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# Strong and Weak Convergence Theorems for the Split Feasibility Problem of $(\beta, k)$-Enriched Strict Pseudocontractive Mappings with an Application in Hilbert Spaces

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**Abstract:** The concept of symmetry has played a major role in Hilbert space setting owing to the structure of a complete inner product space. Subsequently, different studies pertaining to symmetry, including symmetric operators, have investigated real Hilbert spaces. In this paper, we study the solutions to multiple-set split feasibility problems for a pair of finite families of $\beta$-enriched, strictly pseudocontractive mappings in the setup of a real Hilbert space. In view of this, we constructed an iterative scheme that properly included these two mappings into the formula. Under this iterative scheme, an appropriate condition for the existence of solutions and strong and weak convergent results are presented. No sum condition is imposed on the countably finite family of the iteration parameters in obtaining our results unlike for several other results in this direction. In addition, we prove that a slight modification of our iterative scheme could be applied in studying hierarchical variational inequality problems in a real Hilbert space. Our results improve, extend and generalize several results currently existing in the literature.

**Keywords:** strong convergence; variational inequality; enriched nonlinear mapping; split feasibility problem; multiple-set split feasibility problem; fixed point; iterative scheme; hierarchical problem; Hilbert space

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## 1. Introduction

Fixed point theory has no doubt proven to be a rich and complex field, always giving rise to several extensions and applicable results. Nowadays, it has become incredibly convincing that this domain of study is far from reaching its end as regards procreating new ideas or connecting existing ones.

Let $H$ be a real Hilbert space with the inner product $\langle . \rangle$ and the induced norm $\|.\|$. Let $\emptyset \neq K \subset H$ be closed and convex.

**Definition 1** ([1]). *A nonlinear mapping $\Gamma : K \longrightarrow K$ is called $\beta$-enriched Lipschitzian if there exist $\beta \in [0, \infty)$ and $L > 0$ such that the following inequality*

$$\|\beta(\varrho - \omega) + \Gamma\varrho - \Gamma\omega\| \leq (\beta + 1)L\|\varrho - \omega\|, \quad \forall \varrho, \omega \in K. \tag{1}$$

*It is worthy to mention that every Lipschitz mapping is* $0$*-enriched Lipschitzian with* $\beta = 0$*. However, if* $\beta \neq 0$ $\rho \in (0,1)$ *are chosen such that* $\beta = \frac{1}{\rho} - 1$*, then inequality* (1) *becomes*

$$\begin{aligned}
\left\| \frac{1-\rho}{\rho}\Big(\varrho - \omega\Big) + \Gamma\varrho - \Gamma\omega \right\| &\leq \frac{L}{\rho}\|\varrho - \omega\| \\
\Leftrightarrow \|(1-\rho)(\varrho - \omega) + \rho\Gamma\varrho - \rho\Gamma\omega\| &\leq L\|\varrho - \omega\| \\
\Leftrightarrow \|(1-\rho)\varrho + \rho\Gamma\varrho - [(1-\rho)\omega + \rho\Gamma\omega]\| &\leq L\|\varrho - \omega\|. \qquad (2)
\end{aligned}$$

*Set* $\Gamma_\rho = (1-\rho)I + \rho\Gamma$*. Then, the last inequality becomes*

$$\|\Gamma_\rho\varrho - \Gamma_\rho\omega\| \leq \|\varrho - \omega\|. \qquad (3)$$

*Here, the average operator* $\Gamma_\rho$ *is L-Lipschitzian.*

**Remark 1.** *The class of* $\beta$*-enriched Lipschitz mappings is between the class of Lipschitz mappings and the class of* $(\beta, \Phi_\Gamma)$*-enriched Lipschitz mappings studied in* [1]*. (Recall that a nonlinear mapping* $\Gamma : K \longrightarrow K$ *is called a* $(\beta, \Phi_\Gamma)$*-enriched Lipschitz mapping (or* $\Phi_\Gamma$*-enriched Lipschitzian) if for all* $\varrho, \omega \in K$*, there exist* $\beta \in [0, +\infty)$ *and a continuous nondecreasing function* $\Phi_\Gamma : R^+ \longrightarrow R^+$*, with* $\Phi(0) = 0$*, such that* $\|\beta(\varrho - \omega) + \Gamma\varrho - \Gamma\omega\| \leq (\beta+1)\Phi_\Gamma(\|\varrho - \omega\|)$*.) If* $\Phi_\Gamma(r) = r$*, then we recover inequality* (1)*; if* $L \in (0,1)$*, then inequality* (1) *reduces to an important class of nonlinear mappings called enriched contraction mappings, and if* $L = 1$ *in inequality* (1)*, we obtain the class of* $\beta$*-enriched nonexpansive mappings. (Recall that a nonlinear mapping* $\Gamma : K \longrightarrow K$ *is called a* $\beta$*-enriched nonexpansive mapping if for all* $\varrho, \omega \in K$*, there exists* $\beta \in [0, +\infty)$ *such that* $\|\beta(\varrho - \omega) + \Gamma\varrho - \Gamma\omega\| \leq (\beta+1)\|\varrho - \omega\|$*. Every nonexpansive mapping is* $0$*-enriched nonexpansive).*
*These two classes of mappings were introduced in* [2,3] *by Berinde. He proved that if K is a nonempty, bounded, closed and convex subset of a real Hilbert space H and* $\Gamma : K \longrightarrow K$ *is a* $\beta$*-enriched nonexpansive and demicompact mapping, then* $\Gamma$ *has a fixed point in K.*

**Example 1.** *Consider* $R^2$ *denote the* $2$*-dimensional Euclidean plane. Define* $\Gamma : R^2 \longrightarrow R^2$ *by*

$$\Gamma\varrho = \Gamma((\varrho_1, \varrho_2)) = (\varrho_1, \varrho_2) + (\varrho_2, -\varrho_1) = (\varrho_1 + \varrho_2, \varrho_2 - \varrho_1), \quad \forall \varrho = (\varrho_1, \varrho_2) \in R^2.$$

*Then, for all* $\varrho = (\varrho_1, \varrho_2), \omega = (\omega_1, \omega_2) \in R^2$ *and* $\beta = 1$*, we have*

$$\begin{aligned}
\|\beta(\varrho - \omega) + \Gamma\varrho - \Gamma\omega\|^2 &= \|\beta((\varrho_1, \varrho_2) - (\omega_1, \omega_2)) + (\varrho_1 + \varrho_2, \varrho_2 - \varrho_1) \\
&\quad -(\omega_1 + \omega_2, \omega_2 - \omega_1)\|^2 \\
&= \|\beta((\varrho_1 - \omega_1), (\varrho_2 - \omega_2)) + (\varrho_1 + \varrho_2, \varrho_2 - \varrho_1) \\
&\quad -(\omega_1 + \omega_2, \omega_2 - \omega_1)\|^2 \\
&= \|(2(\varrho_1 - \omega_1) + (\varrho_2 - \omega_2)), 2(\varrho_2 - \omega_2 - (\varrho_1 - \omega_1))\|^2 \\
&= (2(\varrho_1 - \omega_1) + (\varrho_2 - \omega_2))^2 + (2(\varrho_2 - \omega_2 - (\varrho_1 - \omega_1))^2 \\
&= 4(\varrho_1 - \omega_1)^2 + (\varrho_2 - \omega_2)^2 + 4(\varrho_2 - \omega_2)^2 + (\varrho_1 - \omega_1)^2 \\
&= 5[(\varrho_1 - \omega_1)^2 + (\varrho_2 - \omega_2)^2] \\
&= 5\|\varrho - \omega\|^2 \\
&= (\beta+1)\|\varrho - \omega\|^2.
\end{aligned}$$

*Hence,* $\Gamma$ *is a* $1$*-enriched* $\frac{\sqrt{5}}{2}$ *Lipschitz mapping.*

If a mapping is $(\beta, k)$-enriched, strictly pseudocontractive (for short, $(\beta, k)$-ESPCM), then for all $\varrho, \omega \in K$, there exist $\beta \in [0, \infty)$ and $k \in [0,1)$ such that the following inequality holds:

$$\|\beta\varrho + \Gamma\varrho - (\beta\omega + \Gamma\omega)\|^2 \leq (\beta+1)^2\|\varrho - \omega\|^2 + k\|(I-\Gamma)\varrho - (I-\Gamma)\omega\|^2. \qquad (4)$$

For some special cases in which $\beta = 0$ in one part and $k = 0$ in another part, inequality (4) reduces to two classes of mappings known as strictly pseudocontractive mappings (recall that a nonlinear mapping $\Gamma : K \longrightarrow K$ is called a strictly pseudocontractive mapping if for all $\varrho, \omega \in K$, there exists$k \in [0,1)$ such that $\|\Gamma\varrho - \Gamma\omega\|^2 \leq \|\varrho - \omega\|^2 + k\|(I-\Gamma)\varrho - (I-\Gamma)\omega\|^2$) and $\beta$-enriched nonexpansive mappings, respectively. Hence, the class of $(\beta, k)$-ESPCM is larger than the class of $\beta$-enriched nonexpansive mappings and the class of $k$-strictly pseudocontractive mappings; see [1,4–15] for more details.

Now, by substituting $\beta = \dfrac{1}{\rho} - 1$ into inequality (4) and simplifying, we obtain

$$\|\Gamma_\rho\varrho - \Gamma_\rho\omega\|^2 \leq \|\varrho - \omega\|^2 + k\|(I-\Gamma_\rho)\varrho - (I-\Gamma_\rho)\omega\|^2, \tag{5}$$

where $\rho \in (0,1]$, and $\Gamma_\rho$ is as defined in inequality (3). Note that the average operator $\Gamma_\rho$ is $k$-strictly pseudocontractive.

In [10], Berinde introduced the concept of $(\beta, k)$-ESPCM and showed that this class of mappings is more general than the class of $k$-strictly pseudocontractive mappings studied in [12,16]. It is of interest to note that the Lipschitz properties enjoyed by the class of strictly pseudocontractive mappings (due to the structure of their definition) are far from the reach of Lipschitz pseudocontractive mappings.

**Example 2.** *Let $X = R^2$ be equipped with the Euclidean norm, and we have the following:*

$$C = \{(\varrho_1, \varrho_2) \in R^2, \varrho_1, \varrho_2 \geq 0, \varrho_1^2 + \varrho_2^2 \leq 1\}.$$

*Define the mapping $\Gamma : C \longrightarrow C$ by*

$$\Gamma(\varrho, \omega) = \left(\frac{\varrho}{2}, \frac{\omega}{2}\right).$$

*It is not difficult to see that $X$ is a uniformly convex Banach space and that $C$ is a bounded, closed and convex subset of $X$. Let $\beta \in [0, \infty)$ and $\alpha \in [0,1)$. It is shown in [1] that $\Gamma$ is a $(\beta, \alpha)$-enriched strictly pseudocontractive mapping and $F(\Gamma) = (0,0)$.*

**Remark 2.** *If, we take $k = 1$ in inequality (4), then we obtain a class of nonlinear mappings called $\beta$-enriched pseudocontraction mappings. Thus, the class of $(\beta, k)$-ESPCM is smaller than the class of $\beta$-enriched pseudocontractive mappings.*

Let $H_a$ and $H_b$ be two Hilbert spaces and $W$ *and* $V$ be nonempty, closed and convex subsets of $H_a$ and $H_b$, respectively. Consider two nonlinear mappings: $\Gamma : H_a \longrightarrow H_b$ and $\Upsilon : H_b \longrightarrow H_b$. The split feasibility problem (for short, SFP) is given as follows: find a point $q \in H_a$ such that

$$q \in W \text{ and } Bq \in V, \tag{6}$$

where $B : H_a \longrightarrow H_b$ is a bounded operator. If the solution of (6) exists, then it can be shown that $\varrho \in W$ solves (6) if and only if it solves the following fixed point equation:

$$\varrho = P_W((I - \lambda B^\star(I - P_V)B)\varrho), \quad \varrho \in W, \tag{7}$$

where $P_W$ and $P_V$ are projections of $W$ *and* $V$, respectively, $\lambda$ is a positive constant, and $B^\star$ represents the adjoint of $B$. When $W$ and $V$ in (6) (where $\emptyset \neq W \subset H_a$ and $\emptyset \neq V \subset H_b$ are closed and convex) are sets of fixed points of nonlinear mappings $\Gamma$ and $\Upsilon$, then the split feasibility problem is also called the common fixed point problem (for short, SCFPP) (see, [17,18]); that is, given $m$ nonlinear operators $\{\Gamma^i\}_{i=1}^m : H_a \longrightarrow H_a$ and $n$ nonlinear operators $\{\Upsilon^j\}_{j=1}^n : H_b \longrightarrow H_b$, the SCFPP for finitely many operators, which is desirable in practical situations, is to find a point

$$\varrho \in \cap_{i=1}^m F(\Gamma^i) \text{ such that } B\varrho \in \cap_{j=1}^n F(\Upsilon^j). \tag{8}$$

In a special case for which $\Gamma^i = P_{W_i}$ and $\Upsilon^j = P_{V_j}$, the SCFPP reduces to the multiple-set split feasibility problem (for short, MSSFP): that is, to find $\varrho \in \cap_i^m W_i$ such that $B\varrho \in \cap_j^n V_i$, where $\{W_i\}_{i=1}^m$ and $\{V_j\}_{j=1}^n$ are nonempty, closed and convex subsets of $H_a$ and $H_b$, respectively. We shall denote the solution to problem (8) in this special case by $D = \{\varrho \in \cap_i^m W_i : B\varrho \in \cap_j^n V_i\}$.

In the setup of a real Hilbert space, problems (6) and (8) have been studied extensively by different authors; see, for example, [17–28].

In [22], Censor and Segal introduced the following algorithm:

$$\varrho_{n+1} = \Upsilon(I - \lambda B^\star)(I - \Gamma)B\varrho_n, \tag{9}$$

which solves problem (6) for directed operators.

Recently, Chang et al. [28] introduced and studied the following fixed point algorithm: for an arbitrary $\varrho_0 \in H_1$, let $\{\varrho_n\}_{n=1}^\infty$ be a sequence generated iteratively as follows:

$$\begin{cases} \varrho_0 \in H_1 \textit{ chosen arbitrarily };\\ \varrho_{n+1} = \delta_{n,0}\omega_n + \sum_{j=1}^\infty \delta_{n,j}\Gamma_{j,\beta}\omega_n;\\ \omega_n = \varrho_n + \lambda B^\star(S_{n(modN)} - I)B\varrho_n, \end{cases} \tag{10}$$

where $\{\delta_{n,j}\}_{n=1}^\infty$ is a countably infinite family of real sequences in $[0,1]$; $\sum_{j=1}^\infty \delta_{n,j} = 1$, $\Gamma_{j,\beta} = \beta I + (1-\beta)\Gamma_j, \beta \in (0,1)$ is a constant; $\{\Gamma_j\}_{j=1}^\infty : H_1 \longrightarrow H_1$ is an infinite family of $\alpha_i$-strictly pseudononspreading mappings; $\{S_j\}_{j=1}^N$ is a finite family of $\gamma_i$-strictly pseudononspreading mappings; and $\lambda > 0$. Using (10), they proved weak and strong convergence theorems.

Subsequently, different researchers have extended and generalized (9) in different directions. Alsulami et al. [19] proved some strong convergence theorems for finding a solution of problem (6) in Banach spaces; in [23], (9) was extended to the case of quasi-nonexpansive mappings, which was later extended to the case of demicontractive mappings in [24,25]; Takahashi generalized the results in [22] to Banach spaces. For more works relating to split feasibility problems, the interested reader is referred to [20,25–27] and the references therein.

Symmetry is an important concept used in Hilbert spaces and plays a crucial role in the structure of a complete inner product space. Also, the concept of symmetry, which includes symmetric operators, has been investigated in real Hilbert spaces. In this paper, inspired and motivated by the results in [29,30], we propose a horizontal iteration technique for solving the multiple-set split feasibility problem in the more general cases of a pair of finite families of $\beta$-enriched strictly pseudocontractive mappings in an infinite-dimensional Hilbert space and establish strong and weak convergence theorems for approximating a common solution for the aforementioned problem. From recent studies, it has been observed (see, for instance, [31]) that iteration techniques involving more than one auxiliary mapping are more robust against certain numerical errors than the ones in which only one auxiliary mapping is used. Consequently, our method is more efficient in application than some of the methods in related works. Finally, it is worth mentioning that the technique presented in this paper does not require a 'sum condition', which has been the case for most of the iterative methods in this direction. Concerning application, we consider the algorithm for hierarchical variational inequality problems through slightly modifying our iterative scheme. Our results improve and generalize several results in the current literature.

The rest of the manuscript is organized as follows: Section 2 is devoted to some preliminary results that will be required to establish our main results; Theorems 1 and 2 will be the subjects of Sections 3–5 and will conclude the paper.

## 2. Preliminary

In the following, we first recall some notations, definitions and known results that are currently in the literature, which will be required to prove the main results of this present paper.

**Assumption 1.** *Throughout the remaining sections, $H$, $K$, $\mathbb{N}, \mathbb{R}, \rightarrow, \rightharpoonup$ and $B : H \longrightarrow H$ shall represent a real Hilbert space, a nonempty closed and convex subset of $H$, the set of natural numbers, the set of real numbers, strong convergence, weak convergence and a bounded linear operator, respectively.*

Also, for the sake of convenience, we restate the following concepts and results.

Let $H$ and $K$ be defined as in Assumption 1. For every $\varrho \in H$, there exists a unique nearest point in $K$, represented as $P_K\varrho$, such that

$$\|\varrho - P_K\varrho\| \leq \|\varrho - \omega\|, \forall \omega \in K$$

and it has been established that for every $\varrho \in H$,

$$\langle \varrho - P_K\varrho, \omega - P_K\varrho \rangle \leq 0, \forall \omega \in K. \quad (11)$$

**Definition 2** ([32])**.** *Let $Z$ be a real Banach space and $\Gamma : Z \longrightarrow Z$ be a self-mapping on $Z$. Then, the following is considered:*

(*i*) *$I - \Gamma$ is said to be demiclosed at zero if for any sequence $\{\varrho_n\}_{n\geq 1} \subset Z$ with $\varrho_n \rightarrow \varrho^\star$ and $\|\varrho_n - \Gamma\varrho_n\| \rightarrow 0$ as $n \rightarrow \infty$, we obtain $\varrho^\star = \Gamma\varrho^\star$;*

(*ii*) *$\Gamma$ is called semicompact if for any bounded sequence $\{\varrho_n\}_{n\geq 1} \subset Z$ with $\|\varrho_n - \Gamma\varrho_n\| \rightarrow 0$ as $n \rightarrow \infty$, there exists a subsequence $\{\varrho_{n_j}\}_{j\geq 1}$ of $\{\varrho_n\}_{n\geq 1}$ such that $\varrho_{n_j} \rightarrow \varrho^\star \in Z$.*

**Definition 3** ([32])**.** *Let $Z$ be a uniformly convex Banach space and $K$ a closed and convex subset of $Z$. A mapping $\Gamma : K \longrightarrow K$ is called asymptotically regular on $K$ if for each $x \in K$,*

$$\|\Gamma^{n+1}x - \Gamma^n x\| \rightarrow 0 \qquad \text{as} \qquad n \rightarrow \infty.$$

**Definition 4** ([32])**.** *Let $Z$ be a uniformly convex Banach space and $C$ a closed and convex subset of $E$. A mapping $\Gamma : K \longrightarrow Z$ is called demicompact if it has the property that if $\{\omega_n\}_{n\geq 1}$ is a bounded sequence in $Z$ and $\{\Gamma\omega_n - \omega_n\}_{n\geq 1}$ is strongly convergent, then there exists a subsequence $\{\omega_{n_k}\}_{k\geq 1}$ of $\{\omega_n\}_{n\geq 1}$ that is strongly convergent.*

**Lemma 1.** *Let $\emptyset \neq K \subset H$, where $H$ is a real Hilbert space, closed and convex, and let $\Gamma : K \longrightarrow K$ be an $\alpha$-strictly pseudocontractive mapping. Then, the following applies:*

(*i*) *If $F(\Gamma) \neq \emptyset$, then $f(\Gamma)$ is closed and convex;*

(*ii*) *$I - \Gamma$ is demiclosed at zero.*

**Lemma 2** ([12])**.** *Let $\{\delta_n\}_{n\geq 1}, \{\tau_n\}_{n\geq 1}, \{\lambda_n\}_{n\geq 1} \subset [0, \infty)$, satisfying the inequality*

$$\delta_{n+1} = (1 - \lambda_n)\delta_n + \tau_n, n \geq 1. \quad (12)$$

*If $\sum_{i=1}^{\infty} \lambda_n < \infty$ and $\sum_{i=1}^{\infty} \tau_n < \infty$, then the $\lim_{n\to\infty} \delta_n$ exists.*

**Lemma 3** ([7,26])**.** *Let $H$ be as in Assumption 1; then, for all $\varrho, \omega \in H$, the following inequality holds:*

$$\|\varrho + \omega\|^2 \quad \leq \quad \|\varrho\|^2 + 2\langle \omega, \varrho + \omega \rangle. \quad (13)$$

**Proposition 1** ([30])**.** *Let $\{\delta_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ be a countable subset of the set of real numbers $\mathbb{R}$, where $k$ is a fixed non-negative integer and $N$ is any integer with $k + 1 \leq N$. Then, the following identity holds:*

$$\delta_k + \sum_{j=k+1}^{\mathbb{N}} \delta_j \prod_{i=k}^{j-1}(1 - \delta_i) + \prod_{i=k}^{\mathbb{N}}(1 - \delta_i) = 1. \quad (14)$$

**Proposition 2** ([30]). *Let $t, u$ and $v$ be arbitrary elements of a real Hilbert space $H$. Let $k$ be any fixed non-negative integer and $N \in \mathbb{N}$ be such that $k+1 \leq N$. Let $\{v_i\}_{i=1}^{N-1} \subseteq H$ and $\{\delta_i\}_{i=1}^{N} \subseteq [0,1]$ be countable finite subsets of $H$ and $\mathbb{R}$, respectively. Define*

$$y = \delta_k t + \sum_{i=k+1}^{N} \delta_i \prod_{j=k}^{i-1}(1-\delta_j)v_{i-1} + \prod_{j=k}^{N}(1-\delta_j)v.$$

*Then,*

$$\begin{aligned}
\|y-u\|^2 \quad = \quad & \delta_k\|t-u\|^2 + \sum_{j=k+1}^{N} \delta_j \prod_{i=k}^{j-1}(1-\delta_i)\|v_{j-1}-u\|^2 + \prod_{i=k}^{N}(1-\delta_i)\|v-u\|^2 \\
& -\delta_k\Big[\sum_{j=k+1}^{N} \delta_j \prod_{i=k}^{j-1}(1-\delta_i)\|t-v_{j-1}\|^2 + \prod_{i=k}^{j-1}(1-\delta_i)\|t-v\|^2\Big] \\
& -(1-\delta_k)\Big[\sum_{j=k+1}^{N} \delta_j \prod_{i=k}^{j-1}(1-\delta_i)\|v_{j-1}-(\delta_{j+1}+w_{j+1})\|^2 \\
& +\delta_N \prod_{i=k}^{j-1}(1-\delta_i)\|v-v_{N-1}\|^2\Big],
\end{aligned}$$

*where $w_k = \sum_{j=k+1}^{N} \delta_j \prod_{i=k}^{j-1}(1-\delta_i)v_{j-1} + \prod_{i=k}^{N}(1-\alpha_i)v, k = 1, 2, \cdots, N$ and $w_n = (1-c_n)v$.*

**Lemma 4** ([2]). *Let $K$ be a nonempty, bounded, closed and convex subset of a real Banach space $Z$, $\Gamma : K \longrightarrow K$ a nonexpansive mapping and $F(\Gamma) \neq \emptyset$; then, for any given $\beta \in (0,1)$, the mapping $\Gamma_\beta = (1-\beta)I + \beta\Gamma$, where $I$ is the identity operator, has the same fixed point as $\Gamma$ and is asymptotically regular.*

**Remark 3.** *When $\Gamma$ is nonexpansive, so is $\Gamma_\beta$, and both have the same fixed point; however, $\Gamma_\beta$ has more felicitous asymptotic behavior than the original mapping (see [2] for details).*

## 3. Main Results

First, we provide an iterative scheme as well as a convergence study regarding this scheme with respect to the solutions to the split feasibility problem for a pair of finite families of $\beta$-enriched strictly pseudocontractive mappings.

**Assumption 2.** *Consider the following:*

($a$) *Let $H_1$ and $H_2$ be two real Hilbert spaces: $B : H_1 \longrightarrow H_2$, a bounded linear operator; and $B^\star : H_2 \longrightarrow H_1$, the adjoint of $B$;*

($b$) *Let $\{\Gamma_i\}_{i=1}^{N} : H_1 \longrightarrow H_1$ be a finite family of $(\alpha_i, \beta)$-enriched strictly pseudocontractive and demicompact mappings with $\alpha = \max\limits_{i\in N}\{\alpha_i\} \in (0,1)$;*

($c$) *Let $\{S_i\}_{i=1}^{N} : H_1 \longrightarrow H_1$ be a finite family of $(\gamma_i, \beta)$-enriched strictly pseudocontractive and demicompact mappings with $\gamma = \max\limits_{i\in N}\{\gamma_i\} \in (0,1)$;*

($d$) *Let $W = \cap_{i=1}^{N} F(\Gamma_i) \neq \emptyset$ and $V = \cap_{i=1}^{N} F(S_i) \neq \emptyset$;*

($e$) *Let $D$ be a set of solutions of (MSSFP); that is, $D = \{\varrho^\star \in W : B\varrho^\star \in V\}$.*

*Now, we present our iteration scheme as follows.*

*Let $H_1, H_2, B, B^\star, \{\Gamma_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty}, W, V, \alpha$ and $\gamma$ be as in Assumption 2. For an arbitrary point $\varrho_1 \in H_1$, construct the sequence $\{\varrho_n\}_{n\geq 1}$ iteratively as follows:*

$$\begin{cases} \varrho_1 \in H_1 \text{ chosen arbitrarily }; \\ \varrho_{n+1} = \delta_{n,1}\omega_n + \sum_{j=2}^{N} \delta_j \prod_{i=1}^{j-1}(1-\delta_i)\Gamma^{j-1}\omega_n + \prod_{i=1}^{N}(1-\delta_i)\Gamma^N\omega_n, \qquad n \geq 1; \\ \omega_n = \varrho_n + \lambda B^\star(S_{n(modN)} - I)B\varrho_n, \end{cases} \tag{15}$$

*where* $\{\{\delta_{n,j}\}_{n=1}^{\infty}\}_{j=1}^{N}$ *is a countably finite family of real sequences in* $[0,1]$.

**Theorem 1.** *Let* $H_1, H_2, B, B^\star, \{\Gamma_i\}_{i=1}^{\infty}, \{S_i\}_{i=1}^{\infty}, W, V, \alpha$ *and* $\gamma$ *be as stated in Assumption 2. Let* $\{\varrho_n\}_{n\geq 1}$ *be a sequence given by (15). If* $\{\{\delta_{n,j}\}_{n=1}^{\infty}\}_{j=1}^{N} \in [0,1]$ *satisfies the following conditions:*

*(1)* $\delta_{n,1} > \alpha > \max\{\alpha_i\}_{i=1}^{N}; \delta_{n,1} < \delta < 1,$ *for each* $i$;
*(2)* $\liminf_{n\to\infty} \prod_{i=1}^{j-1}(1-\delta_i)(\delta_{n,1} - \alpha_{i-1}) > 0, j = 2, \cdots, N;$
*(3)* $\liminf_{n\to\infty} \prod_{i=1}^{j-1}(1-\delta_i)(\delta_{n,1} - \alpha_N) > 0;$
*(4)* $\lambda \in \left(0, \dfrac{1-\gamma}{\|B\|^2}\right).$

*then both* $\{\varrho_n\}_{n\geq 1}$ *and* $\{\omega_n\}_{n\geq 1}$ *converge strongly and weakly to some* $\varrho^\star \in D$.

**Proof.** Since $\{\Gamma_j\}_{i=1}^{N}$ is $(\beta, \alpha_j)$-ESPCM for each $j$, by setting $\beta = \dfrac{1}{\rho} - 1$ for $\beta > 0$ and $\rho \in (0,1]$, we obtain from (5) that

$$\|\frac{1-\rho}{\rho}(\varrho - \omega) + \Gamma^j\varrho - \Gamma^j\omega\|^2 \leq \frac{1}{\rho^2}\|\varrho - \omega\|^2 + \alpha_j\|\varrho - \omega - (\Gamma^j\varrho - \Gamma^j\omega)\|^2,$$

which upon simplifying yields

$$\|\Gamma_\rho^j\varrho - \Gamma_\rho^j\omega\|^2 \quad\leq\quad \|\varrho - \omega\|^2 + \alpha_j\|\varrho - \omega - (\Gamma_\rho^j\varrho - \Gamma_\rho^j\omega)\|^2, \tag{16}$$

where $\Gamma_\rho^j = (1-\rho)I + \rho\Gamma^j$, and $I$ denotes the identity mapping on $H$. It is clear that the finite family of the average operator $\{\Gamma_j\}_{i=1}^{n}$ is an $\alpha_j$-strictly pseudocontractive mapping.

Again, since $\{S_j\}_{i=1}^{N}$ is $(\beta, \gamma_j)$-ESPCM for each $j$, by following a similar approach as in (16), we obtain

$$\|S_\rho^j\varrho - S_\rho^j\omega\|^2 \quad\leq\quad \|\varrho - \omega\|^2 + \gamma_j\|\varrho - \omega - (S_\rho^j\varrho - S_\rho^j\omega)\|^2, \tag{17}$$

where $S_\rho = S^\rho = (1-\rho)I + \rho S$, and $I$ denotes the identity mapping on $H$. It is obvious that the finite family of the average operator $\{S_j\}_{i=1}^{n}$ is again an $\gamma_j$-strictly pseudocontractive mapping.

Recall that for each $j \in \mathbb{N}$,

$$\begin{aligned} \|\Gamma_\rho^j\varrho - \Gamma_\rho^j\omega\|^2 &= \|\varrho - \omega - [\varrho - \Gamma_\rho^j\varrho - (\omega - \Gamma_\rho^j\omega)]\|^2 \\ &= \|\varrho - \omega\|^2 - 2\langle \varrho - \omega, \varrho - \Gamma_\rho^j\varrho - (\omega - \Gamma_\rho^j\omega)\rangle \\ &\quad + \|\varrho - \Gamma_\rho^j\varrho - (\omega - \Gamma_\rho^j\omega)\|^2. \end{aligned} \tag{18}$$

Inequality (16) and Equation (18) imply that for each $j \in \mathbb{N}$,

$$\langle \varrho - \omega, P^j\varrho - P^j\omega\rangle \geq \frac{1-\alpha_j}{2}\|P^j\varrho - P^j\omega\|^2, \tag{19}$$

where $P^j = I - \Gamma_\rho^j$.

Let $Q$ be a convex subset of a linear space $Z$ and $\{\Gamma_\rho^j\}_{j=1}^{N} : Q \longrightarrow Q$ be a given map.

Then, for any $\delta \in \left[\dfrac{1}{\rho+1}, 1\right)$ with $\rho > 0$ and for each $j \in \mathbb{N}$, the mapping $P^j_\delta : Q \longrightarrow Q$ is defined by

$$P^j_\delta = \varrho - \delta P^j \varrho = (1-\delta)\varrho + \delta\Gamma^j_\rho \varrho = (1-\tau)\varrho + \tau\Gamma^j \varrho, \tag{20}$$

where $\tau = \delta\rho \in \left[\dfrac{1}{1+\delta\rho}, 1\right)$ for $\delta\rho > 0$ denotes a translation of $\delta\Gamma_\rho\varrho$ through the vector $(1-\delta)\varrho$.

Now, since

$$\begin{aligned} \|P^j_\delta\varrho - P^j_\delta\omega\|^2 &= \|\varrho - \omega - \delta(P^j\varrho - P^j\omega)\|^2 \\ &= \|\varrho - \omega\|^2 - 2\delta\langle P^j\varrho - P^j\omega, \varrho - \omega + \delta^2\|P^j\varrho - P^j\omega\|^2, \end{aligned}$$

it follows from inequality (19) that

$$\|P^j_\delta\varrho - P^j_\delta\omega\|^2 \leq \|\varrho - \omega\|^2 - \delta(1-\alpha_i)\|P^j\varrho - P^j\omega\|^2 + \delta^2\|P^j\varrho - P^j\omega\|^2,$$

so that for any $\delta$ with $0 < \delta < 1 - \alpha_j$, for each $j \in N$, we obtain

$$\|P^j_\delta\varrho - P^j_\delta\omega\|^2 \leq \|\varrho - \omega\|^2, \quad \forall \varrho, \omega \in W. \tag{21}$$

Using the above information, we restate the iterative scheme defined by (15) as follows:

$$\begin{cases} \varrho_1 \in H_1 \textit{ chosen arbitrarily }; \\ \varrho_{n+1} = \delta_{n,1}\omega_n + \sum_{j=2}^N \delta_j \prod_{i=1}^{j-1}(1-\delta_i)P^{j-1}_\delta\omega_n + \prod_{i=1}^N(1-\delta_i)P^N_\delta\omega_n, \qquad n \geq 1; \\ \omega_n = \varrho_n + \lambda B^\star(S^\rho_{n(modN)} - I)B\varrho_n, \end{cases} \tag{22}$$

with the conditions on the iteration parameters still as in (15).

Now, we show that the sequences $\{\varrho_n\}_{n\geq 1}, \{\omega_n\}_{n\geq 1}$ *and* $\{P^{j-1}_\delta\omega_n\}_{n\geq 1}$ are bounded. By the definition of $D$, for a given $q \in D$, we obtain

$$q \in W = \cap_{j=1}^N F(\Gamma_j) = \cap_{j=1}^N F(P^j_\delta)$$

and

$$q \in V = \cap_{j=1}^N F(S_j) = \cap_{j=1}^N F(S^j_\delta).$$

Thus, $Bq = S_{n(modN)}Bq$.

Since $\{P^j_\delta\}_{j=1}^N$ is a finite family of an $\alpha_j$-strictly pseudocontractive mapping for each $j$, it follows from Lemma 1 that $W = \cap_{j=1}^N F(P^j_\delta)$ is closed and convex. Consequently, using Proposition 2 with $y = \varrho_{n+1}, t = \omega_n, v_{j-1} = P^{j-1}_\delta\omega_n, v = P^N_\delta\omega_n, k = 1$ *and* $u = q$, for each $n \geq 1$ and $q \in D$, we obtain from (22) that

$$\begin{aligned}
\|\varrho_{n+1}-q\|^2 &= \|\delta_{n,1}\omega_n+\sum_{j=2}^{N}\delta_j\prod_{i=1}^{j-1}(1-\delta_i)\Gamma^{j-1}\omega_n+\prod_{i=1}^{N}(1-\delta_i)\Gamma^N\omega_n-q\|^2 \\
&\leq \delta_{n,1}\|\omega_n-q\|^2+\sum_{j=2}^{N}\delta_j\prod_{i=1}^{j-1}(1-\delta_i)\|P_\delta^{j-1}\omega_n-q\|^2+\prod_{i=1}^{N}(1-\delta_i)\|P_\delta^N\omega_n-q\|^2 \\
&\leq \left(\delta_{n,1}-q\|^2+\sum_{j=2}^{N}\delta_j\prod_{i=1}^{j-1}(1-\delta_i)+\prod_{i=1}^{N}(1-\delta_i)\right)\|\omega_n-q\|^2 \\
&= \|\omega_n-q\|^2. \quad (by\ Proposition\ 1)
\end{aligned} \tag{23}$$

Also, from (22), we have

$$\begin{aligned}
\|\omega_n-q\|^2 &= \|\varrho_n-q+\lambda B^\star(S^\rho_{n(modN)}-I)B\varrho_n\|^2 \\
&= \|\omega_n-q\|^2+2\lambda\langle\omega_n-q,B^\star(S^\rho_{n(modN)}-I)B\varrho_n\rangle \\
&\quad +\lambda^2\|B^\star(S^\rho_{n(modN)}-I)B\varrho_n\|^2.
\end{aligned} \tag{24}$$

Since

$$\begin{aligned}
\lambda^2\|B^\star(S^\rho_{n(modN)}-I)B\varrho_n\|^2 &= \lambda^2\langle B^\star(S^\rho_{n(modN)}-I)B\varrho_n,B^\star(S^\rho_{n(modN)}-I)B\varrho_n\rangle \\
&= \lambda^2\langle BB^\star(S^\rho_{n(modN)}-I)B\varrho_n,(S^\rho_{n(modN)}-I)B\varrho_n\rangle \\
&\leq \lambda^2|B\|^2\|(S^\rho_{n(modN)}-I)B\varrho_n\|^2
\end{aligned} \tag{25}$$

and since using inequality (17)

$$\begin{aligned}
\langle\omega_n-q,B^\star(S^\rho_{n(modN)}-I)B\varrho_n\rangle &= \langle B(\omega_n-q),(S^\rho_{n(modN)}-I)B\varrho_n\rangle \\
&= \langle B(\omega_n-q)+(S^\rho_{n(modN)}-I)B\varrho_n \\
&\quad -(S^\rho_{n(modN)}-I)B\varrho_n,(S^\rho_{n(modN)}-I)B\varrho_n\rangle \\
&= \langle(S^\rho_{n(modN)}-I)B\varrho_n-Bq,(S^\rho_{n(modN)}-I)B\varrho_n\rangle \\
&\quad -\|(S^\rho_{n(modN)}-I)B\varrho_n\|^2 \\
&= \frac{1}{2}\Bigg\{\|(S^\rho_{n(modN)}-I)B\varrho_n=Bq\|^2+\|(S^\rho_{n(modN)}-I)B\varrho_n\|^2 \\
&\quad -\|B\varrho_n-Bq\|^2\Bigg\}-\|(S^\rho_{n(modN)}-I)B\varrho_n\|^2 \\
&\leq \frac{1}{2}\left\{\|B\varrho_nBq\|^2+\gamma\|(S^\rho_{n(modN)}-I)B\varrho_n\|^2\right\} \\
&\quad +\frac{1}{2}\left\{\|(S^\rho_{n(modN)}-I)B\varrho_n\|^2-\|B\varrho_n-Bq\|^2\right\}-\|(S^\rho_{n(modN)}-I)B\varrho_n\|^2 \\
&= \frac{\gamma-1}{2}\|(S^\rho_{n(modN)}-I)B\varrho_n\|^2,
\end{aligned}$$

it follows from Equation (24) that

$$\begin{aligned}
\|\omega_n-q\|^2 &= \|\varrho_n-q+\lambda B^\star(S^\rho_{n(modN)}-I)B\varrho_n\|^2 \\
&= \|\varrho_n-q\|^2-\lambda(1-\gamma-\lambda\|B\|^2)\|B^\star(S^\rho_{n(modN)}-I)B\varrho_n\|^2.
\end{aligned} \tag{26}$$

Based on condition 4 from the statement, it is clear that $(1-\gamma-\lambda\|B\|^2) > 0$, and as a consequence, Equation (26) reduces to

$$\|\omega_n - q\| \quad \leq \quad \|\varrho_n - q\|, \qquad \forall n \geq 1. \tag{27}$$

Inequalities (23) and (27) imply that

$$\|\varrho_{n+1} - q\| \quad \leq \quad \|\varrho_n - q\|, \qquad \forall n \geq 1. \tag{28}$$

The last inequality implies that the $\lim_{n\to\infty} \|\varrho_n - q\|$ exists; from (27), it again follows that the $\lim_{n\to\infty} \|\omega_n - q\|$ exists. Thus, the sequences $\{\varrho_n\}_{n\geq 1}$ *and* $\{\omega_n\}_{n\geq 1}$ are bounded. Since for each $j \geq 1$, $\{P_\delta^j\}_{j=1}^N$ is nonexpansive, we have

$$\|P_\delta^j \omega_n - q\| \leq \|\omega_n - q\|.$$

Therefore, $\{P_\delta^j\}_{j=1}^N$ is also bounded for each $j \in \mathbb{N}$.

For each $j = 1, 2, \cdots, N$, denote $\eta_\mu^j = (1-\mu)I + \mu P_\delta^j$. Since $P_\delta^j$ is nonexpansive for each $j = 1, 2, \cdots, N$, it follows from Lemma 4 that $\eta_\mu$ is asymptotically regular. That is,

$$\|\varrho_n - \eta_\mu^j \varrho_n\| \to 0 \; as \; n \to \infty. \tag{29}$$

Also, for each $j \in \mathbb{N}$, we have

$$\eta_\mu^j \varrho - \varrho = \mu(P_\delta^j \varrho - \varrho) = \delta\rho\mu(\Gamma^j \varrho - \varrho). \tag{30}$$

Hence, for each $j \in \mathbb{N}$,

$$\|\varrho_n - P_\delta^j \varrho_n\| \to 0 \; as \; n \to \infty. \tag{31}$$

Next, we show that for each $j = 1, 2, \cdots, N$,

$$\lim_{n\to\infty} \|\omega_n - P_\delta^j \omega_n\| = 0 \; and \; \lim_{n\to\infty} \|S^\rho_{n(modN)} - I)B\varrho_n\| = 0. \tag{32}$$

Now, for any given $q \in D$, we obtain, using (22) and Proposition 2 with $y = \varrho_{n+1}$, $t = \omega_n, v_{j-1} = P_\delta^{j-1}\omega_n, v = P_\delta^N \omega_n, k = 1$ and $u = q$, that

$$\begin{aligned}
\|\varrho_{n+1} - q\|^2 \quad &= \quad \|\delta_{n,1}\omega_n + \sum_{j=2}^{N} \delta_j \prod_{i=1}^{j-1} (1-\delta_i)\Gamma^{j-1}\omega_n + \prod_{i=1}^{N}(1-\delta_i)\Gamma^N \omega_n - q\|^2 \\
&\leq \quad \delta_{n,1}\|\omega_n - q\|^2 + \sum_{j=2}^{N} \delta_j \prod_{i=1}^{j-1}(1-\delta_i)\|P_\delta^{j-1}\omega_n - q\|^2 + \prod_{i=1}^{N}(1-\delta_i)\|P_\delta^N \omega_n - q\|^2 \\
&\quad -\delta_{n,1}\left[\sum_{j=2}^{N} \delta_j \prod_{i=1}^{j-1}(1-\delta_i)\|\omega_n - P_\delta^{j-1}\omega_n\|^2 + \prod_{i=1}^{N}(1-\delta_i)\|\omega_n - P_\delta^N \omega_n\|^2\right].
\end{aligned} \tag{33}$$

Using a strict pseudocontraction condition on each $\{P_\delta^j\}_{j=1}^N$, we obtain

$$\begin{aligned}
\|\varrho_{n+1}-q\|^2 \leq\ & \delta_{n,1}\|\omega_n-q\|^2+\sum_{j=2}^N\delta_j\prod_{i=1}^{j-1}(1-\delta_i)\left[\|\omega_n-q\|^2+\alpha_j\|\omega_n-P_\delta^{j-1}\omega_n\|^2\right] \\
&+\prod_{i=1}^N(1-\delta_i)\left[\|\omega_n-q\|^2+\|\alpha_N\|\omega_n-P_\delta^N\omega_n\|^2\right] \\
&-\delta_{n,1}\left[\sum_{j=2}^N\delta_j\prod_{i=1}^{j-1}(1-\delta_i)\|\omega_n-P_\delta^{j-1}\omega_n\|^2+\prod_{i=1}^N(1-\delta_i)\|\omega_n-P_\delta^N\omega_n\|^2\right] \\
=\ & \left(\delta_{n,1}+\sum_{j=2}^N\delta_j\prod_{i=1}^{j-1}(1-\delta_i)+\prod_{i=1}^N(1-\delta_i)\right)\|\omega_n-q\|^2 \\
&-\left[(\delta_{n,1}-\alpha_j)\sum_{j=2}^N\delta_j\prod_{i=1}^{j-1}(1-\delta_i)\|\omega_n-P_\delta^{j-1}\omega_n\|^2+(\delta_{n,1}-\alpha_N)\prod_{i=1}^N(1-\delta_i)\|\omega_n-P_\delta^N\omega_n\|^2\right],
\end{aligned}$$

which by Proposition 1 and Equation (26) yields

$$\begin{aligned}
\|\varrho_{n+1}-q\|^2 \leq\ & \|\varrho_n-q\|^2-\lambda(1-\gamma-\lambda\|B\|^2)\|B^\star(S^\rho_{n(modN)}-I)B\varrho_n\|^2 \\
&-\Bigg[(\delta_{n,1}-\alpha_j)\sum_{j=2}^N\delta_j\prod_{i=1}^{j-1}(1-\delta_i)\|\omega_n-P_\delta^{j-1}\omega_n\|^2 \\
&+(\delta_{n,1}-\alpha_N)\prod_{i=1}^N(1-\delta_i)\|\omega_n-P_\delta^N\omega_n\|^2\Bigg],
\end{aligned}$$

Set

$$\begin{aligned}
M =\ & \left(\sum_{j=2}^N\delta_j\prod_{i=1}^{j-1}(1-\delta_i)(\delta_{n,1}-\alpha_j)\|\omega_n-P_\delta^{j-1}\omega_n\|^2+(\delta_{n,1}-\alpha_N)\prod_{i=1}^N(1-\delta_i)\|\omega_n-P_\delta^N\omega_n\|^2\right) \\
&+\lambda(1-\gamma-\lambda\|B\|^2)\|B^\star(S^\rho_{n(modN)}-I)B\varrho_n\|^2.
\end{aligned}$$

Then, we obtain from the last inequality that

$$M \leq \|\varrho_n-q\|^2-\|\varrho_{n+1}-q\|^2. \tag{34}$$

Applying conditions 2 and 3 from the statement and the fact that $\lambda(1-\gamma-\lambda\|B\|^2)>0$ in inequality (34), we obtain

$$\lim_{n\to\infty}\|\omega_n-P_\delta^j\omega_n\|=0 \textit{ and } \lim_{n\to\infty}\|(S^\rho_{n(modN)}-I)B\varrho_n\|=0. \tag{35}$$

Furthermore, we show that

$$\lim_{n\to\infty}\|\omega_{n+1}-\omega_n\|=0 \textit{ and } \lim_{n\to\infty}\|\omega_{n+1}-\omega_n\|=0.$$

Using (22) and Proposition 2 with $y=\varrho_{n+1}, t=\omega_n, v_{j-1}=P_\delta^{j-1}\omega_n, v=P_\delta^N\omega_n$, $k=1$ *and* $u=\varrho_n$, we have

$$\begin{aligned}
\|\varrho_{n+1}-\varrho_n\|^2 &= \|\delta_{n,1}\omega_n + \sum_{j=2}^{N}\delta_j \prod_{i=1}^{j-1}(1-\delta_i)\Gamma^{j-1}\omega_n + \prod_{i=1}^{N}(1-\delta_i)\Gamma^N\omega_n - \varrho_n\|^2 \\
&\leq \delta_{n,1}\|\omega_n-\varrho_n\|^2 + \sum_{j=2}^{N}\delta_j\prod_{i=1}^{j-1}(1-\delta_i)\|P_\delta^{j-1}\omega_n-\varrho_n\|^2 + \prod_{i=1}^{N}(1-\delta_i)\|P_\delta^N\omega_n-\varrho_n\|^2 \\
&\leq \delta_{n,1}\|\lambda B^\star(S^\rho_{n(modN)}-I)B\varrho_n\|^2 + \sum_{j=2}^{N}\delta_j\prod_{i=1}^{j-1}(1-\delta_i)\Big[\|P_\delta^{j-1}\omega_n-\omega_n\|+\|\omega_n-\varrho_n\|\Big]^2 \\
&\quad + \prod_{i=1}^{N}(1-\delta_i)\Big[\|P_\delta^N\omega_n-\omega_n\|+\|\omega_n\varrho_n\|\Big]^2. \qquad (36)
\end{aligned}$$

Since

$$\|\omega_n-\varrho_n\| = \lambda\|B^\star(S^\rho_{n(modN)}-I)B\varrho_n\| \to 0 \; as \; n\to\infty \quad (by\ (32)), \qquad (37)$$

it follows from Equation (35), inequality (36) and Equation (37) that

$$\lim_{n\to\infty}\|\varrho_{n+1}-\varrho_n\| = 0. \qquad (38)$$

Also, observe from (22) that

$$\begin{aligned}
\|\omega_{n+1}-\omega_n\| &\leq \|\varrho_{n+1}-\varrho_n\| + \lambda\|B^\star(S^\rho_{n(modN)}-I)B\varrho_{n+1}\| \\
&\quad +\lambda\|B^\star(S^\rho_{n(modN)}-I)B\varrho_n\| \to 0 \; as \; n\to\infty. \qquad (39)
\end{aligned}$$

Considering the above information, we are ready to present our strong and weak convergent results.

Now, since $\{\Gamma^j\}_{j=1}^N$ is demicompact (by hypothesis) for each $j$, it follows from (30) that $\{\eta_\mu^j\}_{j=1}^N$ is demicompact for each $j$. Therefore, using (29), we can find a subsequence $\{\varrho_{n_k}\}_{k\geq 1}$ of $\{\varrho_n\}_{n\geq 1}$ such that $\varrho_{n_j}\to q$ as $j\to\infty$. Further, by the continuity of $\{P_\delta^j\}_{j=1}^N$, for each $j$, it follows that $\{\eta_\mu^j\}_{j=1}^N$ is also continuous for each $j$, and hence,

$$\eta_\mu^j\varrho_{n_k} \to \eta_\mu^j q \; as \; k\to\infty.$$

Thus, $\{\varrho_{n_k}-\eta_\mu^j\varrho_{n_k}\}\to 0`as`k\to\infty$. Using the above information, we have $\eta_\mu^j q = q$ for all $j=1,2,\cdots,N$. To be precise,

$$q\in\cap_{j=1}^N F(\eta_\mu^j) = \cap_{j=1}^N F(P_\delta^j) = \cap_{j=1}^N F(\Gamma_\rho^j) = \cap_{j=1}^N F(\Gamma^j) = W. \qquad (40)$$

Using (28), we obtain that $\{\varrho_n\}_{n=1}^\infty$ converges strongly to $q\in D$.
Again, from (36), we obtain

$$\lim_{n\to\infty}\|(S^\rho_{n_k(modN)}-I)B\varrho_{n_k}\| = 0. \qquad (41)$$

Thus, for any $\tau\in\mathbb{N}$, there exists a subsequence $n_{j_k}\in n_j$ with $n_{j_k}(modN)=\tau$ such that

$$\lim_{n\to\infty}\|S_\tau^\rho B\varrho_{n_{j_k}} - B\varrho_{n_{j_k}}\| = 0. \qquad (42)$$

Obviously, from the boundedness of $B$ and decompactness and continuity property of $S_\tau^\rho$, it is easy to see from (42), by following the same reasoning as in (40), that

$$Bq\in\cap_{\tau=1}^N F(S_\tau^\rho) = \cap_{\tau=1}^N F(S_\tau) = V. \qquad (43)$$

holds.

Finally, we show that every cluster point $\varrho^\star$ of the sequence $\{\varrho_n\}_{n\geq 1}$ is a member of $D$.

Now, since $\{\omega_n\}_{n\geq 1}$ is a bounded sequence in $H_1$, this means that we can find a subsequence $\{\omega_{n_k}\}_{k\geq 1}$ of the sequence $\{\omega_n\}_{n\geq 1}$ such that $\omega_{n_k} \to \varrho^\star \in H_1$.

Using (35), we have

$$\lim_{n\to\infty} \|\omega_{n_k} - P_\delta^j \omega_{n_k}\| = 0 \tag{44}$$

for each $j \in \mathbb{N}$. Observe from (20) that for each $j \in \mathbb{N}$,

$$(I - P_\delta^j) = \delta(I - \Gamma_\rho^j), \tag{45}$$

which immediately guarantees that $(I - P_\delta^j)$ is also demiclosed at zero by the demiclosedness of $\Gamma_\rho$ (see Lemma 1). Consequently, $\varrho^\star \in F(P_\delta^j)$ for each $j \in \mathbb{N}$. Since $j$ is arbitrary, it follows that

$$\varrho^\star \in \cap_{j=1}^N F(P_\delta^j) = \cap_{j=1}^N F(\Gamma_\rho^j) = \cap_{j=1}^N F(\Gamma^j) = W.$$

Conversely, from (22) and (35), we obtain

$$\varrho_{n_k} = \omega_{n_k} - \lambda B^\star (S^\rho_{n_k(modN)} - I) B\varrho_{n_k} \rightharpoonup \varrho^\star. \tag{46}$$

In view of the boundedness of the linear operator $B$, we obtain

$$B\varrho_{n_k} \rightharpoonup B\varrho^\star. \tag{47}$$

Again, from (35), we have

$$\lim_{k\to\infty} \|(S^\rho_{n_k(modN)} - I)B\varrho_{n_k}\| = 0.$$

Thus, for any $\tau \in \mathbb{N}$, there exists a subsequence $n_{k_j} \in n_k$ with $n_{k_j}(modeN) = \tau$ such that

$$\lim_{kj\to\infty} \|S_\tau^\rho B\varrho_{n_{k_j}} - B\varrho_{n_{k_j}}\| = 0.$$

Following the demiclosedness of $\Gamma = S$ (see Lemma 1), we are guaranteed that $(I - S_\tau^\rho) = \rho(I - S)$ is also demiclosed at zero. From the above information and (47), we obtain that $\beta\varrho^\star \in F(S_\tau^\rho)$. By the arbitrariness of $\tau \in \mathbb{N}$, we have

$$B\varrho^\star \in \cap_{\tau=1}^N F(S_\tau^\rho) = \cap_{\tau=1}^N F(S_\tau) = V.$$

This completes the proof.

□

If $\lambda = 0$ in Theorem 1, then the following corollary emerges.

**Corollary 1.** *Let $H_1$, $\{\Gamma_i\}_{i=1}^\infty$, $W$ and $\alpha$ be as in Assumption 2. Let $\{\varrho_n\}_{n\geq 1}$ be a sequence given by*

$$\begin{cases} \varrho_1 \in H_1 \text{ chosen arbitrarily}; \\ \varrho_{n+1} = \delta_{n,1}\varrho_n + \sum_{j=2}^N \delta_j \prod_{i=1}^{j-1}(1-\delta_i)\Gamma^{j-1}\varrho_n + \prod_{i=1}^N (1-\delta_i)\Gamma^N \varrho_n, \qquad n \geq 1; \end{cases} \tag{48}$$

*If $\{\{\delta_{n,j}\}_{n=1}^\infty\}_{j=1}^N \in [0,1]$ satisfies following the conditions:*

*(1)* $\delta_{n,1} > \alpha > \max\{\alpha_i\}_{i=1}^N; \delta_{n,1} < \delta < 1,$ *for each $i$;*

*(2)* $\liminf_{n\to\infty} \prod_{i=1}^{j-1}(1-\delta_j)(\delta_{n,1} - \alpha_{i-1}) > 0, j = 2, \cdots, N;$

*(3)* $\liminf_{n\to\infty} \prod_{i=1}^{j-1}(1-\delta_i)(\delta_{n,1} - \alpha_N) > 0,$

*then $\{\varrho_n\}_{n\geq 1}$ converges strongly and weakly to some $\varrho^\star \in W$.*

## 4. Application

In this section, following the same approach as in [33,34], we shall make use of the results of Section 3 to study the hierarchical variational inequality problem.

Let $H$ *and* $\{\Gamma^j\}_{j=1}^N$ be as in Assumption Q with $\mathcal{F} \cap_{j=1}^N F(\Gamma^j) \neq \emptyset$. Let $S : H \longrightarrow H$ be a nonexpansive mapping. The well-known hierarchical variational inequality problem for the countably finite family of the mappings $\{\Gamma^j\}_{j=1}^N$ with respect to the mapping $S$ is to find a point $\varrho^\star \in \mathcal{F}$ such that

$$\langle \varrho^\star - S\varrho^\star, \varrho^\star - \varrho \rangle \leq 0, \quad \forall \varrho \in \mathcal{F}. \tag{49}$$

It is not difficult to see that (49) is equivalent to the fixed point problem below: find $\varrho^\star \in \mathcal{F}$ such that

$$\varrho^\star = P_{\mathcal{F}} S \varrho^\star, \tag{50}$$

where $P_{\mathcal{F}}$ is the metric projection of $H$ onto $\mathcal{F}$. In setting $W = \mathcal{F}$ and $V = F(P_{\mathcal{F}}S)$ (the set of fixed point of $P_{\mathcal{F}}S$) and $B = I$ (the identity mapping on $H$), then the problem (50) is equivalent to the multiple-set split feasibility problem defined as follows: find $\varrho^\star \in W$ such that

$$\varrho^\star \in V. \tag{51}$$

Consequently, Theorem 2 below follows immediately from Theorem 1.

**Theorem 2.** *Let $H_1, H_2, B, B^\star, \{\Gamma_i\}_{i=1}^\infty, \{S_i\}_{i=1}^\infty, W, V, \alpha$ and $\gamma$ be as stated in Theorem 1. Let $\{\varrho_n\}_{n\geq 1}$ and $\{\omega_n\}_{n\geq 1}$ be the sequences are given by*

$$\begin{cases} \varrho_1 \in H_1 \text{ chosen arbitrarily}; \\ \varrho_{n+1} = \delta_{n,1}\omega_n + \sum_{j=2}^N \delta_j \prod_{i=1}^{j-1}(1-\delta_i)\Gamma^{j-1}\omega_n + \prod_{i=1}^N (1-\delta_i)\Gamma^N \omega_n, & n \geq 1; \\ \omega_n = \varrho_n + \lambda(S - I)\varrho_n, \end{cases} \tag{52}$$

*where $\{\{\delta_{n,j}\}_{n=1}^\infty\}_{j=1}^N$ is a countably finite family of real sequences in $[0,1]$, and $\lambda > 0$, satisfying the following conditions:*

*(1)* $\delta_{n,1} > \alpha > \max\{\alpha_i\}_{i=1}^N; \delta_{n,1} < \delta < 1$, *for each* $i$;

*(2)* $\liminf\limits_{n\to\infty} \prod_{i=1}^{j-1}(1-\delta_i)(\delta_{n,1} - \alpha_{i-1}) > 0, j = 2, \cdots, N$;

*(3)* $\liminf\limits_{n\to\infty} \prod_{i=1}^{j-1}(1-\delta_i)(\delta_{n,1} - \alpha_N) > 0$;

*(4)* $\lambda \in (0,1)$.

*If $W \cap V \neq \emptyset$, then $\{\varrho_n\}_{n\geq 1}$ converges weakly to a solution of the hierarchical variational inequality problem (49). Further, if one of the mappings $\{\Gamma^j\}_{j=1}^N$ is demicompact, then both $\{\varrho_n\}_{n\geq 1}$ and $\{\omega_n\}_{n\geq 1}$ converge strongly to a solution of the hierarchical variational inequality problem (49).*

**Proof.** Based on the fact that $S$ is nonexpansive, by Remark 1, $S$ is a 0-enriched nonexpansive mapping (and, by extension, a 0-enriched pseudocontracive mapping with $\gamma = 0$). In taking $\mathbb{N} = 1$ *and* $B = I$ (where $I$ is the identity mapping on $H$) in Theorem 1, then all the conditions of Theorem 1 are satisfied. Hence, the conclusion of Theorem 2 immediately follows from that of Theorem 1. □

## 5. Numerical Example

In this section, we illustrate the convergence result of Theorem 1.

The following are examples of $(0, \alpha_i)$-enriched strictly pseudocontractive mappings and $(0, \gamma_i)$-enriched strictly pseudocontractive mappings.

**Example 3.** *Let $H_1 = \ell_2 = H_2$. For each $i \in \{1, 2, \cdots, N\}$, let $\Gamma_i, S_i : \ell_2 \longrightarrow \ell_2$ be defined by*

$$\Gamma_i = -(i+1)\varrho$$

*and*

$$S_i = -2\varrho$$

*for all* $\varrho = (\varrho_1, \varrho_2, \cdots,) \in \ell_2$. *Then,*

$$D = (\bigcap_{1=1}^{N} F(\Gamma_i)) \cap (\bigcap_{1=1}^{N} F(S_i)) = \{0\}. \tag{53}$$

*Further, for each* $i \in \{1, 2, \cdots, N\}$, $\{\Gamma_i\}_{i=1}^N$ *is* $(0, \alpha_i)$*-enriched strictly pseudocontractive mappings. Indeed, for any* $\varrho, \omega \in \ell_2$ *and* $\beta = 0$, *we have*

$$\begin{aligned} \langle \varrho - \Gamma_i - (\omega - \Gamma_i\omega), (\beta+1)(\varrho - \omega)\rangle &= \langle \varrho - \Gamma_i - (\omega - \Gamma_i\omega), \varrho - \omega\rangle \\ &= \langle (i+1)(\varrho - \omega), \varrho - \omega\rangle = (i+2)\|\varrho - \omega\|^2. \end{aligned}$$

*Now, since*

$$\|\varrho - \Gamma_i - (\omega - \Gamma_i\omega)\|^2 = (i+2)^2\|\varrho - \omega\|^2,$$

*it follows that*

$$\langle \varrho - \Gamma_i - (\omega - \Gamma_i\omega), (\beta+1)(\varrho - \omega)\rangle \geq \alpha_i\|\varrho - \Gamma_i - (\omega - \Gamma_i\omega)\|^2,$$

*where* $\alpha_i = \dfrac{1}{(i+2)}$.

*Similarly,*

$$\langle \varrho - S_i - (\omega - S_i\omega), (\beta+1)(\varrho - \omega)\rangle \geq \gamma_i\|\varrho - \Gamma_i - (\omega - \Gamma_i\omega)\|^2,$$

*where* $\gamma_i = \dfrac{1}{3}$.

*Thus,* $\{\Gamma_i\}_{i=1}^N$ *and* $\{S_i\}_{i=1}^N$ *are* $(0, \alpha_i)$*-enriched strictly pseudocontractive mappings and* $(0, \gamma_i)$*-enriched strictly pseudocontractive mappings.*

**Example 4.** *Let* $H_1 = \ell_2 = H_2, C \subset H_1$ *and* $Q \subset H_2$. *For each* $i \in \{1, 2, \cdots, N\}$, *let* $\Gamma_i, S_i : \ell_2 \longrightarrow \ell_2$ *be defined by*

$$\Gamma_i = -(i+1)\varrho, \forall \varrho \in C$$

*and*

$$S_i = -2\varrho, \forall \varrho \in Q.$$

*Let* $\lambda = \dfrac{1}{4}, B\varrho = \varrho, \delta_{n,1} = \dfrac{1}{4}, \delta_{n,2} = \delta_{n,3} = \dfrac{1}{n}$ *and* $\{\varrho_n\}_{i=1}^{\infty}$ *be a sequence defined by*

$$\begin{cases} \varrho_1 \in H_1 \text{ chosen arbitrarily ;} \\ \varrho_{n+1} = \delta_{n,1}\omega_n + \sum_{j=2}^{N} \delta_j \prod_{i=1}^{j-1}(1-\delta_i)\Gamma^{j-1}\omega_n + \prod_{i=1}^{N}(1-\delta_i)\Gamma^N\omega_n, \qquad n \geq 1; \\ \omega_n = \varrho_n + \lambda B^\star(S_{n(modN)} - I)B\varrho_n, \end{cases} \tag{54}$$

*where* $\{\{\delta_{n,j}\}_{n=1}^{\infty}\}_{j=1}^N$ *is a countably finite family of real sequences in* $[0,1]$. *Then,* $\{\varrho_n\}_{i=1}^{\infty}$ *converges to an element of* $D$.

**Proof.** By Example 3, $\{\Gamma_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ are $(0, \alpha_i)$-enriched strictly pseudocontractive mappings and $(0, \gamma_i)$-enriched strictly pseudocontractive mappings with $\bigcap_{1=1}^{N} F(\Gamma_i) = 0 = \bigcap_{1=1}^{N} F(S_i)$, respectively. Clearly, $B$ is a bounded linear operator on $\ell_2$, and $B = B^\star = 1$.

Hence,

$$D = \{0 \in \bigcap_{1=1}^{N} F(\Gamma_i) : B(0) = \bigcap_{1=1}^{N} F(S_i)\} = \{0\}. \tag{55}$$

After simplifying (54) for $N = 3$ with $S_{n(modN)} = S_i$, we have

$$\begin{cases} \varrho_1 \in H_1 \text{ chosen arbitrarily };\\ \varrho_{n+1} = \delta_{n,1}\omega_n + V_n \\ \omega_n = \dfrac{\varrho_n}{4}, \end{cases} \tag{56}$$

where $V_n = (1-\delta_{n,1})\delta_{n,2}\Gamma^1\omega_n + (1-\delta_{n,1})(1-\delta_{n,2})\delta_{n,3}\Gamma^2\omega_n + (1-\delta_{n,1})(1-\delta_{n,2})(1-\delta_{n,3})\Gamma^3\omega_n$. Set $\alpha = \frac{1}{8}, \delta_{n,1} = \frac{1}{4}, \delta_{n,2} = \delta_{n,3} = \frac{1}{n}, \Gamma^1\omega_n = -2\omega_n, \Gamma^2\omega_n = -3\omega_n$ and $\Gamma^3\omega_n = -4\omega_n$. Then, (56) reduces to

$$\begin{cases} \varrho_1 \in H_1 \text{ chosen arbitrarily };\\ \varrho_{n+1} = \dfrac{1}{4}\left(1 - \dfrac{n(4n-1)+1}{4n^2}\right)\varrho_n, n \in \mathbb{N}. \end{cases} \tag{57}$$

Now, all the assumptions of Theorem 1 are satisfied. Thus, by Theorem 1, the sequence $\{\varrho_n\}_{i=1}^{\infty}$ defined by (57) converges to a unique element of $D$.
□

## 6. Conclusions

Finding the fixed points of nonlinear mappings (especially nonexpansive mappings) has received unprecedented attention due to its numerous applications in a variety of inverse problems, partial differential equations, image recovery, hierarchical variational inequality problems and signal processing. Interestingly, strictly pseudocontractive mappings (a subclass of the class of $(\beta, \alpha)$-enriched strictly pseudocontractive mappings, which we considered in this paper) have more powerful applications (see [29]) than nonexpansive mappings. Also, Theorem 3.1 complements and improves the corresponding results in [28] in the following ways:

(1) For the mapping, we replaced the mapping from a strictly pseudononspreading mapping to a $(\beta, \alpha)$-enriched strictly pseudocontractive mapping.
(2) For the fixed point iterative scheme, we propose a new horizontal iterative scheme for which the sum condition required for the main results in [28] is not needed. Under appropriate conditions, strong and weak convergent results are proven.

As an application, a slight modification of our iterative method was shown to be suitable for the approximation of hierarchical variational inequality problems.

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