} G(B_3), \quad \Phi_{C_4} = \Phi_{A_4} G(\dot{W}) G(B_4), \\ \Phi_{C_5} &= \Phi_{A_5} G(\dot{X}) + \Phi_{\dot{Y}} G(B_5) + \Phi_{A_6} G(\dot{Z}) G(B_6) + \Phi_{A_7} G(\dot{W}) G(B_7). \end{split}$$

From MATLAB and Algorithms 1 and 2, we can obtain

$$r \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 112 = 8n^2 - 4n, \quad \theta_2 = 7.0360 \times 10^{-15}.$$

According to Algorithm 2, the system of matrix Equation (1) has a unique solution $(X, Y, Z, W) \in \Lambda$, and we derive $\theta_1 = 1.7493 \times 10^{-14}$, $\theta_3 = 2.3111 \times 10^{-14}$ and $\|\Phi_{(X,Y,Z,W)} - \Phi_{(\dot{X},\dot{Y},\dot{Z},\dot{W})}\| = 1.1628 \times 10^{-14}$.

Example 2. Let m = 2, n = 2, k = 2, s = 2, t = 2, and

$$\begin{aligned} A_1 &= \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 &= \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}, \quad A_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ A_5 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_6 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_7 &= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad B_1 &= \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_4 &= \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \quad B_5 &= \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}, \\ B_6 &= O_2, \quad B_7 &= \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad C_1 &= \begin{pmatrix} i & -1+i \\ 0 & 0 \end{pmatrix}, \quad C_2 &= \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \\ C_3 &= \begin{pmatrix} 0 & 2+i \\ 2-i & 0 \end{pmatrix}, \quad C_4 &= \begin{pmatrix} 0 & 0 \\ 0 & j+2k \end{pmatrix}, \quad C_5 &= \begin{pmatrix} 1 & 1+i+3j+k \\ 1-i & -k \end{pmatrix}, \\ D_1 &= \begin{pmatrix} i & 0 \\ 1 & 0 \end{pmatrix}, \quad D_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_3 &= \begin{pmatrix} 0 & 0 \\ 2-i & 0 \end{pmatrix}, \end{aligned}$$

taking

$$X = \begin{pmatrix} 1 & 1+i \\ 1-i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 2+i \\ 2-i & 0 \end{pmatrix}.$$

From MATLAB and Algorithm 2, we obtain

$$r\binom{U_1}{U_2} = 23, \quad \theta_2 = 2.1412 \times 10^{-15}.$$

According to Algorithm 2, the system (1) has infinite solutions $(X, Y, Z, W) \in \Lambda$. We can also obtain $\theta_1 = 3.3532 \times 10^{-15}$, $\theta_3 = 4.4730 \times 10^{-15}$. Then, the optimization problem

$$\min_{(X,Y,Z,W)\in\Lambda} \left(\left\| \Phi_{(X,Y,Z,W)} \right\| \right)$$

has a unique minimizer $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W})$; it can also be expressed as

$$\begin{pmatrix} \operatorname{vec}(\tilde{X}) \\ \operatorname{vec}(\check{Y}) \\ \operatorname{vec}(\check{Z}) \\ \operatorname{vec}(\check{W}) \end{pmatrix} = \mathfrak{W} \Big(U_1^{\dagger} - J^T U_2 U_1^{\dagger}, J^T \Big) \epsilon.$$

Therefore, we can obtain $\|\Phi_{(X,Y,Z,W)} - \Phi_{(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W})}\| = 1$, and

$$\tilde{X} = \begin{pmatrix} 1 & 1+i \\ 1-i & 0 \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{W} = \begin{pmatrix} 0 & 2+i \\ 2-i & 0 \end{pmatrix}.$$

5. Conclusions

In this paper, we have provided the necessary and sufficient conditions for the existence of the Hermitian solutions to the system of commutative quaternion matrix Equation (1), and we have also established an expression of the Hermitian solutions to the system (1) when it is consistent. We have also investigated the least squares solution when the system (1) is not consistent. Some numerical algorithms and examples are provided to illustrate our results. In the future, we will investigate the ϕ -Hermitian solution for such a system of matrix equations over commutative quaternion algebra.

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