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Article

Convex-Faced Combinatorially Regular Polyhedra of Small Genus

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Abstract: Combinatorially regular polyhedra are polyhedral realizations (embeddings) in Euclidean 3-space $\mathbb{E}^3$ of regular maps on (orientable) closed compact surfaces. They are close analogues of the Platonic solids. A surface of genus $g \geqslant 2$ admits only finitely many regular maps, and generally only a small number of them can be realized as polyhedra with convex faces. When the genus g is small, meaning that g is in the historically motivated range $2 \leqslant g \leqslant 6$, only eight regular maps of genus g are known to have polyhedral realizations, two discovered quite recently. These include spectacular convex-faced polyhedra realizing famous maps of Klein, Fricke, Dyck, and Coxeter. We provide supporting evidence that this list is complete; in other words, we strongly conjecture that in addition to those eight there are no other regular maps of genus g, with $2 \leqslant g \leqslant 6$, admitting realizations as convex-faced polyhedra in $\mathbb{E}^3$. For all admissible maps in this range, save Gordan's map of genus 4, and its dual, we rule out realizability by a polyhedron in $\mathbb{E}^3$.

Keywords: Platonic solids; regular polyhedra; regular maps; Riemann surfaces; polyhedral embeddings; automorphism groups

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1. Introduction

A *polyhedron* P, for the purpose of this paper, is a closed compact surface in Euclidean 3-space $\mathbb{E}^3$ made up from finitely many *convex* polygons, called the *faces* of P, such that any two polygons intersect, if at all, in a common vertex or a common edge (see McMullen, Schulz and Wills [1,2], Brehm and Wills [3], and Brehm and Schulte [4]). The *vertices* and *edges* of P are the vertices and edges of the faces of P, respectively. Throughout we insist that any two *adjacent* faces, which share a common edge, do not lie in the same plane. We usually identify P with the underlying map (cell complex) on the surface, or with the abstract polyhedron consisting of the vertices, edges, and faces (as well as $\emptyset$ and P as improper elements), partially ordered by inclusion (see Coxeter and Moser [5] and McMullen and Schulte [6]). In any case, since P is embedded in $\mathbb{E}^3$, the underlying surface is free of self-intersections and is necessarily orientable; and since the faces of P are convex, the underlying abstract polyhedron is necessarily a lattice, meaning here that any two distinct faces meet, if at all, in a common vertex or a common edge.

We are particularly interested in higher-genus analogues of the Platonic solids, the combinatorially regular polyhedra in $\mathbb{E}^3$. A polyhedron P is said to be *combinatorially regular* if its combinatorial automorphism group $\Gamma(P)$ is transitive on the flags (incident triples consisting of a vertex, an edge, and a face) of P. Thus a combinatorially regular polyhedron is a polyhedral realization in $\mathbb{E}^3$ of a regular map on an orientable surface of some genus g (see [5]). Each such polyhedron or map has a (*Schläfli*) *type* $\{p, q\}$ for some $p, q \geqslant 3$, describing the fact that the faces are p-gons, q meeting at each vertex. In constructing polyhedral realizations P of a given regular map we are most interested in those that have a large Euclidean symmetry group $G(P)$. Ideally we wish to achieve maximum possible geometric symmetry among polyhedral embeddings. Clearly, $G(P)$ is a subgroup of $\Gamma(P)$. Generally we must expect $G(P)$ to be rather small compared with $\Gamma(P)$.

Regular maps on surfaces have been studied for well over 120 years, and deep connections have been discovered between maps and other branches of mathematics, including hyperbolic geometry, Riemann surfaces, automorphic functions, number fields, and Galois theory. They can be studied from a combinatorial and topological viewpoint as cell-complexes (tessellations) on surfaces, essentially as abstract polyhedra (see [6]), or be viewed as complex algebraic curves over algebraic number fields (see Jones and Singerman [7]).

The Platonic solids provide the only regular maps on the 2-sphere, with $g = 0$. It is well-known that there are infinitely many regular maps on the 2-torus, with $g = 1$, each of type $\{3, 6\}$, $\{6, 3\}$, or $\{4, 4\}$ (see [5]). However, a surface of genus $g \geqslant 2$ can support only finitely many regular maps. For $g \leqslant 6$, a full classification was first obtained by Sherk [8] (for genus 3) and Garbe [9]. Historically, genus 6 provides an important upper bound on the genera for two reasons: first, two of Coxeter's [10] classical regular skew polyhedra discovered in the early days of the study of maps on surfaces have genus 6; and second, and more importantly, genus 6 turned out to be the threshold for a by-hand enumeration of regular maps by genus.

In the past few years there has been great progress in the computer-aided enumeration of regular maps by genus, leading in particular to the creation of a complete census of regular maps on orientable surfaces of genus up to 101 (see Conder [11]). It is quite surprising that there exist infinitely many genera g, beginning with $g = 2$, for which the surface of genus g does not admit any regular map with a simple underlying graph at all (see Conder, Siran and Tucker [12] and Breda d'Azevedo, Nedela and Siran [13]).

In this paper we study combinatorially regular polyhedra in $\mathbb{E}^3$ of small genus g, meaning that g lies in the historically motivated range $2 \leqslant g \leqslant 6$. There are only finitely many regular maps in this genus range, and we wish to determine those which admit realizations as (convex-faced, combinatorially regular) polyhedra. Only eight regular maps in this range are known to have polyhedral realizations, and two were only discovered quite recently. Some of these polyhedra are quite spectacular and realize famous maps of Klein, Fricke, Dyck, and Coxeter (see [5,10,14–19]). In Section 2 we briefly review these polyhedra and their maps. Then, in Section 3, we provide supporting evidence that this list is complete; in other words, we strongly conjecture that in addition to those eight there are no other regular maps of genus $2 \leqslant g \leqslant 6$ admitting realizations as (convex-faced) polyhedra in $\mathbb{E}^3$. In particular, for all admissible maps in this range, save Gordan's [20] classical map of genus 4, and its dual, we rule out realizability by a polyhedron in $\mathbb{E}^3$. We conclude the paper with a brief discussion of some open problems. For a discussion of regular toroidal polyhedra see also Schwörbel [21].

2. The Eight Maps and Their Polyhedra

For a regular map P of type $\{p, q\}$ with f_0 vertices, f_1 edges, and f_2 faces on a closed surface, the order of its automorphism group $\Gamma(P)$ is linked to the Euler characteristic χ of the surface by the equation

$$\chi = f_0 - f_1 + f_2 = \frac{|\Gamma(P)|}{2}(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}) \tag{1}$$

If $\{p, q\}$ is of hyperbolic type, this immediately leads to the classical Hurwitz inequality,

$$|\Gamma(P)| \leqslant 84|\chi| \tag{2}$$

with equality occurring if and only if P is of type $\{3, 7\}$ or $\{7, 3\}$. In particular, this shows that there can only be finitely many regular maps on a given closed surface of non-zero Euler characteristic. For the genus range under consideration, the inequality in Equation (2) also establishes, a priori, an upper bound for the order of the automorphism group of the map, and hence for the geometric symmetry group of any of its polyhedral realizations in $\mathbb{E}^3$.

The combinatorial automorphism group $\Gamma(P)$ is generated by three involutions ρ_0, ρ_1, ρ_2 satisfying the Coxeter relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = 1 \tag{3}$$

but in general also some further, independent relations. On the underlying surface, these generators can be viewed as "combinatorial reflections" in the sides of a fundamental triangle of the "barycentric subdivision" of P. For a finite regular map, the relations in Equation (3) suffice for a presentation of $\Gamma(P)$ if and only if P is a Platonic solid; that is, if and only if $g = 0$. Thus at least one, but generally more additional relations are needed if $g > 0$. Two kinds of extra relations, usually occurring separately, are of particular importance to us and suffice to describe seven of the eight regular maps under consideration, namely the *Petrie relation*

$$(\rho_0\rho_1\rho_2)^r = 1 \tag{4}$$

and the *hole relation*

$$(\rho_0\rho_1\rho_2\rho_1)^h = 1 \tag{5}$$

These relations are inspired by the notions of a Petrie polygon and of a hole of a regular map, respectively. Recall that a *Petrie polygon* of a regular map (on any surface) is a zigzag along its edges such that every two, but no three, successive edges are the edges of a common face. A *hole* of a regular map (on any surface) is a path along the edges which successively takes the second exit on the right (in a local orientation), rather than the first, at each vertex. The automorphism $\rho_0\rho_1\rho_2$ of P occurring in Equation (4) shifts a certain Petrie polygon of P one step along itself, and hence has period r if the Petrie polygon is of length r. Similarly, the automorphism $\rho_0\rho_1\rho_2\rho_1$ of Equation (5) shifts a certain hole of P one step along itself, and hence has period h if the hole is of length h. Thus, if the relation in Equation (4) or Equation (5) holds, with r or h giving the correct period of $\rho_0\rho_1\rho_2$ or $\rho_0\rho_1\rho_2\rho_1$, respectively, then P has Petrie polygons of length r or holes of length h.

There are a number of standard operations on maps which create new regular maps from old (see [6,22]). The *duality operation* δ replaces a map P by its dual P^δ (often denoted P^*); algebraically this corresponds to reversing the order of the generators ρ_0, ρ_1, ρ_2 of the group. The *Petrie operation* π preserves the edge graph of the map P but replaces the faces by the Petrie polygons; the resulting map is the *Petrie dual* P^π of P. Algebraically, π corresponds to replacing ρ_0, ρ_1, ρ_2 by the new generators $\rho_0\rho_2, \rho_1, \rho_2$. There is also an operation that substitutes the holes for the faces, but we will not require it here (see [6] Section 7B).

It is well-known that every orientable regular map of type $\{p, q\}$ is a quotient of the corresponding regular tessellation $\{p, q\}$ of the 2-sphere (if $g = 0$), the Euclidean plane (if $g = 1$), or the hyperbolic plane (if $g \geqslant 2$). If a map is the quotient of the regular tessellation $\{p, q\}$ obtained by identifying those pairs of vertices which are separated by r steps along a Petrie polygon, for a specified value of r, then it is denoted $\{p, q\}_r$. Similarly, $\{p, q|h\}$ denotes the map derived from the tessellation $\{p, q\}$ by identifying those pairs of vertices which are separated by h steps along a hole, for a specified value of h. These identification processes generally do not result in finite maps, but when they do, the corresponding maps and their automorphism groups usually have nice properties. The Platonic solids are particular instances arising as the maps $\{3, 3\}_4$, $\{3, 4\}_6$, $\{4, 3\}_6$, $\{3, 5\}_{10}$, and $\{5, 3\}_{10}$. The duals of $\{p, q\}_r$ and $\{p, q|h\}$ are $\{q, p\}_r$ and $\{q, p|h\}$, respectively. The Petrie dual of $\{p, q\}_r$ is $\{r, q\}_p$ (there is no simple rule for the Petrie dual of $\{p, q|h\}$).

It is important to point out here that our notation $\{p, q\}_r$ designates only those regular maps for which the relations in Equations(3) and (4) actually form a complete presentation for the automorphism group; in other words, $\{p, q\}_r$ is the "universal" regular map of type $\{p, q\}$ with Petrie polygons of length r. A similar remark applies to $\{p, q|h\}$.

We now discuss the eight regular maps that can be realized as polyhedra in $\mathbb{E}^3$. A summary of their basic properties is given in Table 1. In each case we list the genus g, the name of the map (if any) or simply its Schläfli type $\{p, q\}$, the order of the combinatorial automorphism group, the face vector (f_0, f_1, f_2), and the original reference. The third last column refers to the complete list of all regular maps of genus at most 101 mentioned earlier. In particular, $Rg.k$ designates the k^{th} map among the orientable (reflexible) regular maps of genus g, in the labeling of [11], and $Rg.k^*$ denotes its dual (note that only one map from a pair of duals is listed in [11]). Thus $R3.1$ is the 1st orientable regular map of genus 3 in this list.

Table 1. The Eight Regular Maps.

Genus g	**Name or Type**	**Group Order**	**Face Vector** (f_0, f_1, f_2)	**References**	**Map of [11]**	**Symmetry Group**	**Max.**
3	$\{3,7\}_8$	336	(24,84,56)	Klein [17,18]	$R3.1$	$[3,3]^+$	
3	$\{3,8\}_6$	192	(12,48,32)	Dyck [15,16]	$R3.2$	D_3	$*$
5	$\{3,8\}$	384	(24,96,64)	Klein and Fricke [19]	$R5.1$	$[3,4]^+$	$*$
5	$\{4,5\|4\}$	320	(32,80,40)	Coxeter [10]	$R5.3$	C_2^3	
5	$\{5,4\|4\}$	320	(40,80,32)	Coxeter [10]	$R5.3^*$	C_2^3	
6	$\{3,10\}_6$	300	(15,75,50)	Coxeter and Moser [5]	$R6.1$	D_3	$*$
6	$\{4,6\|3\}$	240	(20,60,30)	Coxeter [10], Boole Stott [23]	$R6.2$	$[3,3]$	$*$
6	$\{6,4\|3\}$	240	(30,60,20)	Coxeter [10], Boole Stott [23]	$R6.2^*$	$[3,3]$	$*$

The next-to-last column of Table 1 records the geometric symmetry group of the most symmetric polyhedral embedding of the map currently known, with an asterisk in the last column indicating if this group has maximum possible order for a polyhedral embedding. Here we let $[3,3]^+$ and $[3,4]^+$, respectively, denote the rotation subgroup of the full symmetry group $[3,3]$ and $[3,4]$ of the tetrahedron $\{3,3\}$ and octahedron $\{3,4\}$.

There are two pairs of duals among the eight maps in Table 1. The other four maps have triangular faces; our analysis in Section 3 will imply that their duals do not admit a polyhedral embedding with convex faces (though at least one, namely $\{7,3\}_8$, does with non-convex faces).

Note that the Klein–Fricke map in the third row of Table 1 does not have a simple designation as a map $\{p,q\}_r$ or $\{p,q|h\}$. The length of its Petrie polygons or 2-holes is not sufficient to define it.

2.1. Klein's Map

Klein's map $\{3,7\}_8$ of genus 3 is arguably the most famous regular map of positive genus. It was constructed by Klein [17] to illustrate the importance of the simple transformation group $PSL(2,7)$ for the solution of equations of degree 7, as well as to highlight the analogy with the appearance of the icosahedron and icosahedral group in the study of quintic equations (see [18]). The group $PSL(2,7)$ occurs as the group of orientation preserving automorphisms of $\{3,7\}_8$, and its order 168 maximizes the order among groups of orientation preserving automorphisms on a Riemann surface of genus 3 (see Hurwitz [24] and Jones and Singerman [7]). The surface can also be represented by Klein's quartic

$$x^3y + y^3z + z^3x = 0$$

a plane algebraic curve of order 4 in homogeneous complex variables (see [17]).

A beautiful polyhedron for $\{3,7\}_8$, discovered by the present authors in [25], can be constructed from a pair of homothetic vertex-truncated tetrahedra by removing their hexagonal faces and connecting each of the four pairs of resulting hexagonal circuits by a suitable tunnel made up of triangles. The resulting polyhedron has 56 triangular faces, 7 meeting at each of the 24 vertices, and the geometric symmetry group is the tetrahedral rotation group (isomorphic to the alternating group A_4). For illustrations see [26–28]. (Since the symmetry group is that of a Platonic solid, the polyhedron is a Leonardo polyhedron in the sense of [29].)

There are a number of other interesting polyhedral models for Klein's map or its dual, which either relax the strong assumption on the faces to be convex, or have self-intersections (like the Kepler–Poinsot polyhedra) and hence no longer give an embedding of the underlying surface in $\mathbb{E}^3$. Particularly remarkable is the polyhedral model for the dual map $\{7,3\}_8$ discovered in McCooey [30]. This has 24 heptagonal faces, all simply-connected but non-convex, and its symmetry group is the tetrahedral rotation group.

In realizing regular maps, shortcomings such as self-intersections can occasionally be compensated by higher symmetry. For example, in [25] we describe a polyhedral immersion for $\{3,7\}_8$ with self-intersections featuring octahedral rotation symmetry. Other polyhedral immersions of $\{3,7\}_8$ with high symmetry are discussed in [31]. Polyhedral models with self-intersections and non-convex faces are also known for the dual map $\{7,3\}_8$ (see [28]).

2.2. Dyck's Map

Dyck's map $\{3,8\}_6$ of genus 3 is closely related to the torus map $\{3,6\}_8$ with 16 vertices, which is just $\{3,6\}_{(4,0)}$ in the notation of [5]. In fact, $\{3,8\}_6 = (\{3,6\}_8)^{\delta\pi\delta}$, where again δ and π, respectively, denote the duality operation and Petrie operation. Thus $\{3,8\}_6$ is dual to the Petrie dual of the dual of $\{3,6\}_8$. It also can be represented by Dyck's quartic

$$x^4 + y^4 + z^4 = 0$$

Dyck's map is the smallest regular map of genus $g \geqslant 2$ for which a polyhedral embedding with convex faces is known. As the map is quite small, having just 12 vertices and 32 faces, its polyhedral realizability is far from being obvious. The challenge here is the polyhedral embeddability; the convexity condition on the faces is trivial since they are triangles. The first realization of Dyck's map as a polyhedron (with trivial geometric symmetry group) was discovered by Bokowski [32]. A more symmetric polyhedron, due to Brehm [33], has a symmetry group D_3 and exhibits maximum possible geometric symmetry for a polyhedral embedding.

An appealing polyhedral model for $\{3,8\}_6$ with self-intersections but full tetrahedral symmetry can be obtained from a pair of homothetic octahedra by first omitting on each octahedron a set of alternate faces, different sets on the two octahedra, and then joining the resulting triangular circuits by tunnels, one tunnel for each pair of circuits in parallel planes (see [34]). This model has maximum possible symmetry among all polyhedral models of $\{3,8\}_6$ and is significantly more symmetric than the polyhedral embedding with maximal symmetry. For the dual map $\{8,3\}_6$, which is a dodecahedron with octagonal faces, a polyhedral model with self-intersecting faces was described in [35]. It does not seem to be known if a polyhedral embedding of $\{8,3\}_6$ with non-convex faces is possible.

2.3. Coxeter's Geometric Skew Polyhedra

The pair of dual regular maps $\{4,6|3\}$ and $\{6,4|3\}$ of genus 6 can be embedded in Euclidean 4-space $\mathbb{E}^4$ such that all combinatorial symmetries are realized as geometric symmetries. The corresponding 4-dimensional polyhedra are two of Coxeter's regular skew polyhedra in $\mathbb{E}^4$ and have (convex) squares or regular hexagons as faces. These polyhedra in $\mathbb{E}^4$ actually trace back to Alicia Boole Stott (see Stott [23] and also p. 45 of Reference [10]), but it was Coxeter who fully explored and popularized them. They are closely related to the regular 4-simplex. Their underlying maps inherit their combinatorial symmetries from two sources, the symmetries and the dualities of the (self-dual) 4-simplex; in fact, the combinatorial automorphism group is isomorphic to $S_5 \times C_2$. It was observed by McMullen, Schulz and Wills [1,2] that, via projection from $\mathbb{E}^4$, they can be polyhedrally embedded with convex faces in $\mathbb{E}^3$. For illustrations of these polyhedra in $\mathbb{E}^3$ see [27,36].

There are similar such polyhedral embeddings for Coxeter's regular skew polyhedra $\{4,8|3\}$ and $\{8,4|3\}$, a dual pair of genus 73, but they fall outside the genus range under consideration (see [1,2,10,23,36]).

2.4. Coxeter's Topological Analogues of Skew Polyhedra

The pair of dual maps $\{4,5|4\}$ and $\{5,4|4\}$ are the first instances, with $g \geqslant 2$, in two infinite series of regular maps of type $\{4,q\}$ and $\{q,4\}$, with $q \geqslant 3$, where the member of type $\{4,q\}$ in the first series is dual to the member of type $\{q,4\}$ in the second series. (The members for $q = 3,4$ are the 3-cube $\{3,3\}$ and the torus map $\{4,4\}_{(4,0)}$.) For each $q \geqslant 3$, the combinatorial automorphism group of either map is $C_2^q \rtimes D_q$ ($\cong C_2 \wr D_q$), and the genus is $2^{q-3}(q-4)+1$. The two series were first discovered by Coxeter [10,14] in 1937, but have been rediscovered several times since then (see pp. 260, 261 of Reference [6] for more details). The map of type $\{4,q\}$ can be realized (with square faces) in the 2-skeleton of the q-dimensional cube in $\mathbb{E}^q$. In McMullen, Schulz and Wills [1,2], two remarkable infinite series of polyhedra in $\mathbb{E}^3$ of types $\{4,q\}$ and $\{q,4\}$ were described, and in [37] were proved to be polyhedral embeddings of Coxeter's regular maps of these types. Thus all of Coxeter's maps can be realized as polyhedra in $\mathbb{E}^3$. Each of these polyhedra, with $q \geqslant 5$, has a geometric symmetry group C_2^3 generated by three reflections in mutually orthogonal planes. (For $q = 3,4$, the symmetry group is larger.) For illustrations of the polyhedra $\{4,5|4\}$ and $\{5,4|4\}$ see [38].

The maps in each series also admit polyhedral embeddings in $\mathbb{E}^4$ as subcomplexes of the boundary complexes of certain convex 4-polytopes: more precisely, of weakly neighborly polytopes, for the maps of type $\{4,q\}$ (see [39]); and of wedge polytopes, for the dual maps of type $\{q,4\}$ (see [40,41]).

2.5. The Klein–Fricke Map

The Klein–Fricke map of type $\{3,8\}$ and genus 5 is a double cover of Dyck's map $\{3,8\}_6$ of genus 3, and has 24 vertices and a group of order 384 (see [19]). The Petrie polygons have length 12; however, this is not the universal map $\{3,8\}_{12}$, which is known to be infinite (see p. 399 of Reference [6]).

In their search for vertex-transitive polyhedra in 3-space in [42], Grünbaum and Shephard discovered an equivelar polyhedron of type $\{3,8\}$ with 24 vertices and octahedral rotation symmetry. This shares the same numbers of vertices, edges, and faces, the same genus, and the same group order, but not the full combinatorics, with the Klein–Fricke map, and in particular is not combinatorially regular. Grünbaum [43] describes how this equivelar polyhedron can be altered (by changing 12 edges) to obtain a polyhedral embedding of the Klein–Fricke map. This new polyhedron again has octahedral rotation symmetry and is vertex-transitive. Just recently, this same embedding of the Fricke–Klein map was rediscovered by Brehm and Wills; more details, along with a proof of isomorphism with the Fricke–Klein map, will appear in a forthcoming article [44].

The polyhedron admits a nice description based on the geometry of the snub cube. Its vertices are those of the snub cube, and the boundaries of the six square faces of the snub cube appear as the holes of the polyhedron. The polyhedron consists of an outer shell and an inner shell connected at the holes (but nowhere else). The entire polyhedron can be pieced together from the orbits of four particular triangles, two adjacent triangles in each shell, under the octahedral symmetry group; the triangles in these orbits then are the faces of the polyhedron. The vertices of the polyhedron can be represented by small integer coordinates. The smallest integer coordinates arise when the four triangles are chosen as follows (see the forthcoming article by Brehm, Grünbaum and Wills [44]. The two basic triangles for the outer shell have vertices

$$(1,2,6),\ (2,6,1),\ (6,1,2)$$

and

$$(1,2,6),\ (2,6,1),\ (-2,1,6)$$

respectively; under the standard octahedral rotation group (generated by 4-fold rotations about the coordinate axes and 3-fold rotation about the main space diagonals), these triangles determine 8 regular triangles and 24 non-regular triangles making up the outer shell. The inner shell similarly consists of 8 regular triangles and 24 non-regular triangles obtained under the standard octahedral rotation group from the two basic triangles for the inner shell with vertices

$$(2,-1,6),\ (-1,6,2),\ (6,2,-1)$$

and

$$(2,-1,6),\ (-1,6,2),\ (-2,6,-1)$$

respectively.

The Klein–Fricke map also admits a polyhedral realization with self-intersections that highlights the relationship with Dyck's map. The model is constructed from a pair of homothetic icosahedra by removing all their faces and joining suitable hexagonal circuits by tunnels (see [45]); identifying antipodal vertices in this realization gives exactly the previously mentioned polyhedral model with self-intersections for Dyck's map derived from a pair of homothetic octahedra.

2.6. The Map $\{3,10\}_6$

Just like Dyck's map, $\{3,10\}_6$ is closely related to a torus map, in this case the map $\{3,6\}_{10} = \{3,6\}_{(5,0)}$ with 25 vertices (see [5]). Now $\{3,10\}_6 = (\{3,6\}_{10})^{\delta\pi\delta}$, with δ and π as before, so $\{3,10\}_6$ is dual to the Petrie dual of the dual of $\{3,6\}_{10}$.

A polyhedral embedding for $\{3,10\}_6$ in $\mathbb{E}^3$ with maximum possible symmetry group D_3 was only recently discovered [46]. It is the only polyhedron among the eight with an odd number of vertices.

The map $\{3, 10\}_6$ is the fourth in an infinite series of finite regular maps $\{3, 2s\}_6$, with $s \geqslant 2$, closely related to the unitary reflection groups $[1\,1\,1]^s$ in complex 3-space (see pp. 389, 399 of Reference [6]). The automorphism group of $\{3, 2s\}_6$ is the semidirect product of $[1\,1\,1]^s$ by C_2 and has order $12s^2$. The first three maps in this sequence are the octahedron $\{3, 4\}_6 = \{3, 4\}$, the torus map $\{3, 6\}_{(3,0)}$, and Dyck's map $\{3, 8\}_6$. Note that, as above, each map $\{3, 2s\}_6$ is dual to the Petrie dual of the dual of $\{3, 6\}_{2s}$. The sequence of maps $\{3, 2s\}_6$ was first discovered by Coxeter [47].

3. Completeness of the List

In this section we give supporting evidence for our conjecture that the eight maps listed in Table 1 are the only (orientable) regular maps of genus $2 \leqslant g \leqslant 6$ that admit realizations as (convex-faced) polyhedra in $\mathbb{E}^3$. In fact, the only maps we have not been able to conclusively exclude as possibly admitting a polyhedral embedding are Gordan's map $\{4, 5\}_6$ and its dual $\{5, 4\}_6$ (see [20]).

We rely on the recent enumeration in [11] of regular maps of genus up to 101. It is immediately clear that we can eliminate maps with multiple edges from consideration, since those certainly cannot be realized as polyhedra. Similarly, by the convexity assumption on the faces, adjacent faces cannot share more than one edge. Using the notation of [11], then this leaves only maps with $m_V = m_F = 1$, where m_V and m_F, respectively, denote the edge-multiplicities of the underlying graph of the map and its dual. Now inspection of the list for genus $2 \leqslant g \leqslant 6$ shows that the only possible candidates for polyhedral embeddings are the maps

$$R3.1,\ R3.2,\ R4.2,\ R4.3,\ R4.6,\ R5.1,\ R5.3,\ R5.4,\ R5.9,\ R6.1,\ R6.2$$

and their duals; since $R4.6$ and $R5.9$ are self-dual, this gives a total of twenty maps.

Among those twenty, the maps $R3.1$, $R3.2$, $R5.1$ and $R6.1$ have triangular faces, so their duals have 3-valent vertices. However, a polyhedron (without self-intersection) in $\mathbb{E}^3$ with only 3-valent vertices and convex faces is necessarily a convex polyhedron and hence has genus 0 (see [3]). Thus the duals of $R3.1$, $R3.2$, $R5.1$ and $R6.1$ can be eliminated as well, leaving at most sixteen possible candidates, including the eight maps of Table 1 admitting polyhedral embeddings as described in the previous section. Finally, then, this leaves the following eight maps

$$R4.2,\ R4.2^*,\ R4.3,\ R4.3^*,\ R4.6,\ R5.4,\ R5.4^*,\ R5.9$$

which need to be ruled out by other means.

First observe that none of the maps $R4.6$, $R5.4$ and $R5.4^*$ has a face-poset which is a lattice; in other words, none of these maps is a cell-complex. Thus polyhedral embeddings cannot exist. In particular, $R4.6$ of type $\{5, 5\}$ is the underlying (combinatorially self-dual) map of the (geometrically) dual pair of Kepler–Poinsot polyhedra $\{5, \frac{5}{2}\}$ and $\{\frac{5}{2}, 5\}$ in $\mathbb{E}^3$, which have self-intersections (see [48]). Here it is easily seen that, for example, two non-adjacent pentagonal faces of $\{5, \frac{5}{2}\}$ with a common vertex, share in fact another vertex which is not adjacent to the first vertex. Hence $R4.6$ is not a lattice. For $R5.4$ of type $\{4, 6\}$, discovered by Sherk [49], we can appeal to the planar diagram of the map given on p. 161 of Reference [45] to show that opposite square faces in the vertex-star of a vertex also share the vertex opposite the first vertex in these faces. Hence neither $R5.4$ nor its dual $R5.4^*$ is a lattice.

The three maps $R4.3$ of type $\{4, 6\}$, $R4.3^*$ of type $\{6, 4\}$, and $R5.9$ of type $\{5, 5\}$ can be eliminated by the following lemma, since they have Petrie polygons of length 4. This lemma is also of independent interest.

Lemma 3.1. *If P is a combinatorially regular polyhedron in $\mathbb{E}^3$ (with convex faces and without self-intersection), and if P has Petrie polygons of length 4, then P is the tetrahedron $\{3, 3\}$.*

Proof. The proof is simple. Let e_1, e_2, e_3, e_4 denote the edges of a Petrie polygon, taken in order. Then any pair e_j, e_{j+1} of successive edges determines a face F_j of P (indices taken modulo 4). Clearly, the Petrie polygon cannot lie in a plane, since otherwise (for example) F_1 and F_2 would have relative interior points in common. Let T denote the convex hull of e_1, e_2, e_3, e_4, which is a tetrahedron. Then the affine hulls of the faces F_j of P are just the faces of T. Now, if P does not have triangular faces, then (for example) F_1 and F_3 would have to intersect in relative interior points, namely in one of the two edges of T not among e_1, e_2, e_3, e_4. Hence the faces of P must be triangles. However, then the two edges of T not among e_1, e_2, e_3, e_4 must also be edges of P, so P must reduce to the tetrahedron T. □

Finally, then, we need to exclude the possibility of a polyhedral embedding for the two well-known regular maps $R4.2$ and $R4.2^*$ of genus 4. These two maps, respectively, are Gordan's $\{4, 5\}_6$ with 24 vertices, 60 edges, and 30 faces, and its dual $\{5, 4\}_6$, with automorphism groups isomorphic to a semidirect product of the full icosahedral group by C_2 (see [20]). In this case it is considerably more difficult to prove non-existence of a polyhedral embedding with convex faces. Here we have the following conjecture. If confirmed, this would establish definitively that the eight maps in Table 1 are the only regular maps of genus $2 \leqslant g \leqslant 6$ admitting realizations as convex-faced polyhedra in $\mathbb{E}^3$.

Conjecture 3.2. *Gordan's maps $\{4, 5\}_6$ and $\{5, 4\}_6$ do not admit a realization as a (convex-faced) polyhedron in $\mathbb{E}^3$.*

In fact, it may be conjectured more strongly that $\{4, 5\}_6$ and possibly also $\{5, 4\}_6$ do not even admit a polyhedral embedding with simply-connected planar faces.

On the other hand, Gordan's maps do admit beautiful polyhedral realizations of Kepler–Poinsot type (that is, with self-intersections) with high symmetry. In our paper [45], we described a realization of $\{4, 5\}_6$ with full octahedral symmetry based on the geometry of the truncated octahedron. However, even self-intersecting polyhedral models with full icosahedral symmetry were discovered for both $\{4, 5\}_6$ and $\{5, 4\}_6$, and were described as "skeletal polyhedra of index 2" in Wills [50], Cutler and Schulte [51], and Cutler [52] (see also [53,54]). More details will appear in a forthcoming paper [55]. The high degree of self-intersections in these models, as well as the smallness of the underlying maps, are strong indications that non-self-intersecting polyhedral embeddings with convex faces are impossible. However, we have not been able to confirm this rigorously.

4. Open Problems

In addition to settling Conjecture 3.2, the following problems seem particularly interesting in connection with the topics discussed here.

Genus 7 or 8. The maps $R7.1$ of genus 7, as well as $R8.1$ and $R8.2$ of genus 8, are the only regular maps of genus 7 or 8 that could possibly admit a realization as a convex-faced polyhedron in $\mathbb{E}^3$ (all other maps violate the condition that $m_V = m_F = 1$). The map $R7.1$ of type $\{3,7\}$ has Petrie polygons of length 18 and a group of order 1008; however, most likely, $R7.1$ itself is not the universal map $\{3,7\}_{18}$. By contrast, $R8.1$ is the universal map $\{3,8\}_8$; its automorphism group is a semidirect product of the unitary reflection group $[1\,1\,1^4]^4$ in complex 3-space by C_2, and has order 672 (see pp. 296, 399 of Reference [6]). The map $R8.2$ of type $\{3,8\}$ has Petrie polygons of length 14 and again a group of order 672; however, $R8.2$ itself is not the universal map $\{3,8\}_{14}$, which is known to be infinite (see p. 399 of Reference [6]). Thus all three maps have much larger automorphism groups than the maps in Table 1. It would be desirable to decide whether or not $R7.1$, $R8.1$, and $R8.2$ admit a polyhedral embedding in $\mathbb{E}^3$, and if so, to explicitly construct a polyhedron with maximum possible geometric symmetry.

Leonardo polyhedra. Among the eight combinatorially regular maps of Table 1, exactly four admit realizations as *Leonardo polyhedra* in the sense of [29], meaning that the symmetry group of the polyhedron is either the full symmetry group of a Platonic solid or its rotation subgroup. Very few Leonardo polyhedra with large combinatorial automorphism groups seem to be known, and it would be a worthwhile task to find more, particularly Leonardo polyhedra which are combinatorially regular.

Icosahedral symmetry. Finally, among all combinatorially regular polyhedra in $\mathbb{E}^3$ discovered so far, none except the icosahedron has full icosahedral symmetry or icosahedral rotation symmetry. Any combinatorially regular polyhedra with icosahedral symmetry would likely be very interesting. As mentioned earlier, when self-intersections are permitted, full icosahedral symmetry can be obtained, for example, for Gordan's maps.

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