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Article

Duffin–Kemmer–Petiau and Dirac Equations —A Supersymmetric Connection

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Abstract: In the present paper we study subsolutions of the Dirac and Duffin–Kemmer–Petiau equations in the interacting case. It is shown that the Dirac equation in longitudinal external fields can be split into two covariant subequations (Dirac equations with built-in projection operators). Moreover, it is demonstrated that the Duffin–Kemmer–Petiau equations in crossed fields can be split into two 3×3 subequations. We show that all the subequations can be obtained via minimal coupling from the same 3×3 subequations which are thus a supersymmetric link between fermionic and bosonic degrees of freedom.

Keywords: relativistic wave equations; supersymmetry

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1. Introduction

Recently, several supersymmetric systems, concerned mainly with anyons in $2+1$ dimensions [1–5] as well as with the $3+1$ dimensional Majorana–Dirac–Staunton theory [6], uniting fermionic and bosonic fields, have been described. Furthermore, bosonic symmetries of the Dirac equation have been found in the massless [7] as well as in the massive case [8]. Our results derived lately fit into this broader picture. We have demonstrated that certain subsolutions of the free Duffin–Kemmer–Petiau (DKP) and the Dirac equations obey the same Dirac equation with some built-in projection operators [9]. We shall refer to this equation as supersymmetric since it has bosonic (spin 0 and 1) as well as fermionic $\left(\text{spin } \frac{1}{2}\right)$ degrees of freedom. In the present paper we extend our results to the case of interacting fields.

The paper is organized as follows. In Section 2 relativistic wave equations as well as conventions and definitions used in the paper are described. In particular, several classical and not-so-classical subsolutions of the free Dirac equation are reviewed in Subsection 2.2. The notion of supersymmetry is invoked since some subequations arising in the context of the Dirac equation appear also in the Duffin–Kemmer–Petiau theory of massive bosons. In Section 3 the Dirac equation in longitudinal fields is split into two 3×3 subequations which can be written as two Dirac equations with built-in projection operators. In the next Section variables are separated in the subequations to yield 2D Dirac equations in (x^0, x^3) subspace and 2D Pauli equations in (x^1, x^2) subspace. In Section 5 the Duffin–Kemmer–Petiau equation for spin 0 in crossed fields is split into two 3×3 subequations—these equations have the same structure as subequations arising in the Dirac theory. It follows that the free 3×3 equations provide a supersymmetric link between the Dirac and DKP theories—this is described in Section 6. In the last Section we discuss our results in a broader context of supersymmetry and Lorentz covariance.

2. Relativistic Wave Equations

In what follows tensor indices are denoted with Greek letters: $\mu = 0, 1, 2, 3$. We shall use the following convention for the Minkowski space-time metric tensor: $g^{\mu\nu} = \mathrm{diag}\ (1, -1, -1, -1)$ and we shall always sum over repeated indices. For example, $a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}$. Four-momentum operators are defined as $p^\mu = i\frac{\partial}{\partial x_\mu}$ where natural units have been used: $c = 1$, $\hbar = 1$. The interaction will be introduced via minimal coupling,

$$p^\mu \longrightarrow \pi^\mu = p^\mu - qA^\mu \tag{1}$$

with a four-potential A^μ and a charge q. In what follows we shall work with external fields of special configuration, so-called crossed and longitudinal fields, non-standard but Lorentz covariant, see [10]. We shall also need elements of spinor calculus. Four-vectors $\zeta^\mu = \left(\zeta^0, \vec{\zeta}\right)$ and spinors $\zeta^{A\dot{B}}$ are related by the formula $\zeta^{A\dot{B}} = \left(\sigma^0 \zeta^0 + \vec{\sigma} \cdot \vec{\zeta}\right)^{A\dot{B}}$:

$$\zeta^{A\dot{B}} = \begin{pmatrix} \zeta^{1\dot{1}} & \zeta^{1\dot{2}} \\ \zeta^{2\dot{1}} & \zeta^{2\dot{2}} \end{pmatrix} = \begin{pmatrix} \zeta^0 + \zeta^3 & \zeta^1 - i\zeta^2 \\ \zeta^1 + i\zeta^2 & \zeta^0 - \zeta^3 \end{pmatrix} \tag{2}$$

where A, $\dot{B}$ number rows and columns, respectively, $\vec{\sigma}$ denotes vector built of the Pauli matrices and σ^0 is the 2×2 unit matrix. Spinor with lowered indices $\zeta_{C\dot{D}}$ reads:

$$\zeta_{C\dot{D}} = \begin{pmatrix} \zeta_{1\dot{1}} & \zeta_{1\dot{2}} \\ \zeta_{2\dot{1}} & \zeta_{2\dot{2}} \end{pmatrix} = \begin{pmatrix} \zeta^0 - \zeta^3 & -\zeta^1 - i\zeta^2 \\ -\zeta^1 + i\zeta^2 & \zeta^0 + \zeta^3 \end{pmatrix} \tag{3}$$

For details of the spinor calculus reader should consult [11–13].

2.1. The Dirac Equation

The Dirac equation is a relativistic quantum mechanical wave equation formulated by Paul Dirac in 1928 providing a description of elementary spin $\frac{1}{2}$ particles, such as electrons and quarks, consistent with both the principles of quantum mechanics and the theory of special relativity [14,15]. The Dirac Equation is [11,16,17]:

$$\gamma^\mu p_\mu \Psi = m\Psi \tag{4}$$

where m is the rest mass of the elementary particle. The γ's are 4×4 anticommuting Dirac matrices: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I$ where I is the 4×4 unit matrix. In the spinor representation of the Dirac matrices we have:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \; \gamma^j = \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix} \; (j = 1, 2, 3) \tag{5}$$

where σ^j are the Pauli matrices and σ^0 is again the 2×2 unit matrix. The wave function is a bispinor, *i.e.*, consists of 2 two-component spinors ξ, η: $\Psi = (\xi, \eta)^T$ where ^{T} denotes transposition of a matrix. Sometimes it is more convenient to use the standard representation:

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \; \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \; (j = 1, 2, 3) \tag{6}$$

2.2. Subsolutions of the Dirac Equation and Supersymmetry

In the $m = 0$ case it is possible to obtain two independent equations for spinors ξ, η by application of projection operators $Q_\pm = \frac{1}{2}\left(1 \pm \gamma^5\right)$ to Equation (4) since $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with $\gamma^\mu p_\mu$:

$$Q_\pm \gamma^\mu p_\mu \Psi = \gamma^\mu p_\mu \left(Q_\mp \Psi\right) = 0 \tag{7}$$

In the spinor representation of the Dirac matrices [11] we have $\gamma^5 = \text{ diag }(-1, -1, 1, 1)$ and thus $Q_-\Psi = (\xi, 0)^T$, $Q_+\Psi = (0, \eta)^T$ and separate equations for ξ, η follow:

$$\left(p^0 + \vec{\sigma} \cdot \vec{p}\right)\eta = 0 \tag{8}$$
$$\left(p^0 - \vec{\sigma} \cdot \vec{p}\right)\xi = 0 \tag{9}$$

Equations (8) and (9) are known as the Weyl equations and are used to describe massless left-handed and right-handed neutrinos. However, since the experimentally established phenomenon of neutrino oscillations requires non-zero neutrino masses, theory of massive neutrinos, which can be based on the Dirac equation, is necessary [18–21]. Alternatively, a modification of the Dirac or Weyl equation, called the Majorana equation, is thought to apply to neutrinos. According to Majorana theory, neutrino and antineutrino are identical and neutral [22].

Although the Majorana equations can be introduced without any reference to the Dirac theory, they are subsolutions of the Dirac Equation [18]. Indeed, demanding in Equation (4) that $\Psi = \mathcal{C}\Psi$ where $\mathcal{C}$ is the charge conjugation operator, $\mathcal{C}\Psi = i\gamma^2\Psi^*$, we obtain in the spinor representation $\xi = -i\sigma^2\eta^*$, $\eta = i\sigma^2\xi^*$ and the Dirac Equation (4) reduces to two separate Majorana equations for two-component spinors:

$$\left(p^0 + \vec{\sigma}\cdot\vec{p}\right)\eta = -im\sigma^2\eta^* \tag{10}$$

$$\left(p^0 - \vec{\sigma}\cdot\vec{p}\right)\xi = +im\sigma^2\xi^* \tag{11}$$

It follows from the condition $\Psi = \mathcal{C}\Psi$ that Majorana particle has zero charge built-in condition. The problem whether neutrinos are described by the Dirac equation or the Majorana equations is still open [18–21].

Let us note that the Dirac Equation (4) in the spinor representation of the γ^μ matrices can be also separated in form of second-order Equations:

$$\left(p^0 + \vec{\sigma}\cdot\vec{p}\right)\left(p^0 - \vec{\sigma}\cdot\vec{p}\right)\xi = m^2\xi \tag{12}$$

$$\left(p^0 - \vec{\sigma}\cdot\vec{p}\right)\left(p^0 + \vec{\sigma}\cdot\vec{p}\right)\eta = m^2\eta \tag{13}$$

Such equations, valid also in the interacting case, were used by Feynman and Gell-Mann to describe weak decays in terms of two-component spinors [23].

More exotic subsolutions of the Dirac equation, related to supersymmetry, are also possible. In the massless case Simulik and Krivsky demonstrated that the following substitution,

$$\Psi = \left(iE_3 - H_0,\ iE_1 - E_2,\ iE_0 - H_3,\ -iH_2 - H_1\right)^T \tag{14}$$

when introduced into the Dirac Equation (4), converts it for $m = 0$ and standard representation of the Dirac matrices Equation (6) into the set of Maxwell equations [7]. In the massive case the Dirac Equation (4) can be written as a set of two Equations:

$$\gamma^\mu p_\mu P_4 \Psi = m P_4 \Psi \tag{15}$$

$$\gamma^\mu p_\mu P_3 \Psi = m P_3 \Psi \tag{16}$$

with $P_4 = \mathrm{diag}\ (1,1,1,0)$, $P_3 = \mathrm{diag}\ (1,1,0,1)$ and spinor representation of the γ^μ matrices Equation (5). Equations analogous to (15,16) appear also in the Duffin–Kemmer–Petiau theory of massive bosons [9].

Let us note finally that as shown in [24] the *square* of the Dirac operator is indeed supersymmetric, and this can be used for a convenient description of fluctuations around a self-dual monopole. Similar behavior has also been observed in the Taub-NUT case, see [25].

2.3. The Duffin–Kemmer–Petiau Equations

The DKP equations for spin 0 and 1 are written as:

$$\beta^\mu p_\mu \Psi = m\Psi \tag{17}$$

with 5×5 and 10×10 matrices β^μ, respectively, which fulfill the following commutation relations [26–29]:

$$\beta^\lambda\beta^\mu\beta^\nu + \beta^\nu\beta^\mu\beta^\lambda = g^{\lambda\mu}\beta^\nu + g^{\nu\mu}\beta^\lambda \tag{18}$$

In the case of 5×5 (spin 0) representation of β^μ matrices Equation (17) is equivalent to the following set of equations:

$$\left.\begin{array}{rcl} p^\mu \psi & = & m\psi^\mu \\ p_\nu \psi^\nu & = & m\psi \end{array}\right\} \tag{19}$$

if we define Ψ in Equation (17) as:

$$\Psi = (\psi^\mu, \psi)^T = \left(\psi^0, \psi^1, \psi^2, \psi^3, \psi\right)^T \tag{20}$$

Let us note that Equation (19) can be obtained by factorizing second-order derivatives in the Klein–Gordon equation $p_\mu p^\mu \psi = m^2 \psi$.

In the case of 10×10 (spin 1) representation of matrices β^μ Equation (17) reduces to:

$$\left.\begin{array}{rcl} p^\mu \psi^\nu - p^\nu \psi^\mu & = & m\psi^{\mu\nu} \\ p_\mu \psi^{\mu\nu} & = & m\psi^\nu \end{array}\right\} \tag{21}$$

with Ψ in Equation (17) defined as $\Psi = \left(\psi^{\mu\nu}, \psi^\lambda\right)^T$:

$$\Psi = \left(\psi^{01}, \psi^{02}, \psi^{03}, \psi^{23}, \psi^{31}, \psi^{12}, \psi^0, \psi^1, \psi^2, \psi^3\right)^T \tag{22}$$

where ψ^λ are real and $\psi^{\mu\nu}$ are purely imaginary (in alternative formulation we have $-\partial^\mu \psi^\nu + \partial^\nu \psi^\mu = m\psi^{\mu\nu}$, $\partial_\mu \psi^{\mu\nu} = m\psi^\nu$, where ψ^λ, $\psi^{\mu\nu}$ are real). Because of antisymmetry of $\psi^{\mu\nu}$ we have $p_\nu \psi^\nu = 0$ what implies spin 1 condition. The set of Equation (21) was first written by Proca [30,31] and in a different context by Lanczos, see [32] and references therein. More on the history of the formalism of Duffin, Kemmer and Petiau can be found in [33].

3. Splitting the Dirac Equation in Longitudinal External Fields

The interaction is introduced into the Dirac Equation (4) via minimal coupling Equation (1). We consider a special class of four-potentials obeying the condition:

$$\left[\pi^0 \pm \pi^3, \pi^1 \pm i\pi^2\right] = 0 \tag{23}$$

where $[X, Y] = XY - YX$ is a commutator. The condition Equation (23) is fulfilled in the Abelian case for

$$A^\mu = A^\mu\left(x^0, x^3\right), \ A^i = A^i\left(x^1, x^2\right), \ \mu = 0, 3, \ i = 1, 2 \tag{24}$$

This is the case of longitudinal potentials for which several exact solutions of the Dirac equation were found [10].

The Dirac Equation (4) can be written in spinor notation as [11]:

$$\left.\begin{array}{l} \pi^{1\dot{1}}\eta_{\dot{1}} + \pi^{1\dot{2}}\eta_{\dot{2}} = m\xi^1 \\ \pi^{2\dot{1}}\eta_{\dot{1}} + \pi^{2\dot{2}}\eta_{\dot{2}} = m\xi^2 \\ \pi_{1\dot{1}}\xi^1 + \pi_{2\dot{1}}\xi^2 = m\eta_{\dot{1}} \\ \pi_{1\dot{2}}\xi^1 + \pi_{2\dot{2}}\xi^2 = m\eta_{\dot{2}} \end{array}\right\} \tag{25}$$

where $\pi^{A\dot{B}}$, $\pi_{A\dot{B}}$ are given by Equations (2) and (3) (note that $\pi_{1\dot{1}} = \pi^{2\dot{2}}$, $\pi_{1\dot{2}} = -\pi^{2\dot{1}}$, $\pi_{2\dot{1}} = -\pi^{1\dot{2}}$, $\pi_{2\dot{2}} = \pi^{1\dot{1}}$). Obviously, due to relations between components of $\pi^{A\dot{B}}$ and $\pi_{C\dot{D}}$ the Equation (25) can be rewritten in terms of components of $\pi^{A\dot{B}}$ only. Equation (25) corresponds to Equation (4) in the spinor representation of γ matrices and $\Psi = (\xi^1, \xi^2, \eta_{\dot{1}}, \eta_{\dot{2}})^T$. We assume here that we deal with four-potentials fulfilling condition Equation (23).

In this Section we shall investigate a possibility of finding subsolutions of the Dirac equation in longitudinal external field, analogous to subsolutions found for the free Dirac equation in ([9]). For $m \neq 0$ we can define new quantities:

$$\pi^{1\dot{1}}\eta_{\dot{1}} = m\xi^1_{(1)}, \quad \pi^{1\dot{2}}\eta_{\dot{2}} = m\xi^1_{(2)} \tag{26}$$

$$\pi^{2\dot{1}}\eta_{\dot{1}} = m\xi^2_{(1)}, \quad \pi^{2\dot{2}}\eta_{\dot{2}} = m\xi^2_{(2)} \tag{27}$$

where we have:

$$\xi^1_{(1)} + \xi^1_{(2)} = \xi^1 \tag{28}$$

$$\xi^2_{(1)} + \xi^2_{(2)} = \xi^2 \tag{29}$$

In spinor notation $\xi^1_{(1)} = \psi^{1\dot{1}}_{\dot{1}}$, $\xi^1_{(2)} = \psi^{1\dot{2}}_{\dot{2}}$, $\xi^2_{(1)} = \psi^{2\dot{1}}_{\dot{1}}$, $\xi^2_{(2)} = \psi^{2\dot{2}}_{\dot{2}}$.

The Dirac Equation (25) can be now written with help of Equations (26) and (27) as (we are now using components $\pi^{A\dot{B}}$ throughout):

$$\left.\begin{aligned}
\pi^{1\dot{1}}\eta_{\dot{1}} &= m\xi^1_{(1)} \\
\pi^{1\dot{2}}\eta_{\dot{2}} &= m\xi^1_{(2)} \\
\pi^{2\dot{1}}\eta_{\dot{1}} &= m\xi^2_{(1)} \\
\pi^{2\dot{2}}\eta_{\dot{2}} &= m\xi^2_{(2)} \\
\pi^{2\dot{2}}\left(\xi^1_{(1)} + \xi^1_{(2)}\right) - \pi^{1\dot{2}}\left(\xi^2_{(1)} + \xi^2_{(2)}\right) &= m\eta_{\dot{1}} \\
-\pi^{2\dot{1}}\left(\xi^1_{(1)} + \xi^1_{(2)}\right) + \pi^{1\dot{1}}\left(\xi^2_{(1)} + \xi^2_{(2)}\right) &= m\eta_{\dot{2}}
\end{aligned}\right\} \tag{30}$$

It follows from Equations (26) and (27) *and* Equation (23) that the following identities hold:

$$\pi^{2\dot{1}}\xi^1_{(1)} = \pi^{1\dot{1}}\xi^2_{(1)} \tag{31}$$

$$\pi^{2\dot{2}}\xi^1_{(2)} = \pi^{1\dot{2}}\xi^2_{(2)} \tag{32}$$

Taking into account the identities Equations (31) and (32) we can decouple Equation (30) and write it as a system of the following two Equations:

$$\left.\begin{aligned}
\pi^{1\dot{1}}\eta_{\dot{1}} &= m\xi^1_{(1)} \\
\pi^{2\dot{1}}\eta_{\dot{1}} &= m\xi^2_{(1)} \\
\pi^{2\dot{2}}\xi^1_{(1)} - \pi^{1\dot{2}}\xi^2_{(1)} &= m\eta_{\dot{1}}
\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned}
\pi^{1\dot{2}}\eta_{\dot{2}} &= m\xi^1_{(2)} \\
\pi^{2\dot{2}}\eta_{\dot{2}} &= m\xi^2_{(2)} \\
-\pi^{2\dot{1}}\xi^1_{(2)} + \pi^{1\dot{1}}\xi^2_{(2)} &= m\eta_{\dot{2}}
\end{aligned}\right\} \tag{34}$$

System of Equations (33) and (34) is equivalent to the Dirac Equation (25) if the definitions Equations (28) and (29) are invoked.

Due to the identities, Equations (31–34) can be cast into form:

$$\begin{pmatrix} 0 & 0 & \pi^{1\dot{1}} & \pi^{1\dot{2}} \\ 0 & 0 & \pi^{2\dot{1}} & \pi^{2\dot{2}} \\ \pi^{2\dot{2}} & -\pi^{1\dot{2}} & 0 & 0 \\ -\pi^{2\dot{1}} & \pi^{1\dot{1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^1_{(1)} \\ \xi^2_{(1)} \\ \eta_{\dot{1}} \\ 0 \end{pmatrix} = m \begin{pmatrix} \xi^1_{(1)} \\ \xi^2_{(1)} \\ \eta_{\dot{1}} \\ 0 \end{pmatrix} \tag{35}$$

$$\begin{pmatrix} 0 & 0 & \pi^{2\dot{2}} & \pi^{2\dot{1}} \\ 0 & 0 & \pi^{1\dot{2}} & \pi^{1\dot{1}} \\ \pi^{1\dot{1}} & -\pi^{2\dot{1}} & 0 & 0 \\ -\pi^{1\dot{2}} & \pi^{2\dot{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^2_{(2)} \\ \xi^1_{(2)} \\ \eta_{\dot{2}} \\ 0 \end{pmatrix} = m \begin{pmatrix} \xi^2_{(2)} \\ \xi^1_{(2)} \\ \eta_{\dot{2}} \\ 0 \end{pmatrix} \tag{36}$$

Let us consider Equation (35). It can be written as:

$$\gamma^\mu \pi_\mu P_4 \Psi_{(1)} = m P_4 \Psi_{(1)} \tag{37}$$

where P_4 is the projection operator, $P_4 = \text{diag}\ (1, 1, 1, 0)$ in the spinor representation of the Dirac matrices and $\Psi_{(1)} = \left(\xi^1_{(1)}, \xi^2_{(1)}, \eta_{\dot{1}}, \eta_{\dot{2}}\right)^T$. There are also other projection operators which lead to analogous three component equations, $P_1 = \text{diag}\ (0, 1, 1, 1)$, $P_2 = \text{diag}\ (1, 0, 1, 1)$, $P_3 = \text{diag}\ (1, 1, 0, 1)$. Acting from the left on Equation (37) with P_4 and $(1 - P_4)$ we obtain two Equations:

$$P_4 \left(\gamma^\mu_\mu \pi\right) P_4 \Psi_{(1)} = m P_4 \Psi_{(1)} \tag{38}$$

$$(1 - P_4) \left(\gamma^\mu_\mu \pi\right) P_4 \Psi_{(1)} = 0 \tag{39}$$

In the spinor representation of γ^μ matrices, Equation (38) is equivalent to Equation (33) while Equation (39) is equivalent to the identity Equation (31), respectively. The operator P_4 can be written as $P_4 = \frac{1}{4}(3+\gamma^5 - \gamma^0\gamma^3 + i\gamma^1\gamma^2)$ where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ (similar formulae can be given for other projection operators P_1, P_2, P_3, see [13] where another convention for γ^μ matrices was however used). It thus follows that Equation (37) is given representation independent form and is Lorentz covariant (in [9] subsolutions of form Equation (37) were obtained for the free Dirac equation).

Let us note finally that Equation (36) can be alternatively written as

$$\gamma^\mu \pi_\mu P_3 \Psi_{(2)} = m P_3 \Psi_{(2)} \tag{40}$$

where $\Psi_{(2)} = \left(\xi^1_{(2)}, \xi^2_{(2)}, \eta_{\dot{1}}, \eta_{\dot{2}}\right)^T$, $P_3 = \frac{1}{4}(3+\gamma^5 + \gamma^0\gamma^3 - i\gamma^1\gamma^2)$, note that $\Psi = P_4\Psi_{(1)} + P_3\Psi_{(2)}$.

4. Separation of Variables in Subequations

It is possible to separate variables in Equations (33) and (34) following procedures described in [10]. Substituting $\xi^1_{(1)}$ and $\xi^2_{(1)}$ from the first two equations into the third in Equation (33) we get:

$$\pi^{2\dot{2}}\pi^{1\dot{1}}\eta_{\mathrm{i}} - \pi^{1\dot{2}}\pi^{2\dot{1}}\eta_{\mathrm{i}} = m^2\eta_{\mathrm{i}} \tag{41}$$

Taking into account definition of $\pi^{A\dot{B}}$ and property Equation (24) we obtain:

$$\left(\pi_\mu\pi^\mu + iqE\left(x^0, x^3\right) + qH\left(x^1, x^2\right)\right)\eta_{\mathrm{i}} = m^2\eta_{\mathrm{i}} \tag{42}$$

where $E = \partial_0 A_3 - \partial_3 A_0$, $H = \partial_2 A_1 - \partial_1 A_2$.

To achieve separation of variables we put:

$$\eta_{\mathrm{i}}(x) = \varphi_{\mathrm{i}}\left(x^0, x^3\right)\psi_{\mathrm{i}}\left(x^1, x^2\right) \tag{43}$$
$$\xi^1_{(1)}(x) = \alpha_{\mathrm{i}}\left(x^0, x^3\right)\psi_{\mathrm{i}}\left(x^1, x^2\right) \tag{44}$$
$$\xi^2_{(1)}(x) = \varphi_{\mathrm{i}}\left(x^0, x^3\right)\beta_{\mathrm{i}}\left(x^1, x^2\right) \tag{45}$$

We now substitute Equation (43) into Equation (42) to get:

$$\left(\left(\pi^0\right)^2 - \left(\pi^3\right)^2 + iqE\left(x^0, x^3\right)\right)\varphi_{\mathrm{i}}\left(x^0, x^3\right) = \left(m^2 + \lambda_{\mathrm{i}}^2\right)\varphi_{\mathrm{i}}\left(x^0, x^3\right) \tag{46a}$$
$$\left(\left(\pi^1\right)^2 + \left(\pi^2\right)^2 - qH\left(x^1, x^2\right)\right)\psi_{\mathrm{i}}\left(x^1, x^2\right) = \lambda_{\mathrm{i}}^2\psi_{\mathrm{i}}\left(x^1, x^2\right) \tag{46b}$$

where λ_{i}^2 is the separation constant and we note that Equations (46a) and (46b) are analogous to Equations (12.15) and (12.19) in [10].

Combining now Equation (46a) with the first of Equation (33) and rescaling, $\alpha_{\mathrm{i}}\left(x^0, x^3\right) = \sqrt{1 + \frac{\lambda_{\mathrm{i}}^2}{m^2}}\tilde{\alpha}_{\mathrm{i}}\left(x^0, x^3\right)$, we obtain 2D Dirac Equation:

$$\left(\pi^0 + \pi^3\right)\varphi_{\mathrm{i}}\left(x^0, x^3\right) = \tilde{m}\tilde{\alpha}_{\mathrm{i}}\left(x^0, x^3\right) \tag{47a}$$
$$\left(\pi^0 - \pi^3\right)\tilde{\alpha}_{\mathrm{i}}\left(x^0, x^3\right) = \tilde{m}\varphi_{\mathrm{i}}\left(x^0, x^3\right) \tag{47b}$$

with effective mass $\tilde{m} = \sqrt{m^2 + \lambda_{\mathrm{i}}^2}$.

On the other hand, combining Equation (46b) with the second of Equation (33) we get equations:

$$\left(\pi^1 - i\pi^2\right)\psi_{\mathrm{i}}\left(x^1, x^2\right) = m\beta_{\mathrm{i}}\left(x^1, x^2\right) \tag{48a}$$
$$\left(\pi^1 + i\pi^2\right)\beta_{\mathrm{i}}\left(x^1, x^2\right) = \frac{\lambda_{\mathrm{i}}^2}{m}\psi_{\mathrm{i}}\left(x^1, x^2\right) \tag{48b}$$

which can be written as the Pauli Equation:

$$\left[\left(\left(\pi^1\right)^2 + \left(\pi^2\right)^2\right)\sigma^0 - qH\left(x^1, x^2\right)\sigma^3\right]\begin{pmatrix}\psi_{\mathrm{i}}\\ \beta_{\mathrm{i}}\end{pmatrix} = \lambda_{\mathrm{i}}^2\begin{pmatrix}\psi_{\mathrm{i}}\\ \beta_{\mathrm{i}}\end{pmatrix} \tag{49}$$

The same procedure applied to Equation (34) yields the equation for $\eta_{\dot{2}}$:

$$\left(\pi_{\mu}\pi^{\mu} - iqE\left(x^{0}, x^{3}\right) - qH\left(x^{1}, x^{2}\right)\right)\eta_{\dot{2}} = m^{2}\eta_{\dot{2}} \tag{50}$$

Carrying out separation of variables we get 2D Dirac Equation:

$$\begin{aligned}\left(\pi^{0} - \pi^{3}\right)\varphi_{\dot{2}}\left(x^{0}, x^{3}\right) &= \hat{m}\hat{\alpha}_{\dot{2}}\left(x^{0}, x^{3}\right) &\text{(51a)}\\ \left(\pi^{0} + \pi^{3}\right)\hat{\alpha}_{\dot{2}}\left(x^{0}, x^{3}\right) &= \hat{m}\varphi_{\dot{2}}\left(x^{0}, x^{3}\right) &\text{(51b)}\end{aligned}$$

with effective mass $\hat{m} = \sqrt{m^{2} + \lambda_{\dot{2}}^{2}}$ and $\alpha_{\dot{2}}\left(x^{0}, x^{3}\right) = \sqrt{1 + \frac{\lambda_{\dot{2}}^{2}}{m^{2}}}\hat{\alpha}_{\dot{2}}\left(x^{0}, x^{3}\right)$ and equation:

$$\begin{aligned}\left(\pi^{1} + i\pi^{2}\right)\psi_{\dot{2}}\left(x^{1}, x^{2}\right) &= m\beta_{\dot{2}}\left(x^{1}, x^{2}\right) &\text{(52a)}\\ \left(\pi^{1} - i\pi^{2}\right)\beta_{\dot{2}}\left(x^{1}, x^{2}\right) &= \frac{\lambda_{\dot{2}}^{2}}{m}\psi_{\dot{2}}\left(x^{1}, x^{2}\right) &\text{(52b)}\end{aligned}$$

which is written as the Pauli Equation

$$\left[\left(\left(\pi^{1}\right)^{2} + \left(\pi^{2}\right)^{2}\right)\sigma^{0} + qH\left(x^{1}, x^{2}\right)\sigma^{3}\right]\begin{pmatrix}\psi_{\dot{2}}\\ \beta_{\dot{2}}\end{pmatrix} = \lambda_{\dot{2}}^{2}\begin{pmatrix}\psi_{\dot{2}}\\ \beta_{\dot{2}}\end{pmatrix} \tag{53}$$

where the following definitions were used:

$$\begin{aligned}\eta_{\dot{2}}(x) &= \varphi_{\dot{2}}\left(x^{0}, x^{3}\right)\psi_{\dot{2}}\left(x^{1}, x^{2}\right) &\text{(54)}\\ \xi_{(2)}^{1}(x) &= \alpha_{\dot{2}}\left(x^{0}, x^{3}\right)\psi_{\dot{2}}\left(x^{1}, x^{2}\right) &\text{(55)}\\ \xi_{(2)}^{2}(x) &= \varphi_{\dot{2}}\left(x^{0}, x^{3}\right)\beta_{\dot{2}}\left(x^{1}, x^{2}\right) &\text{(56)}\end{aligned}$$

5. Splitting the Spin 0 Duffin–Kemmer–Petiau Equations in Crossed Fields

We introduce interaction into DKP Equation (19) via minimal coupling Equation (1). We consider four-potentials obeying the condition:

$$\left[\pi^{0}, \pi^{3}\right] = \left[\pi^{1}, \pi^{2}\right] = 0 \tag{57}$$

The condition Equation (57) means that $E^{3} = H^{3} = 0$ and is fulfilled by crossed fields [10]:

$$\vec{E}\cdot\vec{n} = \vec{H}\cdot\vec{n} = \vec{E}\cdot\vec{H} = 0, \quad |\vec{E}| = |\vec{H}| \tag{58}$$

with $\vec{n} = [0, 0, 1]$.

Equation (19) in the interacting case can be written within spinor formalism (*cf.* Equations (2) and (3)) as:

$$\left.\begin{aligned}\pi^{A\dot{B}}\psi &= m\psi^{A\dot{B}}\\ \pi_{A\dot{B}}\psi^{A\dot{B}} &= 2m\psi\end{aligned}\right\} \tag{59}$$

Indeed, it follows from Equation (59) that $\pi_{A\dot{B}}\psi^{A\dot{B}} = \frac{1}{m}\pi_{A\dot{B}}\pi^{A\dot{B}}\psi$ and $\pi_{A\dot{B}}\pi^{A\dot{B}}\psi = 2m^2\psi$. We have $\pi_{A\dot{B}}\pi^{A\dot{B}} = \pi_{1\dot{1}}\pi^{1\dot{1}} + \pi_{2\dot{1}}\pi^{2\dot{1}} + \pi_{1\dot{2}}\pi^{1\dot{2}} + \pi_{2\dot{2}}\pi^{2\dot{2}} = 2\pi_\mu\pi^\mu$ and the Klein–Gordon Equation $\pi_\mu\pi^\mu\psi = m^2\psi$ follows.

Let us note now that for fields obeying Equation (57), the following spinor identities hold:

$$\pi_{1\dot{1}}\pi^{1\dot{1}} + \pi_{2\dot{1}}\pi^{2\dot{1}} = \pi_\mu\pi^\mu, \quad \pi_{1\dot{2}}\pi^{1\dot{2}} + \pi_{2\dot{2}}\pi^{2\dot{2}} = \pi_\mu\pi^\mu \tag{60}$$

Due to identities Equation (60) we can split the last of Equation (59) and write Equation (59) as a set of two equations:

$$\left.\begin{aligned} \pi^{1\dot{1}}\psi &= m\psi^{1\dot{1}} \\ \pi^{2\dot{1}}\psi &= m\psi^{2\dot{1}} \\ \pi_{1\dot{1}}\psi^{1\dot{1}} + \pi_{2\dot{1}}\psi^{2\dot{1}} &= m\psi \end{aligned}\right\} \tag{61}$$

$$\left.\begin{aligned} \pi^{1\dot{2}}\psi &= m\psi^{1\dot{2}} \\ \pi^{2\dot{2}}\psi &= m\psi^{2\dot{2}} \\ \pi_{1\dot{2}}\psi^{1\dot{2}} + \pi_{2\dot{2}}\psi^{2\dot{2}} &= m\psi \end{aligned}\right\} \tag{62}$$

each of which describes particle with mass m (we check this by substituting e.g. $\psi^{1\dot{1}}$, $\psi^{2\dot{1}}$ or $\psi^{1\dot{2}}$, $\psi^{2\dot{2}}$ into the third equations). Equation (59) and the set of two Equations (61) and (62) are equivalent. We described Equations (61) and (62) in non-interacting case in [34,35]. Equations (61) and (62) and Equations (33) and (34) have the same structure (recall that $\pi_{1\dot{1}} = \pi^{2\dot{2}}$, $\pi_{1\dot{2}} = -\pi^{2\dot{1}}$, $\pi_{2\dot{1}} = -\pi^{1\dot{2}}$, $\pi_{2\dot{2}} = \pi^{1\dot{1}}$). However these equations cannot be written in the form of the Dirac Equations (35) and (36) because identities analogous to Equations (31) and (32) do not hold, *i.e.*, $\pi^{2\dot{1}}\psi^{1\dot{1}} \neq \pi^{1\dot{1}}\psi^{2\dot{1}}$, $\pi^{2\dot{2}}\psi^{1\dot{2}} \neq \pi^{1\dot{2}}\psi^{2\dot{2}}$.

Substituting first two equations into the third one in Equation (61), we get the Klein–Gordon equation $\pi_\mu\pi^\mu\psi = m^2\psi$, which can be solved via separation of variables for the case of crossed fields, see Chapter 3 in [10] (the same can be done in Equation (62)).

6. A Supersymmetric Link between Dirac and DKP Theories

We have shown that subsolutions of the Dirac equation as well as of the DKP equations for spin 0 obey analogous pairs of 3×3 Equations (33–62), respectively.

More exactly, Equations (33) and (34) can be written as:

$$\begin{aligned} \rho^\mu\pi_\mu\Psi &= m\Psi, \quad \Psi = \left(\xi^1_{(1)}, \xi^2_{(1)}, \eta_{\dot{1}}\right)^T, \qquad \left(\xi^1_{(1)} = \psi^{1\dot{1}}_{\dot{1}}, \xi^2_{(1)} = \psi^{2\dot{1}}_{\dot{1}}\right) && (63) \\ \tilde{\rho}^\mu\pi_\mu\tilde{\Psi} &= m\tilde{\Psi}, \quad \tilde{\Psi} = \left(\xi^1_{(2)}, \xi^2_{(2)}, \eta_{\dot{2}}\right)^T, \qquad \left(\xi^1_{(2)} = \psi^{1\dot{2}}_{\dot{2}}, \xi^2_{(2)} = \psi^{2\dot{2}}_{\dot{2}}\right) && (64) \end{aligned}$$

with

$$\rho^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \rho^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \rho^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \rho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{65}$$

$$\tilde{\rho}^0 = i\rho^2, \quad \tilde{\rho}^1 = \rho^3, \quad \tilde{\rho}^2 = i\rho^0, \quad \tilde{\rho}^3 = i\rho^2 \tag{66}$$

and $\pi^\mu = p^\mu - qA^\mu$, A^μ obeying condition of longitudinality Equation (23).

On the other hand, Equations (61) and (62) can be written in analogous form:

$$\rho^\mu \pi_\mu \Phi = m\Phi, \quad \Phi = \left(\psi^{1\dot{1}},\, \psi^{2\dot{1}},\, \psi\right)^T \tag{67}$$

$$\tilde{\rho}^\mu \pi_\mu \tilde{\Phi} = m\tilde{\Phi}, \quad \tilde{\Phi} = \left(\psi^{1\dot{2}},\, \psi^{2\dot{2}},\, \psi\right)^T \tag{68}$$

with the same matrices ρ^μ, $\tilde{\rho}^\mu$, *cf.* Equations (65) and (66), and $\pi^\mu = p^\mu - qA^\mu$, A^μ obeying condition Equation (57)—fulfilled by crossed fields.

It thus follows that the 3×3 free equations described in [34,35]:

$$\rho^\mu p_\mu \Theta = m\Theta \tag{69}$$

$$\tilde{\rho}^\mu p_\mu \tilde{\Theta} = m\tilde{\Theta} \tag{70}$$

provide a link between solutions of the Dirac and DKP equations. Namely, Equations (69) and (70) in the interacting case, $p^\mu \longrightarrow \pi^\mu = p^\mu - qA^\mu$, lead to subsolutions of the Dirac Equations (63) and (64) in the case of longitudinal fields Equation (23), while for crossed fields Equation (57) yield DKP subsolutions Equations (67) and (68).

7. Discussion

We have shown that the Dirac equation in longitudinal external fields is equivalent to a pair of 3×3 subequations (33) and (34) which can be further written as Dirac equations with built-in projection operators, Equations (37) and (40). Furthermore, we have demonstrated that the Duffin–Kemmer–Petiau equations for spin 0 in crossed fields can be split into two 3×3 subequations (61) and (62) (subequations of the DKP equations for spin 1 were discussed in [36]). It was also shown that all the subequations can be obtained via minimal coupling from the same 3×3 subequations (69) and (70), which are thus a supersymmetric link between fermionic and bosonic degrees of freedom. It can be expected that for a combination of crossed and longitudinal potentials these subequations should describe interaction of fermionic and bosonic degrees of freedom. We shall investigate this problem in our future work.

Finally, we shall address problem of Lorentz covariance of the subequations. Let us have a closer look at a single subequation of spin 0 DKP equation, say Equation (67). Although both equations, Equation (67) and (68), are covariant as a whole, this subequation alone is not Lorentz covariant. Moreover, it cannot be written as manifestly covariant Dirac equation, *cf.* the end of Section 5. There is however another possibility of introducing full covariance. Let us consider left and right eigenvectors of the operator $\rho^\mu \pi_\mu$:

$$\rho^\mu \overrightarrow{\pi_\mu} \Phi_R = m\Phi_R \tag{71a}$$

$$\Phi_L \rho^\mu \overleftarrow{\pi_\mu} = m\Phi_L \tag{71b}$$

where symbols $\overrightarrow{\pi_\mu}$, $\overleftarrow{\pi_\mu}$ mean action of π_μ to the right or to the left, respectively (left solutions are actually used in the Dirac theory, where they are denoted as $\bar{\Psi}$, they are however related to the right solutions by the formula $\bar{\Psi} = \Psi^\dagger \gamma^0$ (symbol † denotes Hermitian conjugation) [11]).

It turns out that Equation (71), with $\Phi_R = \left(\psi^{1\dot{1}},\ \psi^{2\dot{1}},\ \psi\right)^T$ and $\Phi_L = (\psi_{1\dot{1}},\ \psi_{2\dot{1}},\ \psi)$, are equivalent to Equations (61) and (62) respectively (note that $\Phi_R \equiv \Phi$) and involve components of the whole spinor $\psi^{A\dot{B}}$ since $\Phi_L = \left(\psi^{2\dot{2}},\ -\psi^{1\dot{2}},\ \psi\right)$. The same analysis applies to Equation (68), *i.e.*, $\tilde{\rho}^\mu \overrightarrow{\pi_\mu} \tilde{\Phi}_R = m\tilde{\Phi}_R$, $\tilde{\Phi}_L \tilde{\rho}^\mu \overleftarrow{\pi_\mu} = m\tilde{\Phi}_L$ and $\tilde{\Phi}_R = \left(\psi^{1\dot{2}},\ \psi^{2\dot{2}},\ \psi\right)^T$, $\tilde{\Phi}_L = (\psi_{1\dot{2}},\ \psi_{2\dot{2}},\ \psi) = \left(-\pi^{2\dot{1}},\ \pi^{1\dot{1}},\ \psi\right)$ (note that $\tilde{\Phi}_R$ and Φ_L, as well as $\tilde{\Phi}_L$ and Φ_R are algebraically related).

We shall now discuss problem of Lorentz covariance of subequations of the Dirac equation, Equations (63) and (64). Let first note that Equations (69) and (70), as well as Equations (63) and (64), can be written in covariant form as the Dirac equation with one zero component as Equations (15,16,37,40), respectively. However, solutions of Equations (63) and (64) do not involve the whole spinor $\psi^{A\dot{B}}_{\dot{C}}$. We might consider left eigensolutions of the operator $\rho^\mu \pi_\mu$ again but this does not change the picture—Equations (63) and (64) involve components $\psi^{1\dot{1}}_{\dot{1}}$, $\psi^{2\dot{1}}_{\dot{1}}$, $\psi^{1\dot{2}}_{\dot{2}}$, $\psi^{2\dot{2}}_{\dot{2}}$ only as well as the whole spinor $\eta_{\dot{C}}$. It follows that in Equations (63) and (64) we deal with Lorentz symmetry breaking—a hypothetical phenomenon considered in some extensions of the Standard Model [37–39].

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