Article

Supersymmetric Sigma Model Geometry

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Abstract: This is a review of how sigma models formulated in Superspace have become
important tools for understanding geometry. Topics included are: The (hyper)kähler
reduction; projective superspace; the generalized Legendre construction; generalized Kähler
geometry and constructions of hyperkähler metrics on Hermitian symmetric spaces.

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1. Introduction

Sigma models take their name from a phenomenological model of beta decay introduced more than
fifty years ago by Gell-Mann and Lévy [1]. It contains pions and a new scalar meson that they called
sigma. A generalization of this model is what is nowadays meant by a sigma model. It will be described
in detail below.

Non-linear sigma models arise in a surprising number of different contexts. Examples are effective
field theories (coupled to gauge fields), the scalar sector of supergravity theories, *etc*. Not the least
concern to modern high energy theory is the fact that the string action has the form of a sigma model
coupled to two dimensional gravity and that the compactified dimensions carry the target space geometry
dictated by supersymmetry.

The close relation between supersymmetric sigma models and complex geometry was first observed
more than thirty years ago in [2] where the target space of $\mathcal{N} = 1$ models in four dimensions is

shown to carry Kähler geometry. For $\mathcal{N} = 2$ models in four dimensions the target space geometry was subsequently shown to be hyperkähler in [3]. This latter fact was extensively exploited in a $\mathcal{N} = 1$ superspace formulation of these models in [4], where two new constructions were presented; the Legendre transform construction and the hyperkähler quotient construction. The latter reduction was developed and given a more mathematically stringent formulation in [5] where we also elaborated on a manifest $\mathcal{N} = 2$ formulation, originally introduced in [6] based on observations in [7].

A $\mathcal{N} = 2$ superspace formulation of the $\mathcal{N} = 2$, four dimensional sigma model is obviously desirable, since it will automatically lead to hyperkähler geometry on the target space. The $\mathcal{N} = 2$ Projective Superspace which makes this possible grew out of the development mentioned last in the preceding paragraph. Over the years it has been developed and refined in, e.g., [6–19]. In this article we report on some of that development along with some more recent development, such as projective superspace for supergravity [20–25], and applications such as the construction of certain classes of hyperkähler metrics [26–28].

The target space geometry depends on the number of supersymmetries as well as on the dimension of the domain. There are a number of features peculiar to sigma models with a two dimensional domain (2D sigma models). Here the target space geometry can be torsionful and generalizes the Kähler and hyperkähler geometries. This has been exploited to give new and interesting results in generalized geometry [29–41].

Outside the scope of this report lies, e.g., Kähler geometries with additional structure, such as special geometry relevant for four dimensional $N = 2$ sigma models, (see, e.g., [42]).

All our presentations will concern the classical theory. We shall not discuss the interesting and important question of quantization.

2. Sigma Models

A non-linear sigma model is a theory of maps from a (super) manifold $\Sigma^{(d,\mathcal{N})}$ to a target space (we shall mostly avoid global issues and assume that all of $\mathcal{T}$ can be covered by such maps in patches) $\mathcal{T}$:

$$\Phi : \Sigma^{(d,\mathcal{N})} \to \mathcal{T}$$
$$\Phi(z) \longmapsto Z \in \mathcal{T} \tag{1}$$

Denoting the coordinates on $\Sigma^{(d,\mathcal{N})}$ by $z = (\xi, \theta)$, the maps are derived by extremizing an action

$$S = \int_\Sigma dz \mathcal{L}(\Phi) + \int_{\partial\Sigma} .... \tag{2}$$

The actual form of the action depends on the bosonic and fermionic dimensions d_B, d_F of Σ. We have temporarily included a boundary term, which is sometimes needed for open models to have all the symmetries of the bulk-theory [43–45], or when there are fields living only on the boundary coupling to the bulk fields as is the well known case for (stacks of) D-branes. For a discussion of the latter in a sigma model context, see [46]. Expanding the superfield as $\Phi(\xi, \theta) = X + \theta\Psi + ...$, where $X(\xi)$ and $\Psi(\xi)$ are bosonic and fermionic fields over the even part of Σ (coordinatized by ξ), the action Equation (2) becomes

$$S = \mu^{d-2} \int_{\Sigma_B} d\xi \left\{ \partial X^\mu G_{\mu\nu}(X) \partial X^\nu + \ldots \right\} + \int_{\partial\Sigma} .... \tag{3}$$

where $d = d_B$ is the bosonic dimension of Σ and we have rescaled the X's to make them dimensionless, thus introducing the mass-scale μ.

Let us make a few general comments about the action Equation (3), mostly quoted from Hull [47]:

1. The mass-scale μ shows that the model typically will be non-renormalizable for $d \geqslant 3$ but renormalizable and classically conformally invariant in $d = 2$.
2. We have not included a potential for X and thus excluded Landau–Ginsburg models.
3. There is also the possibility to include a Wess–Zumino term. We shall return to this when discussing $d = 2$.
4. From a quantum mechanical point of view it is useful to think of $G_{\mu\nu}(X)$ as an infinite number of coupling constants:

$$G_{\mu\nu}(X) = G^0_{\mu\nu} + G^1_{\mu\nu,\rho}X^\rho + \dots \tag{4}$$

5. Classically, it is more rewarding to emphasize the geometry and think of $G_{\mu\nu}(X)$ as a metric on the target space $\mathcal{T}$. This is the aspect we shall be mainly concerned with.
6. The invariance of the action S under *Diff*($\mathcal{T}$),

$$X^\mu \to X^{\mu'}(X)\,, \quad G_{\mu\nu}(X) \to G_{\mu'\nu'}(X') \tag{5}$$

(field-redefinitions from the point of view of the field theory on Σ), implies that the sigma model is defined by an equivalence class of metrics. *N.B.* This is not a symmetry of the model since the "coupling constants" also transform. It is an important property, however. Classically it means that the model is extendable beyond a single patch in $\mathcal{T}$, and quantum mechanically it is needed for the effective action to be well defined.

That the geometry of the target space $\mathcal{T}$ is inherently related to the sigma model is clear already from the preceding comments. Further, the maps extremizing S satisfy

$$\nabla_i\partial^i X^\mu := \partial_i X^\nu \nabla_\nu \partial^i X^\mu = 0 \tag{6}$$

where $\partial_i := \frac{\partial}{\partial\xi^i}$, the operator ∇ is the Levi-Civita connection for $G_{\mu\nu}$ and we have ignored the fermions. This is the pull-back of the covariant Laplacian on $\mathcal{T}$ to Σ and hence the maps are sometimes called harmonic maps.

The geometric structure that has emerged from the bosonic part shows that the target space geometry must be Riemannian (by which we mean that it comes equipped with a metric and corresponding Levi-Civita connection). Further restrictions arise from supersymmetry.

3. Supersymmetry

This section provides a very brief summary of some aspects of supersymmetry. For a thorough introduction the reader should consult a textbook, e.g., [48–50].

At the level of algebra, supersymmetry is an extension of the d-dimensional Poincaré-algebra to include anticommuting charges Q. The form of the algebra depends on d. In $d_B = 4$ the additional (anti-)commutators satisfy

$$\{Q^a_\alpha, Q^b_\beta\} = 2\delta^{ab}(\gamma^i C)_{\alpha\beta}P_i + C_{\alpha\beta}Z^{ab} + (\gamma_5 C)_{\alpha\beta}Y^{ab} \tag{7}$$

Here $\alpha, \beta, ..$ are four dimensional spinor indices, γ^i are Dirac gamma matrices, C is the (electric) charge conjugation matrix and the charges Q^a are spinors that satisfy a Majorana reality condition and transform under some internal symmetry group $\mathcal{G} \subset O(N)$ (corresponding to the index a). The generators of the graded Poincaré-algebra are thus the Lorentz-generators M, the generators of translations P and the supersymmetry generators Q. In addition, for non-trivial $\mathcal{G}$, there are central charge generators Z and Y that commute with all the others.

Representations of supersymmetry are most economically collected into superfields $\Phi(\xi, \theta)$, where θ^a are Grassmann valued spinorial "coordinates" to which one can attach various amounts of importance. We may think of them as a book-keeping device, much as collecting components into a column-vector. But thinking about (ξ, θ) as coordinates on a supermanifold $\mathcal{M}^{(d,\mathcal{N})}$ [51,52] and investigating the geometry of this space has proven a very fruitful way of generating interesting results.

The index a on θ^a is the same as that on Q^a and thus corresponds to the number $\mathcal{N}$ of supersymmetries. Let us consider the case $\mathcal{N} = 1$, $d_B = 4$, which implies $Z = Y = 0$. In a Weyl-representation of the spinors and with the usual identification of the translation generator as a differential operator $P_{\alpha\dot{\alpha}} = i\partial_{\alpha\dot{\alpha}} := i\sigma^i_{\alpha\dot{\alpha}}\partial_i$, the algebra Equation (7) becomes

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\partial_{\alpha\dot{\alpha}} \tag{8}$$

Introducing Berezin integration/derivation [53], $\partial_\alpha := \frac{\partial}{\partial\theta^\alpha}$, the supercharges Q may also be represented (in one of several possible representations) as differential operators acting on superfields:

$$Q_\alpha = i\partial_\alpha + \frac{1}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}$$
$$\bar{Q}_{\dot{\alpha}} = i\partial_{\dot{\alpha}} + \frac{1}{2}\theta^\alpha\partial_{\alpha\dot{\alpha}} \tag{9}$$

Apart from the various representations alluded to above (chiral, antichiral and vector in four dimensions [48]) there are two basic ways that supercharges can act on superfields, corresponding to left and right group action. This means that, given Equation (9), there is a second pair of differential operators

$$D_\alpha = \partial_\alpha + i\frac{1}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}$$
$$\bar{D}_{\dot{\alpha}} = \partial_{\dot{\alpha}} + i\frac{1}{2}\theta^\alpha\partial_{\alpha\dot{\alpha}} \tag{10}$$

which also generate the algebra Equation (8) and that anticommute with the Q's:

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i\partial_{\alpha\dot{\alpha}}\ , \quad \{Q_\alpha, \bar{D}_{\dot{\alpha}}\} = \{Q_\alpha, D_\beta\} = 0 \tag{11}$$

Often the supersymmetry algebra is given only in terms of the D's. From a geometrical point of view, these are covariant derivatives in superspace and may be used to impose invariant conditions on superfields.

In general, covariant derivatives $\nabla_{\mathcal{A}}$ in a curved superspace space satisfy

$$[\nabla_{\mathcal{A}}, \nabla_{\mathcal{B}}\} = R_{\mathcal{A}\mathcal{B}} \cdot \mathbb{M} + 2T_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}\nabla_{\mathcal{C}} \tag{12}$$

where the left hand side contains a graded commutator $T_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}$ is the torsion tensor and $R_{\mathcal{A}\mathcal{B}} \cdot \mathbb{M}$ the curvature with $\mathbb{M}$ the generators of the structure group. The indices $\mathcal{A}$ *etc.* run over both bosonic and

fermionic indices. Comparision to Equation (11), with $\nabla_{\mathcal{A}} = (D_\alpha, \bar{D}_{\dot{\alpha}}, i\nabla_{\alpha\dot{\alpha}})$, shows that even "flat" $(R_{\mathcal{AB}} \cdot \mathbb{M} = 0)$ superspace has torsion.

In four dimensions the D's may be used to find the smallest superfield representation. Such a chiral superfield ϕ and its complex conjugate antichiral field $\bar{\phi}$ are required to satisfy

$$\bar{D}_{\dot{\alpha}}\phi = D_\alpha\bar{\phi} = 0 \tag{13}$$

The Minkowski-field content of this may be read off from the θ-expansion. However, this expansion depends on the particular representation of the D's and Q'. For this reason it is preferable to define the components in the following representation independent form:

$$X := \phi| \,, \quad \chi := \bar{D}\phi| \,, \quad \bar{\chi} := D\phi| \,, \quad \mathcal{F} := \tfrac{1}{2}D\bar{D}\phi| \tag{14}$$

where a vertical bar denotes "the θ-independent part of". In a chiral representation where $\bar{D}_\alpha = \partial_\alpha$ the θ-expansion of a chiral field reads

$$\phi(\xi, \theta) = X(\xi) + \bar{\theta}\chi(\xi) + \theta\bar{\chi}(\xi) + \bar{\theta}\theta\mathcal{F} \tag{15}$$

but its complex conjugate involves a θ dependent shift in ξ and looks more complicated.

A dimensional analysis shows that if X is a physical scalar, χ is a physical spinor and $\mathcal{F}$ has to be a non-propagating (auxiliary) field. This is the smallest multiplet that contains a scalar, and thus suitable for constructing a supersymmetric extension of the bosonic sigma models we looked at so far. (All other superfields will either be equivalent to (anti) chiral ones or contain additional bosonic fields of higher spin.) Denoting a collection of chiral fields by $\phi = (\phi^\mu)$, the most general action we can write down

$$S = \int d^4\xi d^2\theta d^2\bar{\theta} K(\phi, \bar{\phi}) \tag{16}$$

reduces to the bosonic integral

$$\int d^4\xi \left\{ \partial^i X^b \partial_b \partial_{\bar{a}} K(X, \bar{X}) \partial_i \bar{X}^{\bar{a}} + \ldots \right\} \tag{17}$$

The most direct way to perform the reduction is to write

$$S = \int d^4\xi D^2 \bar{D}^2 K(\phi, \bar{\phi})| \tag{18}$$

and then to use Equation (13) when acting with the covariant spinor derivatives. Note that K in Equation (16) is only defined up to a term $\eta(\phi) + \bar{\eta}(\bar{\phi})$ due to the chirality conditions as in Equation (13).

We immediately learn additional things about the geometry of the target space $\mathcal{T}$:

1. It must be even-dimensional.
2. The metric is Hermitian with respect to the canonical complex structure

$$J := \begin{pmatrix} i\delta_b^a & 0 \\ 0 & -i\delta_{\bar{b}}^{\bar{a}} \end{pmatrix} \tag{19}$$

   for which X and $\bar{X}$ are canonical coordinates.
3. The metric has a potential $K(\phi, \bar{\phi})$. In fact, the geometry is Kähler and the ambiguity in the Lagrangian in Equation (16) is known as a Kähler gauge transformation.

4. Complex Geometry I

Let us interrupt the description of sigma models to recapitulate the essentials of Kähler geometry.

Consider $(\mathcal{M}, G, J)$ where G is a metric on the manifold $\mathcal{M}$ and J is an almost complex structure, *i.e.*, an endomorphism(A $(1,1)$ tensor J^μ_ν.) $J : TM \hookleftarrow$ such that $J^2 = -1$ (only possible if $\mathcal{M}$ is even dimensional). This is an almost Hermitian space if (as matrices)

$$J^t G J = G \tag{20}$$

Construct the projection operators

$$\pi_\pm := \tfrac{1}{2}(1 \pm iJ) \tag{21}$$

If the vectors $\pi_\pm V$ in T are in involution, *i.e.*, if

$$\pi_\mp [\pi_\pm V, \pi_\pm U] = 0 \tag{22}$$

which implies that the Nijenhuis torsion vanishes,

$$N_J(U,V) = [JU, JV] - J[JU,V] - J[U,JV] - [U,V] = 0 \tag{23}$$

then the distributions defined by $\pi_\pm$ are integrable and J is called a complex structure and G Hermitian. To evaluate the expression in Equation (22) in index notation, think of $\pi_\mp$ as matrices acting on the components of the Lie bracket between the vectors $\pi_\pm V$ and $\pi_\pm U$. This leads to the following expression corresponding to Equation (23):

$$\mathcal{N}(J)^\mu_{\nu\rho} = J^\sigma_{[\nu} J^\mu_{\rho],\sigma} + J^\mu_\sigma J^\sigma_{[\nu,\rho]} = 0 \tag{24}$$

The fundamental two-form ω defined by J and G is

$$\omega := G_{\mu\nu} J^\nu_\sigma dX^\mu \wedge dX^\sigma \tag{25}$$

If it is closed for a Hermitian complex space, then the metric has a Kähler potential

$$G = \begin{pmatrix} 0 & \partial\bar\partial K \\ \bar\partial\partial K & 0 \end{pmatrix} \tag{26}$$

and the geometry is Kähler.

An equivalent characterization is as an almost Hermitian manifold with

$$\nabla J = 0 \tag{27}$$

where ∇ is the Levi-Civita connection.

We shall also need the notion of hyperkähler geometry. Briefly, for such a geometry there exists an $SU(2)$-worth of complex structures labeled by $A \in \{1,2,3\}$

$$J^{(A)} J^{(B)} = -\delta^{AB} + \epsilon^{ABC} J^{(C)} , \quad \nabla J^{(A)} = 0 \tag{28}$$

with respect to all of which the metric is Hermitian

$$J^{(A)t} G J^{(A)} = G \tag{29}$$

5. Sigma Model Geometry

We have already seen how $\mathcal{N} = 1$ (one supersymmetry) in $d = 4$ requires the target space geometry to be Kähler. To investigate how the geometry gets further restricted for $\mathcal{N} = 2$, there are various options: (i) Discuss the problem entirely in components (no supersymmetry manifest); (ii) Introduce projective (or harmonic) superspace and write models with manifest $\mathcal{N} = 2$ symmetry; (iii) Add a non-manifest supersymmetry to the model already described and work out the consequences. The last option requires the least new machinery, so we first follow this. The question is thus under what conditions

$$S = \int d^4\xi d^2\theta d^2\bar{\theta} K(\phi, \bar{\phi}) \tag{30}$$

can support an additional supersymmetry. The most general ansatz for such a symmetry is

$$\delta\phi^\mu = \bar{D}^2(\bar{\varepsilon}\bar{\Omega}^\mu) \tag{31}$$

One finds closure of the additional supersymmetry algebra (on-shell) and invariance of the action provided that the target space geometry is hyperkähler with the non-manifest complex structures formed from Ω [54]:

$$J^{(1)} = \begin{pmatrix} 0 & \Omega^{\bar{a}}{}_{,b} \\ \bar{\Omega}^a{}_{,\bar{b}} & 0 \end{pmatrix} \quad J^{(2)} = \begin{pmatrix} 0 & i\Omega^{\bar{a}}{}_{,b} \\ -i\bar{\Omega}^a{}_{,\bar{b}} & 0 \end{pmatrix} \quad J^{(3)} = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix} \tag{32}$$

Further, the parameter superfield ε obeys

$$\bar{D}_{\dot{\alpha}}\varepsilon = D^2\varepsilon = \partial_{\alpha\dot{\alpha}}\varepsilon = 0 \tag{33}$$

It contains the parameter for central charge transformations along with the supersymmetry parameters. The full table of geometries for supersymmetric non-linear sigma models without Wess–Zumino term reads

d =	6	4	2	Geometry
N =	1	2	4	Hyperkähler
N =		1	2	Kähler
N =			1	Riemannian

(Odd dimensions have the same structure as the even dimension lower.) When we specialize to two or six dimensions, we have the additional possibility of having independent left and right supersymmetries; the $\mathcal{N} = (p, q)$ supersymmetries of Hull and Witten [55]. We shall return to this possibility when we discuss $d = 2$, but now we turn to the question of how to gauge isometries on Kähler and hyperkähler manifolds.

6. Gauging Isometries and the HK Reduction

This section is to a large extent a review of [5,54].

6.1. Gauging Isometries of Bosonic Sigma Models

The table in the previous section describing the target-space geometry shows that constructing, e.g., new $\mathcal{N} = 2, d = 4$ nonlinear sigma models is tantamount to finding new hyper-kähler geometries. A systematic method for doing this involves isometries of the target space, which we now discuss.

Consider again the bosonic action

$$S = \int d\xi \partial_i \phi^\mu G_{\mu\nu}(\phi) \partial^i \phi^\nu \tag{34}$$

As noted in Section 2, a target space diffeomorphism leaves this action invariant and corresponds to a field redefinition. As also pointed out, this is not a symmetry of the field theory. A symmetry of the field theory involves a transformation of ϕ only;

$$\delta\phi^\mu = \lambda^A k_A^\mu(\phi) = [\lambda k, \phi]^\mu \equiv \mathcal{L}_{\lambda k} \phi^\mu \tag{35}$$

where $\mathcal{L}_{\lambda k}$ denotes the Lie derivative along the vector λk. Under such a transformation the action varies as

$$\delta S = \int d\xi \partial_i \phi^\mu \mathcal{L}_{\lambda k} G_{\mu\nu}(\phi) \partial^i \phi^\nu \tag{36}$$

The transformation thus gives an invariance of the action if

$$\mathcal{L}_{\lambda k} G_{\mu\nu} = 0 \tag{37}$$

i.e., if the transformation is an isometry and hence if the k_A^μ's are Killing-vectors.

We take the k_A's to generate a Lie algebra $\mathfrak{g}$

$$[k_A, k_B] = c_{AB}{}^C k_C \tag{38}$$

with

$$k_A \equiv k_A^\mu \frac{\partial}{\partial \phi^\mu} \tag{39}$$

In what follows, we assume that $\mathfrak{g}$ can be exponentiated to a group $\mathcal{G}$.

One way to construct a new sigma model from one which has isometries is to gauge the isometries and then find a gauge connection that extremizes the action [54]. The new sigma model will be a quotient of the original one. Briefly, this goes as follows:

The isometries generated by k_A are gauged introducing a gauge field A_i^A using minimal coupling,

$$\partial_i \phi^\mu \to \partial_i \phi^\mu - A_i^A k_A^\mu = (\partial_i - A_i^A k_A)\phi^\mu \equiv \nabla_i \phi^\mu \tag{40}$$

in the action Equation (34):

$$S = \int d\xi \nabla_i \phi^\mu G_{\mu\nu}(\phi) \nabla^i \phi^\nu \tag{41}$$

This action is now locally invariant under the symmetries defined by the algebra Equation (38). Note that there is no kinetic term for the gauge-field. Extremizing Equation (41) with respect to A_i^A singles out a particular gauge-field:

$$\delta S = 0 \Rightarrow A_i^A = \mathbb{H}^{-1AB} G_{\mu\nu} k_B^\mu \partial_i \phi^\nu \tag{42}$$

where

$$\mathbb{H}_{AB} \equiv k_A^\mu G_{\mu\nu} k_B^\nu \tag{43}$$

In terms of this particular connection, the action Equation (41) now reads

$$S = \int d\xi \partial_i \phi^\mu \left(G_{\mu\nu} - \mathbb{H}^{-1AB} k_{\mu_A} k_{\nu_B} \right) \partial^i \phi^\nu \tag{44}$$

where indices have been lowered using the metric G. Since $\partial_i \phi^\mu \approx \partial_i \phi^\mu + v_i^A k_A^\mu$ in the action Equation (44), this is a new sigma model defined on the space of orbits of the group $\mathcal{G}$, *i.e.*, on the quotient space $\mathcal{T}/\mathcal{G}$. The metric on this space is $\tilde{G}_{\mu\nu} \equiv G_{\mu\nu} - \mathbb{H}^{-1AB} k_{\mu_A} k_{\nu_B}$.

To apply this construction to Kähler manifolds, or equivalently, supersymmetric models, in such a way as to preserve the Kähler properties, more restrictions are required. First, the isometries we need to gauge are holomorphic and the gauge group we have to consider is the complexification of the group relevant to the bosonic part.

6.2. Holomorphic Isometries

A holomorphic isometry on a Kähler manifold satisfies

$$\mathcal{L}_{\lambda k} J = 0, \quad \mathcal{L}_{\lambda k} \omega = 0 \tag{45}$$

where J is the complex structure and ω is the Kähler two-form defined in Equation (25). The fact that a Kähler manifold is symplectic makes it possible to consider the moment map for the Hamiltonian vector field λk. The corresponding Hamiltonian function $\mu^{\lambda k}$ is defined by

$$\imath_{\lambda k} \omega =: d\mu^{\lambda k} \tag{46}$$

In holomorphic coordinates $(\phi^q, \bar{\phi}^{\bar{q}})$ where $\lambda k = \lambda^A (k_A + \bar{k}_A)$, this reads

$$\omega_{\bar{q}p} \lambda^A \bar{k}_A^{\bar{q}} \equiv -i K_{\bar{q}p} \lambda^A \bar{k}_A^{\bar{q}} = \frac{\partial \mu^{\lambda k}}{\partial \phi^p} =: \mu_p^{\lambda k} \tag{47}$$

From these relations it is clear why $\mu^{\lambda k}$ is sometimes referred to as a Killing potential. Now μ defines a map from the target space of the sigma model into the dual of the Lie-algebra generated by k_A:

$$\mathcal{T} \mapsto {}^*\mathfrak{g}, \quad \mu^{\lambda k} = \lambda^A \mu_A \tag{48}$$

where μ_A is the basis for ${}^*\mathfrak{g}$ that corresponds to the basis k_A for $\mathfrak{g}$. When the action of the Hamiltonian field can be made to agree with the natural action of the group $\mathcal{G}$ on $\mathcal{T}$ and on ${}^*\mathfrak{g}$, the μ_A's are called moment maps (We use to the (US) East cost nomenclature as opposed to the West coast "momentum map".) This is the case when $\mu^{\lambda k}$ is equivariant, *i.e.*, when

$$\lambda k \mu^{\lambda' k} = \mu^{[\lambda k, \lambda' k]}$$
$$\iff \quad k_A^p \mu_{B,p} + \bar{k}_A^{\bar{p}} \mu_{B,\bar{p}} = c_{AB}^C \mu_C \tag{49}$$

where the equivalence refers to holomorphic isometries.

The Kähler metric is the Hessian of the Kähler potential K. As mentioned in Section 3, this leaves an ambiguity in the potential; it is only defined up to the sum of a holomorphic and an antiholomorphic term.

$$K(\phi, \bar{\phi}) \approx K(\phi, \bar{\phi}) + \eta(\phi) + \bar{\eta}(\bar{\phi}) \tag{50}$$

An isometry thus only has to preserve K up to such terms:

$$\mathcal{L}_{\lambda k} K = \eta^{\lambda k} + \bar{\eta}^{\lambda k} \tag{51}$$

For a holomorphic isometry $\lambda k = \lambda^A (k_A(\phi) + \bar{k}_A(\bar{\phi}))$ this implies for the projection

$$\tfrac{1}{2}(1 - iJ)\lambda k K = \lambda^A k_A(\phi)\partial_p K = -i\mu^{\lambda k} + \eta^{\lambda k} \tag{52}$$

(Recall that $\lambda^A \bar{k}_A^{\bar{q}} K_{p\bar{q}} = i\lambda^A \mu_{A,p}$ and $\lambda^A k_A^q K_{q\bar{p}} = -i\lambda^A \mu_{A,\bar{p}}$, so that $\lambda^A k_A^q K_q = -i\lambda^A \mu_A$+hol. and $\lambda^A \bar{k}_A^{\bar{q}} K_{\bar{q}} = i\lambda^A \mu_A$+antihol.)

6.3. Gauging Isometries of Supersymmetric Sigma Models

Due to the chiral nature of superspace, the isometries act through the complexification of the isometry group $\mathcal{G}$. Explicitly, the parameter λ gets replaced by superfield parameters Λ and $\bar{\Lambda}$, while the bosonic gauge field A_i^A becomes one of the components of a real superfield V^A. (This is the last occurrence of i as a world volume index. Below $i, j, ...$ denote gauge indies.) In the simplest case of isotropy, $k_A^i(\phi) = (T_A)_j^i \phi^j$, the coupling to the chiral fields in the sigma model is

$$\bar{\phi}^i \to \tilde{\phi}^i := \bar{\phi}^j (e^V)_j^i, \qquad (V)_j^i = V^A (T_A)_j^i \tag{53}$$

At the same time, a gauge transformation with (chiral) parameter Λ acts on the chiral and antichiral fields as

$$\bar{\phi}^i \to \bar{\phi}^j (e^{-i\bar{\Lambda}})_j^i, \qquad (\bar{\Lambda})_j^i = \bar{\Lambda}^A (T_A)_j^i \tag{54}$$

The relation Equation (53) can thus be interpreted as a gauge transformation of $\bar{\phi}$ with parameter iV. For general isometries a gauge transformation with parameter iV is

$$\bar{\phi}^i \to \tilde{\phi}^i \equiv e^{\mathcal{L}_{iVk}} \bar{\phi} \tag{55}$$

and this is the form we need to use when defining $\tilde{\phi}^i$. The coupling is thus

$$K(\phi, \bar{\phi}) \to K(\phi, \tilde{\phi}) \tag{56}$$

Under a global isometry transformation, according to Equation (51), there may arise terms such as

$$\delta S = \int \lambda^A \bar{\eta}_A(\bar{\phi}) = 0 \tag{57}$$

whose vanishing ensures the invariance. This is no-longer true in the local case where the corresponding term

$$\delta S = \int \Lambda^A \tilde{\eta}_A \tag{58}$$

will not vanish in general. The remedy is to introduce auxiliary coordinates $(\zeta, \bar{\zeta})$ and assign transformations to them that make the modified Kähler potential $\tilde{K}$ invariant (rather than invariant up to (anti) holomorphic terms):

$$\tilde{K} := K(\phi, \bar{\phi}) - \zeta - \bar{\zeta} \tag{59}$$

The action involving $\tilde{K}$ may now be gauged using the prescription Equation (56) and takes the form (dropping the irrelevant $\zeta + \bar{\zeta}$ term after gauging)

$$\tilde{S} = \int \hat{K}(\phi, \bar{\phi}, V) \tag{60}$$

where

$$\hat{K} \equiv K(\phi, \tilde{\phi}) - i \frac{e^{\mathcal{L}_{iVk}} - 1}{\mathcal{L}_{iVk}} \bar{\eta}_A V^A \tag{61}$$

Using the definition of $\tilde{\phi}$ and the relation to the moment maps (see Equation (52) and below), we rewrite this as

$$\hat{K} = K(\phi, \bar{\phi}) + \left(\frac{e^{\mathcal{L}_{iVk}} - 1}{\mathcal{L}_{iVk}} \right) V^A \mu_A \tag{62}$$

A more geometric form of the gauged Lagrangian is

$$\hat{K} = K(\phi, \bar{\phi}) + \int_0^1 dt\, e^{-\frac{1}{2} t \mathcal{L}_{VJk}} \mu^V \tag{63}$$
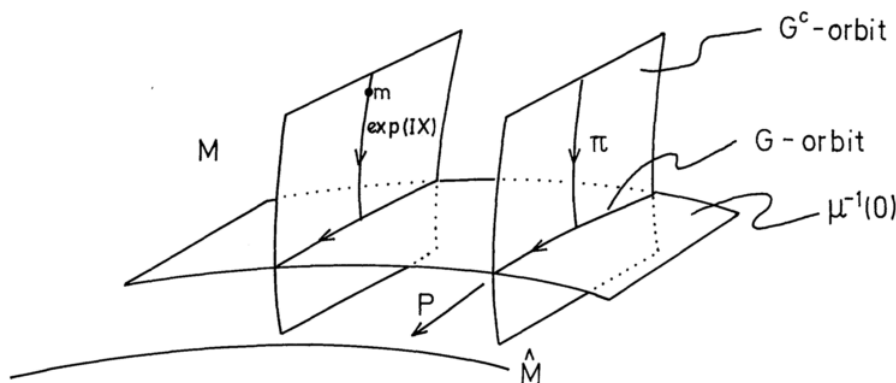
where we recall that

$$\mathcal{L}_k K = \eta + \bar{\eta}$$
$$\mathcal{L}_{Jk} K = i(\eta - \bar{\eta}) + 2\mu \tag{64}$$

The form of the action that follows from Equation (63) directly leads to the symplectic quotient as applied to a Kähler manifold [56]: Eliminating V^A results in

$$e^{-\frac{1}{2} t \mathcal{L}_{VJk}} \mu_A = 0$$
$$\Longleftrightarrow \quad \mu_A = 0 \tag{65}$$

The Kähler quotient is illustrated in the following picture, taken from [5] (with permission from the publisher Springer Verlag) :

The isometry group $\mathcal{G}$ acts on $\mu^{-1}(0)$ and produces the quotient $\hat{M}$. The same space is obtained if one considers the extension of $\mu^{-1}(0)$ by $exp(JX)$ and takes the quotient by the complexified group $\mathcal{G}^{\mathcal{C}}$.

If we start from a hyperkähler manifold with triholomorphic isometries, there will also be complex moment maps corresponding to the two non-canonical complex structures

$$\omega^\pm \leftrightarrow \mu^\pm \tag{66}$$

In addition to Equation (65), we will then have the conditions

$$\mu_A^+ = \mu_A^- = 0 \tag{67}$$

defining holomorphic subspaces. This hyperkähler quotient prescription [4,5] gives a new hyperkähler space from an old one. The $\mathcal{N} = 2$ sigma model action that encodes this (in $d = 4, \mathcal{N} = 1$ language) reads

$$S = \int d^4\xi \left(d^2\theta d^2\bar{\theta} \hat{K}(\phi, \bar{\phi}, V) + \tfrac{1}{2} d^2\theta S^A \mu_A^+ + \tfrac{1}{2} d^2\bar{\theta}\bar{S}^A \mu_A^- \right) \tag{68}$$

where $\hat{K}$ is defined in Equation (61) and $(V, S, \bar{S})$ is the $\mathcal{N} = 2$ vector (gauge) multiplet. The latter consists of a chiral superfield S and its complex conjugate $\bar{S}$ in addition to V.

7. Two Dimensional Models and Generalized Kähler Geometry

The quotient constructions just discussed are limited to backgrounds with only a metric present. Recently extensions of such geometries to include also an antisymmetric B-field have led to a number of new results in $d = 2$ which are expected to contribute to new quotients involving such geometries.

Two (bosonic) dimensional domains Σ are interesting in that they support sigma models with independent left and right supersymmetries. Such (p, q) models were introduced by Hull and Witten in [55], and also have analogues in $d = 6$. There is a wealth of results on the target-space geometry for (p, q)-models. The geometry is typically a generalization of Kähler geometry with vector potential for the metric instead of a scalar potential *etc*. See, e.g., [57,58]. Here we first focus on $(1, 1)$ and $(2, 2)$ models in $d = 2$.

The $(1, 1)$ supersymmetry algebra is

$$D_\pm^2 = i\partial_{\underline{\pm}} \tag{69}$$

where $+$ and $-$ are spinor indices and $\xi^{+\!\!+}, \xi^=$ are light-cone coordinates in $d = 2$ Minkowski space.

A general sigma model written in terms of real $\mathcal{N} = (1, 1)$ superfields Φ is

$$S = \int_\Sigma d^2\xi d^2\theta D_+ \Phi^\mu E_{\mu\nu}(\Phi) D_- \Phi^\nu \tag{70}$$

where the metric G and B-field have been collected into

$$E_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu} \tag{71}$$

Here $\mathcal{N} = (1, 1)$ supersymmetry is manifest by construction and we shall see that additional non-manifest ones will again restrict the target space geometry. In fact, the geometry is already modified due to the presence of the B-field. The field-equations now read

$$\nabla_+^{(+)} D_- \Phi^\mu = 0 \tag{72}$$

where

$$\nabla^{(\pm)} \equiv \nabla^{(0)} \pm T \tag{73}$$

is the sum of the Levi-Civita connection and a torsion-term formed from the field-strength for the B-field (The anti-symmetrization does not include a combinatorial factor):

$$T_{\mu\nu\rho} \equiv H_{\mu\nu\rho} = \tfrac{1}{2}\partial_{[\mu}B_{\nu\rho]} \tag{74}$$

Only when $H = 0$ do we recover the non-torsionful Riemann geometry. The full picture is given in the following table:

Table 1. The geometries of sigma-models with different supersymmetries.

Supersymmetry	(0,0) or (1,1)	(2,2)	(2,2)	(4,4)	(4,4)
Background	G, B	G	G, B	G	G, B
Geometry	Riemannian	Kähler	bi-Hermitian	hyperkähler	bihypercomplex

We first look at the Gates–Hull–Roček (GHR) bi-Hermitian geometry, or Generalized Kähler geometry as is its modern guise. Starting from the $\mathcal{N} = (1,1)$ action Equation (7), one can ask for additional, non-manifest supersymmetries. By dimensional arguments, such a symmetry must act on the superfields as

$$\delta^{(\pm)}\Phi^\mu = \epsilon^\pm D_\pm \Phi^\nu J^{(\pm)\mu}_\nu(\Phi) \tag{75}$$

where $(\pm)$ correspond to left or right symmetries. It was shown by GHR in [7] that invariance of the action Equation (7) and closure of the algebra require that $J^{(\pm)}$ are complex structures that are covariantly constant with respect to the torsionful connections

$$\nabla^{(\pm)} J^{(\pm)} = 0 \tag{76}$$

and that the metric is Hermitian with respect to both these complex structures

$$J^{(\pm)t} G J^{(\pm)} = G \tag{77}$$

In addition, the B-field field-strength (torsion) must obey

$$H = H^{(2,1)} + H^{(1,2)} = d^c_{(+)}\omega_{(+)} = -d^c_{(-)}\omega_{(-)} \tag{78}$$

where the chirality assignment refers to *both* complex structures. Here we have introduced $d^c := J(d) = i(\bar{\partial} - \partial)$, where the last equality holds in canonical coordinates, the two-forms are $\omega_{(\pm)} \equiv G J^{(\pm)}$ as tensors and $dd^c_{(\pm)}\omega_{(\pm)} = 0$. When $J^{(+)} = \pm J^{(-)}$ this geometry reduces to Kähler geometry.

Gualtieri gives a nice interpretation of the full bi-Hermitian geometry in the context of Generalized Complex Geometry [59] and calls it Generalized Kähler Geometry [60], which we now briefly describe.

8. Complex Geometry II

The definition of Generalized Complex Geometry (GCG) parallels that of complex geometry but is based on the sum of the tangent and cotangent bundle instead of just the tangent bundle. Hence we consider a section $\mathcal{J}$ of $End(\mathcal{T} \oplus \mathcal{T}^*)$ such that $\mathcal{J}^2 = -1$. To define integrability we again use projection operators

$$\Pi_{\pm} \equiv \tfrac{1}{2}\left(1 \pm i\mathcal{J}\right) \tag{79}$$

but now we require that the subspaces of $\mathcal{T} \oplus \mathcal{T}^*$ defined by $\Pi_{\pm}$ are in involution with respect to a bracket defined on that bundle. Denoting an element of $\mathcal{T} \oplus \mathcal{T}^*$ by $v + \xi$ with $v \in \mathcal{T}, \xi \in \mathcal{T}^*$, we thus require

$$\Pi_{\mp}\left[\Pi_{\pm}(v + \xi), \Pi_{\pm}(u + \chi)\right]_C = 0 \tag{80}$$

where the bracket is the Courant bracket defined by

$$[v + \xi, u + \chi]_C = [v, u] + \mathcal{L}_v \chi - \mathcal{L}_u \xi - \tfrac{1}{2} d(\imath_v \chi - \imath_u \xi) \tag{81}$$

where $[\,,\,]$ is the Lie bracket and, e.g., $\imath_v \chi = \chi(v) = v \cdot \chi$. The full definition of GCG also requires the natural pairing metric $\mathcal{I}$ to be preserved. The natural pairing is

$$< v + \xi, u + \chi >= \imath_v \chi + \imath_u \xi \tag{82}$$

In a coordinate basis (∂_μ, dx^ν) where we represent $v + \xi$ as $(v, \xi)^t$, the relation in Equation (82) may be written as [30]:

$$(v, \xi)\,\mathcal{I}\begin{pmatrix} u \\ \chi \end{pmatrix} = \begin{pmatrix} 0 & v\chi \\ u\xi & 0 \end{pmatrix} \tag{83}$$

so that

$$\mathcal{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{84}$$

and preservation of $\mathcal{I}$ by $\mathcal{J}$ means

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I} \tag{85}$$

The specialization to Generalized Kähler Geometry (GKG) occurs when we have *two* commuting GCS's

$$\left[\mathcal{J}^{(1)}, \mathcal{J}^{(2)}\right] = 0\,. \tag{86}$$

This allows the definition of a metric (this is really a local product structure as defined but appropriate contractions with $\mathcal{I}$ makes it a metric) $\mathcal{G} \equiv -\mathcal{J}^{(1)}\mathcal{J}^{(2)}$ which satisfies

$$\mathcal{G}^2 = 1 \tag{87}$$

The definition of GKG requires this metric to be positive definite.

The relation of GKG to bi-Hermitian geometry is given by the following "Gualtieri map":

$$\mathcal{J}^{(1,2)} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J^{(+)} + \pm J^{(-)} & -(\omega_{(+)}^{-1} \mp \omega_{(-)}^{-1}) \\ \omega_{(+)} \mp \omega_{(-)} & -(J^{t(+)} + \pm J^{t(-)}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \tag{88}$$

which maps the bi-Hermitian data into the GK data. When $H = 0$ the first and last matrix on the right hand side represent a "B-transform" which is one of the automorphisms of the Courant bracket, but here $H \neq 0$ in general.

For the Kähler case, $\mathcal{J}^{(1,2)}$ in Equation (88), and $\mathcal{G}$ reduce to

$$\mathcal{J}^{(1)} = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} , \quad \mathcal{J}^{(2)} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} , \quad \mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{89}$$

where J is the complex structure, ω is the corresponding Kähler form and g is the Hermitian metric. This fact is the origin of the name Generalized Kähler coined by Gualtieri [60].

9. $\mathcal{N} = (2,2) , d = 2$ Sigma Models Off-Shell

The $\mathcal{N} = (1,1)$ discussion of GHR identified the geometry of the sigma models that could be extended to have $\mathcal{N} = (2,2)$ supersymmetry, but they found closure of the algebra only when the complex structures commute, *i.e.*, on $ker[J^{(+)}, J^{(-)}]$, in which case they gave a full $\mathcal{N} = (2,2)$ description in terms of chiral and twisted chiral fields. In this section we extend the discussion to the non-commuting case and describe the general situation following [32]. Earlier relevant discussions may be found in [61–64].

The $d = 2, \mathcal{N} = (2,2)$ algebra of covariant derivatives is

$$\begin{aligned}
\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} &= \pm i \partial_{\pm} , \quad \{\mathbb{D}_\pm, \mathbb{D}_\pm\} = 0 \\
\{\mathbb{D}_\pm, \mathbb{D}_\mp\} &= 0 , \qquad \{\mathbb{D}_\pm, \bar{\mathbb{D}}_\mp\} = 0
\end{aligned} \tag{90}$$

A chiral superfield ϕ satisfies the same constraints as in $d = 4$:

$$\bar{\mathbb{D}}_\pm \phi = \mathbb{D}_\pm \bar{\phi} = 0 \tag{91}$$

but in $d = 2$ we may also introduce twisted chiral fields χ that satisfy

$$\begin{aligned}
\bar{\mathbb{D}}_+ \chi &= \mathbb{D}_+ \bar{\chi} = 0 \\
\mathbb{D}_- \chi &= \bar{\mathbb{D}}_- \bar{\chi} = 0
\end{aligned} \tag{92}$$

The sigma model action

$$S = \int d^2\theta d^2\bar{\theta} K(\phi, \bar{\phi}, \chi, \bar{\chi}) \tag{93}$$

then precisely yields the GKG on $ker[J^{(+)}, J^{(-)}]$, as may be seen by reducing the action to a $\mathcal{N} = (1,1)$-formulation. Denoting the $(1,1)$ covariant derivatives by $D_\pm$ and the generators of the second supersymmetry $Q_\pm$ we have

$$\begin{aligned}
D_\pm &:= \frac{1}{\sqrt{2}}(\mathbb{D}_\pm + \bar{\mathbb{D}}_\pm) \\
Q_\pm &:= i\frac{1}{\sqrt{2}}(\mathbb{D}_\pm - \bar{\mathbb{D}}_\pm)
\end{aligned} \tag{94}$$

Note that this formulation shows that both the metric and the H-field have K as a potential

$$\begin{aligned}
G_{\phi\bar{\phi}} &= K_{\phi\bar{\phi}} , \quad G_{\chi\bar{\chi}} = -K_{\chi\bar{\chi}} \\
H &= \bar{\partial}_\chi \partial_- \phi \bar{\partial}_\phi K + \partial_\chi \partial_\phi \bar{\partial}_\phi K + \bar{\partial}_\phi \partial_\phi \bar{\partial}_\chi K - \partial_\phi \partial_\phi \bar{\partial}_\chi K
\end{aligned} \tag{95}$$

where derivatives on K are understood in the first line, and the second line is a three-form written in terms of the holomorphic differentials:

$$d = \partial_\phi + \bar{\partial}_\phi + \partial_\chi + \bar{\partial}_\chi \tag{96}$$

See [65] for a more detailed discussion of the above coordinatization.

Furthermore, as discussed for GKG in the previous section, the commuting complex structures imply the existence of a local product structure

$$\mathfrak{P} = -J^{(+)}J^{(-)}, \quad \nabla\mathfrak{P} = 0 \quad \mathfrak{P}^2 = 1 \tag{97}$$

To discriminate it from the general GK case we call this a "Bi-Hermitian Local Product" (BiLP) geometry.

The general case with $(ker[J^{(+)}, J^{(-)}])^\perp \neq \emptyset$ was long a challenge. (In a number of publications co-authored by me, $(ker[J^{(+)}, J^{(-)}])^\perp$ was incorrectly denoted $coker[J^{(+)}, J^{(-)}]$. I apologize for participating in this misuse). The key issue here is what additional $\mathcal{N} = (2,2)$ superfields (if any) would suffice to describe the geometry. The available fields are complex linear Σ_ϕ, twisted complex linear Σ_χ and semichiral superfields. Of these Σ_ϕ are dual to chirals and Σ_χ to twisted chirals (See appendix A). The candidate superfields are thus left and right semi-(anti)chirals $\mathbb{X}_{L.R}$ which obey

$$\bar{\mathbb{D}}_+\mathbb{X}_L = \mathbb{D}_+\bar{\mathbb{X}}_L = 0$$
$$\bar{\mathbb{D}}_-\mathbb{X}_R = \mathbb{D}_-\bar{\mathbb{X}}_L = 0 \tag{98}$$

A $\mathcal{N} = (2,2)$ model written in terms of these fields reads

$$S = \int d^2\theta d^2\bar{\theta} K(\mathbb{X}_{L.R}, \bar{\mathbb{X}}_{L.R}) \tag{99}$$

and an equal number of left and right fields are needed to yield a sensible sigma model. When reduced to $\mathcal{N} = (1,1)$ superspace, this action gives a more general model than what we have considered so far. Using Equation (94) we have the following $\mathcal{N} = (1,1)$ superfield content;

$$X_L \equiv \mathbb{X}_L|, \quad \Psi_{L-} \equiv Q_-\mathbb{X}_L|$$
$$X_R \equiv \mathbb{X}_R|, \quad \Psi_{R+} \equiv Q_+\mathbb{X}_R| \tag{100}$$

where the vertical bar now denotes setting *half* the fermi-coordinates to zero. Clearly, $X_{L,R}$ are scalar superfields and hence suitable for the $\mathcal{N} = (1,1)$ sigma model, but $\Psi_{L,R\pm}$ are spinorial fields. They enter the reduced action as auxiliary fields and are the auxiliary $\mathcal{N} = (1,1)$ superfields needed for closure of the $\mathcal{N} = (2,2)$ algebra when $[J^{(+)}, J^{(-)}] \neq 0$ (see [12]). The structure of such an a $\mathcal{N} = (1,1)$ action is schematically [29]

$$S = \int d^2\theta d^2\bar{\theta}(D_+XE^{-1}D_-X + D_{(+}X\Psi_{-)}) \tag{101}$$

where $E = G + B$ as before.

In [32] we show that a sigma model fully describing GKG, *i.e.*, $ker[J^{(+)}, J^{(-)}] \oplus (ker[J^{(+)}, J^{(-)}])^\perp$, is (away from irregular points, *i.e.*, points where the Poisson structures Equation (103), Equation (108) change rank)

$$S = \int d^2\theta^2 d^2\bar{\theta} K(\mathbb{X}_L, \bar{\mathbb{X}}_L\mathbb{X}_R, \bar{\mathbb{X}}_R, \phi, \bar{\phi}, \chi, \bar{\chi}) \tag{102}$$

where K acts as a generalized Kähler potential in terms of derivatives of which all geometric quantities can be expressed (locally). This K has the additional interpretation as a generating function for symplectomorphisms between certain sets of coordinates on $(ker[J^{(+)}, J^{(-)}])^{\perp}$, the canonical coordinates for $J^{(+)}$ and $J^{(-)}$, respectively. The proof of these statements relies heavily on Poisson geometry [32] and is summarized in what follows.

First, the fact that (ϕ, χ) and their Hermitian conjugates are enough to describe $ker[J^{(+)}, J^{(-)}]$ may be reformulated using the Poisson-structures [66]

$$\pi_{\pm} \equiv (J^{(+)} \pm J^{(-)})G^{-1} \tag{103}$$

In a neighborhood of a regular point, coordinates may be chosen such that

$$\pi^{A\mu} = 0, \Rightarrow J_{\mu}^{(+)A} = J_{\mu}^{(-)A}$$

$$\pi^{A'\mu} = 0, \Rightarrow J_{\mu}^{(+)A'} = -J_{\mu}^{(-)A'}$$

$$\tag{104}$$

It can be shown that $A \neq A'$ and that we have coordinates labeled $(a, a', A.A')$ adapted to

$$ker(J^{(+)} - J^{(-)}) \oplus ker(J^{(+)} + J^{(-)}) \oplus (ker[J^{(+)}, J^{(-)}])^{\perp} \tag{105}$$

where

$$J^{\pm} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & \pm I_t \end{pmatrix} \tag{106}$$

Here I_c and I_t have the canonical form

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{107}$$

We thus have nice coordinates for $ker(J^{(+)} - J^{(-)}) \oplus ker(J^{(+)} + J^{(-)}) = ker[J^{(+)}, J^{(-)}]$, but $(ker[J^{(+)}, J^{(-)}])^{\perp}$ remains to be described. Here a third Poisson structure turns out to be useful;

$$\sigma \equiv [J^{(+)}, J^{(-)}]G^{-1} = \pm(J^{(+)} \mp J^{(-)})\pi_{\pm} \tag{108}$$

Now $ker\,\sigma = ker\,\pi_+ \oplus ker\,\pi_-$ so we focus on $(ker\,\sigma)^{\perp}$. The symplectic leaf for σ is $(ker[J^{(+)}, J^{(-)}])^{\perp}$ and the third Poisson structure also has the following useful properties [67]:

$$J^{\pm}\sigma J^{\pm t} = -\sigma$$
$$\sigma = \sigma^{(2,0)} + \bar{\sigma}^{(0,2)}$$
$$\bar{\partial}\sigma^{(2,0)} = 0 \tag{109}$$

where the holomorphic types are with respect to *both* complex structures. To investigate the consequences of Equation (109) it is advantageous to first consider the case when $ker[J^{(+)}, J^{(-)}] = \emptyset$. It then follows that σ is invertible and its inverse Ω is a symplectic form;

$$\Omega = \sigma^{-1}, \quad d\Omega = 0 \tag{110}$$

We may chose coordinates adapted to $J^{(+)}$

$$J^{(+)} = \begin{pmatrix} I_s & 0 \\ 0 & I_s \end{pmatrix} \tag{111}$$

In those coordinates we have from Equation (109) that

$$\Omega = \Omega^{(2,0)}_{(+)} + \bar{\Omega}^{(0,2)}_{(+)}$$
$$\partial \Omega^{(2,0)}_{(+)} = 0 = \bar{\partial} \bar{\Omega}^{(0,2)}_{(+)} \tag{112}$$

which identifies $\Omega^{(2,0)}_{(+)}$ as a holomorphic symplectic structure. The coordinates may then be further specified to be Darboux coordinates for this symplectic structure

$$\Omega = dq^a \wedge dp^a + c.c. \tag{113}$$

The same derivation with $J^{(+)}$ replaced by $J^{(-)}$ gives a second set of Darboux coordinates which are canonical coordinates for $J^{(-)}$ and where

$$\Omega = dQ^{a'} \wedge dP^{a'} + c.c. \tag{114}$$

Clearly the two sets of canonical coordinates are related by a symplectomorphism. Let $K(q, P)$ denote a generating function for this symplectomorphism. Expressing all our quantities in the mixed coordinates (q, P), we discover that the expressions for $J^{(\pm)}$, $\Omega, G = \Omega[J^{(+)}, J^{(-)}], \dots$ are precisely what we (see also [12,64] for partial results) derived from the sigma model action Equation (102) provided that we identify the coordinates (q, P) with $(\mathbb{X}_L, \mathbb{X}_R)$ (and the same for the Hermitian conjugates).

In the general case when $[J^{(+)}, J^{(+)}] \neq \emptyset$, we again get agreement, provided that the coordinates indexed A and A' in Equation (106) are identified with the chiral and twisted chiral fields (ϕ, χ). We thus have a one to one correspondence between the description covered by the sigma model and all of $[J^{(+)}, J^{(+)}]$, *i.e.*, for all possible cases.

10. Linearization of Generalized Kähler Geometry

The generalized Kähler potential $K(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R, \phi, \bar{\phi}, \chi, \bar{\chi})$ yields all geometric quantities, but as non-linear expressions (dualizing a BiLP to (twisted) complex linear fields yields a model with similar nonlinearities) in derivatives of K (unlike the case Equation (95)). In [34] we show that these non-linearities can be viewed as arising from a quotient of a higher dimensional model with certain null Kac–Moody symmetries. To illustrate the idea, we first consider an example.

10.1. A Bosonic Example

Consider a Lagrangian of the form

$$L_1 := A^\mu A_\mu \tag{115}$$

Following Stückelberg, we may think of this as a gauge fixed version of the gauge-invariant Lagrangian

$$L_2 := D^\mu \varphi D_\mu \varphi \tag{116}$$

with $D_\mu \varphi := \partial_\mu \varphi + A_\mu$ and gauge invariance

$$\delta\varphi = \epsilon$$
$$\delta A_\mu = -\partial_\mu \epsilon \tag{117}$$

Finally, the Lagrangian L_2 can in turn be thought of as arising through gauging of the global translational symmetry $\delta\varphi = \epsilon$ in a third Lagrangian

$$L_3 := \partial^\mu \varphi \partial_\mu \varphi \tag{118}$$

A slightly more elaborate example is provided by the following sigma model Lagrangian;

$$\tilde{L}_1 := G_{ab}(\phi)\partial^\mu \phi^a \partial\mu \phi^b + 2G_a(\phi)\partial^\mu \phi^a A_\mu + G(\phi)A^\mu A_\mu \tag{119}$$

where A_μ is an auxiliary field. Following the line of reasoning above, this Lagrangian may be thought of as a gauge fixed version of

$$\tilde{L}_2 := G_{ab}(\phi)\partial^\mu \phi^a \partial\mu \phi^b + 2G_{a0}(\phi)\partial^\mu \phi^a D_\mu \varphi + G_{00}(\phi)D^\mu \varphi D_\mu \varphi \tag{120}$$

where D_μ is as defined above, $G_{a0} \equiv G_a$, $G_{00} \equiv G$ and the Stückelberg field $\varphi \equiv \phi^0$. In turn $\tilde{L}_2$ is the gauged version (in adapted coordinates) of the Lagrangian

$$\tilde{L}_3 := G_{ij}(\phi)\partial^\mu \phi^i \partial_\mu \phi^j \,, \quad i = 0, a \tag{121}$$

whose global symmetry is given by the isometry

$$\partial_0 G_{ij} = 0 \tag{122}$$

We see that eliminating the auxiliary field in $\tilde{L}_1$ is tantamount to extremizing $\tilde{L}_2$ with respect to the gauge field, *i.e.*, to constructing a quotient of the Lagrangian $\tilde{L}_3$ with respect to its isometry. The only remaining question seems to be if varying the gauge-fixed $\tilde{L}_1$ is the same as varying $\tilde{L}_2$ (modulo gauge-fixing). The resulting A_μ's differ by a gauge-transformation $\partial_\mu \varphi$. Explicitly:

$$\delta\tilde{L}_1 = 0 \Rightarrow -A_\mu = \frac{G_a \partial_\mu \phi^a}{G}$$
$$\delta\tilde{L}_2 = 0 \Rightarrow -(\partial_\mu \varphi + A_\mu) = \frac{G_{a0} \partial_\mu \phi^a}{G_{00}} \tag{123}$$

10.2. *The Generalized Kähler Potential*

We apply the procedure described above to a semichiral sigma model. Most of the rest of this section is taken directly from [34,35] where more details may be found.

Consider the generalized Kähler potential

$$K(\mathbb{X}_{L,R}, \bar{\mathbb{X}}_{\bar{L},\bar{R}}) \tag{124}$$

where $\mathbb{X}_{L,R}$ are left and right semi-chiral $N = (2,2)$ superfields:

$$\bar{\mathbb{D}}_+ \mathbb{X}_L = 0 = \bar{\mathbb{D}}_- \mathbb{X}_R \tag{125}$$

We descend to $N = (1, 1)$ as in Equation (100) by defining components

$$X_L = \mathbb{X}_L|, \quad \Psi_{L-} = Q_- \mathbb{X}_L|$$
$$X_R = \mathbb{X}_R|, \quad \Psi_{R+} = Q_+ \mathbb{X}_R| \tag{126}$$

which satisfy

$$Q_+ \Psi = JD_+ \Psi_{L-}, \quad Q_- \Psi_{L-} = -i\partial_= X_L \tag{127}$$
$$Q_- \Psi_{R+} = JD_- \Psi_{R+}, \quad Q_+ \Psi_{R+} = -i\partial_= X_R \tag{128}$$

The $N = (1, 1)$ form of the Lagrangian is

$$\begin{pmatrix} D_+X_L & \Psi_{L+} & D_+X_R & \Psi_{R+} \end{pmatrix} E \begin{pmatrix} D_-X_L \\ \Psi_{L-} \\ D_-X_R \\ \Psi_{R-} \end{pmatrix} \tag{129}$$

where

$$E = \begin{pmatrix} 0 & K_{LL} + IK_{LL}I & IK_{LR}J & 0 \\ 0 & 0 & 0 & 0 \\ 0 & K_{RL} & 0 & 0 \\ K_{RL} & IK_{RL}I & K_{RR} + IK_{RR}I & 0 \end{pmatrix} \tag{130}$$

Here we use a short hand notation where, e.g., K_{LR} denotes the matrix of second derivatives of the potential Equation (124) with respect to both bared and un-bared left and right fields. Also the canonical complex structures I is defined in Equation (107). Notice that neither Ψ_{L+} nor Ψ_{R-} occur in the action.

10.3. ALP and Kac–Moody Quotient

The procedure $\tilde{L}_1 \to \tilde{L}_2 \to \tilde{L}_3$ outlined in Section 10.1 applied to the present case entails the replacements $(\Psi_{L-}, \Psi_{R+}) \to (\nabla_-\varphi_L, \nabla_+\varphi_R) \to (D_-\varphi_L, D_+\varphi_R)$ with $\tilde{L}_1$ given by Equation (129), and where

$$\nabla_{\pm}\varphi_{R/L} := D_{\pm}\varphi_{R/L} + \Psi_{R/L\pm} \tag{131}$$

The gauge invariance of $\tilde{L}_2$ is

$$\delta\varphi_{R/L} = \varepsilon_{R/L} , \quad \delta\Psi_{R/L\pm} = -D_{\pm}\varepsilon_{R/L} \tag{132}$$

which gauges the following "global" invariance of $\tilde{L}_3$:

$$\delta\varphi_{R/L} = \varepsilon_{R/L} , \quad D_{\pm}\varepsilon_{R/L} = 0 \tag{133}$$

Since the matrix E in Equation (130) is independent of $\varphi_{R/L}$, invariance under Equations (117) and (133) is immediate. Using the metric $G := \frac{1}{2}(E + E^t)$ one also verifies that the Killing-vectors

$$k_R := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varepsilon_R \end{pmatrix}, \qquad k_L := \begin{pmatrix} 0 \\ \varepsilon_L \\ 0 \\ 0 \end{pmatrix} \tag{134}$$

are null-vectors. Furthermore, the constraints in Equation (133) imply

$$
\begin{aligned}
D_+ k_R &= 0 \\
D_- k_L &= 0 \\
\partial_{+\!\!+} k_R &= 0 \\
\partial_{=} k_L &= 0
\end{aligned}
$$

(135)

or covariantly [68]

$$
\nabla^{(+)} k_{LA} = 0 = \nabla^{(-)} k_{RA}
$$

(136)

with $\nabla^{(\pm)}$ defined in Equation (73). These relations identify the global symmetries as null Kac–Moody isometries.

After applying the procedure outlined above, we obtain a Lagrangian $\tilde{L}_3$. It is then useful to introduce a definition from [34]:

The space corresponding to $\tilde{L}_3$ is the $N = (1,1)$ form of the Auxiliary Local Product space (ALP) for the $N = (2,2)$ Lagrangian $\tilde{L}_1$ in Equation (129).

In other words, the ALP is given by the action

$$
\begin{pmatrix} D_+X_L & \bullet & D_+X_R & D_+\varphi_R \end{pmatrix} E \begin{pmatrix} D_-X_L \\ D_-\varphi_L \\ D_-X_R \\ \bullet \end{pmatrix}
$$

(137)

where E is the matrix given in Equation (130), the bullets denote the decoupled Ψ_{L+}, Ψ_{R-}, and the Lagrangian is invariant under the global Kac–Moody isometry Equation (134).

10.3.1. Kac–Moody Quotient in $(1,1)$

The Lagrangian Equation (137) is an equivalent starting point for deriving the GK geometry for the target space of Equation (129): To recapitulate from Section 10.1, this proceeds by gauging the isometry to obtain the $\tilde{L}_2$ Lagrangian

$$
\begin{pmatrix} D_+X_L & \bullet & D_+X_R & D_+\varphi_R + \Psi_{R+} \end{pmatrix} E \begin{pmatrix} D_-X_L \\ D_-\varphi_L + \Psi_{L-} \\ D_-X_R \\ \bullet \end{pmatrix}
$$

(138)

Elimination of the gauge fields (*cf.* Equation (123));

$$
\delta\Psi_{L-} \Rightarrow D_-\varphi_L + \Psi_{L-} = -JK_{LR}^{-1}J(K_{RR} + JK_{RR}J)D_-\mathbb{X}_R - JK_{LR}^{-1}JK_{RL}D_-\mathbb{X}_L
$$

$$
\delta\Psi_{R+} \Rightarrow D_-\varphi_R + \Psi_{R+} = -JK_{RL}^{-1}J(K_{LL} + JK_{LL}J)D_-\mathbb{X}_L - JK_{RL}^{-1}JK_{LR}D_-\mathbb{X}_R
$$

(139)

yields the quotient metric and B-field from **E**:

$$\mathbf{E} = \begin{pmatrix} C_{LL} K_{LR}^{-1} J K_{RL} & J K_{LR} J + C_{LL} K_{LR}^{-1} C_{RR} \\ -K_{RL} J K_{LR}^{-1} J K_{RL} & -K_{RL} J K_{LR}^{-1} C_{RR} \end{pmatrix} \tag{140}$$

where, suppressing indices on the two by two complex matrices,

$$C := [J, K] \tag{141}$$

The corresponding Lagrangian is

$$\begin{pmatrix} D_+ X_L & D_+ X_R \end{pmatrix} \mathbf{E} \begin{pmatrix} D_- X_L \\ D_- X_R \end{pmatrix} \tag{142}$$

10.3.2.   Kac–Moody quotient in $(2, 2)$

As an alternative, we may perform the Kac–Moody quotient in $(2, 2)$ superspace. Very briefly, this goes as follows:

In the generalized potential we replace the semi-chiral fields by sums of chiral and twisted chiral fields according to

$$K(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R, \phi, \bar{\phi}, \chi, \bar{\chi})$$
$$\longrightarrow \quad K(\phi_L + \chi_L, \bar{\phi}_L + \bar{\chi}_L, \phi_R + \bar{\chi}_R, \bar{\phi}_R + \chi_R, \ldots) \tag{143}$$

This doubles the degrees of freedom in the semi sector but the corresponding action has a Kac–Moody symmetry

$$\begin{aligned} \delta \phi_{L,R} &= \lambda_{L,R} \\ \delta \chi_L &= -\lambda_L \\ \delta \chi_R &= -\bar{\lambda}_R \end{aligned} \tag{144}$$

where the parameters satisfy

$$\bar{\mathbb{D}}_\pm \lambda_L = \mathbb{D}_- \lambda_L = 0, \quad \partial_= \lambda_L = 0$$
$$\bar{\mathbb{D}}_\pm \lambda_R = \mathbb{D}_+ \lambda_R = 0, \quad \partial_{+\!\!+} \lambda_R = 0 \tag{145}$$

To keep the same degrees of freedom as in the original model, we gauge the Kac–Moody symmetry which reintroduces semi-chiral fields:

$$K(\phi_L + \chi_L, \bar{\phi}_L + \bar{\chi}_L, \phi_R + \bar{\chi}_R, \bar{\phi}_R + \chi_R, \ldots)$$
$$\longrightarrow \quad K(\phi_L + \chi_L + \mathbb{X}_L, \bar{\phi}_L + \bar{\chi}_L + \bar{\mathbb{X}}_L, \phi_R + \bar{\chi}_R + \mathbb{X}_R, \bar{\phi}_R + \chi_R + \bar{\mathbb{X}}_R, \ldots) \tag{146}$$

The local complex Kac–Moody symmetry is now

$$\begin{aligned} \delta \phi_{L,R} &= \Lambda_{L,R} \\ \delta \chi_L &= -\tilde{\Lambda}_L \\ \delta \chi_R &= -\bar{\tilde{\Lambda}}_R \end{aligned}$$

$$\delta\mathbb{X}_{L,R} = -\Lambda_{L,R} + \tilde{\Lambda}_{L,R} \tag{147}$$

The equivalence to the generalized potential is seen by going to a gauge where the "$\phi+\chi$" terms are zero.

For comparison, we descend to $\mathcal{N} = (1,1)$ via the identification

$$\begin{aligned} X_L &= (\phi_L + \chi_L)| \,, \quad \psi_{L-} = Q_-(\phi_L + \chi_L)| = ID_-(\phi_L - \chi_L) \equiv ID_-\varphi_L \\ X_R &= (\phi_R + \bar{\chi}_R)| \,, \quad \psi_{R+}| \equiv ID_+\varphi_R \end{aligned} \tag{148}$$

This gives the $\mathcal{N} = (1,1)$ action in terms of the complex scalar fields $X_{L,R}$ and $\varphi_{L,R}$, with the Kac–Moody generated by the null Killing vectors Equation (134). These corresponding isometries may be used in a quotient to give precisely the nonlinear expressions in terms of derivatives of K that we found in Equation (140). They arise from

$$\mathbf{E}_{\mu\nu} = E_{\mu\nu} - k_{L\mu}h^{-1}k_{R\nu}, \quad h \equiv k_R G k_L \tag{149}$$

where $E_{\mu\nu}$ are the $X_{L,R}$ components in Equation (130). Note that the existence of a left and a right isometry generalizes the construction in Equation (44) slightly, to allow for a B-field.

11. Projective Superspace

Typically, the $\mathcal{N} = (2,2)$ formulation of the $\mathcal{N} = (4,4)$ models require explicit transformations on the $\mathcal{N} = (2,2)$ superfields that close to the supersymmetry algebra on-shell. This non-manifest formulation makes the construction of new models difficult. Below follows a brief description of a superspace where all supersymmetries are manifest. This projective superspace (The name refers to the projective coordinates on $\mathbb{CP}^1 =: \mathbb{P}^1$ being used. It is really a misnomer in that it is unrelated to the usual definition of projective spaces) [6–19] has been developed independent of harmonic superspace [69]. The relation between the two approaches was first discussed in [70] and more recently in [71]. A key reference for this section is [72] and the review [73].

A hyperkähler space $\mathcal{T}$ supports three globally defined integrable complex structures I, J, K obeying the quaternion algebra: $IJ = -JI = K$, plus cyclic permutations. Any linear combination of these $aI + bJ + cK$ is again a complex structure on $\mathcal{T}$ if $a^2 + b^2 + c^2 = 1$, *i.e.*, if $\{a, b, c\}$ lies on a two-sphere $S^2 \simeq \mathbb{P}^1$. The Twistor space $\mathcal{Z}$ of a hyperkähler space $\mathcal{T}$ is the product of $\mathcal{T}$ with this two-sphere $\mathcal{Z} = \mathcal{T} \times \mathbb{P}^1$. The two-sphere thus parametrizes the complex structures and we choose projective coordinates ζ to describe it (in a patch including the north pole). It is an interesting and remarkable fact that the very same S^2 arises in an extension of superspace to accommodate manifest $\mathcal{N} = (4,4)$ models.

Although projective superspace can be defined for different bosonic dimensions, we shall remain in two. Here the algebra of $\mathcal{N} = (4,4)$ superspace derivatives is

$$\begin{aligned} \{\mathbb{D}_{a\pm}, \bar{\mathbb{D}}^b_\pm\} &= \pm i\delta^b_a \partial_{\underset{=}{\pm}} \,, \quad \{\mathbb{D}_{a\pm}, \mathbb{D}_{b\pm}\} = 0 \\ \{\mathbb{D}_{a\pm}, \mathbb{D}_{b\mp}\} &= 0 \,, \quad\quad \{\mathbb{D}_{a\pm}, \bar{\mathbb{D}}^b_\mp\} = 0 \end{aligned} \tag{150}$$

We may parameterize a $\mathbb{P}^1$ of maximal graded Abelian sub-algebras as (suppressing the spinor indices)

$$\nabla(\zeta) = \mathbb{D}_2 + \zeta\mathbb{D}_1 \,, \quad \bar{\nabla}(\zeta) = \bar{\mathbb{D}}^1 - \zeta\bar{\mathbb{D}}^2 \tag{151}$$

where ζ is the coordinate introduced above, and the bar on ∇ denotes conjugation with respect to a real structure $\mathfrak{R}$ defined as complex conjugation composed with the antipodal map on $\mathbb{P}^1 \simeq S^2$. The two new covariant derivatives in Equation (151) anti-commute

$$\{\nabla, \bar{\nabla}\} = 0 \tag{152}$$

They may be used to introduce constraints on superfields similarly to how the $\mathcal{N} = (2, 2)$ derivatives are used to impose chirality constraints in Section 9. Superfields now live in an extended superspace with coordinates ξ, ζ, θ. The superfields Υ we shall be interested in satisfy the projective chirality constraint

$$\nabla \Upsilon = \bar{\nabla} \Upsilon = 0 \tag{153}$$

and are taken to have the following ζ-expansion:

$$\Upsilon = \sum_i \Upsilon_i \zeta^i \tag{154}$$

When the index $i \in [0, \infty)$ the field Υ is analytic around the north pole of the $\mathbb{P}^1$ and consequently called an arctic multiplet. For tropical and antarctic multiplets see [17]. We use the real structure acting on superfields, $\mathfrak{R}(\Upsilon) \equiv \bar{\Upsilon}$, to impose reality conditions on the superfields. An $\mathcal{O}(2n)$ multiplet is thus defined via

$$\Upsilon \equiv \eta_{(2n)} = (-)^n \zeta^{2n} \bar{\Upsilon} \tag{155}$$

The expansion Equation (154) is useful in displaying the $\mathcal{N} = (2, 2)$ content of the multiplets. Using the relation Equation (151) to the $\mathcal{N} = (2, 2)$ derivatives in Equation (153) we read off the following expansion for an $\mathcal{O}(4)$ multiplet Equation (155):

$$\eta_{(4)} = \phi + \zeta \Sigma + \zeta^2 X - \zeta^3 \overline{\Sigma} + \zeta^4 \bar{\phi} \tag{156}$$

with the component $\mathcal{N} = (2, 2)$ fields being chiral ϕ, unconstrained X and complex linear Σ. A complex linear field satisfies

$$\bar{\mathbb{D}}^2 \Sigma = 0 \tag{157}$$

and is dual to a chiral superfield (see the appendix). A general arctic projective chiral Υ has the expansion

$$\Upsilon = \phi + \zeta \Sigma + \sum_{i=2}^{\infty} X_i \zeta^i \tag{158}$$

with all X_i's unconstrained.

11.1. The Generalized Legendre Transform

In this section we review one particular construction of hyperkähler metrics using projective superspace introduced in [11].

An $\mathcal{N} = (4, 4)$ invariant action for the field in Equation (158) may be written as

$$S = \int \mathbb{D}^2 \bar{\mathbb{D}}^2 F \tag{159}$$

with

$$F \equiv \oint_C \frac{d\zeta}{2\pi i\zeta} f(\Upsilon, \bar{\Upsilon}; \zeta) \tag{160}$$

for some suitably defined contour C. Eliminating the auxiliary fields X_i by their equations of motion will yield an $\mathcal{N} = (2, 2)$ model defined on the tangent bundle $T(\mathcal{T})$ parametrized by (ϕ, Σ). Dualizing the complex linear fields Σ to chiral fields $\tilde{\phi}$ the final result is a supersymmetric $\mathcal{N} = (2, 2)$ sigma model in terms of $(\phi, \tilde{\phi})$ which is guaranteed by construction to have $\mathcal{N} = (4, 4)$ supersymmetry, and thus to define a hyperkähler metric. In equations, these steps are:

Solve the equations of motion for the auxiliary fields (techniques for this were developed in [74]):

$$\frac{\partial F}{\partial \Upsilon_i} = \oint_C \frac{d\zeta}{2\pi i\zeta} \zeta^i \left(\frac{\partial}{\partial \Upsilon} f(\Upsilon, \bar{\Upsilon}; \zeta) \right) = 0 \ , \quad i \geq 2 \tag{161}$$

Solving these equations puts us on $\mathcal{N} = 2$-shell, which means that only the $\mathcal{N} = (2, 2)$ component symmetry remains off-shell. (In fact, insisting on keeping the $\mathcal{N} = (4, 4)$ constraints Equation (153) will put us totally on-shell.) In $\mathcal{N} = (2, 2)$ superspace the resulting model, after eliminating X_i, is given by a Lagrangian $K(\phi, \bar{\phi}, \Sigma, \bar{\Sigma})$. This is finally dualized to $\tilde{K}(\phi, \bar{\phi}, \tilde{\phi}, \bar{\tilde{\phi}})$ via a Legendre transform

$$\tilde{K}(\phi, \bar{\phi}, \tilde{\phi}, \bar{\tilde{\phi}}) = K(\phi, \bar{\phi}, \Sigma, \bar{\Sigma}) - \tilde{\phi}\Sigma - \bar{\tilde{\phi}}\bar{\Sigma}$$

$$\tilde{\phi} = \frac{\partial K}{\partial \Sigma} \ , \quad \bar{\tilde{\phi}} = \frac{\partial K}{\partial \bar{\Sigma}} \tag{162}$$

11.2. Hyperkähler Metrics on Hermitian Symmetric Spaces

This section contains an introduction to [26] where the generalized Legendre transform described in the previous section is used to find metrics on the Hermitian symmetric spaces listed in the following table:

Compact	Non-Compact
$U(n + m)/U(n) \times U(m)$	$U(n, m)/U(n) \times U(m)$
$SO(2n)/U(n); \ Sp(n)/U(n)$	$SO^*(2n)/U(n); \ Sp(n, \mathbb{R})/U(n)$
$SO(n + 2)/SO(n) \times SO(2)$	$SO_0(n + 2)/SO(n) \times SO(2)$

The special features of these quotient spaces that allow us to find a hyperkähler metric on their co-tangent bundle is the existence of holomorphic isometries and that we are able to find convenient coset representatives.

A simple example of how the coset representative enters in understanding a quotient is given, e.g., in [75]. In $\mathbb{R}^{n+1}$ the sphere S^n forms a representation of $SO(n + 1)$. The isotropy subgroup at the north pole p_0 of S^n is $SO(n)$. Consider another point p on S^n and let $g_p \in SO(n+1)$ be an element that maps $p_0 \to p$. The complete set of elements of $SO(n + 1)$ which map $p_0 \to p$ is thus of the form $g_pSO(n)$, or in other words $S^n = SO(n + 1)/SO(n)$. A coset representative is a choice of element in $g_pSO(n)$, and that choice can make the transport of properties defined at the north pole to an arbitrary point more or less transparent.

An important step in the generalized Legendre transform is to solve the auxiliary field Equation (161). As outlined in [74] and further elaborated in [76], for Hermitian symmetric spaces

the auxiliary fields may be eliminated exactly. In the present case, we start from a solution at the origin $\phi = 0$,

$$\Upsilon^{(0)} = \zeta\Sigma^{(0)} \tag{163}$$

We then extend this solution to a solution Υ^* at an arbitrary point using a coset representative. We illustrate the method in an example due to S. Kuzenko.

Ex. (Kuzenko)

The Kähler potential for $\mathbb{P}^1$ is given by

$$K(\phi, \bar{\phi}) = ln(1 + \phi\bar{\phi}) \tag{164}$$

and we denote the metric that follows from this by $g_{\phi,\bar{\phi}}$. Here ϕ is a holomorphic coordinate which we extend to an $\mathcal{N} = (2,2)$ chiral superfield. To construct a hyperkähler metric we first replace $\phi \to \Upsilon$, and then solve the auxiliary field equation as in Equation (163). Thinking of $\mathbb{CP}^n$ as the quotient $G_{1,n+1}(\mathbb{C}) = \mathrm{U}(n+1)/\mathrm{U}(n) \times \mathrm{U}(1)$, we use a carefully chosen coset representative $L(\phi, \bar{\phi})$ to extend the solution from the origin to an arbitrary point. The result is

$$\Upsilon^* = \frac{\Upsilon^{(0)} + \phi}{1 - \Upsilon^{(0)}\bar{\phi}} = \frac{\zeta\Sigma^{(0)} + \phi}{1 - \zeta\Sigma^{(0)}\bar{\phi}} \tag{165}$$

To find the chiral multiplet Σ that parametrizes the tangent bundle, we use the definition

$$\Sigma \equiv \frac{d\Upsilon^*}{d\zeta}|_{\zeta=0} = (1 + \phi\bar{\phi})\Sigma^{(0)} \tag{166}$$

yielding

$$\Upsilon^* = \frac{(1 + \phi\bar{\phi})\phi + \zeta\Sigma}{(1 + \phi\bar{\phi}) - \zeta\Sigma\bar{\phi}} \tag{167}$$

The $\mathcal{N} = (2,2)$ superspace Lagrangian on the tangent bundle is then

$$K(\Upsilon^*, \bar{\Upsilon}^*) = K(\phi, \bar{\phi}) + ln(1 - g_{\phi\bar{\phi}}\Sigma\bar{\Sigma}) \tag{168}$$

The final Legendre transform replacing the linear multiplet by a new chiral field $\Sigma \to \tilde{\phi}$ produces the Kähler potential $K(\phi, \bar{\phi}, \tilde{\phi}, \bar{\tilde{\phi}})$ for the Eguchi–Hanson metric.

The $\mathbb{P}^1$ example captures the essential idea in our construction. The reader is referred to the papers [26–28] for more examples.

11.3. Other Alternatives in Projective Superspace

Of the two methods for constructing hyperkähler metrics introduced in [4], we have dwelt on the Legendre transform generalized to projective superspace. The hyperkähler reduction discussed in Section 6 may also be lifted to projective superspace. Both these methods involve only chiral $\mathcal{N} = (2,2)$ superfields. When a nonzero B-field is present, the $\mathcal{N} = (2,2)$ sigma models involve chiral, twisted chiral and semichiral superfields, as discussed in Section 2. For a full description of (generalizations of) hyperkähler metrics on such spaces, the doubly projective superspace [12] is required. We now briefly touch on this construction.

In the doubly projective superspace, at each point in ordinary superspace we introduce one $\mathbb{P}^1$ for each chirality and denote the corresponding coordinates by ζ_L and ζ_R. The condition Equation (151) turns into

$$\nabla_+(\zeta_L) = \mathbb{D}_{2+} + \zeta_L \mathbb{D}_{1+}$$

$$\nabla_-(\zeta_R) = \mathbb{D}_{2+} + \zeta_R \mathbb{D}_{1-} \tag{169}$$

with the conjugated operators defined with respect to the real structure $\mathfrak{R}$ acting on both ζ_L and ζ_R. A superfield has the expansion

$$\Upsilon = \sum_{i,j} \Upsilon_{i,j} \zeta_L^i \zeta_R^j \tag{170}$$

and is taken to be both left and right projectively chiral. We may also impose reality conditions using $\mathfrak{R}$, as well as particular conditions on the components, such as the "cylindrical" condition

$$\Upsilon_{i,j+k} = \Upsilon_{i,j} \tag{171}$$

for some k. Actions are formed in analogy to Equations (159) and (160). The $\mathcal{N} = (2,2)$ components of such a model include twisted chiral fields χ, as well as semi-chiral ones $\mathbb{X}_{L,R}$. In fact this is the context in which the semi-chiral $\mathcal{N} = (2,2)$ superfields were introduced [12]. Hyperkähler metrics derived in this superspace are discussed in [14]. An exciting project is to merge this picture with the results in [34].

Acknowledgment

Appendices

A. Chiral-Complex linear duality

In two dimensions, chiral superfields Equation (13) obey

$$\bar{\mathbb{D}}_\pm \phi = 0 \tag{A1}$$

and twisted chiral superfields χ obey

$$\bar{\mathbb{D}}_+ \chi = \mathbb{D}_- \chi = 0 \tag{A2}$$

and the complex conjugate relations. They are related via Legendre transformations to complex linear Σ_ϕ and twisted complex linear Σ_χ superfields obeying

$$\bar{\mathbb{D}}^2 \Sigma_\phi = 0 = \bar{\mathbb{D}}_+ \mathbb{D}_- \Sigma_\chi \tag{A3}$$

and the complex conjugate relations.

A parent action which relates a BiLP generalized Kähler potential $K(\phi, \bar{\phi}, \chi, \bar{\chi})$ to its dual $U(\Sigma_\phi, \overline{\Sigma}_\phi, \Sigma_\chi, \overline{\Sigma}_\chi)$ is

$$K(X, \bar{X}, Y, \bar{Y}) - \Sigma_\phi X - \overline{\Sigma}_\phi \bar{X} - \Sigma_\chi Y - \overline{\Sigma}_\chi \bar{Y} \tag{A4}$$

Variation of the (twisted) complex linear fields constrains $X, \bar{X}, Y, \bar{Y}$ to be $\phi, \bar{\phi}, \chi, \bar{\chi}$ and $K(\phi, \bar{\phi}, \chi, \bar{\chi})$ is recovered. On the other hand, the $X, \bar{X}, Y, \bar{Y}$ field equations are

$$K_X - \Sigma_\phi = 0 \,, \quad K_{\bar{X}} - \overline{\Sigma}_\phi = 0 \,, \quad K_Y - \Sigma_\chi = 0 \,, \quad K_{\bar{Y}} - \overline{\Sigma}_\chi = 0 \tag{A5}$$

Assuming that they can be solved for $X, \bar{X}, Y, \bar{Y}$ as functions of the (twisted) complex linear fields we find the Legendre transformed potential $U(\Sigma_\phi, \overline{\Sigma}_\phi, \Sigma_\chi, \overline{\Sigma}_\chi)$ when the solutions are plugged back into Equation (A4).

The above discussion is purely local. To consider global issues, one must take into account gluing of the potential between patches. In the BiLP case the allowed change between patches $\mathcal{O}_a$ and $\mathcal{O}_b$ is given by holomorphic coordinate transformations the (generalized) Kähler gauge transformations.

Let us look at Kähler gauge transformations, restricting to the case with no twisted chiral fields for simplicity. We thus have

$$\tilde{K}(\phi, \bar{\phi}) = K(\phi, \bar{\phi}) - F(\phi) - \bar{F}(\bar{\phi}) \tag{A6}$$

which via a holomorphic coordinate transformation $\phi' = F(\phi)$ is equivalent to

$$\mathcal{K}(\phi', \bar{\phi}') = K(F^{-1}(\phi'), \bar{F}^{-1}(\bar{\phi}')) - \phi' - \bar{\phi}' \tag{A7}$$

One may ask what this freedom corresponds to in the dual model where no ambiguity of the same type exists.

The dual to K is found from the Legendre transform with parent action

$$K(X, \bar{X}) - \Sigma_\phi X - \overline{\Sigma}_\phi \bar{X} \tag{A8}$$

and reads

$$U(\Sigma_\phi, \overline{\Sigma}_\phi) = K\left(X(\Sigma_\phi, \overline{\Sigma}_\phi), \bar{X}(\Sigma_\phi, \overline{\Sigma}_\phi)\right) - \Sigma_\phi X(\Sigma_\phi, \overline{\Sigma}_\phi) - \overline{\Sigma}_\phi \bar{X}(\Sigma_\phi, \overline{\Sigma}_\phi) \tag{A9}$$

after solving

$$K_X - \Sigma_\phi = 0 \,, \quad K_{\bar{X}} - \overline{\Sigma}_\phi = 0 \tag{A10}$$

The parent action to $\tilde{K}$ is best considered after the coordinate transformation Equation (A7):

$$\begin{aligned} & \mathcal{K}(X, \bar{X}) - \Sigma_\phi X - \overline{\Sigma}_\phi \bar{X} \\ &= K\left(F^{-1}(X), \bar{F}^{-1}(\bar{X})\right) - X - \bar{X} - \Sigma_{\phi'} X - \overline{\Sigma}_{\phi'} \bar{X} \\ &= K\left(F^{-1}(X), \bar{F}^{-1}(\bar{X})\right) - \Sigma'_{\phi'} X - \overline{\Sigma}'_{\phi'} \bar{X} \end{aligned} \tag{A11}$$

where $\Sigma'_{\phi'} := 1 + \Sigma_{\phi'}$. We find the corresponding dual potential

$$\begin{aligned} & \mathcal{U}(\Sigma_{\phi'}, \overline{\Sigma}'_{\phi'}) \\ &= K\left(F^{-1}(X(\Sigma'_{\phi'}, \overline{\Sigma}'_{\phi'})), \bar{F}^{-1}(\bar{X}(\Sigma'_{\phi'}, \overline{\Sigma}'_{\phi'}))\right) - \Sigma'_{\phi'} X(\Sigma'_{\phi'}, \overline{\Sigma}'_{\phi'}) - \overline{\Sigma}'_{\phi'} \bar{X}(\Sigma'_{\phi'}, \overline{\Sigma}'_{\phi'}) \end{aligned} \tag{A12}$$

after solving

$$K_X \frac{\partial F^{-1}}{\partial X} - \Sigma'_{\phi'} = 0 \, , \quad K_{\bar{X}} \frac{\partial \bar{F}^{-1}}{\partial \bar{X}} - \overline{\Sigma}'_{\phi'} = 0 \tag{A13}$$

Comparing to Equation (A10) we see that $\Sigma'_{\phi'}$ as a functions of X is related to Σ_ϕ as a function of X via a holomorphic coordinate transformation depending on the Kähler gauge transformation F and similarly for their complex conjugate. Explicitly

$$\Sigma_\phi = K_X(X, \bar{X})$$
$$\Sigma'_{\phi'} = K_X \left(F^{-1}(X), \bar{F}^{-1}(\bar{X}) \right) \frac{\partial F^{-1}(X)}{\partial X} \tag{A14}$$

The relation between $\mathcal{U}(\Sigma_{\phi'}, \overline{\Sigma'}_{\phi'})$ and $U(\Sigma_\phi, \overline{\Sigma}_\phi)$ is more complicated due to the linear X terms, but can be worked out from Equations (A12) and (A9).

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