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A Symmetric Non-Stationary Loop Subdivision with Applications in Initial Point Interpolation

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Abstract: Loop subdivision is a significant surface scheme with wide applications in fields like computer graphics and wavelet. As a type of stationary scheme, Loop subdivision cannot adjust the limit surface directly. In this paper, we present a new way to solve this problem by proposing a symmetric non-stationary Loop subdivision based on a suitable iteration. This new scheme can be used to adjust the limit surfaces freely and thus can generate surfaces with different shapes. For this new scheme, we show that it is C^2 convergent in the regular part of mesh and is at least tangent plane continuous at the limit positions of the extraordinary points. Additionally, we present a non-uniform generalization of this new symmetric non-stationary subdivision so as to locally control the shape of the limit surfaces. More interestingly, we present the limit positions of the initial points, both for the symmetric non-stationary Loop subdivision and its non-uniform generalization. Such limit positions can be used to interpolate the initial points with different valences, generalizing the existing result. Several numerical examples are given to illustrate the performance of the new schemes.

Keywords: symmetric non-stationary Loop subdivision; shape control; smoothness; limit position; interpolation; local control



Citation: Zhang, B.; Zhang, Y.; Zheng, H. A Symmetric Non-Stationary Loop Subdivision with Applications in Initial Point Interpolation. *Symmetry* **2024**, *16*, 379. <https://doi.org/10.3390/sym16030379>

Academic Editor: Alessandro Sarracino

Received: 15 February 2024

Revised: 12 March 2024

Accepted: 19 March 2024

Published: 21 March 2024



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1. Introduction

As an efficient tool to generate smooth surfaces, subdivision schemes have been widely used in fields like computer graphics, animation, and games. In general, subdivision schemes can be divided into stationary and non-stationary ones, depending on whether the subdivision rules are relevant with the recursion level or not. Compared with the stationary ones, like the schemes in [1], the non-stationary schemes have the advantage of being able to generate richer function spaces and more flexible surfaces with different shapes.

In connection with the design and application of non-stationary subdivision, there have been interesting works. In fact, apart from the non-stationary curve subdivision, like those in [2,3], there have been significant works on non-stationary surface subdivision. For this, Novara et al. [4] proposed a non-stationary interpolatory scheme reproducing surfaces like the torus and the ellipsoid, which improves the smoothness and accuracy of the modified butterfly subdivision [5]. Fang et al. [6] presented a generalized order d exponential spline surface subdivision which can generate surfaces like the torus and the sphere. Badoual et al. [7] constructed a non-stationary generalized Loop subdivision for chemical imaging. For other non-stationary surface subdivisions, see also [8–11] and references therein.

The above schemes can be seen as being obtained based on a fixed point iteration coming from the generation of exponential polynomials [10] and thus can generate this kind of polynomial. In fact, besides this kind of non-stationary subdivision, there are also non-stationary subdivisions which are based on other kinds of iterations, such as those

in [12,13]. Yet, such schemes are curve but not surface subdivisions. Zhang et al. [10] presented the first surface subdivision from this point of view and derived a non-stationary Catmull–Clark subdivision with shape control. However, the obtained non-stationary Catmull–Clark subdivision lacks further analysis like the analysis of the limit positions of initial points.

Therefore, in this paper, we try to construct and analyze a new non-stationary surface subdivision, i.e., a symmetric non-stationary Loop subdivision [14]. This new scheme can adjust the shape of the limit surface due to its free parameter. For such a scheme, we show that it is C^2 convergent in the regular part of the mesh and has tangent plane continuity at the limit positions of extraordinary points. Thus, this new scheme can generate surfaces with different shapes from a simple initial mesh while keeping a satisfactory smoothness property. Such a non-stationary subdivision is obtained in a way similar to the non-stationary scheme in [10]. Specifically speaking, with a suitably chosen iteration and a function of this iteration, we properly modify the Loop subdivision rules in the regular part of the mesh and design the subdivision rules in the neighborhoods of extraordinary points in order to obtain the desired symmetric non-stationary Loop subdivision. More interestingly, in the spirit of the push-back operation [15], we also derive the limit positions of the initial points, which generalizes the existing result and can be used to interpolate the initial points with certain valence. Furthermore, we also present a non-uniform generalization which can locally adjust the shape of the limit surface and also derive the corresponding limit positions of the initial points. With such limit positions, this non-uniform generalization can be used to interpolate points with different valences.

Therefore, the contribution of this paper is as follows: (1) a non-stationary Loop subdivision with a non-uniform generalization is presented. Such schemes can adjust the shapes of the limit surfaces freely, which cannot be achieved using Loop subdivision [14] or other stationary schemes; (2) compared with the existing non-stationary schemes, like the one in [10], the limit positions of the initial points are derived in this paper, and these generalize the results from the stationary case [16] to non-stationary and even non-uniform cases; (3) compared with the existing methods used to interpolate initial points, like [17], the new schemes in this paper can interpolate initial points directly and do not require additional adjustment of the initial points, making them more efficient.

The rest of this paper is organized as follows. Section 2 is devoted to the review of some basic knowledge about subdivision. Section 3 is devoted to the construction of this symmetric non-stationary Loop subdivision and its analysis, including smoothness analysis and the derivation of the limit positions of the initial points. Section 4 is devoted to local control discussion of the symmetric non-stationary Loop subdivision. In Section 5, we discuss the interpolation of control points with different valences. Section 6 concludes this paper.

2. Preliminaries

This section is devoted to some basic knowledge about subdivision, which is needed in the rest of this paper.

Given the initial data sequence $q^0 = \{q_\alpha^0, \alpha \in \mathbb{Z}^2\} \in l_0(\mathbb{Z}^2)$, we consider the binary non-stationary subdivision scheme in this paper as

$$(S_{a^k} q^k)_\alpha = q_\alpha^{k+1} = \sum_{\beta \in \mathbb{Z}^2} a_{\alpha-2\beta}^k q_\beta^k, \quad \alpha \in \mathbb{Z}^2,$$

where S_{a^k} is the k -level subdivision operator, and the sequence $a^k = \{a_\alpha^k, \alpha \in \mathbb{Z}^2\}$ is the k -level mask with finite support. We denote this scheme by $\{S_{a^k}\}_{k \geq 0}$, and the corresponding k -level symbol is the Laurent polynomial $a^k(z) = \sum_{\alpha \in \mathbb{Z}^2} a_\alpha^k z^\alpha$.

In order to present the discussion of the smoothness of the new non-stationary Loop subdivision clearly, we need the following definitions and results.

Definition 1 ([18]). A non-stationary subdivision scheme $\{S_{u^k}\}_{k \geq 0}$ with the k -level mask u^k is said to be asymptotically similar to the stationary subdivision scheme S_u with the mask u , if the k -level mask u^k and the mask u have the same support U and satisfy

$$\lim_{k \rightarrow \infty} u_{\alpha}^k = u_{\alpha}, \quad \alpha \in U.$$

Definition 2 ([19]). A non-stationary subdivision scheme $\{S_{u^k}\}_{k \geq 0}$ with the k -level symbol $u^k(z)$ is said to satisfy approximate sum rules of order $r + 1$ if

$$\mu^k = |u^k(\mathbf{1}) - 2|, \quad \delta^k = \max_{|\eta| \leq r} |2^{-k|\eta|} D^{\eta} u^k(-\mathbf{1})|$$

with $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}_0^s$ and $\eta \in \mathbb{N}_0^s$ satisfy

$$\sum_k \mu^k < \infty, \quad \sum_k 2^{kr} \delta^k < \infty.$$

With the above two definitions, the smoothness of non-stationary subdivision in the regular portion of the mesh can be investigated using the results from the following.

Theorem 1 ([18]). Assume that the non-stationary subdivision scheme $\{S_{u^k}\}_{k \geq 0}$ satisfies approximate sum rules of order $r + 1$ and is asymptotically similar to a convergent stationary subdivision scheme S_u who is C^r -convergent. Then, the non-stationary scheme $\{S_{u^k}\}_{k \geq 0}$ is C^r -convergent.

As for the smoothness of the non-stationary scheme near the extraordinary points, we recall the definition of asymptotic equivalence as follows:

Definition 3 ([20]). The schemes $\{S_{u^k}\}_{k \geq 0}$ and $\{S_{v^k}\}_{k \geq 0}$ are asymptotically equivalent if

$$\sum_k \|S_{u^k} - S_{v^k}\|_{\infty} < \infty, \quad (1)$$

where

$$\|S_{u^k} - S_{v^k}\|_{\infty} = \max_{\alpha \in \mathbb{Z}^2} \left\{ \sum_{\beta} |u_{\alpha-2\beta}^k - v_{\alpha-2\beta}^k| : \alpha \in \Omega \right\},$$

with Ω being the set of extreme vertices of $[0, 1]^2$.

Remark 1. If the condition in (1) is replaced by

$$\sum_k 2^k \|S_{u^k} - S_{v^k}\|_{\infty} < \infty,$$

then the two schemes $\{S_{u^k}\}_{k \geq 0}$ and $\{S_{v^k}\}_{k \geq 0}$ are asymptotically equivalent of order 1.

To discuss the smoothness of the non-stationary schemes, let $\mathcal{M}^{(0)}$ denote the initial mesh of arbitrary topology and $\mathcal{M}^{(0)}$ consist of $\mathcal{R}^{(0)}$ and $\mathcal{E}^{(0)}$, which denote the neighborhood of a regular vertex and an extraordinary vertex, respectively. Then, with Definition 3, we can analyze the smoothness of the new non-stationary Loop subdivision scheme in the neighborhoods of extraordinary points using the following result.

Theorem 2 ([7]). Let $\mathcal{S}$ be a non-stationary subdivision scheme whose action in $\mathcal{E}^{(0)}$ is described by the matrix sequence $\{S^k\}_{k \geq 0}$. Moreover, let $\bar{\mathcal{S}}$ be a stationary subdivision scheme that in $\mathcal{E}^{(0)}$ is identified by the matrix S . Assume that

- (i) $\bar{\mathcal{S}}$ is C^1 -convergent in $\mathcal{R}^{(0)}$ with symbol $c(z)$ containing the factor $(1 + z_1)(1 + z_2)$ and G^1 convergent in $\mathcal{E}^{(0)}$;

- (ii) S^k is defined in $\mathcal{R}^{(0)}$ by the symbol $\{c^k(z)\}_{k \geq 0}$, where each $c^k(z)$ contains the factor $(1 + z_1)(1 + z_2)$;
- (iii) S is asymptotically equivalent of order 1 to $\tilde{S}$ in $\mathcal{R}^{(0)}$;
- (iv) In $\mathcal{E}^{(0)}$, the matrices $\{S^k\}_{k \geq 0}$ and S satisfy, for all $k \geq 0$, $\|S_k - S\|_\infty \leq \frac{C}{\sigma^k}$, where C is some finite positive constant and $\sigma > \frac{1}{\lambda_1} > 1$ with $\lambda_1 \in \mathbb{R}_+$ being the subdominant eigenvalue of S , which is double and non-defective.

Then, the subdivision surface generated by S is convergent in $\mathcal{E}^{(0)}$ and produces tangent plane continuous surfaces at the limit positions of the extraordinary points.

3. The Symmetric Non-Stationary Loop Subdivision

In this section, we construct and analyze the new symmetric non-stationary Loop subdivision. To this purpose, we first give a brief review on the Loop subdivision.

3.1. Review of Loop Subdivision

Let S_a denote the Loop subdivision scheme. In fact, in the regular part of the mesh, the Loop subdivision S_a can be given in terms of the following symmetric mask (symmetric about the origin)

$$a = \begin{pmatrix} 0 & 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{1}{16} & \frac{3}{8} & \frac{5}{8} & \frac{3}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{8} & \frac{5}{8} & \frac{1}{8} & 0 \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & 0 & 0 \end{pmatrix},$$

with the corresponding symbol

$$a(z) = 4\left(\frac{1+z_1}{2}\right)^2\left(\frac{1+z_2}{2}\right)^2\left(\frac{1+z_2z_1}{2}\right)^2z_1^{-2}z_2^{-2}. \quad (2)$$

For the subdivision rules in the neighborhood of an extraordinary point of valence n , by p^0 we denote the vector that collects the points in the 1-ring neighborhood of such an extraordinary point. Then, the Loop subdivision scheme can be described by $p^{k+1} = \tilde{S}p^k$ [21]. Here, $\tilde{S}$ is the local subdivision matrix

$$\tilde{S} = \begin{pmatrix} 1 - n\alpha & \alpha \\ \beta & \tilde{B} \end{pmatrix}, \quad (3)$$

where

$$\alpha = \begin{cases} \frac{3}{16}, & n = 3 \\ \frac{3}{8n}, & n > 3 \end{cases}, \quad \alpha = (\alpha, \dots, \alpha), \quad \beta = \left(\frac{3}{8}, \dots, \frac{3}{8}\right)^\top,$$

and $\tilde{B}$ is the $n \times n$ circulant matrix $\tilde{B} = \text{circ}(b_0, b_1, \dots, b_{n-1})$ with $b_0 = \frac{3}{8}$, $b_1 = \frac{1}{8}$, $b_2 = \dots = b_{n-2} = 0$, $b_{n-1} = \frac{1}{8}$.

As is known, the Loop subdivision scheme S_a is C^2 convergent in the regular part of the mesh and C^1 convergent at the limit positions of extraordinary points.

3.2. Construction of the Symmetric Non-Stationary Loop Subdivision

Now, we present the construction of the new symmetric non-stationary Loop subdivision, which is denoted by $\{S_{a^k}\}_{k \geq 0}$.

Since we see the non-stationary subdivision $\{S_{a^k}\}_{k \geq 0}$ as one based on some iteration and a function of this iteration, we first give the function $\varphi(x) = \frac{x}{5}$ with the fixed point $x^* = 0$. The corresponding iteration process is

$$v^{k+1} = \varphi(v^k) = \frac{v^k}{5}, \quad v^0 \in \mathbb{R}.$$

Let $h(v^k) = \frac{1}{8}(1 + v^k)$ be the function of this iteration. Then, in the regular part of the mesh, the new non-stationary Loop subdivision $\{S_{a^k}\}_{k \geq 0}$ can be characterized in terms of the following k -level symmetric mask (symmetric about the origin)

$$a^k = \begin{pmatrix} 0 & 0 & \frac{h(v^{k+1})}{2} & h(v^{k+1}) & \frac{h(v^{k+1})}{2} \\ 0 & h(v^{k+1}) & \frac{1}{2} - h(v^{k+1}) & \frac{1}{2} - h(v^{k+1}) & h(v^{k+1}) \\ \frac{h(v^{k+1})}{2} & \frac{1}{2} - h(v^{k+1}) & 1 - 3h(v^{k+1}) & \frac{1}{2} - h(v^{k+1}) & \frac{h(v^{k+1})}{2} \\ h(v^{k+1}) & \frac{1}{2} - h(v^{k+1}) & \frac{1}{2} - h(v^{k+1}) & h(v^{k+1}) & 0 \\ \frac{h(v^{k+1})}{2} & h(v^{k+1}) & \frac{h(v^{k+1})}{2} & 0 & 0 \end{pmatrix},$$

with the corresponding k -level symbol

$$a^k(z_1, z_2) = 4\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right)(h(v^{k+1}))(1+z_1+z_2 + z_1z_2^2 + z_1^2z_2 + z_1^2z_2^2) + (1-6h(v^{k+1}))z_1z_2z_1^{-2}z_2^{-2}. \quad (4)$$

For the subdivision rules in the neighborhood of an extraordinary point of valence n , the corresponding local subdivision matrix $\tilde{S}^k$ can be written down as

$$\tilde{S}^k = \begin{pmatrix} 1 - n\alpha^k & \alpha^k \\ \beta^k & \tilde{B}_j^k \end{pmatrix}, \quad (5)$$

where

$$\alpha^k = \frac{3}{m}h(v^{k+1}), \alpha^k = (\alpha^k, \dots, \alpha^k), \beta^k = \left(\frac{1}{2} - h(v^{k+1}), \dots, \frac{1}{2} - h(v^{k+1})\right)^\top,$$

with $m = \begin{cases} 2, & n = 3 \\ n, & n > 3 \end{cases}$, and $\tilde{B}^k$ is the $n \times n$ circulant matrix $\tilde{B}^k = \text{circ}(b_0^k, b_1^k, \dots, b_{n-1}^k)$ with $b_0^k = \frac{1}{2} - h(v^{k+1}), b_1^k = h(v^{k+1}), b_2^k = \dots = b_{n-2}^k = 0, b_{n-1}^k = h(v^{k+1})$. The corresponding stencils used to generate a 'vertex' point with valence n and an 'edge' point are as shown in Figure 1.

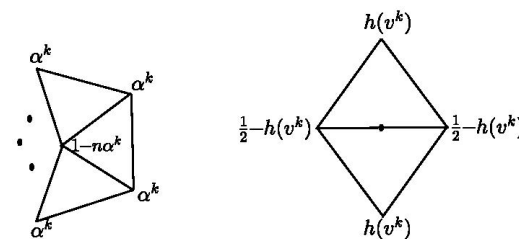


Figure 1. Stencils for the scheme $\{S_{a^k}\}_{k \geq 0}$ to generate a 'vertex' point (left) and an 'edge' point (right).

Note that, when $n = 6$, the stencil used for generating new 'vertex' points reduces to the one used in the regular part. When $v^0 = 0$, we have $v^k = 0$ for $k \in \mathbb{N}_0$. Thus, the k -level symbol $a^k(z)$ in (4) and the local subdivision matrix in (5) are actually the stationary counterparts in (2) and (3), respectively, meaning that the new scheme $\{S_{a^k}\}_{k \geq 0}$ becomes the classical Loop subdivision S_a . Thus, since $\lim_{k \rightarrow \infty} v^k = 0$, the scheme $\{S_{a^k}\}_{k \geq 0}$ reduces to the Loop subdivision S_a as $k \rightarrow \infty$. Since $h(v^k) = \frac{1}{8}(1 + v^k)$, for the k -level mask a^k , we have $(a^k)_{i,j} = a_{i,j} + sv^{k+1}$ with $s \in \mathbb{R}$. The local subdivision matrix $\tilde{S}^k$ has the same property. In this way, for the new scheme $\{S_{a^k}\}_{k \geq 0}$, it can actually be seen as the Loop subdivision S_a plus some perturbation.

3.3. Smoothness Analysis

For the smoothness of the scheme $\{S_{a^k}\}_{k \geq 0}$, we first analyze its smoothness in the regular part of the mesh and then near the extraordinary points.

In fact, based on the definitions and theorems in Section 2, we can show the following result for the smoothness of the scheme $\{S_{a^k}\}_{k \geq 0}$ in the regular part of the mesh.

Theorem 3. *The new symmetric non-stationary Loop subdivision $\{S_{a^k}\}_{k \geq 0}$ is C^2 convergent in the regular part of the mesh.*

Proof. Since $\lim_{k \rightarrow \infty} v^k = 0$, it can be seen that $\lim_{k \rightarrow \infty} a^k(z) = a(z)$. Thus, the scheme $\{S_{a^k}\}_{k \geq 0}$ is asymptotically similar to S_a .

Now, we show that the scheme $\{S_{a^k}\}_{k \geq 0}$ has approximate sum rules of order 3, and then, by Theorem 1, it is C^2 convergent, since the Loop subdivision S_a is C^2 convergent in the regular part of the mesh.

Let $\mathcal{Z} = \{(1, -1), (-1, 1), (-1, -1)\}$. Then, from the k level symbol $a^k(z)$ in (4), it can be computed that for $\eta \in \mathbb{N}^2$,

$$\delta^k = \max_{|\eta| \leq 2} |D^\eta a^k(\xi_1, \xi_2)| = 2|h(v^{k+1}) - \frac{1}{8}| = \frac{1}{4}|v^{k+1} - 0|, \quad (\xi_1, \xi_2) \in \mathcal{Z}.$$

For the iteration $v^{k+1} = \varphi(v^k)$, we have

$$|v^{k+1} - 0| = |\varphi(v^k) - \varphi(0)| \leq L|v^k - 0| \leq \dots L^{k-1}|v^1 - 0| \leq c_1 L^k, \quad (6)$$

where $L = \frac{1}{5}$, and c_1 is independent of k . Therefore, it can be computed that

$$\sum_{k=1}^{\infty} 2^{2k} \delta^k = c_1 \sum_{k=1}^{\infty} (2L)^k < \infty.$$

Thus, by Definition 2, the scheme $\{S_{a^k}\}_{k \geq 0}$ has approximate sum rules of order 3, and it is C^2 convergent in the regular part of mesh. $\square$

As for the smoothness of the new scheme $\{S_{a^k}\}_{k \geq 0}$ near an extraordinary point of valence n , we have the following result:

Theorem 4. *The scheme $\{S_{a^k}\}_{k \geq 0}$ is tangent plane continuous at the limit position of the extraordinary point of valence n .*

Proof. We prove this result by verifying all the conditions in Theorem 2. The Loop subdivision scheme S_a is C^2 convergent in the regular part of the mesh and G^1 convergent in the neighborhood of extraordinary points. The symbol $a(z)$ contains $(1 + z_1)(1 + z_2)$. Thus, condition (i) is verified. The k -level symbol $a^k(z)$ also contains $(1 + z_1)(1 + z_2)$, and thus, condition (ii) is verified.

Now, we verify condition (iii). In fact, following (6), it can be seen that there exists a constant c_2 independent of k such that $|v^k - 0| \leq c_2 L^k |v^0 - 0|$. Thus, it can be computed that

$$|h(v^{k+1}) - \frac{1}{8}|, \quad |\alpha^k - \alpha| \leq c_3 L^k, \quad (7)$$

where c_3 is a constant independent of k . Then, we have

$$\|S_{a^k} - S_a\|_{\infty} \leq c_3 L^k, \quad \sum_k 2^k \|S_{a^k} - S_a\| \leq c_3 \sum_k (2L)^k < \infty.$$

Thus, $\{S_{a^k}\}_{k \geq 0}$ is asymptotically equivalent to S_a of order 1 and condition (iii) is verified.

Now, we verify condition (iv) of Theorem 6. Let

$$B_j = \begin{pmatrix} \frac{1}{n} - \alpha & \alpha \\ \frac{3}{8n} & b_j \end{pmatrix}, \quad B_j^k = \begin{pmatrix} \frac{1}{n} - \alpha^k & \alpha^k \\ \frac{1}{2n} - \frac{h(v^{k+1})}{n} & b_j^k \end{pmatrix}, \quad j = 0, \dots, n-1.$$

Then, the local subdivision matrices $\tilde{S}$ and $\tilde{S}^k$ can be transformed into the block-circulant ones [21],

$$S := \text{circ}(B_0, \dots, B_{n-1}) \quad \text{and} \quad S^k := \text{circ}(B_0^k, \dots, B_{n-1}^k).$$

For B_j^k , $j = 0, \dots, n-1$, following the proof of Theorem 4.1 in [10] and by (7), we can compute that

$$\|B_i^k - B_i\|_\infty \leq c_5 |v^k - 0| \leq c_5 L^k, \quad i = 0, \dots, n-1,$$

where c_5 is a constant independent of k . Thus, for $S^k - S$, there exists a constant c_6 independent of n and k such that

$$\|S^k - S\|_\infty \leq \sum_{i=0}^{n-1} \|B_i^k - B_i\|_\infty \leq c_6 |v^k - 0| = c_7 L^k.$$

Since the matrix $\tilde{S}$ has a subdominant eigenvalue $\frac{1}{4} \leq \lambda < 1$ with algebraic and geometric multiplicity 2, we have $\frac{1}{L} > 4 \geq \frac{1}{\lambda}$, and thus, condition (iv) of Theorem 6 can be verified. Therefore, $\{S_{a^k}\}_{k \geq 0}$ is tangent plane continuous at the limit position of this extraordinary point. $\square$

Figure 2 gives an example showing the surfaces obtained by the scheme $\{S_{a^k}\}_{k \geq 0}$ with different values of v^0 from the initial mesh (left). From Figure 2, we can see the change of the shape of the obtained surface with the change of v^0 .

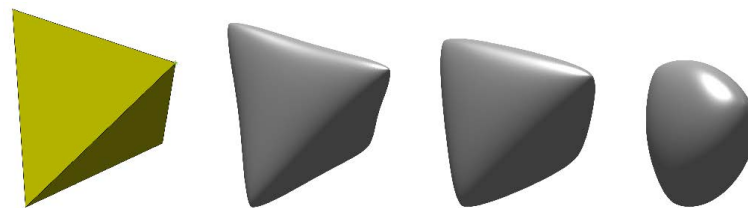


Figure 2. Initial mesh with control points of valence 3 (left) and the corresponding limit surfaces generated by $\{a^k\}_{k \geq 0}$ with $v^0 = -8, -4, 0$ (left to right for the right 3 surfaces).

3.4. Limit Positions of Initial Points

Now, we derive the limit positions of initial points in the spirit of the push-back method [10]. In fact, for the Loop subdivision S_a , the limit position of initial points is [16]

$$P_0^\infty = (1 - n\gamma)P_0^0 + \gamma \sum_{i=1}^n P_i^0, \quad (8)$$

where $\gamma = (\frac{3}{8\alpha} + n)^{-1}$. Next, we generalize this result to the non-stationary case by deriving the limit positions of initial points for the new scheme $\{S_{a^k}\}_{k \geq 0}$. From this we obtain the following result:

Theorem 5. For the new scheme $\{S_{a^k}\}_{k \geq 0}$, the limit position of the initial point P_0^0 with 1-ring neighborhood points P_j^0 , $j = 1, \dots, n$, is

$$P_0^\infty = \lambda \sum_{j=1}^n P_j^0 + (1 - n\lambda)P_0^0, \quad (9)$$

where

$$\lambda = \sum_{k=0}^{\infty} \alpha^k \prod_{j=1}^k \left(\frac{5m-3n}{8m} - \frac{3n-m}{8m} \frac{v^0}{5^j} \right), \quad \text{with } m = \begin{cases} 2, & n=3 \\ n, & n>3. \end{cases} \quad (10)$$

Proof. In fact, following the subdivision rules of $\{S_{a^k}\}_{k \geq 0}$, we can compute the distance between P_0^{k+1} and P_0^k as

$$P_0^{k+1} - P_0^k = (1 - n\alpha^k)P_0^k + \alpha^k \sum_{j=1}^k P_j^k - P_0^k = \alpha^k \sum_{j=1}^n (P_j^k - P_0^k) = \alpha^k A^k,$$

where $A^k := \sum_{j=1}^n (P_j^k - P_0^k)$.

For A^k , it can be computed that

$$\begin{aligned} A^k &= \sum_{j=1}^n P_j^k - nP_0^k \\ &= \left[\left(\frac{1}{2} - h(v^k) \right) (P_0^{k-1} + P_1^{k-1}) + h(v^k) (P_2^{k-1} + P_n^{k-1}) \right] + \dots \\ &\quad + \left[\left(\frac{1}{2} - h(v^k) \right) (P_0^{k-1} + P_i^{k-1}) + h(v^k) (P_{i+1}^{k-1} + P_{i-1}^{k-1}) \right] + \dots \\ &\quad + \left[\left(\frac{1}{2} - h(v^k) \right) (P_0^{k-1} + P_{n-1}^{k-1}) + h(v^k) (P_n^{k-1} + P_{n-2}^{k-1}) \right] \\ &\quad + \left[\left(\frac{1}{2} - h(v^k) \right) (P_0^{k-1} + P_n^{k-1}) + h(v^k) (P_1^{k-1} + P_{n-1}^{k-1}) \right] \\ &\quad - n \left[(1 - n\alpha^{k-1})P_0^{k-1} + \alpha^{k-1} \sum_{j=1}^n P_j^{k-1} \right] \\ &= n \left(\frac{1}{2} - h(v^k) \right) P_0^{k-1} + \left(\frac{1}{2} + h(v^k) \right) \sum_{j=1}^n P_j^{k-1} - n \left[(1 - n\alpha^{k-1})P_0^{k-1} + \alpha^{k-1} \sum_{j=1}^n P_j^{k-1} \right] \\ &= \left(\frac{5m-3n}{8m} - \frac{3n-m}{8m} \frac{v^0}{5^k} \right) A^{k-1}. \end{aligned}$$

In this way, we have

$$P_0^{k+1} - P_0^k = \alpha^k A^k = \dots = \alpha^k \prod_{j=1}^k \left(\frac{5m-3n}{8m} - \frac{3n-m}{8m} \frac{v^0}{5^j} \right) A^0.$$

Therefore, the corresponding limit position is

$$\begin{aligned} P_0^\infty &= P_0^0 + \sum_{k=0}^{\infty} (P_0^{k+1} - P_0^k) = P_0^0 + \sum_{k=0}^{\infty} \alpha^k A^k = P_0^0 + \sum_{k=0}^{\infty} \alpha^k \prod_{j=1}^k \left(\frac{5m-3n}{8m} - \frac{3n-m}{8m} \frac{v^0}{5^j} \right) A^0 \\ &= \lambda \sum_{j=1}^n P_j^0 + (1 - n\lambda)P_0^0, \end{aligned}$$

where $\lambda = \sum_{k=0}^{\infty} \alpha^k \prod_{j=1}^k \left(\frac{5m-3n}{8m} - \frac{3n-m}{8m} \frac{v^0}{5^j} \right)$. $\square$

From Theorem 5, it can be seen that when $v^0 = 0$, the corresponding limit position in (9) actually reduces to the one in (8).

4. Discussion on the Local Control

In this section, we provide a non-uniform version of the scheme $\{S_{a^k}\}_{k \geq 0}$ to locally control the limit surface.

To achieve this, we use a method similar to the local control discussion in [10]. To be more specific, we set different initial parameters v_i^0 to different initial points and use the average of them $v_e^0 := \frac{1}{n} \sum_{i=0}^{n-1} v_i^0$ as the initial parameter for a new edge point generation. As for the correspondence of the points in the coarse mesh and the ones in the refined mesh, we use the iteration $v^{k+1} = \frac{v^k}{5}$ for the parameters of new vertex points and new edge points. Then, we replace α^k with $\alpha_i^k = \frac{3}{m} h(v_i^{k+1})$ and $h(v^k)$ with $h(v_e^k)$ in Figure 1. In this way, we can achieve the desired non-uniform version of the scheme $\{S_{a^k}\}_{k \geq 0}$, which is not symmetric.

For such a non-uniform scheme, we point out that when $k \rightarrow \infty$, it still tends to the Loop subdivision S_a . The smoothness of this non-uniform scheme in the regular part of the mesh can be calculated as in [10]. As for the smoothness near the extraordinary points, it can be seen that all the conditions of Theorem 2 are satisfied, and thus, it still has tangent plane continuity at the limit positions of extraordinary points.

Figure 3 shows the surfaces generated by this new non-uniform scheme starting from the initial mesh in Figure 4 (left) with different values of the parameter $v_{*,2}^0$ corresponding to the point indexed by * and 2, while others are the same. From Figure 3, we can see the change of the limit surface with the change of $v_{*,2}^0$ and find that the limit surface tends to generate a cusp as $v_{*,2}^0$ increases.



Figure 3. Surfaces generated by the new non-uniform scheme with $v_{*,2}^0 = 4, 2, 0, -4, -8$ (left to right) with other initial parameters set to 0 from the initial mesh in Figure 4 (left).

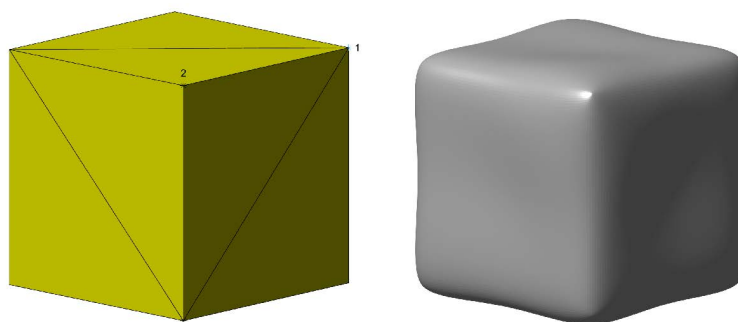


Figure 4. Initial mesh with indexed point (left) and the corresponding surface (right) using $\{S_{a^k}\}_{k \geq 0}$ with $v^0 = -8.5229$.

5. Interpolation of Initial Points

Based on the analysis in Sections 3, we now move a further step to discuss the interpolation of initial points.

In connection with the initial points interpolation, several works have been conducted using approximating subdivision. For example, Sun and Lu [17] presented a progressive interpolation, while Zheng and Cai [22] proposed a two-phase subdivision to interpolate the initial mesh based on the Catmull–Clark subdivision. Other similar works can be found in [23,24] and the references therein.

Differently from these works, we use the limit position in (9) to interpolate initial control points. To do this, we just need to let the initial point P_0^0 equal its limit point P_0^∞ ,

i.e., $P_0^0 = P_0^\infty$, so as to obtain the value of v^0 . Specifically, to interpolate an initial point with valence n from the limit position in (9), let $v^0 = -8.5529$ when $n = 3$ and $v^0 = -8.7187$ when $n \neq 3$. Then, the corresponding value of λ in (10) is about 10^{-6} .

As an example, we derive the surface crossing the initial points in the initial mesh in Figure 2 (left). Note that any point of this initial mesh has a valence of 3, and thus, we set $v^0 = -8.5529$. Figure 5 shows the surface interpolating the initial mesh in Figure 2 (left) using the new non-uniform subdivision.

Remark 2. We can also use the iteration coming from the generation of exponential polynomials, i.e., $\varphi(v^{k+1}) = \sqrt{\frac{v^k}{2} + \frac{1}{2}}$ with $v^0 \in (-1, \infty)$, and obtain the corresponding limit position of initial points. However, the corresponding scheme cannot reach the initial points, as this requires λ in (10) to be 0, which is impossible for $v^0 \in (-1, \infty)$.

In fact, it can be seen that λ in (10) depends on the valence n . Thus, with a certain initial parameter, the subdivision $\{S_{a^k}\}_{k \geq 0}$ can only interpolate points with certain valence but not all of them when the initial points have different valences. Figure 4 shows the surface (right) starting from the mesh (left) using $\{S_{a^k}\}_{k \geq 0}$ with $v^0 = -8.5229$. From Figure 4, it can be seen that the point indexed by * and 1 with valence 5 cannot be interpolated.

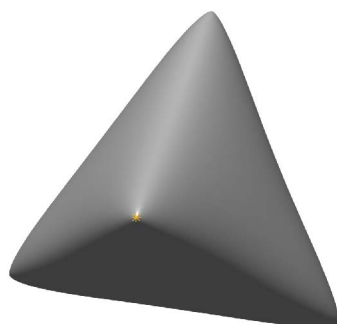


Figure 5. Surface obtained from the initial mesh in Figure 2 (left) interpolating the initial points using the new non-uniform subdivision with $v^0 = -8.5529$.

Now, with the local control discussion, we provide a way to interpolate all the initial points. To achieve this, similar to Theorem 5, we first provide the corresponding limit positions of the initial points for this new non-uniform scheme as follows.

Theorem 6. With the non-uniform generalization of the scheme $\{S_{a^k}\}_{k \geq 0}$, the limit position of the initial point P_0^0 with 1-ring neighborhood points $P_j^0, j = 1, \dots, n$, is

$$P_0^\infty = (1 - n\lambda)P_0^0 + \lambda \sum_{j=1}^n P_j^0,$$

with

$$\lambda = \sum_{k=0}^{\infty} \alpha_0^{k+1} \prod_{j=1}^k \left(\frac{5}{8} - \frac{3n}{8m} + \frac{v_e^0}{8} \frac{1}{5j} - \frac{3n}{8m} \frac{v_0^0}{5j} \right), \quad \text{with } m = \begin{cases} 2, & n = 3 \\ n, & n > 3. \end{cases}$$

Proof. In fact, from the proof of Theorem 5, the distance between P_0^{k+1} and P_0^k can be written as

$$P_0^{k+1} - P_0^k = \alpha^k A_0^k,$$

where $A_0^k = \sum_{j=1}^n (P_j^k - P_0^k)$.

For A_0^k , it can be computed that

$$\begin{aligned}
 A_0^k &= \sum_{j=1}^n P_j^k - nP_0^k \\
 &= \left[\left(\frac{1}{2} - h(v_e^k) \right) (P_0^{k-1} + P_1^{k-1}) + h(v_e^k) (P_2^{k-1} + P_n^{k-1}) \right] + \dots \\
 &\quad + \left[\left(\frac{1}{2} - h(v_e^k) \right) (P_0^{k-1} + P_i^{k-1}) + h(v_e^k) (P_{i+1}^{k-1} + P_{i-1}^{k-1}) \right] + \dots \\
 &\quad + \left[\left(\frac{1}{2} - h(v_e^k) \right) (P_0^{k-1} + P_{n-1}^{k-1}) + h(v_e^k) (P_n^{k-1} + P_{n-2}^{k-1}) \right] \\
 &\quad + \left[\left(\frac{1}{2} - h(v_e^k) \right) (P_0^{k-1} + P_n^{k-1}) + h(v_e^k) (P_1^{k-1} + P_{n-1}^{k-1}) \right] \\
 &\quad - n \left[(1 - n\alpha_0^{k-1}) P_0^{k-1} + \alpha_0^{k-1} \sum_{j=1}^n P_j^{k-1} \right] \\
 &= n \left(\frac{1}{2} - h(v_e^k) \right) P_0^{k-1} + \left(\frac{1}{2} + h(v_e^k) \right) \sum_{j=1}^n P_j^{k-1} - n \left[(1 - n\alpha_0^{k-1}) P_0^{k-1} + \alpha_0^{k-1} \sum_{j=1}^n P_j^{k-1} \right], \\
 &= \left(\frac{1}{2} + h(v_e^k) - n\alpha_0^{k-1} \right) A_0^{k-1}, \\
 &= \left(\frac{5}{8} - \frac{3n}{m} + \frac{v_e^k}{8} - \frac{3nv_0^k}{8m} \right) A_0^{k-1}.
 \end{aligned}$$

In this way, we have

$$P_0^{k+1} - P_0^k = \alpha_0^{k-1} A_0^k = \dots = \alpha_0^k \prod_{j=1}^k \left(\frac{5}{8} - \frac{3n}{m} + \frac{v_e^j}{8} - \frac{3nv_0^j}{8m} \right) A_0^{k-1}.$$

Therefore, the corresponding limit position is

$$\begin{aligned}
 P_0^\infty &= P_0^0 + \sum_{k=0}^\infty (P_0^{k+1} - P_0^k) = P_0^0 + \sum_{k=0}^\infty \alpha_0^k A_0^k = P_0^0 + \sum_{k=0}^\infty \alpha_0^k \prod_{j=1}^k \left(\frac{5}{8} - \frac{3n}{m} + \frac{v_e^j}{8} - \frac{3nv_0^j}{8m} \right) A_0^0 \\
 &= \lambda \sum_{j=1}^n P_j^0 + (1 - n\lambda) P_0^0,
 \end{aligned}$$

$$\text{where } \lambda = \sum_{k=0}^\infty \alpha_0^k \prod_{j=1}^k \left(\frac{5}{8} - \frac{3n}{m} + \frac{v_e^j}{8} \frac{1}{5^j} - \frac{3nv_0^j}{8m} \frac{1}{5^j} \right). \quad \square$$

As an example, we take the mesh with 8 points and 2 of them with valence 3, i.e., the one in Figure 4 (left). According to Theorem 6, to interpolate the initial points, we need the corresponding initial parameters v_1^0 (to the initial points with valence $n = 3$) and v_2^0 (to the initial points with valence $n > 3$) to satisfy

$$\begin{cases} f_1 := \sum_{k=0}^\infty \left(1 + \frac{v_1^0}{5^{k+1}} \right) \prod_{j=1}^k \left[\frac{1}{16} + \frac{1}{5^j} \frac{pv_1^0}{8\tilde{n}} + \frac{1}{5^j} \frac{(\tilde{n}-p)v_2^0}{8\tilde{n}} - \frac{9}{16} \frac{v_1^0}{5^j} \right] = 0, \\ f_2 := \sum_{k=0}^\infty \left(1 + \frac{v_2^0}{5^{k+1}} \right) \prod_{j=1}^k \left[\frac{1}{4} + \frac{1}{5^j} \frac{pv_1^0}{8\tilde{n}} + \frac{1}{5^j} \frac{(\tilde{n}-p)v_2^0}{8\tilde{n}} - \frac{3}{8} \frac{v_2^0}{5^j} \right] = 0. \end{cases} \quad (11)$$

Here, $p = 2$, and $\tilde{n} = 8$. It can be computed that, when $(v_1^0, v_2^0) = (-8.52466, -8.72849)$, f_1, f_2 in (11) is about 10^{-7} .

Figure 6 shows the surface obtained by this new non-uniform scheme starting from the mesh in Figure 4 (left). The initial parameters are set to be -8.52466 for points with

valence 3 and -8.72849 for points with valence $n > 3$. From Figure 6, we can see that the limit surface indeed interpolates initial points with different valences.

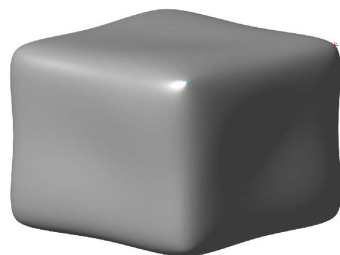


Figure 6. Surface obtained by this new non-uniform scheme starting from the mesh in Figure 4 (left) with initial parameter -8.52466 for points with valence 3 and -8.72849 for points with valence $n \neq 3$.

In fact, compared with the existing methods to interpolate initial points, a non-stationary Loop subdivision with a non-uniform generalization is used directly to interpolate the initial points. Thus, the new method in this paper has all of the subdivision's advantages, such as locality and efficiency. Therefore, this new method can be used to construct smooth interpolating surfaces with different shapes and can even be applied in fields like scientific visualization.

6. Conclusions

This paper proposed a symmetric non-stationary Loop subdivision controlling the shape of the limit surface, which is based on a suitable iteration. For such a scheme, we show that it is C^2 convergent in the regular part of the mesh and tangent plane continuous at the limit positions of extraordinary points. Additionally, the limit position of the initial points is also given, which can be used to interpolate initial points. We also provide a non-uniform generalization to locally control the shape of the limit surface together with the corresponding limit position. Although this scheme can interpolate the initial points, it still requires computation of the initial value of parameters. We hope to provide a more simple and efficient way to interpolate the whole initial mesh and inflect the initial mesh well. Furthermore, we also hope to discuss the corresponding applications in fields like chemical modeling.

Author Contributions: Conceptualization, B.Z. and H.Z.; methodology, B.Z. and H.Z.; formal analysis, Y.Z. and B.Z.; writing—original draft preparation, Y.Z.; writing—review and editing, B.Z. and H.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Science and Technology Project of Hebei Education Department (QN2024065) and the Scientific Research and Development of Hebei University of Economics and Business (2022QN15).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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