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Recent Developments in Iterative Algorithms for Digital Metrics

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Abstract: This paper aims to provide a comprehensive analysis of the advancements made in understanding Iterative Fixed-Point Schemes, which builds upon the concept of digital contraction mappings. Additionally, we introduce the notion of an Iterative Fixed-Point Schemes in digital metric spaces. In this study, we extend the idea of Iteration process Mann, Ishikawa, Agarwal, and Thakur based on the $\mathcal{F}$-Stable Iterative Scheme in digital metric space. We also design some fractal images, which frame the compression of Fixed-Point Iterative Schemes and contractive mappings. Furthermore, we present a concrete example that exemplifies the motivation behind our investigations. Moreover, we provide an application of the proposed Fractal image and Sierpinski triangle that compress the works by storing images as a collection of digital contractions, which addresses the issue of storing images with less storage memory in this paper.

Keywords: fixed point; $\mathcal{F}$-stable iterative scheme; digital metric space; fractal image; symmetry; sierpinski triangle

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1. Introduction and Preliminaries

In recent years, Rosenfeld [1] introduced the concept of the digital image, laying the foundation for Boxer's [2] development of the topological notion of digital representation. Subsequently, Ege and Karaca [3–6] defined a digital metric space, offering a unified approach that has shed new light on the Banach contraction principle. This framework is particularly useful for measuring distances and similarities between points or patterns within a digital image. As a result, in this research, iterative schemes are employed to reduce data size or the dimension of picture files in digital contraction mappings and its applications. These schemes efficiently compress an image by iteratively refining approximations until a close match to the original image is achieved.

Now, we review some fundamental aspects of digital images, digital metric space, and $\mathcal{F}$-stable iterative scheme for the main continuation of our theoretical and geometrically analysis.

Definition 1 ([6]). *Let r, n be positive integers with $1 \leq r \leq n$ in such a way that*

$$\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n), \kappa = (\chi_1, \chi_2, \ldots, \chi_n) \in \mathbb{Z}^n.$$

where ζ and κ are ϱ_r -adjacent, if there is r indices τ such that $|\zeta_\tau - \chi_\tau| = 1$ and for every other indices l such that

$$|\zeta_\tau - \chi_\tau| \neq 1, \ \zeta_l = \chi_l.$$

There are some facts can be derived from Definition 1:

- Two points ζ and χ in Z are 2-adjacent, if $|\zeta - \chi| = 1$ (see Figure 1).

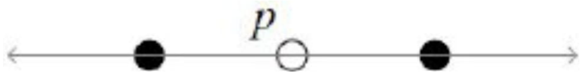

Figure 1. 2-adjacent.

- Two points ζ and χ in Z^2 are 8-adjacent, if they are distinct and differ by at most 1 in each coordinate.
- Two points ζ and χ in Z^2 are 4-adjacent, if they are 8-adjacent and differ in exactly one coordinate (see Figure 2).

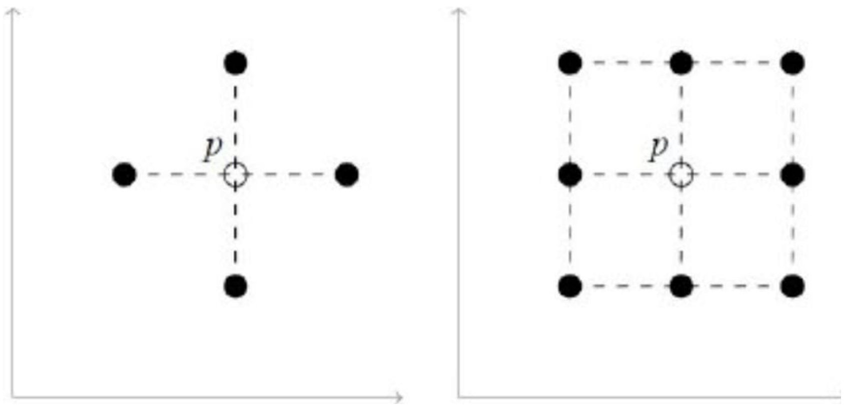

Figure 2. 4-adjacent and 8-adjacent.

- Two points ζ and χ in Z^3 are 26-adjacent, if they are distinct and differ by at most 1 in each coordinate.
- Two points ζ and χ in Z^3 are 18-adjacent, if they are 26-adjacent and differ at most two coordinates.
- Two points ζ and χ in Z^3 are 6-adjacent, if they are 18-adjacent and differ in exactly one coordinate (see Figure 3).

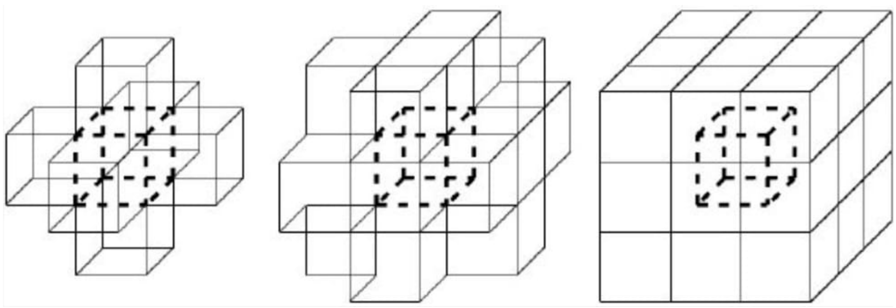

Figure 3. 6-adjacent, 18-adjacent, and 26-adjacent.

Definition 2 ([6]). *Digital image is a graph (D, ϱ), where ϱ is an adjacency relation, and D is a finite subset of Z^n for some positive integer n.*

Definition 3 ([6]). *A ϱ-neighbor of $\zeta \in Z^n$ is a point of Z^n that is ϱ-adjacent to ζ, where $\varrho \in \{2, 4, 6, 8, 18, 26\}$ and $n \in \{1, 2, 3\}$.*

Definition 4 ([6]). *The set $N_\varrho(\zeta) = \{\chi \mid \chi$ is ϱ-neighbor of $\zeta\}$ is called the ϱ-neighborhood of ζ.*

Definition 5 ([6]). *A digital interval is defined by $[i, j]_Z = \{\kappa \in Z \mid i \leq \kappa \leq j\}$, where $i, j \in Z$ and $i < j$.*

A digital image $D \subset Z^n$ is $\varrho-$connected [6] if and only if for every pair of different points $\imath, \jmath \in D$, there is a set $\{\imath_0, \imath_1, \ldots, \imath_r\}$ of points of a digital image X such that $\imath = \imath_0$, $\jmath = \imath_r$ and $\imath_i$ and $\imath_{i+1}$ are ϱ-neighbors where $i = 0, 1, \ldots, r-1$.

Definition 6 ([7])**.** *Let $(D, \varrho_0) \subset Z^{p_0}$, and $(M, \varrho_1) \subset Z^{p_1}$ be digital images and $\mu : D \to M$ be a function'*

- *If for every ϱ_0-connected of $X \subset D$, $\mu(X)$ is a ϱ_1-connected of $M \subset D$, then μ is said to be (ϱ_0, ϱ_1)-continuous [8].*
- *μ is (ϱ_0, ϱ_1)-continuous [8] if and only if for every ϱ_0-adjacent points $\{\imath_0, \imath_1\}$ of D, either $\mu(\imath_0) = \mu(\imath_1)$ or $\mu(\imath_0)$ and $\mu(\imath_1)$ are ϱ_1-adjacent in M.*
- *If μ is (ϱ_0, ϱ_1)-continuous, bijective, and μ^{-1} is (ϱ_1, ϱ_0)-continuous, then μ is called (ϱ_0, ϱ_1)-isomorphism [2], that is $D \cong_{(\varrho_0, \varrho_1)} M$.*

Definition 7 ([7])**.** *Let $D \subset Z^n$, g be the Euclidean metric on Z^n. Suppose metric space (D, g) and (D, ϱ) is a digital image with ϱ- adjaceny, then(D, g, ϱ) is called a digital metric space.*

Definition 8 ([6])**.** *A sequence $\{\imath_p\}$ in digital metric space (D, g, ϱ) converges to $q \in D$, if for all $\epsilon > 0$, there exists $\eta \in N$ such that for all $p > \eta$, then*

$$g(\imath_p, q) < \epsilon.$$

Definition 9 ([6])**.** *Let (D, g, ϱ) be a digital metric space and $\mu : (D, g, \varrho) \to (D, g, \varrho)$ be a self digital mapping. If there is $L \in (0, 1)$ such that for all $\imath, \jmath \in D$,*

$$g(\mu(\imath), \mu(\jmath)) \leq Lg(\imath, \jmath),$$

then μ is called a digital contraction map.

Proposition 1 ([6])**.** *Every digital contraction map is digitally continuous.*

Theorem 1 ([9])**.** *Digital metric space (D, g, ϱ) is a complete metric space.*

Hicks and Harder [10] presented a stability result for iterative processes in a complete metric space, which can be restated as follows:

Definition 10 ([10])**.** *Let (D, g) be a complete metric space, $\digamma : D \to D$ be a self mapping, and $\imath_{n+1} = f(\digamma, \imath_n)$ be a iterative procedure. Suppose that $Fix(\digamma) \neq \phi$, the set of fixed point, and sequence $\imath_n$ converges to $l \in Fix(\digamma)$.*

Let a sequence $\{\jmath_n\} \subset D$ and $\epsilon_n = g(\jmath_{n+1}, f(\digamma, \jmath_n))$. If $\lim_{n\to\infty} \epsilon_n = 0$ such that $\lim_{n\to\infty} \jmath_n = l$, then $\imath_{n+1} = f(\digamma, \imath_n)$ is said to be $\digamma$-stable. See more details in [10,11].

Osilike and Udomene [12] designed the following contractive condition, that is, for each $\imath, \jmath \in D$ there is constants $\eta \in [0, 1)$ and $\hat{L} \geq 0$ such that

$$g(\digamma\imath, \digamma\jmath) \leq \hat{L}(g(\imath, \digamma\imath)) + \eta g(\imath, \jmath). \tag{1}$$

After, Osilike [12] established various stability results that generalize and extend many of the findings of Rhoades [11].

If $\hat{L} = 2\xi$ and $\eta = \xi$, where $\xi = max\{\eta, \frac{\mu}{1-\mu}, \frac{\nu}{1-\nu}\}$ and $0 \leq \xi < 1$ then the contractive condition (1) restated to the Zamfirscu [13] contraction condition, that is,

$$g(\digamma\imath, \digamma\jmath) \leq \xi g(\imath, \jmath) + 2\xi(g(\imath, \digamma\imath)). \tag{2}$$

Furthermore, if $\hat{L} = 0$ then (1) yield as

$$g(\digamma\imath, \digamma\jmath) \leq \eta g(\imath, \jmath). \tag{3}$$

Berinde [14–17] established several generalizations of Banach's fixed point theorem. In one of the results of Berinde [17], designed that ideas in the following frame of extension, that is, for each $\imath, \jmath \in D$, there exist $\eta \in [0,1)$ and $\hat{L} \geq 0$ such that

$$g(F\imath, F\jmath) \leq \hat{L}(\alpha(\imath, F\imath)) + \eta A(\imath, \jmath). \tag{4}$$

where

$$A(\imath, \jmath) = max\{g(\imath, \jmath), g(\imath, F\imath), g(\jmath, F\jmath), \frac{1}{2}[g(\imath, F\jmath) + g(\jmath, F\imath)]\}$$

and

$$\alpha(\imath, \jmath) = min\{g(\imath, F\imath), g(\jmath, F\jmath), g(\imath, F\jmath), g(\jmath, F\imath)\}.$$

Pradip Debnath [18] introduced the following generalized Boyd–Wong-type contractive conditions and proved fixed-figure theorems in metric spaces. Suppose $\alpha : D \to D$ and upper semi-continuous functions $\Xi : [0,\infty) \to [0,\infty)$ with $0 \leq \Xi(s) < s$ for $s > 0$ such that $\Xi(0) = 0$. If there is $\beta_0 \in D$ such that

$$\chi(\alpha\beta, \beta) > 0 \Rightarrow \chi(\alpha\beta, \beta) \leq \Xi(\chi(\beta, \beta_0)) \tag{5}$$

for all $\beta \in D$, then α is called a Boyd and Wong type β_0-contraction.

Amid the years, which have been failed since this hypothsis, a number of iteration techniques have been established to approximate non-expasive mappings. Mann's iteration system [19] has been substantially used to approximate fixed point of non-expasive mappings. In this fashion of iterative system, the sequence $\{\imath_n\}$is procreated from an arbitrary point $\imath_0 \in D$; by the technique as follow:

$$\imath_{n+1} = (1-\eta_n)\imath_n + \eta_n F\imath_n, \eta_n \in [0,1], \tag{6}$$

where $n = 0,1,2,\ldots,\infty$.

Later on, Ishikawa [20] investigated the new iterative system which has been broadly used to approximate fixed point of non-expasive mappings. In this regard of iterative system the sequence $\{\imath_n\}$ is given iteratively from $\imath_0 \in D$ by

$$\begin{aligned} \imath_{n+1} &= (1-\eta_n)\imath_n + \eta_n F\jmath_n; \\ \jmath_n &= (1-\mu_n)\imath_n + \mu_n F\imath_n, \end{aligned} \tag{7}$$

where $\eta_n, \mu_n \in [0,1]$ and n = 0,1, 2, ...

Hereafter, Agrawal et al. [21] provided the iteration system and they declared that the process of converges rate of analysis same as that of the Picard iterative system and faster than the Mann iterative system for contractions, where the sequence $\{\imath_n\}$ is generated by

$$\begin{aligned} \imath_{n+1} &= (1-\eta_n)F\imath_n + \eta_n F\jmath_n; \\ \jmath_n &= (1-\mu_n)\imath_n + \mu_n F\imath_n, \end{aligned} \tag{8}$$

where η_n, $\mu_n \in [0,1]$ for all $n \geq 0$ values.

B. S. Thakur et al. [22] introduce a new iteration process to approximate fixed point of nonexpasive mappings, where for any fixed value $\imath_0 \in D$ and the sequence $\{\imath_n\}$ is construct by

$$\begin{aligned} \imath_{n+1} &= (1-\eta_n)F\zeta_n + \eta_n F\jmath_n; \\ \jmath_n &= (1-\mu_n)\zeta_n + \mu_n F\zeta_n; \\ \zeta_n &= (1-\nu_n)\imath_n + \nu_n F\imath_n, \end{aligned} \tag{9}$$

for all $n \geq 0$, where η_n, μ_n, and ν_n are in $[0,\ 1]$.

Lemma 1 ([14]). *Let $\{\varepsilon_n\}$ where $n = 0, 1, 2, \ldots, \infty$ such that $\lim_{n \longrightarrow \infty} \nu_n = 0$ and $0 \leq \xi < 1$, then for any sequence of positive numbers $\{b_n\}$ where $n = 0, 1, 2, \ldots,$ satisfying*

$$b_{n+1} \leq \xi(b_n) + \varepsilon_n, \qquad \forall\, n = 0, 1, 2, \ldots \tag{10}$$

then we have $\lim_{n \to \infty} b_n = 0$.

Definition 11. *Let $(D, \tilde{\mu}, \varrho)$ be a digital metric space and $\diamondsuit : D \times [0,1] \to D$ be a mapping such that $\diamondsuit(l, \eta + \mu) = \diamondsuit(l, \eta) + \diamondsuit(l, \mu)$ and $\diamondsuit(l, 1) = l$. We say that a digital metric space have linear digital structure if for all $l, m, h, t \in D$ and $\eta, \mu \in [0,1]$, if it satisfies,*

$$\tilde{\mu}(\diamondsuit(l, \eta) + \diamondsuit(m, \mu), \diamondsuit(h, \eta) + \diamondsuit(t, \mu)) \leq \eta\tilde{\mu}(l, h) + \mu\tilde{\mu}(m, t). \tag{11}$$

Definition 12 ([23]). *Let $(D, \tilde{\mu}, \varrho)$ be a digital metric space and $\digamma$ a self-map of D. Let $\imath_{n+1}$ be a iteration procedure defined as*

$$\imath_{n+1} = f_{\digamma, \alpha_n}(\imath_n) \tag{12}$$

where $\alpha_n \in [0,1]$ and f is a function involving the digital structure. Suppose that $Fix(\digamma)$, the fixed point set of $\digamma$, is nonempty and that $\imath_n$ converges to a point $l \in Fix(\digamma)$. Let $\{\jmath_n\} \subset D$ and define $\epsilon_n = \tilde{\mu}(\jmath_{n+1}, f_{\digamma, \alpha_n}(\jmath_n))$. If $\lim_{n \to \infty} \epsilon_n = 0$ implies that $\lim_{n \to \infty} \jmath_n = l$, then the iteration procedure $\imath_{n+1} = f_{\digamma, \alpha_n}(\imath_n)$ is said to be $\digamma$-stable.

Motivated by the work of Berinde [17], we will use the following contractive condition: For a mapping $\digamma : D \longrightarrow D$, there exists $\eta \in [0,1)$ and $L \geq 0$ such that for all $\imath, \jmath \in D$

$$\tilde{\mu}(\digamma\imath, \digamma\jmath) \leq \; \eta\tilde{\mu}(\imath, \jmath) + L(m(\imath, \jmath)) \tag{13}$$

where,

$$\begin{aligned} m(\imath, \jmath) =& min\{\tilde{\mu}(\imath, \digamma\imath), \tilde{\mu}(\jmath, \digamma\jmath), \tilde{\mu}(\imath, \digamma\jmath), \tilde{\mu}(\jmath, \digamma\imath), \\ & \frac{1}{2}[\tilde{\mu}(\imath, \digamma\jmath) + \tilde{\mu}(\jmath, \digamma\imath)], \frac{1}{2}[\tilde{\mu}(\imath, \digamma\imath) + \tilde{\mu}(\jmath, \digamma\jmath)]\}. \end{aligned} \tag{14}$$

The contractive condition (13) is more general than the contractive conditions (1), (3), (2) and (4).

In this study, inspired by the concept of Mann, Ishikawa, Agarwal, and Thakur iterative procedure in the class of Banch spaces, we develop a new iterative procedure and design $\digamma$-stability in the context of digital metric space. We also develop several fractal images to illustrate the compression of Fixed-Point Iterative Schemes and contractive mappings. Additionally, we present a specific example to demonstrate the motivation behind our investigations. Furthermore, we provide an application of the proposed Fractal image and Sierpinski triangle, which compress works by storing images as a collection of digital contractions, addressing the issue of storing images with less storage memory.

2. Main Results

First, in order to give our new extended iterative procedure in the class of digital metric space:

- Extended Mann (6) iteration process:

$$\imath_{n+1} = \diamondsuit(\imath_n, (1 - \eta_n)) + \diamondsuit(\digamma(\imath_n), \eta_n), \; \eta_n \in [0,1]. \tag{15}$$

- Extended Ishikawa (7) iteration process:

$$\begin{aligned} \imath_{n+1} &= \diamondsuit(\imath_n, (1 - \eta_n)) + \diamondsuit(\digamma(\jmath_n), \eta_n), \; \eta_n \in [0,1], \\ \jmath_n &= \diamondsuit(\imath_n, (1 - \mu_n)) + \diamondsuit(\digamma(\imath_n), \mu_n), \; \mu_n \in [0,1]. \end{aligned} \tag{16}$$

- Extended Agarwal (8) iteration process:

$$\begin{aligned} \imath_{n+1} &= \diamondsuit(F\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), \eta_n \in [0,1], \\ \jmath_n &= \diamondsuit(\imath_n, (1-\mu_n)) + \diamondsuit(F(\imath_n), \mu_n), \mu_n \in [0,1]. \end{aligned} \tag{17}$$

- Extended Thakur (9) iteration process:

$$\begin{aligned} \imath_{n+1} &= \diamondsuit(Fz_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), \eta_n \in [0,1], \\ \jmath_n &= \diamondsuit(z_n, (1-\mu_n)) + \diamondsuit(F(z_n), \mu_n), \mu_n \in [0,1]. \\ \zeta_n &= \diamondsuit(\imath_n, (1-\nu_n)) + \diamondsuit(F(\imath_n), \nu_n), \nu_n \in [0,1]. \end{aligned} \tag{18}$$

Theorem 2. *Let $(D, \tilde{\mu}, \varrho)$ be a digital metric space with linear digital structure $\diamondsuit$ and $F : D \longrightarrow D$ be a mapping that satisfies contractive condition (13). Suppose that F has a fixed point l. For arbitrary setting $\imath_0 \in D$, let the sequence $\{\imath_n\}_{n=0}^{\infty}$ is generated by the extended Mann iterative procedure (15), where $\eta_n \in [0,1]$ such that $0 < \eta \leq \eta_n$. Then, the extended Mann iteration is F-stable.*

Proof. Let $\{\imath_n\}$ be the sequence in $\mathcal{D}$, where $n = 0, 1, 2, \dots$ and define $\varepsilon_n = \tilde{\mu}(\imath_{n+1}, \diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(F(\imath_n), \eta_n))$. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Then, we establish that $\lim_{n\to\infty} \imath_n = l$. By using condition (13). Thus, we have

$$\begin{aligned} \tilde{\mu}(\imath_{n+1}, l) &\leq \tilde{\mu}(\imath_{n+1}, \diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(F(\imath_n), \eta_n)) \\ &\quad + \tilde{\mu}(\diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(F(\imath_n), \eta_n), l) \\ &= \tilde{\mu}(\diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(F(\imath_n), \eta_n), l) + \varepsilon_n \\ &= \tilde{\mu}(\diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(F(\imath_n), \eta_n), \diamondsuit(l, 1)) + \varepsilon_n \\ &= \tilde{\mu}(\diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(F(\imath_n), \eta_n), \diamondsuit(l, (1-\eta_n) + \eta_n) + \varepsilon_n \\ &= \tilde{\mu}(\diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(F(\imath_n), \eta_n), \diamondsuit(l, (1-\eta_n)) + \diamondsuit(l, \eta_n) + \varepsilon_n \\ &\leq (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n \tilde{\mu}(F(\imath_n), l) + \varepsilon_n \\ &= (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n \tilde{\mu}(F(\imath_n), F(l)) + \varepsilon_n. \end{aligned}$$

Using (13), we have

$$\tilde{\mu}(\imath_{n+1}, l) \leq (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n[\xi \tilde{\mu}(\imath_n, l) + \mu m(\imath_n, l)] + \varepsilon_n$$

Now, $m(\imath_n, l) = 0$, and using (14), so

$$\begin{aligned} \tilde{\mu}(\imath_{n+1}, l) &\leq (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n(\xi \tilde{\mu}(\imath_n, l)) + \varepsilon_n \\ &= [(1-\eta_n) + \eta_n \xi]\tilde{\mu}(\imath_n, l) + \varepsilon_n \\ &= [1 - (1-\xi)\eta_n]\tilde{\mu}(\imath_n, l) + \varepsilon_n. \end{aligned}$$

Therefore, we have

$$\tilde{\mu}(\imath_{n+1}, l) \leq [1 - (1-\xi)\eta]\tilde{\mu}(\imath_n, l) + \varepsilon_n. \tag{19}$$

Therefore, since $0 \leq 1-(1-\xi)\eta < 1$, applying Lemma 1 in (19), which yields $\lim_{n\longrightarrow\infty} \tilde{\mu}(\imath_n, l) = 0$, that is, $\lim_{n\longrightarrow\infty} \imath_n = l$. Conversely, $\lim_{n\longrightarrow\infty} \imath_n = l$. Then, we have to prove that $\lim_{n\longrightarrow\infty} \varepsilon_n = 0$. Next,

$$\begin{aligned}
\varepsilon_n &= \tilde{\mu}(\imath_{n+1}, \diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(\imath_n),\eta_n)) \\
&\leq \tilde{\mu}(\imath_{n+1}, l) + \tilde{\mu}(l, \diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(\imath_n),\eta_n)) \\
&= \tilde{\mu}(\imath_{n+1}, l) + \tilde{\mu}(\diamondsuit(l,1), \diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(\imath_n),\eta_n)) \\
&\leq \tilde{\mu}(\imath_{n+1}, l) + \tilde{\mu}(\diamondsuit(l,(1-\eta_n)+\eta_n), \diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(\imath_n),\eta_n)) \\
&= \tilde{\mu}(\imath_{n+1}, l) + \tilde{\mu}(\diamondsuit(l,(1-\eta_n)) + \diamondsuit(l,\eta_n), \diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(\imath_n),\eta_n)) \\
&\leq \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\tilde{\mu}(F(\imath_n), l) \\
&= \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\tilde{\mu}(F(\imath_n), F(l)) \\
&\tilde{\mu}(\imath_{n+1}, l) + [1-(1-\xi)\eta_n]\tilde{\mu}(\imath_n, l) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

□

Theorem 3. *Let $(D, \tilde{\mu}, \varrho)$ be a digital metric space with linear digital structure $\diamondsuit$ and $F : D \longrightarrow D$ be a mapping that satisfies contractive condition (13). Suppose that F has a fixed point l. For arbitrary setting $\imath_0 \in D$, let the sequence $\{\imath_n\}_{n=0}^{\infty}$ is generated by the extended Ishikawa iterative procedure (16), where $\eta_n, \mu_n \in [0,1]$ such that $0 < \eta \leq \eta_n$ and $0 < \mu \leq \mu_n$. Then, the extended Ishikawa iteration is F-stable.*

Proof. Let $\{\imath_n\}$ be the sequence in $\mathcal{D}$, where $n = 0,1,2,\ldots$ and define

$$\begin{aligned}
b_n &= \diamondsuit(\imath_n,(1-\mu_n)) + \diamondsuit(F(\imath_n),\mu_n) \\
\varepsilon_n &= \tilde{\mu}(\imath_{n+1}, \diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(b_n),\eta_n)).
\end{aligned}$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Then, we establish that $\lim_{n\to\infty} \imath_n = l$ by using condition (13). Thus, we have

$$\begin{aligned}
\tilde{\mu}(\imath_{n+1}, l) &\leq \tilde{\mu}(\imath_{n+1}, \diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(b_n),\eta_n)) \\
&\quad + \tilde{\mu}(\diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(b_n),\eta_n), l) \\
&= \tilde{\mu}(\diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(b_n),\eta_n), l) + \varepsilon_n \\
&= \tilde{\mu}(\diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(b_n),\eta_n), \diamondsuit(l,1)) + \varepsilon_n \\
&= \tilde{\mu}(\diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(b_n),\eta_n), \diamondsuit(l,(1-\eta_n)+\eta_n)) + \varepsilon_n \\
&= \tilde{\mu}(\diamondsuit(\imath_n,(1-\eta_n)) + \diamondsuit(F(b_n),\eta_n), \diamondsuit(l,(1-\eta_n)) + \diamondsuit(l,\eta_n)) + \varepsilon_n \\
&\leq (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\tilde{\mu}(F(b_n), l) + \varepsilon_n \\
&= (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\tilde{\mu}(F(b_n), F(l)) + \varepsilon_n.
\end{aligned}$$

Using (13), we have,

$$\tilde{\mu}(\imath_{n+1}, l) \leq (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n[\xi\tilde{\mu}(b_n, l) + \nu m(b_n, l)] + \varepsilon_n.$$

Now $m(b_n, l) = 0$ using (14), so

$$\tilde{\mu}(\imath_{n+1}, l) \leq (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\xi\tilde{\mu}(b_n, l) + \varepsilon_n. \tag{20}$$

Now

$$\begin{aligned}
\tilde{\mu}(b_n, l) &= \tilde{\mu}(\diamondsuit(\imath_n,(1-\mu_n)) + \diamondsuit(F(\imath_n,\mu)), \diamondsuit(l,1)) \\
&= \tilde{\mu}(\diamondsuit(\imath_n,(1-\mu_n)) + \diamondsuit(F(\imath_n,\mu), \diamondsuit(l,(1-\mu_n)) + \diamondsuit(F(l,\mu)) \\
&\leq (1-\mu_n)\tilde{\mu}(\imath_n, l) + \mu_n\tilde{\mu}(F\imath_n, l) \\
&\leq [1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l).
\end{aligned}$$

Therefore, we have
$$\tilde{\mu}(b_n, l) \leq [1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l). \tag{21}$$
Using (21) in (20), we have
$$\begin{aligned}\tilde{\mu}(\imath_{n+1}, l) &\leq (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\xi([1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l)) + \varepsilon_n \\ &= [1-(1-\xi)\eta_n - (1-\xi)\xi\eta_n\mu_n]\tilde{\mu}(\imath_n, l) + \varepsilon_n \\ &\leq [1-(1-\xi)\eta - (1-\xi)\xi\eta\mu]\tilde{\mu}(\imath_n, l) + \varepsilon_n.\end{aligned}$$
Therefore, since $0 \leq 1-(1-\xi)\eta - (1-\xi)\xi\eta\mu < 1$, applying Lemma 1 in above yields $\lim_{n\longrightarrow\infty} \tilde{\mu}(\imath_n, l) = 0$, that is, $\lim_{n\longrightarrow\infty} \imath_n = l$.

Conversely, $\lim_{n\longrightarrow\infty} \imath_n = l$. Then, we have to prove that $\lim_{n\longrightarrow\infty} \varepsilon_n = 0$. We have
$$\begin{aligned}\varepsilon_n &= \tilde{\mu}(\imath_{n+1}, \diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(\digamma b_n, \eta_n)) \\ &= \tilde{\mu}(\imath_{n+1}, l) + \tilde{\mu}(l, \diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(\digamma b_n, \eta_n)) \\ &= \tilde{\mu}(\imath_{n+1}, l) + \tilde{\mu}(\diamondsuit(l, (1-\eta_n)+\eta_n), \diamondsuit(\imath_n, (1-\eta_n)) + \diamondsuit(\digamma(b_n, \eta_n)) \\ &= \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\tilde{\mu}(\digamma(b_n), \digamma(l)) \\ &\leq \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n[\xi\tilde{\mu}(b_n, l) + \nu m(b_n, l)].\end{aligned}$$
Now, $m(b_n, l) = 0$ using (14), so
$$\varepsilon_n \leq \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\xi\tilde{\mu}(\jmath_n, l).$$
Using (21)
$$\begin{aligned}\varepsilon_n &\leq \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\tilde{\mu}(\imath_n, l) + \eta_n\xi[1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l) \\ &\leq \tilde{\mu}(\imath_{n+1}, l) + [(1-\eta_n) + \eta_n\xi(1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l) \\ &\leq \tilde{\mu}(\imath_{n+1}, l) + [1-\eta_n(1-\xi) + \eta_n\mu_n\xi(1-\xi)]\tilde{\mu}(\imath_n, l) \\ &\leq \tilde{\mu}(\imath_{n+1}, l) + [1-\eta(1-\xi) + \eta\mu\xi(1-\xi)]\tilde{\mu}(\imath_n, l).\end{aligned}$$
Now, since $0 \leq 1-(1-\xi)\eta - (1-\xi)\xi\eta\mu < 1$. Using Lemma 1, we have
$$\varepsilon_n \leq \tilde{\mu}(\imath_{n+1}, l) + [1-\eta(1-\xi) + \eta\mu\xi((1-\xi)]\tilde{\mu}(\imath_n, l) \longrightarrow 0 \; as \; n \longrightarrow \infty.$$
□

Theorem 4. *Let $(D, \tilde{\mu}, \varrho)$ be a digital metric space with linear digital structure $\diamondsuit$ and $\digamma : D \longrightarrow D$ be a mapping that satisfies contractive condition (13). Suppose that $\digamma$ has a fixed point l. For arbitrary setting $\imath_0 \in D$, let the sequence $\{\imath_n\}_{n=0}^{\infty}$ is generated by the extended Agarwal iterative procedure (17), where $\eta_n, \mu_n \in [0,1]$ such that $0 < \eta \leq \eta_n$ and $0 < \mu \leq \mu_n$. Then, the extended Agarwal iteration is $\digamma$-stable.*

Proof. Let $\{\imath_n\}$ be the sequence in $\mathcal{D}$, where $n = 0, 1, 2, \ldots$ and define
$$\begin{aligned}\jmath_n &= \diamondsuit(\imath_n, (1-\mu_n)) + \diamondsuit(\digamma(\imath_n), \mu_n) \\ \varepsilon_n &= \tilde{\mu}(\imath_{n+1}, \diamondsuit(\digamma\imath_n, (1-\eta_n)) + \diamondsuit(\digamma(\jmath_n), \eta_n)).\end{aligned}$$
Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$.

Then, we establish that $\lim_{n\to\infty} \imath_n = l$ by using condition (13). Thus, we have

$$\begin{aligned}
\tilde{\mu}(\imath_{n+1}, l) &\leq \tilde{\mu}(\imath_{n+1}, \diamondsuit(F\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n)) \\
&\quad + \tilde{\mu}(\diamondsuit(F\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), l) \\
&= \tilde{\mu}(\diamondsuit(F\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), l) + \varepsilon_n \\
&= \tilde{\mu}(\diamondsuit(F\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), \diamondsuit(l, 1)) + \varepsilon_n \\
&= \tilde{\mu}(\diamondsuit(F\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), \diamondsuit(l, (1-\eta_n) + \eta_n)) + \varepsilon_n \\
&= \tilde{\mu}(\diamondsuit(F\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), \diamondsuit(l, (1-\eta_n)) + \diamondsuit(l, \eta_n)) + \varepsilon_n \\
&= \tilde{\mu}(\diamondsuit(F\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), \diamondsuit(F\,l, (1-\eta_n)) + \diamondsuit(Fl, \eta_n)) + \varepsilon_n \\
&\leq (1-\eta_n)\tilde{\mu}(F\imath_n, F\,l) + \eta_n\tilde{\mu}(F(\jmath_n), F\,l) + \varepsilon_n \\
&= (1-\eta_n)\tilde{\mu}(F\,\imath_n, F\,l) + \eta_n\tilde{\mu}(F(\jmath_n), F(l)) + \varepsilon_n.
\end{aligned}$$

Using (13), we have

$$\tilde{\mu}(\imath_{n+1}, l) \leq (1-\eta_n)[\xi\tilde{\mu}(\imath_n, l) + \nu m(\imath_n, l)] + \eta_n[\xi\tilde{\mu}(\jmath_n, l) + \nu m(\jmath_n, l)] + \varepsilon_n.$$

Now, $m(\imath_n, l) = 0$ and $m(\jmath_n, l) = 0$. By applying (14)

$$\tilde{\mu}(\imath_{n+1}, l) \leq (1-\eta_n)\xi\tilde{\mu}(\imath_n, l) + \eta_n\xi\tilde{\mu}(\jmath_n, l) + \varepsilon_n. \tag{22}$$

Next,

$$\begin{aligned}
\tilde{\mu}(\jmath_n, l) &= \tilde{\mu}(\diamondsuit(\imath_n, (1-\mu_n)) + \diamondsuit(F(\imath_n, \mu)), \diamondsuit(l, 1)) \\
&= \tilde{\mu}(\diamondsuit(\imath_n, (1-\mu_n)) + \diamondsuit(F(\imath_n, \mu), \diamondsuit(l, (1-\mu_n)) + \diamondsuit(F(l, \mu)) \\
&\leq (1-\mu_n)\tilde{\mu}(\imath_n, l) + \mu_n\tilde{\mu}(F\,\imath_n, l) \\
&\leq [1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l).
\end{aligned}$$

Therefore, we have

$$\tilde{\mu}(\jmath_n, l) \leq [1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l). \tag{23}$$

Using (22) in (23), we have

$$\begin{aligned}
\tilde{\mu}(\imath_{n+1}, l) &\leq (1-\eta_n)\xi\tilde{\mu}(\imath_n, l) + \eta_n\xi([1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l)) + \varepsilon_n \\
&= \xi[1-(1-\xi)\eta_n\mu_n]\tilde{\mu}(\imath_n, l) + \varepsilon_n \\
&\leq \xi[1-(1-\xi)\eta\mu]\tilde{\mu}(\imath_n, l) + \varepsilon_n.
\end{aligned}$$

Since $0 \leq 1-(1-\xi)\eta\mu < 1$, and applying Lemma 1 which yields $\lim_{n\longrightarrow\infty} \tilde{\mu}(\imath_n, l) = 0$, that is, $\lim_{n\longrightarrow\infty} \imath_n = l$.

Conversely, $\lim_{n\longrightarrow\infty} \imath_n = l$. Then, we have to prove that $\lim_{n\longrightarrow\infty} \varepsilon_n = 0$. We have,

$$\begin{aligned}
\varepsilon_n &= \tilde{\mu}(\imath_{n+1}, \diamondsuit(F\,\imath_n, (1-\eta_n)) + \diamondsuit(F\,\jmath_n, \eta_n)) \\
&= \tilde{\mu}(\imath_{n+1}, l) + \tilde{\mu}(l, \diamondsuit(F\,\imath_n, (1-\eta_n)) + \diamondsuit(F\,\jmath_n, \eta_n)) \\
&= \tilde{\mu}(\imath_{n+1}, l) + \tilde{\mu}(\diamondsuit(l, (1-\eta_n) + \eta_n), \diamondsuit(F\,\imath_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n, \eta_n)) \\
&= \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\tilde{\mu}(F\,\imath_n, F\,l) + \eta_n\tilde{\mu}(F\,\jmath_n, F(l)) \\
&\leq \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)[\xi\tilde{\mu}(\imath_n, l) + \nu m(\imath_n, l)] + \eta_n[\xi\tilde{\mu}(\jmath_n, l) + \nu m(\jmath_n, l)]
\end{aligned}$$

Now, $m(\imath_n, l) = 0$ and $m(\jmath_n, l) = 0$ using (14), so

$$\varepsilon_n \leq \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\xi\tilde{\mu}(\imath_n, l) + \eta_n\xi\tilde{\mu}(\jmath_n, l).$$

Using (22)

$$\begin{aligned}\varepsilon_n &\leq \tilde{\mu}(\imath_{n+1}, l) + (1-\eta_n)\xi\tilde{\mu}(\imath_n, l) + \eta_n\xi[1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l)\\ &\leq \tilde{\mu}(\imath_{n+1}, l) + \xi[(1-\eta_n) + \eta_n(1-(1-\xi)\mu_n]\tilde{\mu}(\imath_n, l)\\ &\leq \tilde{\mu}(\imath_{n+1}, l) + \xi[(1-\eta_n\mu_n(1-\xi)]\tilde{\mu}(\imath_n, l)\\ &\leq \tilde{\mu}(\imath_{n+1}, l) + \xi[(1-\eta\mu(1-\xi)]\tilde{\mu}(\imath_n, l).\end{aligned}$$

Now, since $0 \leq 1-(1-\xi)\eta\mu < 1$, using Lemma 1:

$$\varepsilon_n \leq \tilde{\mu}(\imath_{n+1}, l) + [1-\eta\mu(1-\xi)]\tilde{\mu}(\imath_n, l) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad \square$$

Theorem 5. *Let $(D, \tilde{\mu}, \varrho)$ be a digital metric space with linear digital structure $\diamondsuit$ and $F : D \longrightarrow D$ be a mapping that satisfies contractive condition (13). Suppose that F has a fixed point l. For arbitrary setting $\imath_0 \in D$, let the sequence $\{\imath_n\}_{n=0}^{\infty}$ is generated by the extended Thakur iterative procedure (18), where $\eta_n, \mu_n, \nu_n \in [0,1]$ such that $0 < \eta \leq \eta_n$, $0 < \mu \leq \mu_n$ and $0 < \nu \leq \nu_n$. Then, the extended Thakur iteration is F-stable.*

Proof. Let $\{\imath_n\}$ be the sequence in $\mathcal{D}$, where $n = 0, 1, 2, \ldots$ and define

$$\begin{aligned}\zeta_n &= \diamondsuit(\imath_n, (1-\nu_n)) + \diamondsuit(F\imath_n, \nu_n),\\ \jmath_n &= \diamondsuit(\zeta_n, (1-\mu_n)) + \diamondsuit(F\zeta_n, \mu_n),\\ \varepsilon_n &= \tilde{\mu}(\imath_{n+1}, \diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n)).\end{aligned}$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Then, we establish that $\lim_{n\to\infty} \imath_n = l$ by using condition (13). Thus, we have

$$\begin{aligned}\tilde{\mu}(\imath_{n+1}, l) &\leq \tilde{\mu}(\imath_{n+1}, \diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n))\\ &\quad + \tilde{\mu}(\diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n), \eta_n), l)\\ &= \tilde{\mu}(\diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n), l) + \varepsilon_n\\ &= \tilde{\mu}(\diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n), \diamondsuit(l, 1)) + \varepsilon_n\\ &= \tilde{\mu}(\diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n), \diamondsuit(l, (1-\eta_n) + \eta_n)) + \varepsilon_n\\ &= \tilde{\mu}(\diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n), \diamondsuit(l, (1-\eta_n)) + \diamondsuit(l, \eta_n)) + \varepsilon_n\\ &= \tilde{\mu}(\diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n), \diamondsuit(F\,l, (1-\eta_n)) + \diamondsuit(Fl, \eta_n)) + \varepsilon_n\\ &\leq (1-\eta_n)\tilde{\mu}(F\zeta_n, F\,l) + \eta_n\tilde{\mu}(F\jmath_n, F\,l) + \varepsilon_n\\ &= (1-\eta_n)\tilde{\mu}(F\,\zeta_n, F\,l) + \eta_n\tilde{\mu}(F\,\jmath_n, F(l)) + \varepsilon_n.\end{aligned}$$

Using (13), we have,

$$\tilde{\mu}(\imath_{n+1}, l) \leq (1-\eta_n)[\xi\tilde{\mu}(\zeta_n, l) + \eta m(\zeta_n, l)] + \eta_n[\xi\tilde{\mu}(\jmath_n, l) + \eta m(\jmath_n, l)] + \varepsilon_n.$$

Now, $m(\imath_n, l) = 0$ and $m(\jmath_n, l) = 0$ using (2.4), so

$$\tilde{\mu}(\imath_{n+1}, l) \leq (1-\eta_n)\xi\tilde{\mu}(\zeta_n, l) + \eta_n\xi\tilde{\mu}(\jmath_n, l) + \varepsilon_n. \tag{24}$$

Now,

$$\begin{aligned}\tilde{\mu}(\jmath_n, l) &= \tilde{\mu}(\diamondsuit(\zeta_n, (1-\mu_n)) + \diamondsuit(F(\zeta_n, \mu)), \diamondsuit(l, 1))\\ &= \tilde{\mu}(\diamondsuit(\zeta_n, (1-\mu_n)) + \diamondsuit(F(\zeta_n, \mu), \diamondsuit(l, (1-\mu_n)) + \diamondsuit(F(l, \mu))\\ &\leq (1-\mu_n)\tilde{\mu}(\zeta_n, l) + \mu_n\tilde{\mu}(F(\zeta_n, l)\\ &\leq [1-(1-\xi)\mu_n]\tilde{\mu}(\zeta_n, l).\end{aligned}$$

Therefore, we have

$$\tilde{\mu}(\jmath_n, l) \leq [1-(1-\xi)\mu_n]\tilde{\mu}(\zeta_n, l). \tag{25}$$

Now

$$\begin{aligned}\tilde{\mu}(\zeta_n, l) &= \tilde{\mu}(\diamondsuit(\iota_n, (1-\nu_n)) + \diamondsuit(F(\zeta_n, \nu)), \diamondsuit(l, 1)).\\ &= \tilde{\mu}(\diamondsuit(\iota_n, (1-\nu n)) + \diamondsuit(F(\iota_n, \nu), \diamondsuit(l, (1-\nu_n)) + \diamondsuit(F(l, \nu))\\ &\leq (1-\nu_n)\tilde{\mu}(\iota_n, l) + \nu_n \tilde{\mu}(F(\iota_n, l)\\ &\leq [1-(1-\xi)\nu_n]\tilde{\mu}(\iota_n, l).\end{aligned}$$

Therefore, we have

$$\tilde{\mu}(\zeta_n, l) \leq [1-(1-\xi)\nu_n]\tilde{\mu}(\iota_n, l) \tag{26}$$

Using (24) and (25), we have

$$\begin{aligned}\tilde{\mu}(\iota_{n+1}, l) &\leq (1-\eta_n)\xi\tilde{\mu}(\zeta_n, l) + \eta_n\xi(1-(1-\xi)\mu_n)\tilde{\mu}(\zeta_n, l) + \varepsilon_n\\ &= \xi[(1-\eta_n) + \eta_n([1-(1-\xi)\mu_n)]\tilde{\mu}(\zeta_n, l) + \varepsilon_n\\ &= \xi[1-(1-\xi)\eta_n\mu_n]\tilde{\mu}(\zeta_n, l) + \varepsilon_n.\end{aligned}$$

Now, using (26),

$$\begin{aligned}\tilde{\mu}(\iota_{n+1}, l) &\leq \xi(1-(1-\xi)\eta_n\mu_n)(1-(1-\xi)\nu_n)\tilde{\mu}(\iota_n, l) + \varepsilon_n\\ &\leq \xi(1-(1-\xi)\eta\mu)(1-(1-\xi)\nu)\tilde{\mu}(\iota_n, l) + \varepsilon_n.\end{aligned}$$

Therefore, since $0 \leq (1-(1-\xi)\eta\mu)(1-(1-\xi)\nu) < 1$, by applying Lemma 1 yields $\lim_{n\longrightarrow\infty} \tilde{\mu}(\iota_n, l) = 0$, that is, $\lim_{n\longrightarrow\infty} \iota_n = l$.

Conversely, let $\lim_{n\longrightarrow\infty} \iota_n = l$. We have to prove that $\lim_{n\longrightarrow\infty} \varepsilon_n = 0$. We have

$$\begin{aligned}\varepsilon_n &= \tilde{\mu}(\iota_{n+1}, \diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n))\\ &= \tilde{\mu}(\iota_{n+1}, l) + \tilde{\mu}(l, \diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F\jmath_n, \eta_n))\\ &= \tilde{\mu}(\iota_{n+1}, l) + \tilde{\mu}(\diamondsuit(l, (1-\eta_n) + \eta_n), \diamondsuit(F\zeta_n, (1-\eta_n)) + \diamondsuit(F(\jmath_n, \eta_n))\\ &= \tilde{\mu}(\iota_{n+1}, l) + (1-\eta_n)\tilde{\mu}(F\zeta_n, Fl) + \eta_n\tilde{\mu}(F\jmath_n, Fl)\\ &\leq \tilde{\mu}(\iota_{n+1}, l) + (1-\eta_n)[\xi\tilde{\mu}(\zeta_n, l) + \nu m(\zeta_n, l)] + \eta_n[\xi\tilde{\mu}(\jmath_n, l) + \nu m(\jmath_n, l)]\end{aligned}$$

Now, $m(\zeta_n, l) = 0$ and $m(\jmath_n, l) = 0$ using (14), so

$$\varepsilon_n \leq \tilde{\mu}(\iota_{n+1}, l) + (1-\eta_n)\xi\tilde{\mu}(\zeta_n, l) + \eta_n\xi\tilde{\mu}(\jmath_n, l).$$

Using (25) and (26),

$$\begin{aligned}\varepsilon_n &\leq \tilde{\mu}(\iota_{n+1}, l) + (1-\eta_n)\xi\tilde{\mu}(\zeta_n, l) + \eta_n\xi[1-(1-\xi)\mu_n]\tilde{\mu}(\zeta_n, l)\\ &\leq \tilde{\mu}(\iota_{n+1}, l) + \xi[(1-\eta_n) + \eta_n(1-(1-\xi)\mu_n]\tilde{\mu}(\zeta_n, l)\\ &\leq \tilde{\mu}(\iota_{n+1}, l) + \xi[(1-(1-\xi)\eta_n\mu_n][1-(1-\xi)\nu_n]\tilde{\mu}(\iota_n, l)\\ &\leq \tilde{\mu}(\iota_{n+1}, l) + \xi[(1-(1-\xi)\eta\mu][1-(1-\xi)\nu]\tilde{\mu}(\iota_n, l)\end{aligned}$$

Now, since $0 \leq (1-(1-\xi)\eta\mu)(1-(1-\xi)\nu) < 1$, by using Lemma 1 we have $\varepsilon_n \leq \tilde{\mu}(\iota_{n+1}, l) + [(1-(1-\xi)\eta\mu)(1-(1-\xi)\nu)]\tilde{\mu}(\iota_n, l) \longrightarrow 0$ as $n \longrightarrow \infty$. □

Here, we have designed a non trivial example to check the stability of digital contraction mapping and compare the rate of convergence with the different iterative schemes.

Example 1. *Let $\mathcal{D} = [0, \infty]_Z$ and $(\mathcal{D}, \tilde{\mu}, \varrho)$ be the digital metric spaces endowed with the metric $\tilde{\mu}(\iota, \jmath) = |\iota - \jmath|$ and digital structure $\diamondsuit : \mathcal{D} \times [0,1]_Z \to \mathcal{D}$ defined as $\diamondsuit(l, \alpha) = \alpha l$. For $F : (\mathcal{D}, \tilde{\mu}, \varrho) \to (\mathcal{D}, \tilde{\mu}, \varrho)$, define*

$$F\iota = \frac{\iota}{2} + 3,$$

and $\eta_n = \mu_n = \nu_n = \frac{5}{6}$, $n = 1, 2, 3, \ldots$.

From Table 1, it is evident that every iterative algorithm was F*-stable and converges to* $l^\star = 6$. *Table 2 and Figure 4 shows the rate of convergence of Picard-S, K. Ullah, Agarwal and Noor's iterative schemes.*

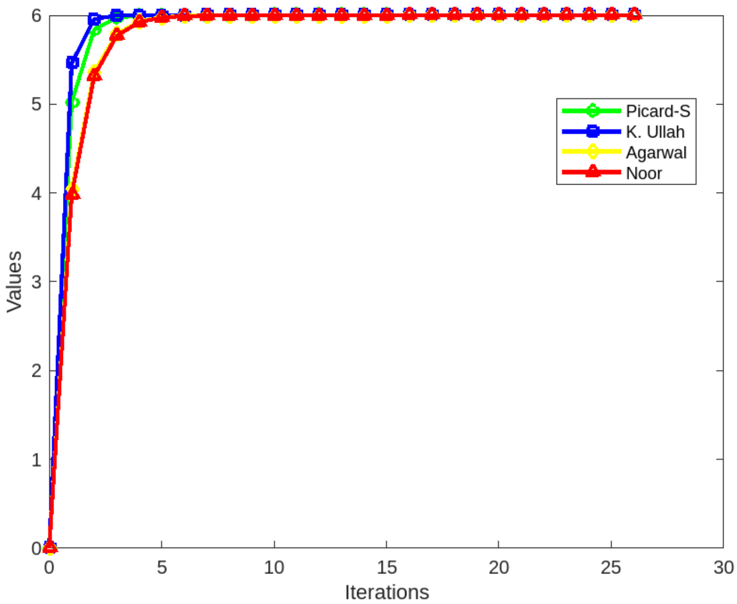

Figure 4. Graphical presentation of Table 1.

Table 1. Numerical values obtained for different initial approximations.

Iterations	Picard-S	K. Ullah	Agarwal	Noor
0	0	0	0	0
1	5.0208333333	5.45958333333	4.04166666667	3.97569444444
2	5.8402054398	5.95697913773	5.36082175926	5.31703116962
3	5.9739224155	5.99968577145	5.79137932420	5.76957706707
4	5.9957442831	5.99981670001	5.93190852943	5.92225892946
5	5.9993054906	5.99997326855	5.97777570058	5.97377138650
6	5.9998866599	5.99999772599	5.99274623560	5.99115087866
7	5.9999815035	5.99999980655	5.99763245190	5.99701444575
8	5.9999969815	5.99999998354	5.99922725861	5.99899272099
9	5.9999995074	5.99999999986	5.99974778579	5.99966015992
10	5.9999999196	5.99999999988	5.99991768009	5.99988534331
11	5.9999999869	5.99999999999	5.99997313169	5.99996131664
12	5.9999999979	6.00000000000	5.99999123048	5.99998694884
13	5.9999999997	6.00000000000	5.99999713773	5.99999559674
14	5.9999999999	6.00000000000	5.99999906579	5.99999851441
15	6.0000000000	6.00000000000	5.99999969508	5.99999949879
16	6.00000000000	6.00000000000	5.99999990048	5.99999983090
17	6.00000000000	6.00000000000	5.99999996752	5.99999994295
18	6.00000000000	6.00000000000	5.99999998940	5.99999998075
19	6.00000000000	6.00000000000	5.99999999654	5.99999999351
20	6.00000000000	6.00000000000	5.99999999887	5.99999999781
21	6.00000000000	6.00000000000	5.99999999963	5.99999999926
22	6.00000000000	6.00000000000	5.99999999988	5.99999999975
23	6.00000000000	6.00000000000	5.99999999996	5.99999999992
24	6.00000000000	6.00000000000	5.99999999999	5.99999999997
25	6.00000000000	6.00000000000	6.00000000000	5.99999999999
26	6.00000000000	6.00000000000	6.00000000000	6.00000000000

Table 2. Comparisions of iterative steps.

Algorithm	Iterations
Mann	48
Ishikawa	32
Noor	26
Agarwal	28
Picard-S	15
K.Ullah	11

3. Application

Recurring patterns up to scale similarity are seen in many natural phenomena at all scales. This gives rise to a novel concept of symmetry. This is also known mathematically as a "fractal", and it occurs when self similarity patterns appear similar at different small scales. For example Mandelbrot set (Figure 5). When a precise and intricate pattern is observed to repeat itself, fractals are employed.

The fractal tree (Figure 6) is another examples of a fractal.

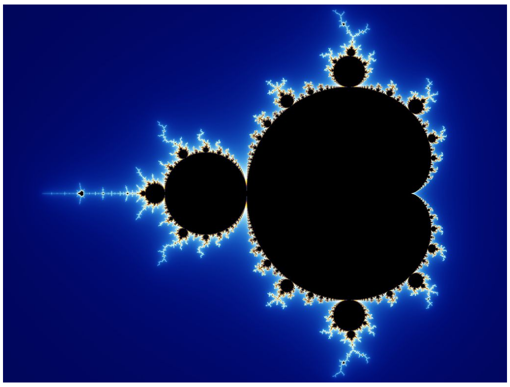

Figure 5. Madelbrot set.

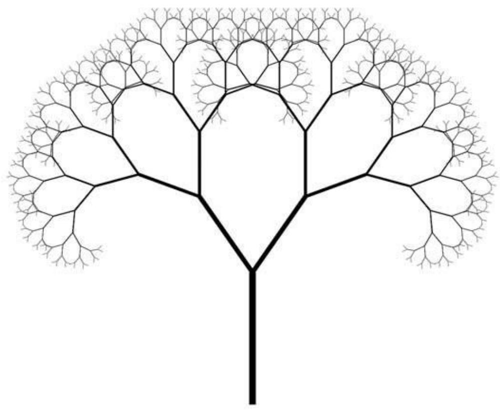

Figure 6. Fractal tree.

Fractal compression uses an image's self-similarity to its advantage in order to compress data. In this technique, the image is divided into smaller blocks known as range blocks, and comparable patterns inside the image known as domain blocks are found. Fractal compression can achieve high compression ratios by identifying these matches and encoding the modifications required to recreate them.

Now, we give an example to illustrate how fractal compression techniques' iterative nature helps in measuring distances and similarities between points or patterns within a digital image and is efficient in the compression of an image by repeatedly improving approximations until a near match to the actual image is achieved.

Example 2. *Let $\mathcal{D} = [0,2]_{\mathbb{Z}}$ be a digital interval with 2-adjacency. LetX_0 be a digital image (see Figure 7,*

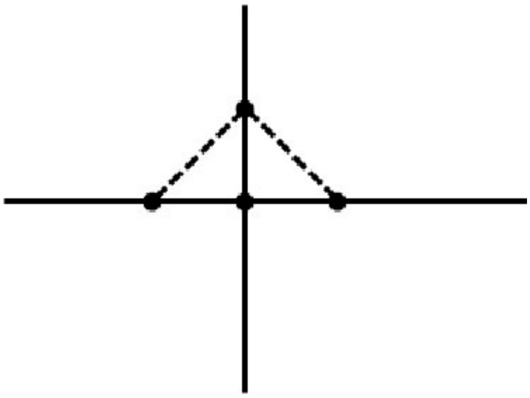

Figure 7. X_0.

Using the Ishikawa (16) iteration scheme and $\eta_n = 0$ and $\mu_n = 1,\ n = 1,2,3,\ldots$, duplicating X_0 and attach one copy to the vertex on the lower left and one to the lower right makes a new digital image X_1 as (see Figure 8),

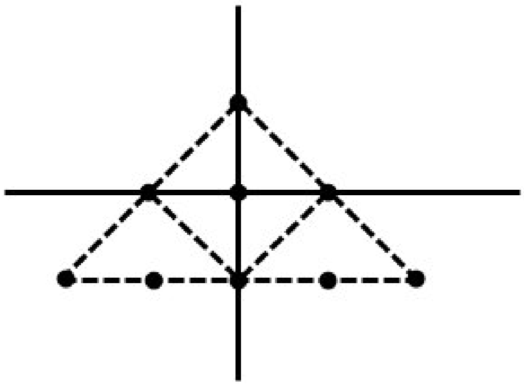

Figure 8. X_1.

Applying the second iteration on X_1, we have again a new digital image X_2, which is similar to X_1.

X_2 (see Figure 9) is therefore the fixed point in this process. We would want to present the mathematical version of the higher process. Give $\digamma$ the function that converts X_i to $\digamma(X_i)$. Thus, we can see that X_2 is a fixed point of this function or that $\digamma(X_2) = X_2$. An infinite sequence is produced if the procedure is repeated on X_n sets. There is a convergence of X_n to X_2. It is impossible to differentiate between X_5 and X_2. Consequently, the computer software uses X_5 rather than X_2 for improved resolution. Simultaneously, the application may quickly determine certain digital image properties by using X_2 instead of X_5.

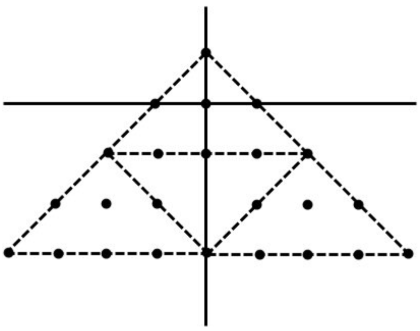

Figure 9. X_2.

Example 3. *(Sierpinski triangle) We took a triangle and cut off its middle, then we repeated it again to generate the Sierpinski triangle. However, an iterative function system can also be used to represent the Sierpinski triangle. Start with a solid triangle with digital image I_0 (see Figure 10).*

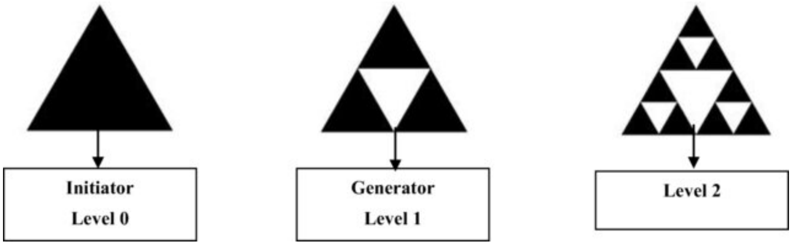

Figure 10. Generators of Sierpinski triangle.

Then, three functions $\{\xi_1, \xi_2, \xi_3\}$ are generators, representing a contractive mapping are used to form I_1. Every mapping reduces the triangle's size by half, placing the reduced triangles in each of I_0's corners.

The corresponding Iterative process is given by $\{R^2 : \xi_1, \xi_2, \xi_3\}$, where the contractive transformations $\xi_1, \xi_2,$ and ξ_3 are given by

$$
\begin{aligned}
\xi_1(\imath, \jmath) &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \imath \\ \jmath \end{bmatrix}, \\
\xi_2(\imath, \jmath) &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \imath \\ \jmath \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \\
xi_3(\imath, \jmath) &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \imath \\ \jmath \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \end{bmatrix}.
\end{aligned}
\tag{27}
$$

The result of this Iterative process is the Sierpinski triangle (see Figure 11) and is given by $I = \lim_{n\to\infty} \xi^n(I_0)$.

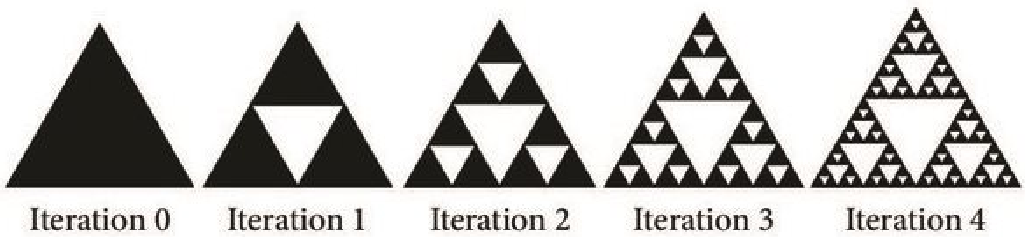

Figure 11. Iterations.

4. Conclusions

In conclusion, this paper has undertaken a thorough examination of the advancements achieved in comprehending Iterative Fixed-Point Schemes, grounded in the concept of digital contraction mappings. Additionally, we have introduced the concept of Iterative Fixed-Point Schemes within digital metric spaces. This study extends the Iteration process of Mann (15), Ishikawa (16), Agarwal (17), and Thakur (18), incorporating the $\mathcal{F}$-Stable Iterative Scheme in the context of digital metric spaces. The design and exploration of fractal images serve to illustrate the compression of Fixed-Point Iterative Schemes and contractive mappings. Furthermore, a concrete example has been presented to elucidate the underlying motivation for our investigations.

Moreover, our paper has demonstrated the practical application of the proposed Fractal image and Sierpinski triangle in compressing works, specifically addressing the challenge of storing images efficiently by representing them as a collection of digital contractions. This approach offers a solution to the problem of conserving storage memory while retaining the essential features of the images discussed in this study.

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