Note

Perlman and Wellner's Circular and Transformed Circular Copulas are Particular Beta and t Copulas

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Abstract: All but one of the copulas in a recent paper in *Symmetry* by Perlman and Wellner can be identified as particular members of either the beta or t families of elliptical copulas.

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1. Introduction

In an interesting paper in this journal, Perlman and Wellner ([?]; henceforth PW) explored the copulas arising from uniform distributions on the unit ball in R^d and, when $d = 2$, the copulas arising from certain transformations thereof. Copulas are probability distributions whose marginal distributions are all uniformly distributed; they have a major role to play in multivariate statistical analysis. Amongst the more prominent examples of copulas in the statistical literature are "elliptical copulas" which are those based on marginal transformations to uniformity of distributions whose densities have elliptically symmetric contours; these, of course, include distributions whose densities have spherically symmetric contours as special cases. Prominent amongst multivariate elliptical/spherical distributions/copulas are multivariate t distributions/copulas and multivariate symmetric beta distributions/copulas; in elliptical distribution form, the latter were introduced as multivariate Pearson Type II distributions ([? ?]), and a scaled version of the former are called multivariate Pearson Type VII distributions. See Fang *et al.* [?].

It turns out that all the copulas in PW can be identified as members of one or other of these families except for their "spherical copula" (PW, Section 4). In two cases (Section 2 below), this is just a not-especially-helpful renaming of the copulas. In the third (Section 3 below), the link is perhaps surprising and worthy of some explication.

2. Immediate Links with Beta Copulas

For clarity, suppose that $d = 2$. The bivariate spherically symmetric beta distribution has the simple density function

$$f_{b,s}(x, y; \beta) = \frac{\beta}{\pi}(1 - x^2 - y^2)^{\beta - 1}, \quad x^2 + y^2 \le 1,$$

for parameter $\beta > 0$. Its marginal densities are both symmetric beta distributions on $[-1, 1]$ with parameter $b = \beta + 1/2$ where, for example,

$$f_b(x; b) = \frac{1}{2^{2b-1}B(b, b)}(1 - x^2)^{b-1}, \quad x^2 \le 1.$$

Clearly, these marginals are uniform if $\beta = 1/2$ in which case the bivariate density is

$$f_{b,s}(x, y; 1/2) = \frac{1}{2\pi}\frac{1}{\sqrt{1 - x^2 - y^2}}, \quad x^2 + y^2 \le 1,$$

which is density (6) of PW. That is, *PW's "circular copula" is the bivariate spherically symmetric beta (or Pearson Type II) copula with parameter $\beta = 1/2$ in the above parametrisation.* (This identification says nothing about its uniqueness, as proved by PW, nor, like the other identifications I make in this note, does it help with obtaining its distribution function—often just called its "copula"—which is a major focus of PW's work.)

Now introduce the elliptically symmetric beta distribution associated with the above spherically symmetric beta distribution. In the usual way (e.g., [?]), it arises by linearly transforming the random variables that follow the spherically symmetric beta distribution by premultiplying their vector $(X, Y)^T$ by a square root of the matrix

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where $-1 < \rho < 1$ turns out to be the correlation coefficient. The resulting density, in a standard parametrisation, is

$$f_{b,e}(x, y; \beta) = \frac{\beta}{\pi\sqrt{1 - \rho^2}}\left(1 - \frac{x^2 - 2\rho xy + y^2}{1 - \rho^2}\right)^{\beta - 1}, \quad x^2 - 2\rho xy + y^2 \le 1 - \rho^2.$$

When $\beta = 1/2$ and $\rho = \sin\gamma$, this can be seen to equate to (32) of PW. That is, *PW's "one-parameter family of elliptical copulas" is the bivariate elliptically symmetric beta (or Pearson Type II) copula with parameter $\beta = 1/2$ in the above parametrisation.* Algebraically, the non-standard derivation of this distribution by PW (Section 5) arises from explicitly working directly with the Cholesky square root of Σ, essentially their (29).

3. Less Immediate Links with t Copulas

In their Section 6, PW observe that if $\{X, Y\}$ are the random variables distributed according to the "circular copula", then the non-linearly transformed variables $\{U, V\}$ where

$$U = \frac{X}{\sqrt{1 - Y^2}}, \quad V = \frac{Y}{\sqrt{1 - X^2}}$$

are marginally uniformly distributed and hence jointly distributed according to a copula. In their Proposition 6.1, PW show this copula to have density

$$c_{PW}(u,v) = \frac{1}{\pi} \frac{\sqrt{(1-u^2)(1-v^2)}}{(1-u^2v^2)^2}, \qquad -1 \le u, v \le 1.$$

I now observe that *this is the bivariate spherically symmetric t copula with degrees of freedom 2.* (I have not seen this simple formula for, in short, the t_2 copula density published anywhere before.)

To verify this claim, start from the density of the bivariate spherically symmetric t distribution, again simple and given by

$$f_{t,s}(x,y;\nu) = \frac{1}{2\pi} \frac{1}{(1+(x^2+y^2)/\nu)^{(\nu/2)+1}}, \qquad x, y \in R^2.$$

Here, the parameter $\nu > 0$, for reasons concerned with sampling derivations of such distributions, is known as the degrees of freedom (henceforth d.f.). As is well known, its marginal densities are both ("Student") t distributions on ν d.f. The t_ν copula density on $[0,1] \times [0,1]$ can then be derived from $f_{t,s}$ via the formula (e.g., Nelsen [**?**])

$$c_t^\dagger(w,z;\nu) = \frac{f_{t,s}(F_\nu^{-1}(w), F_\nu^{-1}(z);\nu)}{f_\nu(F_\nu^{-1}(w))f_\nu(F_\nu^{-1}(z))}, \qquad 0 \le w, z \le 1.$$

Here, f_ν and F_ν are the density and distribution functions of the univariate t_ν distribution. A further linear transformation from $[0,1] \times [0,1]$ to $[-1,1] \times [-1,1]$, as in PW, then sees the t_ν copula density become

$$c_t(u,v;\nu) = \frac{1}{4} c_t^\dagger \left(\frac{1+u}{2}, \frac{1+v}{2}; \nu \right), \qquad -1 \le u, v \le 1.$$

In general, F_ν involves the incomplete beta function but, as is well known in the case of $\nu = 1$ (the Cauchy case) and under-appreciated in the case of $\nu = 2$ (Jones [**?**]), in each of those two cases it reduces to a simple invertible formula. In particular,

$$f_2(x) = \frac{1}{(2+x^2)^{3/2}}, \quad F_2(x) = \frac{1}{2}\left(1+\frac{x^2}{\sqrt{2+x^2}}\right), \quad F_2^{-1}(w) = \frac{2w-1}{\sqrt{2w(1-w)}}.$$

Using these formulae and that of $f_{t,s}(x,y;2)$ in $c_t^\dagger(w,z;2)$ results in $c_t(u,v;2) = c_{PW}(u,v)$ by straightforward algebraic manipulations.

A more interesting route to the same result takes advantage of general relationships between univariate and bivariate spherically symmetric beta and t distributions. First, let B be a random variable following the symmetric beta distribution on $[-1,1]$ with parameter $b = \nu/2$ and let T be a random variable following the t distribution with parameter ν. The following relationship is a version of one that is long established (Cacoullos [**? ?**]) but still relatively little known among statisticians:

$$B = \frac{T}{\sqrt{\nu+T^2}}, \qquad T = \frac{\sqrt{\nu}B}{\sqrt{1-B^2}}.$$

Second, let $\{B_1, B_2\}$ be random variables following the bivariate spherically symmetric beta distribution with density $f_{b,s}(x,y;\nu/2)$, *i.e.*, with $\beta = \nu/2$, and let $\{T_1, T_2\}$ be random variables following the bivariate spherically symmetric t distribution with density $f_{t,s}(x,y;\nu)$. Then, the relationship between

$\{B_1, B_2\}$ and $\{T_1, T_2\}$, which is also known and is a rather easy consequence of the polar representations of these bivariate spherically symmetric distributions, is that

$$B_i = \frac{T_i}{\sqrt{\nu + T_1^2 + T_2^2}}, \quad T_i = \frac{\sqrt{\nu} B_i}{\sqrt{1 - B_1^2 - B_2^2}}, \quad i = 1, 2.$$

Recalling PW's notation that $\{B_1, B_2\} = \{X, Y\}$, it follows that the random variables

$$T_1 = \frac{\sqrt{2}X}{\sqrt{1 - X^2 - Y^2}}, \quad T_2 = \frac{\sqrt{2}Y}{\sqrt{1 - X^2 - Y^2}}$$

follow the bivariate spherically symmetric t_2 distribution. But marginally transforming $\{T_1, T_2\}$ back to beta marginals (in this case, uniform marginals since $b = 1$) via $B_i = T_i / \sqrt{2 + T_i^2}$, $i = 1, 2$, yields a distribution with the same copula as $\{T_1, T_2\}$—the t_2 copula. And

$$B_1 = \frac{T_1}{\sqrt{2 + T_1^2}} = \frac{X}{\sqrt{1 - Y^2}} = U, \quad B_2 = \frac{T_2}{\sqrt{2 + T_2^2}} = \frac{Y}{\sqrt{1 - X^2}} = V.$$

As PW note, their construction of $\{U, V\}$ "extends readily to generate a [d-dimensional] copula". I will now outline briefly how the above argument generalises to show that *this copula is the d-dimensional spherically symmetric t copula with degrees of freedom 2*. Start from $\{X_1, ..., X_d\}$ being uniformly distributed on the unit ball, which still corresponds to a spherically symmetric beta distribution with $\beta = 1$. PW's elegant conditional argument (p. 594) carries through to the random variables

$$U_i = \frac{X_i}{\sqrt{1 - \sum_{j=1, j \neq i}^d X_j^2}}, \quad i = 1, ..., d,$$

following the uniform distribution marginally and hence a copula jointly. The d-dimensional relationship between spherically symmetric beta and t distributions can be readily shown to be

$$T_i = \frac{\sqrt{\nu} B_i}{\sqrt{1 - \sum_{j=1}^d B_j^2}}, \quad i = 1, ..., d.$$

Then, the argument based on the same marginal transformations back to the beta (uniform) hold as for $d = 2$. In d dimensions, the t_2 copula density on $[-1, 1]^d$ actually turns out to be

$$c_t(u_1, ..., u_d) = \frac{\Gamma((d/2) + 1)}{\pi^{d/2}} \frac{\prod_{i=1}^d (1 - u_i^2)^{(d-1)/2}}{\left(\prod_{i=1}^d (1 - u_i^2) + \sum_{i=1}^d u_i^2 \prod_{j=1, j \neq i}^d (1 - u_j^2)\right)^{(d/2)+1}}.$$

(It is not clear how much, if anything, this identification would have helped PW with their task of obtaining its distribution function.)

4. Conclusions

Most of the copulas in PW ([**?**]) have been identified as either beta or t copulas. The simple explicit form of what has been shown in Section 3 to be the spherically symmetric t_2 copula density is an interesting byproduct of PW's work.

Acknowledgements

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