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Three New Classes of *Solvable* *N*-Body Problems of Goldfish Type with *Many Arbitrary* Coupling Constants

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Abstract: Three new classes of N-body problems of goldfish type are identified, with N an *arbitrary* positive integer ($N \geqslant 2$). These models are characterized by nonlinear Newtonian ("accelerations equal forces") equations of motion describing N equal point-particles moving in the complex z-plane. These highly nonlinear equations feature *many arbitrary* coupling constants, yet they can be *solved* by *algebraic* operations. Some of these N-body problems are *isochronous*, their *generic* solutions being *all completely periodic* with an overall period T independent of the initial data (but quite a few of these solutions are actually periodic with smaller periods T/p with p a positive integer); other models are *isochronous* for an *open* region of initial data, while the motions for other initial data are *not periodic*, featuring instead *scattering* phenomena with some of the particles incoming from, or escaping to, infinity in the remote past or future.

Keywords: solvable many-body problems; integrable many-body problems; integrable dynamical systems; solvable many-body problems of goldfish type

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1. Introduction

Recently, a new technique to identify many-body problems *solvable* by algebraic operations has been introduced [1,2], and several examples of such models have been discussed [1–6]. In the present paper, three *new* classes of such models are introduced and discussed (the impatient reader may immediately glance at these findings reported in the next section). We term these models "of goldfish type" because their Newtonian ("accelerations equal forces") equations of motion read as follows:

$$\ddot{z}_n = (2r+1)\,\mathbf{i}\omega\dot{z}_n + r\,(r+1)\,\omega^2 z_n - \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{(\dot{z}_n + \mathbf{i}r\omega z_n)\,(\dot{z}_\ell + \mathbf{i}r\omega z_\ell)}{z_n - z_\ell}\right] + f_n\left(\vec{z},\,\dot{\vec{z}}\right)\,, \tag{1}$$

with the nonlinear functions $f_n\left(\vec{z},\,\dot{\vec{z}}\right)$ appropriately defined (see below). (The original goldfish model is the special case of these equations of motion with $r = 0$ and $f_n\left(\vec{z},\,\dot{\vec{z}}\right) = 0$; after its first identification as a *solvable* model [7], and its tentative recognition as a "goldfish" [8], this N-body problem and some of its extensions have been investigated in several publications (see, for instance, [9–20]).

Notation 1.1. Above, and hereafter, N is an *arbitrary* positive integer ($N \geqslant 2$); indices such as n, m, ℓ run over the *positive integers* from 1 to N (unless otherwise indicated: see, for instance, the restriction on the values of ℓ in (1)); the N *complex* coordinates $z_n \equiv z_n\,(t)$ identify the positions of N points moving in the *complex* z-plane as functions of the time t; below, we also introduce other *complex* variables $\zeta_n \equiv \zeta_n(\tau)$ that depend on the *complex* variable τ related to the *real* variable t ("time") as follows:

$$\tau \equiv \tau(t) = \frac{\exp(\mathbf{i}\omega t) - 1}{\mathbf{i}\omega} \,. \tag{2a}$$

Above, and hereafter, $\mathbf{i}$ is the imaginary unit ($\mathbf{i}^2 = -1$), ω is an *arbitrary* (nonvanishing) *real* constant, so that $\tau(t)$ vanishes at the initial time $t = 0$ and is periodic in time with period T_0,

$$\tau(0) = 0 \,, \quad \tau(t + T_0) = \tau(t) \,, \quad T_0 = \frac{2\pi}{|\omega|} \,. \tag{2b}$$

Above, and hereafter, we adopt the standard notation according to which superimposed dots denote differentiations with respect to the time variable t, so that, for instance,

$$\dot{\tau} \equiv \dot{\tau}(t) = \exp(\mathbf{i}\omega t) \,, \tag{2c}$$

while *primes* appended to a function indicate differentiations with respect to the argument of that function (see, for instance, (3b) and (3c), where the relevant variable is of course τ). The auxiliary *complex* coordinates $\zeta_n \equiv \zeta_n(\tau)$ are related to the N *complex* coordinates $z_n \equiv z_n(t)$—which are the main protagonists of our N-body problems, as they identify the positions of N equal unit-mass point-particles moving in the complex z-plane according to the Newtonian ("accelerations equal forces") equations of motion of type (1)—by the following relations:

$$z_n(t) = \exp(\mathbf{i}r\omega t)\,\zeta_n(\tau) \,, \;\; \zeta_n(\tau) = \exp(-\mathbf{i}r\omega t)\,z_n(t) \,, \tag{3a}$$

implying

$$\begin{aligned} \dot{z}_n &= \mathbf{i}r\omega z_n + \exp[\mathbf{i}(r+1)\omega t]\;\zeta_n' \\ \zeta_n' &= \exp[-\mathbf{i}(r+1)\omega t]\;(\dot{z}_n - \mathbf{i}r\omega z_n) \,, \end{aligned} \tag{3b}$$

$$\begin{aligned} \ddot{z}_n &= \mathbf{i}(2r+1)\,\omega\dot{z}_n + r(r+1)\,\omega^2 z_n + \exp[\mathbf{i}(r+2)\,\omega t]\,\zeta_n'' \\ \zeta_n'' &= \exp[-\mathbf{i}(r+2)\,\omega t]\left[\ddot{z}_n - \mathbf{i}(2r+1)\,\omega\dot{z}_n - r(r+1)\,\omega^2 z_n\right]. \end{aligned} \tag{3c}$$

Above, and hereafter, the number r is required to be *real* and *rational*,

$$r = \frac{q}{p} \,, \tag{3d}$$

where, of course, the numerator q and the denominator p are *coprime integers*, and, for definiteness, we hereafter assume that p is *positive*, $p \geqslant 1$.

Finally, note that above we often omitted indicating *explicitly* the argument of functions, and we will do so throughout whenever this can be done without causing confusion. ■

In this paper, a key role is played by the following monic τ-dependent polynomial of degree N in the (*complex*) variable ζ,

$$p_N\left(\zeta; \underline{\zeta}(\tau); \vec{\gamma}(\tau)\right) = \prod_{n=1}^{N}\left[\zeta - \zeta_n(\tau)\right] = \zeta^N + \sum_{m=1}^{N}\left[\gamma_m(\tau)\,\zeta^{N-m}\right] \,, \tag{4a}$$

which features the N *zeros* $\zeta_n(\tau)$ and the N *coefficients* $\gamma_m(\tau)$; and by its counterpart,

$$P_N\left(z;\underline{z}\left(t\right);\vec{w}\left(t\right)\right)=\prod_{n=1}^{N}\left[z-z_n\left(t\right)\right]=z^N+\sum_{m=1}^{N}\left[w_m\left(t\right)z^{N-m}\right], \tag{4b}$$

featuring the N *zeros* $z_n(t)$ and the N *coefficients* $w_m(t)$ (see **Notation 1.1**).

Notation 1.2. The N *coefficients* $\gamma_m \equiv \gamma_m(\tau)$ of the polynomial (4a) are of course expressed in terms of its N *zeros* $\zeta_n \equiv \zeta_n(\tau)$ via the formulas

$$\gamma_m=\left(-1\right)^m\sigma_m\left(\underline{\zeta}\right), \tag{5a}$$

$$\sigma_m\left(\underline{\zeta}\right)=\frac{1}{m!}\sum_{n_1,n_2,\ldots,n_m=1}^{N}{}^{*}\left(\zeta_{n_1}\,\zeta_{n_2}\cdots\zeta_{n_m}\right). \tag{5b}$$

Above, and hereafter, the symbol $\underline{\zeta}$ denotes the *unordered* set of the N complex numbers ζ_n (and likewise $\underline{z}$ is the *unordered* set of the N complex numbers z_n), while $\vec{\gamma}$ (see (4a) and below) is the N-vector with components γ_m (and likewise $\vec{w}$ is the N-vector with components w_m). The symbol $\sum_{n_1,n_2,\ldots,n_m}^{N}{}^{*}$ denotes the sum from 1 to N over the m indices $n_1, n_2, \ldots, n_m$ with the restriction that these indices be *all different among themselves*, so that for $N=2$,

$$\gamma_1=-\left(\zeta_1+\zeta_2\right),\;\gamma_2=\zeta_1\zeta_2, \tag{5c}$$

for $N=3$,

$$\gamma_1=-\left(\zeta_1+\zeta_2+\zeta_2\right),\;\gamma_2=\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1,\;\gamma_3=-\zeta_1\zeta_2\zeta_3, \tag{5d}$$

and so on. Let us also display the corresponding formulas for the τ-derivative of $\gamma_m \equiv \gamma_m(\tau)$,

$$\gamma_m'=\left(-1\right)^m\sum_{n=1}^{N}\left[\zeta_n'\sigma_{n,m}\left(\zeta_n;\underline{\zeta}\right)\right], \tag{6a}$$

$$\begin{aligned}
&\sigma_{n,1}\left(\zeta_n;\underline{\zeta}\right)=1,\\
&\sigma_{n,m}\left(\zeta_n;\underline{\zeta}\right)=\frac{1}{(m-1)!}\sum_{n_1,n_2,\ldots,n_{m-1}=1}^{N}{}^{*n*}\left(\zeta_{n_1}\,\zeta_{n_2}\cdots\zeta_{n_{m-1}}\right),\\
&m=2,\ldots,N,
\end{aligned} \tag{6b}$$

where the symbol $\sum_{n_1,n_2,\ldots,n_m}^{N}{}^{*n*}$ denotes the sum from 1 to N over the m indices $n_1, n_2, \ldots, n_m$ with the restriction that these indices be *all different among themselves and all different from the index n*, so that for $N=2$,

$$\gamma_1'=-\left(\zeta_1'+\zeta_2'\right),\;\gamma_2'=\zeta_1'\zeta_2+\zeta_1\zeta_2', \tag{6c}$$

for $N=3$,

$$\begin{aligned}
\gamma_1' &= -\left(\zeta_1'+\zeta_2'+\zeta_2'\right),\\
\gamma_2' &= \zeta_1'\left(\zeta_2+\zeta_3\right)+\zeta_2'\left(\zeta_3+\zeta_1\right)+\zeta_3'\left(\zeta_1+\zeta_2\right),\\
\gamma_3' &= -\left(\zeta_1'\zeta_2\zeta_3+\zeta_2'\zeta_3\zeta_1+\zeta_3'\zeta_1\zeta_2\right),
\end{aligned} \tag{6d}$$

and so on. Analogous formulas hold, of course, for the relations among the N *coefficients* $w_m \equiv w_m(t)$ and the N *zeros* $z_n \equiv z_n(t)$ of the polynomial (4b), and their time-derivatives; for instance,

$$w_m = \frac{(-1)^m}{m!} \sum_{n_1,n_2,\ldots,n_m=1}^{N} {}^{*} \left(z_{n_1}\, z_{n_2} \cdots z_{n_m}\right). \tag{7}$$

Let us also report the relations—implied by (3a)—among the N *coefficients* $w_m(t)$ of the polynomial (4b) and the N *coefficients* $\gamma_m(\tau)$ of the polynomial (4a)

$$w_m(t) = \exp(\mathbf{i} m r \omega t)\, \gamma_m(\tau), \tag{8a}$$

with τ related to t via (2) and r defined by (3d), implying

$$\begin{aligned} &\dot{w}_m(t) = \mathbf{i} m r \omega w_m(t) + \exp[\mathbf{i}(mr+1)\omega t]\, \gamma_m'(\tau), \\ &\gamma_m'(\tau) = \exp[-\mathbf{i}(mr+1)\omega t]\,[\dot{w}_m(t) - \mathbf{i} m r \omega w_m(t)]. \end{aligned} \tag{8b}$$

$$\begin{aligned} &\ddot{w}_m(t) = \mathbf{i}(2mr+1)\,\omega \dot{w}_m(t) + mr(mr+1)\,\omega^2 w_m(t) \\ &+ \exp[\mathbf{i}(mr+2)\omega t]\, \gamma_m''(\tau), \\ &\gamma_m''(\tau) = \exp[-\mathbf{i}(mr+2)\omega t]\,\big[\ddot{w}_m(t) \\ &- \mathbf{i}(2mr+1)\,\omega \dot{w}_m(t) - mr(mr+1)\,\omega^2 w_m(t)\big]. \quad \blacksquare \end{aligned} \tag{8c}$$

Remark 1.1. Two comments on the problem to determine the N *zeros* ζ_n, respectively, z_n of a monic polynomial of degree N in ζ, respectively, z from its N *coefficients* γ_m, respectively, w_m (see (4a), respectively, (4b)).

(i) Of course, the assignment of the N *coefficients* of a polynomial defines *uniquely* the corresponding *unordered* set of its N *zeros*, but generally it only allows to compute *explicitly* these N *zeros* for $N \leqslant 4$.

(ii) Moreover—and quite relevantly in our context (see below)—if a polynomial features a dependence on an additional variable (as, for instance, the dependence of the polynomial $P_N(z; \underline{z}(t); \vec{w}(t))$ on the real variable t ("time") (see (4b)), then the *unordered* character of the set of its N zeros $z_n(t)$ is generally only relevant at one value of time, say at the "initial" time $t = 0$, since, at other values of time, the ordering gets generally determined by the natural requirement that the functions $z_n(t)$ evolve *continuously* over time. This prescription then fixes, for all time, the ordering of the *zeros* $z_n(t)$—i.e., the assignment of the value n of its index to each zero $z_n(t)$—as long as the coefficients $w_m(t)$ evolve themselves *continuously* and *unambiguously* over time and moreover no "collision" of two or more zeros occurs over the time evolution, i.e., for all time $z_n(t) \neq z_\ell(t)$ if $n \neq \ell$ (since clearly such collisions imply a *loss of identity* of the coinciding *zeros*). However, this identification requires an analysis of the time evolution of the N zeros $z_n(t)$ not only in the *complex* z-plane, but in fact over the N-sheeted Riemann surface associated to the N roots of the polynomial $P_N(z; \underline{z}(t); \vec{w}(t))$, and/or over the evolution of each coefficient $w_m(t)$ if its time evolution takes itself place on a Riemann surface (as it indeed happens in the cases discussed below). $\blacksquare$

A key formula for the identification and investigation of *solvable* Newtonian N-body problems is the following relation [1] among the τ-evolutions of the N *zeros* and the N *coefficients* of the monic polynomial $p_N(\zeta; \tau)$ (see 4a)) of degree N in its argument ζ and depending on the extra variable τ:

$$\zeta_n'' = \sum_{\ell=1;\, \ell \neq n}^{N} \left(\frac{2\zeta_n' \zeta_\ell'}{\zeta_n - \zeta_\ell}\right) - \left[\prod_{\ell=1;\, \ell \neq n}^{N} (\zeta_n - \zeta_\ell)^{-1}\right] \sum_{m=1}^{N} \left[\gamma_m''(\zeta_n)^{N-m}\right]. \tag{9}$$

In the present paper—which is a sequel to [6], so that we dispense below from some of the remarks reported there about the significance of the *solvable* many-body models treated herein—we focus on the N-body models that obtain via (9) for the coordinates $\zeta_n(\tau)$—and especially for the corresponding coordinates $z_n(t)$ (see (3))—when the coefficients $\gamma_m(\tau)$ evolve according to the following system of *decoupled nonlinear* Ordinary Differential Equations (ODEs):

$$\gamma''_m = g_m \left(\gamma'_m\right)^{a_m} \left(\gamma_m\right)^{b_m}, \tag{10a}$$

where the N "coupling constants" g_m are *arbitrary* (possibly *complex*), while the two (sets of) parameters a_m and b_m are instead hereafter required to be *real rational* numbers and to be related to each other (and to the parameter r, (see (3d)) by the single relation:

$$a_m = 2 - \frac{(1+b_m)\,mr}{1+mr},\; b_m = 1 - a_m + \frac{2-a_m}{mr}, \tag{10b}$$

which—provided neither r nor $1+mr$ vanish, i.e., $r \neq 0$ and $r \neq -1/m$ for m in its range from 1 to N, as we now assume (the case $r = 0$ shall be discussed below separately)—is necessary and sufficient to guarantee that the corresponding system of ODEs satisfied by the coefficients $w_m(t)$ be *autonomous*, reading as follows (as the diligent reader will easily verify via (10a) and (8)) :

$$\begin{aligned} \ddot{w}_m(t) \;\; &= \;\; \mathbf{i}\,(2mr+1)\,\omega\,\dot{w}_m(t) + mr\,(mr+1)\,\omega^2 w_m(t) \\ &\quad + g_m\left(\dot{w}_m - \mathbf{i} mr\omega w_m\right)^{a_m}\left(w_m\right)^{b_m}. \end{aligned} \tag{10c}$$

In fact, we will focus below only on certain specific assignments of the parameters r, a_m, b_m which allow the *explicit* solution of the ODEs (10a) in terms of *elementary* functions. The corresponding *solvable* N-body models satisfied by the coordinates $z_n \equiv z_n(t)$ are displayed—and their properties discussed—in the following Section 2, with the corresponding proofs provided in Section 3, while the special case with $a_m = 2$ and $b_m = -1$ (and with g_m an *arbitrary rational* number) is treated in [6].

2. Results

The Newtonian equations of motion of the *first* class of N-body models treated in this paper read as follows:

$$\begin{aligned} &\ddot{z}_n = \mathbf{i}\,(2r+1)\,\omega\dot{z}_n + r\,(r+1)\,\omega^2 z_n \\ &+ \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{2\,(\dot{z}_n - \mathbf{i} r\omega z_n)\,(\dot{z}_\ell - \mathbf{i} r\omega z_\ell)}{z_n - z_\ell}\right] - \left[\prod_{\ell=1,\,\ell\neq n}^{N}(z_n - z_\ell)^{-1}\right] \\ &\cdot \sum_{m=1}^{N}\left[g_m\left(\dot{w}_m - \mathbf{i} mr w_m\right)^{1+[1/(1+mr)]}(z_n)^{N-m}\right], \end{aligned} \tag{11a}$$

with r an *arbitrary rational* number (see (3d); of course $r \neq -1/m$ for $m = 1, 2, ..., N$ if $g_m \neq 0$), ω an *arbitrary nonvanishing real* number, the N coupling constants g_m *arbitrary complex numbers* (*not all vanishing*) and (see **Notation 1.2**)

$$w_m = \frac{(-1)^m}{m!} \sum_{n_1,n_2,...,n_m=1}^{N} {}^{*}\left(z_{n_1}\, z_{n_2} \cdots z_{n_m}\right), \tag{11b}$$

$$\dot{w}_m = \frac{(-1)^m}{(m-1)!} \sum_{n=1}^{N}\left\{\dot{z}_n\left[\sum_{n_1,n_2,...,n_{m-1}=1}^{N} {}^{*n*}\left(z_{n_1}\, z_{n_2} \cdots z_{n_m}\right)\right]\right\}. \tag{11c}$$

We then assert that this N-body model is *solvable by algebraic operations*, its remarkable properties being detailed by the following

Proposition 2.1. The N *complex* coordinates $z_n(t)$ providing the solution at time t of the initial-values problem of the Newtonian equations of motion (11) are the N *zeros* of the monic polynomial (4b), the N coefficients $w_m(t)$ of which are given, in terms of the initial data $z_n(0)$, $\dot{z}_n(0)$, by the following formulas:

$$w_m(t) = \exp(\mathbf{i} rm\omega t)\, \gamma_m(\tau(t)), \ \tau(t) = \frac{\exp(\mathbf{i}\omega t) - 1}{\mathbf{i}\omega}, \tag{12a}$$

$$\gamma_m(\tau) = \left[w_m(0) + B_m^{(1)}\right]\left(1 - \frac{\tau}{\check{\tau}_m^{(1)}}\right)^{-mr} - B_m^{(1)}, \tag{12b}$$

$$\check{\tau}_m^{(1)} = \frac{mr\left\{A_m\left[w_m(0) + B_m^{(1)}\right]\right\}^{-1/(mr)}}{A_m}, \tag{12c}$$

$$A_m = \frac{g_m mr}{1 + mr} \tag{12d}$$

$$B_m^{(1)} = \frac{\left[\dot{w}(0) - \mathbf{i} mr\omega w_m(0)\right]^{(mr)/(1+mr)}}{A_m} - w_m(0), \tag{12e}$$

$$w_m(0) = \frac{(-1)^m}{m!} \sum_{n_1, n_2, \ldots, n_m = 1}^{N} {}^{*} \left[z_{n_1}(0)\, z_{n_2}(0) \cdots z_{n_m}(0)\right], \tag{12f}$$

$$\dot{w}_m(0) = \frac{(-1)^m}{(m-1)!} \sum_{n=1}^{N} \left\{\dot{z}_n(0) \sum_{n_1, n_2, \ldots, n_{m-1} = 1}^{N} {}^{*n*} \left[z_{n_1}(0)\, z_{n_2}(0) \cdots z_{n_{m-1}}(0)\right]\right\}. \tag{12g}$$

These functions $w_m(t)$ are, of course, defined by continuity in t from their initial values $w_m(0)$ (i.e., this prescription identifies the determination of all the rational roots appearing in the above formulas), and note that, if $g_m = 0$, the formula (12b) must be replaced by

$$\gamma_m(\tau) = w_m(0) + \left[\dot{w}(0) - \mathbf{i} mr\omega w_m(0)\right]\tau. \tag{12h}$$

Let us also report the properties of these solutions, (12), when they are generated by *generic* initial data, $z_n(0)$, $\dot{z}_n(0)$, excluding the *nongeneric* initial data identified in **Proposition 2.2** for which the system of evolution equations (11) runs into a *singularity* at a *finite* time.

These solutions are *all* nonsingular for *all* time, remaining in a *finite* region—the size of which depends on the initial data—of the *complex* z-plane and featuring *no particle collisions*, i.e., for *all* time t, $z_n(t) \neq z_\ell(t)$ if $n \neq \ell$; and they are *all completely periodic* with a period T which is an *integer multiple* of the basic period T_0 (see (2b)),

$$z_n(t + T) = z_n(t), \ T = KT_0, \tag{13}$$

with the *positive integer* K restricted as follows: $p \leqslant K \leqslant p\, N!$, where, of course, p is the denominator of the *rational* number r (see (3d)). ■

Note the *arbitrariness* of the *real (nonvanishing) rational number* r, and especially of the N *complex* coupling constants g_m. In addition, note that **Proposition 2.1** implies that *all generic* solutions of the N-body model characterized by the Newtonian equations of motion (11a)—excluding the *nongeneric* solutions which are *singular* (see below **Proposition 2.2**)—are *completely periodic* with the same period $T_{MAX} = (p\, N!)\ T_0$. However, there are lots of solutions that are *completely periodic* with periods which are *integer submultiples* of T_{MAX}. The detailed identification of these solutions and their periods is a nontrivial matter, as shown, for instance, by the discussion of this phenomenology in the paper [11]—that treats the "periodic goldfish model" (for this terminology, see [8]), which is in fact characterized by the same equations of motions (11a), but with *all* coupling constants vanishing, $g_m = 0$—and by the detailed investigation of the structure of the Riemann surfaces associated with other analogous many-body models [21–25].

Proposition 2.2. The solutions of the Newtonian equations of motion (11) may feature *singularities* via *two phenomena*, both of which correspond to *nongeneric* initial data.

The *first phenomenon* is characterized by initial data satisfying—for at least one value $\bar{m}$ of the index m in its range from 1 to N such that $g_{\bar{m}} \neq 0$ and $-\bar{m}\, r$ is *not a positive integer*—the *equality*

$$\left|1 + \mathbf{i}\omega\check{\tau}_{\bar{m}}^{(1)}\right| = 1, \tag{14}$$

where $\check{\tau}_{\bar{m}}^{(1)}$ is defined in terms of the initial data $z_n(0)$, $\dot{z}_n(0)$ as above (see (12c)).

This singularity occurs at the time $t = \check{t}^{(1)}$ defined as follows: $\check{t}^{(1)} = \min\left[\check{t}_{\bar{m}}^{(1)}\right]$ (with the minimum taken over the values of the index $\bar{m}$ satisfying the condition (14)), with

$$\check{t}_{\bar{m}}^{(1)} = \frac{\theta_{\bar{m}}^{(1)}}{\omega}\,, \quad \check{t}_{\bar{m}}^{(1)} > 0\,, \tag{15a}$$

where $\theta_{\bar{m}}^{(1)}$ is defined *mod* $[2\pi]$ as follows:

$$1 + \mathbf{i}\omega\check{\tau}_{\bar{m}}^{(1)} = \exp\left(\mathbf{i}\theta_{\bar{m}}^{(1)}\right). \tag{15b}$$

Note that the fact that $\theta_{\bar{m}}^{(1)}$ is *real* is implied by (14), and that certainly $0 < \check{t}_{\bar{m}}^{(1)} < T_0$, hence $0 < \check{t}^{(1)} < T_0$.

The *second phenomenon* causing the equations of motion (11a) to hit a singularity is the occurrence of a *collision* of two (or possibly more) particles at some time $t = t_c$; so that, for some indices n, ℓ with $n \neq \ell$ (of course both in their interval from 1 to N) there holds the equality $z_n(t_c) = z_\ell(t_c)$, causing the term $z_n(t) - z_\ell(t)$ appearing in the denominators in the right-hand side of (11a) to vanish. This phenomenon corresponds to the fact that two of the *zeros* of the polynomial (4b) with (12) coincide, a fact that clearly only happens for *nongeneric* initial data (in the *complex* z-plane), although the condition on the initial data that would cause this phenomenon to happen—and the time at which this phenomenon would happen—can be computed *explicitly* only for small values of N (and even then the result is not very enlightening). ■

Let us complete our discussion of the *first* class of *solvable* N-body problems by displaying the equations of motion (11) in the (simplest) $N = 2$ case:

$$\begin{aligned} \ddot{z}_n &= \mathbf{i}\,(2r+1)\,\omega\dot{z}_n + r\,(r+1)\,\omega^2 z_n - (-1)^n\,(z_1 - z_2)^{-1} \\ &\cdot \{2\,(\dot{z}_1 - \mathbf{i}rz_1)\,(\dot{z}_2 - \mathbf{i}rz_2) \\ &+ \left(g_1\,[\dot{z}_1 + \dot{z}_2 - \mathbf{i}r\omega\,(z_1 + z_2)]^{1+[1/(1+r)]}\, z_n \right. \\ &\left. - g_2\,[\dot{z}_1 z_2 + \dot{z}_2 z_1 - 2\mathbf{i}r\omega z_1 z_2]^{1+[1/(1+2r)]}\right)\Big\}\,, \; n = 1, 2, \end{aligned} \tag{16}$$

with r an *arbitrary rational* number ($r \neq 0$, $r \neq -1/2$, $r \neq -1$), and g_1, g_2 two *arbitrary complex* numbers (*not both vanishing*). Note that, for $r = -2/3$, *both* exponents on the right-hand side of these ODEs are *integers*, since then $1/(1+r) = 3$, $1/(1+2r) = -3$, and this is as well the case for $r = -3/4$, since then $1/(1+r) = 4$, $1/(1+2r) = -2$.

Let us end our treatment of the *first* class of *solvable* N-body problems characterized by the Newtonian equations of motion (11) by pointing out that, for $r = -2/3$ and $N = 3$, the exponents on the right-hand side of (11a) are *all integers*, indeed for $r = -2/3$, the three exponents $(2 + mr) / (1 + mr)$ take, for $m = 1, 2, 3$, the three values $4, -2, 0$.

The Newtonian equations of motion of the *second* class of *solvable* N-body models read as follows:

$$\ddot{z}_n = \mathbf{i}\,\omega\,\dot{z}_n + \sum_{\ell=1,\,\ell\neq n}^{N} \left(\frac{2\dot{z}_n\dot{z}_\ell}{z_n - z_\ell}\right) - \left[\prod_{\ell=1,\,\ell\neq n}^{N} (z_n - z_\ell)^{-1}\right] \sum_{m=1}^{N} \left[g_m\,(\dot{w}_m)^2\,(z_n)^{N-m}\right], \tag{17}$$

with the various quantities defined as above (in particular $\dot{w}_m$ defined in terms of $\dot{z}_n$ and z_n by (11c) with **Notation 1.2**). Note that this might be considered the special case of the *first* class of models (see above) with $r = 0$, $a_m = 2$ and $b_m = 0$, which was previously excluded because it requires a special treatment (see Section 3).

The *solvable* character of this N-body system is demonstrated by the following

Proposition 2.3. The N *complex* coordinates $z_n\,(t)$ providing the solution at time t of the initial-values problem of the Newtonian equations of motion (17) are the N *zeros* of the monic polynomial (4b), the N coefficients $w_m\,(t)$ of which are given, in terms of the initial data $z_n\,(0)$, $\dot{z}_n\,(0)$, by the following formula:

$$w_m\,(t) = w_m\,(0) - \frac{1}{g_m} \ln\left[1 - \frac{\tau\,(t)}{\check{\tau}_m^{(2)}}\right],\ \check{\tau}_m^{(2)} = \frac{1}{g_m \dot{w}_m\,(0)}, \tag{18}$$

of course with $\tau\,(t) = [\exp\,(\mathbf{i}\omega t) - 1] / (\mathbf{i}\omega)$ (see (2a)). Here and below, the function $\ln\left\{1 - \left[\tau\,(t) / \check{\tau}_m^{(2)}\right]\right\}$ is defined by continuity in t from its vanishing value at $t = 0$ where $\tau\,(t) = 0$, and, of course, $w_m\,(0)$, respectively, $\dot{w}_m\,(0)$ are defined in terms of the initial data $z_n\,(0)$ and $\dot{z}_n\,(0)$ by (12f), respectively, (12g), and if $g_m = 0$ then (18) becomes $w_m\,(t) = w_m\,(0) + \dot{w}_m\,(0)\,\tau\,(t)$, while if $\dot{w}_m\,(0) = 0$ it yields $w_m\,(t) = w_m\,(0)$. ■

Clearly, these coefficients $w_m\,(t)$ are completely periodic in the time t with period T_0, see (2a),

$$w_m\,(t + T_0) = w_m\,(t)\ , \tag{19a}$$

iff the initial data satisfy the *inequality*

$$\left|1 + \mathbf{i}\omega\check{\tau}_m^{(2)}\right| > 1, \tag{19b}$$

iff instead the initial data satisfy the opposite *inequality*,

$$\left|1 + \mathbf{i}\omega\check{\tau}_m^{(2)}\right| < 1, \tag{20a}$$

they are periodic in t except for a constant shift (independent of the initial data!) over each period T_0, so that

$$w_m\,(t + kT_0) = w_m\,(t) - \frac{2\pi\mathbf{i}k}{g_m},\ k = \pm1, \pm2, \pm3, ..., \tag{20b}$$

hence, in this second case, they diverge as $t \to \pm\infty$. In addition—in the intermediate, *nongeneric* case in which the initial data imply the *equality*

$$\left|1 + \mathbf{i}\omega\check{\tau}_m^{(2)}\right| = 1 \tag{21a}$$

—the *coefficient* $w_m(t)$ diverges at the *finite* times

$$\check{t}_m^{(2)} = (\mathbf{i}\,\omega)^{-1}\ln\left(1+\mathbf{i}\omega\check{\tau}_m^{(2)}\right) \mod(T_0)\,, \tag{21b}$$

which are, of course, *real* thanks to (21a).

Correspondingly, the *particles coordinates* $z_n(t)$—being the N zeros of the polynomial (4b) with the *coefficients* $w_m(t)$ (see (18))—are periodic with period $K\,T_0$—K being a *positive integer* in the range from 1 to $N!$ (see below **Remark 3.1**)—iff the *initial data* satisfy the *inequality* (19b), while iff instead the initial data satisfy the opposite *inequality* (20a) at least one of the particle coordinates $z_n(t)$ comes from or escapes to *infinity* in the remote past and future: see, for instance, the relevant discussion in Appendix G ("Asymptotic behavior of the zeros of a polynomial whose coefficients diverge exponentially") of the book [9]. In addition, of course, if the (*nongeneric*) initial data imply validity of the *equality* (21a), the equations of motion run into a singularity at $t = \check{t}_m^{(2)}$ (see (21b)). Other *nongeneric* initial data causing the equations of motion (17) to run into a singularity at a *finite* time are those leading to particle collisions. Note that generally the *nongeneric* initial data causing singularities are also those that *separate* the regions of initial data associated to *different* behaviors of the model, including the emergence of the higher periodicities associated to values of K larger than *unity* as well as the periodic and nonperiodic cases.

We do not display explicitly the equations of motion of this *second* model in the simplest $N = 2$ case because they can be immediately obtained by setting $r = 0$ in those of the *first* model (see (16)).

We conclude our report on the properties of the *second* class of *solvable* N-body problems with the following remark (proven in Section 3):

Remark 2.1. The system of Newtonian equations of motion (17) is Hamiltonian, albeit with a time-dependent Hamiltonian. ■

The *third* class of *solvable* N-body problems is characterized by the Newtonian equations of motion

$$\begin{aligned}\ddot{z}_n &= \mathbf{i}\,(2r+1)\,\omega\dot{z}_n + r\,(r+1)\,\omega^2 z_n \\ &+ \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{2\,(\dot{z}_n - \mathbf{i}r\omega z_n)\,(\dot{z}_\ell - \mathbf{i}r\omega z_\ell)}{z_n - z_\ell}\right] - \left[\prod_{\ell=1,\,\ell\neq n}^{N}(z_n - z_\ell)^{-1}\right] \\ &\cdot \sum_{m=1}^{N}\left[g_m\,(\dot{w}_m - \mathbf{i}mrw_m)^{2-\frac{1}{1+mr}}\,(w_m)^{-1+\frac{1}{mr}}\,(z_n)^{N-m}\right],\end{aligned} \tag{22}$$

where ω is an *arbitrary nonvanishing real* number, r is an *arbitrary rational* number (of course if $g_m \neq 0$ then $r \neq 0$ and $r \neq 1/m$ for the integer m in its range from 1 to N), the N coupling constants g_m are *arbitrary complex* numbers (*not all vanishing*), and the quantities $w_m \equiv w_m(t)$, respectively, $\dot{w}_m \equiv \dot{w}_m(t)$ are expressed in terms of the coordinates $z_n \equiv z_n(t)$ and $\dot{z}_n \equiv \dot{z}_n(t)$ via (11b), respectively (11c).

The *solvable* character of this N-body problem is demonstrated by the following

Proposition 2.4. The N *complex* coordinates $z_n(t)$ providing the solution at time t of the initial-values problem of the Newtonian equations of motion (22) are the N *zeros* of the monic polynomial (4b), the N coefficients $w_m(t)$ of which are given, in terms of the initial data $z_n(0)$, $\dot{z}_n(0)$, by the following formulas:

$$w_m(t) = \exp(\mathbf{i}mr\omega t)\,\gamma_m(\tau(t))\,,\;\tau(t) = \frac{\exp(\mathbf{i}\omega t)-1}{\mathbf{i}\omega}, \tag{23a}$$

$$\gamma_m(\tau) = \left[B_m^{(3)}\right]^{mr}\left\{\left[\frac{\check{\tau}_m^{(3)} - \hat{\tau}_m^{(3)}}{\check{\tau}_m^{(3)} - \tau}\right]^{\frac{1}{mr}} - 1\right\}^{-mr}, \tag{23b}$$

$$B_m^{(3)} = \frac{\left[\dot{w}_m\left(0\right) - \mathbf{i} m r \omega w\left(0\right)\right]^{\frac{1}{1+mr}}}{A_m} - \left[w\left(0\right)\right]^{\frac{1}{mr}}, \ A_m = \frac{g_m m r}{1+mr}, \tag{23c}$$

$$\check{\tau}_m^{(3)} = -\frac{\left\{1 + B_m^{(3)} \ \left[w\left(0\right)\right]^{-\frac{1}{mr}}\right\}^{-mr}}{\left(A_m\right)^{1+mr} \ B_m^{(3)}}, \tag{23d}$$

$$\hat{\tau}_m^{(3)} = \check{\tau}_m^{(3)} \left\{1 - \left(1 + B_m^{(3)} \ \left[w\left(0\right)\right]^{-\frac{1}{mr}}\right)^{mr}\right\}, \tag{23e}$$

where $w_m\left(0\right)$, respectively, $\dot{w}_m\left(0\right)$, are defined in terms of the initial data $z_n\left(0\right)$ and $\dot{z}_n\left(0\right)$ by (12f), respectively, (12g), and of course the determinations of the *rational* roots are implied by the requirement that these formulas be valid at the initial time $t = 0$ and thereafter by continuity in t, and again if $g_m = 0$, the expression (23b) must be replaced by (12h). ■

It is plain that, if neither mr nor $1/\left(mr\right)$ are *integers*, for *generic* initial data the function $\gamma_m\left(\tau\right)$ features, in the *complex* τ-plane, two *rational* branch points at $\tau = \check{\tau}_m^{(3)}$, respectively, at $\tau = \hat{\tau}_m^{(3)}$, and that the *periodicity* of $\gamma_m\left(t\right) \equiv \gamma_m\left(\tau\left(t\right)\right)$ as function of t (see (23b)), is determined by the location in the *complex* τ-plane of these two branch points with respect to the circle $\tilde{C}$ centered at $\tau = \mathbf{i}/\omega$ and with radius $1/\left|\omega\right|$ on which rotates the point $\tau\left(t\right) = \left[\exp\left(\mathbf{i}\,\omega\,t\right) - 1\right]/\left(\mathbf{i}\,\omega\right)$ as function of the time t. If *both* branch points are located *outside* the circle $\tilde{C}$—and the condition on the initial data determining this outcome is clearly validity of *both inequalities*

$$\left|1 + \mathbf{i}\omega\check{\tau}_m^{(3)}\right| > 1, \ \left|1 + \mathbf{i}\omega\hat{\tau}_m^{(3)}\right| > 1, \tag{24a}$$

with $\check{\tau}_m^{(3)}$, respectively, $\hat{\tau}_m^{(3)}$ defined in terms of the initial data by (23d), respectively, (23e)—then clearly $\gamma_m\left(t\right) \equiv \gamma_m\left(\tau\left(t\right)\right)$ as a function of t is *periodic* with period T_0, see (2a),

$$\gamma_m\left(t + T_0\right) = \gamma_m\left(t\right), \tag{24b}$$

hence $w_m\left(t\right)$ is *periodic* in t with period $T_{\hat{p}_m} = \hat{p}_m T_0$,

$$w_m\left(t + T_{\hat{p}_m}\right) = w_m\left(t\right), \tag{24c}$$

with $\hat{p}_m = p/m$ if this number is *integer*; otherwise, $\hat{p}_m = p$ (where of course p is the denominator of r: see (23a) and (3d)). In addition, as a consequence—if the inequalities (24a) are valid for *all* values of m in the range from 1 to N—then the polynomial (4b) is periodic in t with period

$$T = T_0 \left\{\underset{\bar{m}=1,2,\ldots,N}{\text{MinimumCommonMultiple}}\left[\hat{p}_{\bar{m}}\right]\right\}, \tag{24d}$$

where the MinimumCommonMultiple must be evaluated for the values of $m = \bar{m}$ such that $g_{\bar{m}} \neq 0$; hence (see below **Remark 3.1**), the N coordinates $z_n\left(t\right)$ are periodic with period $Q\,T$, where Q is a *positive integer* in the range from 1 to $N!$.

It is also plain that the solutions $z_n\left(t\right)$ are periodic in t with a period which is a *positive integer multiple* $\tilde{Q}$ of T_0 even if the initial data imply instead that some of the inequalities (24a) are reversed, but, in these cases, the determination of the outcome—in particular, of the value of $\tilde{Q}$—requires, to begin with, a standard analysis of the structure of the Riemann surface associated to the function

$$\varphi_m\left(\tau\right) = \left\{\left[\frac{\check{\tau}_m^{(3)} - \hat{\tau}_m^{(3)}}{\check{\tau}_m^{(3)} - \tau}\right]^{\frac{1}{mr}} - 1\right\}^{-mr} \tag{25}$$

of the *complex* variable τ, and consequently of the periodicity of $\varphi_m\left(t\right) \equiv \varphi_m\left(\tau\left(t\right)\right)$ as a function of the *real* variable t ("time") when $\tau\left(t\right) = \left[\exp\left(\mathbf{i}\omega t\right) - 1\right]/\left(\mathbf{i}\omega\right)$ so that $\tau\left(t\right)$ travels on the circle $\tilde{C}$ with radius $1/|\omega|$ centered at $\tau = \mathbf{i}/\omega$; and then an analysis of the resulting periodicity of the solutions $z_n(t)$ analogous to that made in Section 3 for the *first* class of *solvable* models. We leave this task to the interested reader.

Let us complete this discussion of the *third* class of *solvable* N-body problems by displaying the equations of motion (22) in the (simplest) $N = 2$ case:

$$\begin{aligned}
&\ddot{z}_n = \mathbf{i}\,(2r+1)\,\omega\dot{z}_n + r\,(r+1)\,\omega^2 z_n - (-1)^n\,[z_1 - z_2]^{-1} \\
&\cdot \{2\,(\dot{z}_1 - \mathbf{i}rz_1)\,(\dot{z}_2 - \mathbf{i}rz_2) \\
&-g_1\,[-\dot{z}_1 - \dot{z}_2 + \mathbf{i}r\,(z_1+z_2)]^{2-\frac{1}{1+r}}\,[-(z_1+z_2)]^{-1+\frac{1}{r}}\,z_n \\
&-g_2\,(\dot{z}_1 z_2 + z_1\dot{z}_2 - 2\mathbf{i}rz_1z_2)^{2-\frac{1}{1+2r}}\,(z_1z_2)^{-1+\frac{1}{2r}}\Big\}\,,\; n = 1,2,
\end{aligned} \tag{26}$$

with r an *arbitrary rational* number ($r \neq 0$, $r \neq -1/2$, $r \neq -1$), and g_1, g_2 two *arbitrary complex* numbers (*not both vanishing*).

In addition, let us end this discussion of the *third* class of *solvable* N-body problems characterized by the Newtonian equations of motion (22) by also displaying these equations in the following two special cases:

case (i), with $r = -1/2$ and $g_m = 0$ for $m > 1$, when these equations read

$$\begin{aligned}
\ddot{z}_n &= -\frac{1}{4}\omega^2 z_n + \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{(2\dot{z}_n + \mathbf{i}\omega z_n)\,(2\dot{z}_\ell + \mathbf{i}\omega z_\ell)}{2\,(z_n - z_\ell)}\right] \\
&+g_1\,(z_n)^{N-1}\left[\prod_{\ell=1,\,\ell\neq n}^{N}(z_n - z_\ell)^{-1}\right]\left[\sum_{n_1=1}^{N}(z_{n_1})\right]^{-3};
\end{aligned} \tag{27a}$$

case (ii), with $r = -1/4$ and $g_m = 0$ for $m \neq 2$, when these equations of motion read

$$\begin{aligned}
\ddot{z}_n &= \frac{1}{2}\,\mathbf{i}\omega\dot{z}_n - \frac{3}{16}\omega^2 z_n + \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{(4\dot{z}_n + \mathbf{i}\omega z_n)\,(4\dot{z}_\ell + \mathbf{i}\omega z_\ell)}{8\,(z_n - z_\ell)}\right] \\
&-\frac{1}{2}\,g_2\,(z_n)^{N-2}\left[\prod_{\ell=1,\,\ell\neq n}^{N}(z_n - z_\ell)^{-1}\right]\left[\sum_{n_1,n_2=1;\,n_1\neq n_2}^{N}(z_{n_1}\,z_{n_2})\right]^{-3}.
\end{aligned} \tag{27b}$$

3. Proofs

In this Section 3 we prove the results reported without their proofs in the preceding Section 2. The first task is to integrate once the ODE (10a). This is an easy task, yielding

$$\gamma_m'(\tau) = \left\{A_m\left([\gamma_m(\tau)]^{1+b_m} + B_m\right)\right\}^{\frac{1}{2-a_m}}, \tag{28a}$$

$$A_m = \frac{g_m\,(2-a_m)}{1+b_m} = \frac{g_m mr}{1+mr}, \tag{28b}$$

$$B_m = \frac{[\gamma_m'(0)]^{2-a_m}}{A_m} - [\gamma_m(0)]^{1+b_m}. \tag{28c}$$

Note that, above and hereafter, we exclude from consideration the special case with $a_m = 2$ and $b_m = -1$, which is treated in [6].

The next integration can be performed in terms of *elementary* functions only for special assignments (satisfying the restriction (10b)) of the parameters a_m and b_m, to which we restrict attention in the present paper.

Our *first* assignment is

$$b_m = 0, \tag{29a}$$

implying via (10b)

$$a_m = 2 - \frac{mr}{1+mr} = \frac{2+mr}{1+mr} \tag{29b}$$

and

$$\gamma_m'(\tau) = \left\{ A_m \left[\gamma_m(\tau) + B_m^{(1)} \right] \right\}^{1+\frac{1}{mr}}, \tag{29c}$$

with A_m, respectively, $B_m^{(1)}$ defined by (28b) (or, equivalently, (12d)), respectively, (12e). Here, we assume, of course, that $r \neq 0$ (implying $a_m \neq 2$) and moreover that $r \neq -1/m$ with $m = 1, 2, ..., N$ (implying that a_m is a *finite rational* number for all values of m in its range from 1 to N).

This ODE (29c) can now be easily integrated, yielding (12b) with (12c).

These developments clearly prove the first part of **Proposition 2.1**.

To prove the second part of **Proposition 2.1** we ascertain, to begin with, the periodicity properties as functions of the time variable t of the *coefficients* $\gamma_m(t) \equiv \gamma_m(\tau(t))$ (see (12b)). The starting point is the observation that $\tau(t)$—see (2) or (12a)—is a periodic function of t with period T_0, rotating in the *complex* τ-plane on the circle $\tilde{C}$ centered at the point $\mathbf{i}/\omega$ and having radius $1/|\omega|$. Hence, any *holomorphic* function of τ is as well periodic in t with period T_0; this clearly is (for all values of m in its range from 1 to N) the case of the functions $\gamma_m(\tau)$ (see (12b)), if r is a *negative integer*. If instead r is a *positive integer*, the functions $\gamma_m(\tau)$ are *meromorphic* in τ, featuring a *polar* singularity at $\tau = \check{\tau}_m^{(1)}$(see (12b)). In this case, $\gamma_m(t) \equiv \gamma_m(\tau(t))$ is again generally periodic in t with period T_0, $\gamma_m(t+T_0) = \gamma_m(t)$, but for the *nongeneric* assignments of the *initial data* such that $\check{\tau}_m^{(1)}$ falls on the circle $\tilde{C}$—note that $\check{\tau}_m^{(1)}$ does depend on the initial data (see (12c) and (12e)), and that the condition for this to happen is validity of the equality

$$\left| 1 + \mathbf{i}\omega\check{\tau}_m^{(1)} \right| = 1, \tag{30a}$$

then the function $\gamma_m(t) \equiv \gamma_m(\tau(t))$ diverges at the times

$$\check{t}_m^{(1)} = (\mathbf{i}\,\omega)^{-1} \ln\left(1 + \mathbf{i}\omega\check{\tau}_m^{(1)}\right) \mod (T_0). \tag{30b}$$

Finally, if r is *rational* but *not integer*, i.e., $p > 1$ (see (3d)), and m is *not* an integer multiple of p, $\gamma_m(\tau)$ features a *rational branch point* at $\tau = \check{\tau}_m^{(1)}$ (see (12b)), then the evolution of $\gamma_m(t) \equiv \gamma_m(\tau(t))$ as function of the time t depends on the location of the branch point $\check{\tau}_m^{(1)}$ in the complex τ-plane, whether it falls *outside*, *inside* or just *on* the circle $\tilde{C}$. The latter case requires that the *initial data* satisfy the condition (30a), implying again that they are *not generic*. While clearly the condition that the *branch point* be located *outside* the circle $\tilde{C}$ is validity of the *inequality*

$$\left| 1 + \mathbf{i}\omega\check{\tau}_m^{(1)} \right| > 1, \tag{31}$$

and for the corresponding *nongeneric* initial data the function $\gamma_m(t) \equiv \gamma_m(\tau(t))$ is again periodic with period T_0. If instead the branch point falls *inside* the circle $\tilde{C}$, and the condition on the initial data for this to happen is validity of the opposite *inequality*

$$\left| 1 + \mathbf{i}\omega\check{\tau}_m^{(1)} \right| < 1, \tag{32}$$

the periodicity of $\gamma_m(t) \equiv \gamma_m(\tau(t))$ gets modified: the period is then $\tilde{T}_p = pT_0$ (see (12b) and (3d)), unless $p/m = \check{p}_m$ is an *integer*, in which case the period is $\check{T}_{\check{p}_m} = \check{p}_m T_0$.

Next, let us discuss the t-periodicity of $w(t)$, related to $\gamma(t) \equiv \gamma(\tau(t))$ by (8a) or (12a). We then note that the prefactor $\exp(\mathbf{i} r m \omega t)$ in (8a) or (12a) is also periodic in t (see (3d)), with the same period $\check{T}_{\check{p}_m} = \check{p}_m T_0$, where $\check{p}_m = p$ unless p/m is an integer, in which case $\check{p}_m = p/m$. We may therefore conclude that, for *all generic initial data*, the functions $w_m(t)$ are periodic with period $\check{T}_{\check{p}_m}$,

$$w_m\left(t + \check{T}_{\check{p}_m}\right) = w_m(t)\,. \tag{33}$$

Next, let us discuss the periodicity of the *zeros* $z_n(t)$ of the polynomial (4b) with *coefficients* $w_m(t)$ periodic as indicated just above. In this context, the following **Remark 3.1** is relevant.

Remark 3.1. If a time-dependent polynomial $P_N(z;t)$, of degree N in z, is time-periodic with period $\tilde{T}$, $P_N\left(z;t+\tilde{T}\right) = P_N(z;t)$, the *unordered* set $\underline{z}(t)$ of its N *zeros* $z_n(t)$ is of course periodic with the same period $\tilde{T}$, $\underline{z}\left(t+\tilde{T}\right) = \underline{z}(t)$ (since after a period the polynomial is unchanged); however, due to the possibility that these *zeros*, as it were, "exchange their places" over their time evolution, the period of each individual *zero* $z_n(t)$, considered as a *continuous* function of time, may be a *positive integer multiple* of $\tilde{T}$; although of course that *multiple* cannot exceed the number $N!$ of permutations of the N elements of the unordered set $\underline{z}(t)$ (for a detailed discussion of this phenomenology in analogous many-body contexts see [11,21–25]). ■

We can therefore conclude that the N *zeros* $z_n(t)$ of the monic polynomial (4b), the N *coefficients* $w_m(t)$ of which are *periodic* as described above, are also *periodic*, for *generic* initial data, with periods KT_0, with the *positive integer* K restricted as follows:

$$p \leqslant K \leqslant N!p, \tag{34}$$

since the very definition of $\check{p}_m$ (see above) implies that p is the Minimum Common Multiple of the N parameters $\check{p}_m$.

This ends the proofs of the findings reported in **Propositions 2.1** and **2.2**.

The proofs of the other two Propositions reported in Section 2 follow, but below we omit the proofs of some aspects of the results reported in Section 2, which are analogous to those detailed above in the formulations and proofs of **Propositions 2.1** and **2.2**.

Hence, the only aspect relevant to prove **Proposition 2.3** that we do report is the derivation of the formula (18). The starting point is the ODE

$$\gamma_m'' = g_m\left(\gamma_m'\right)^2, \tag{35a}$$

which is the special case of (10a) with $r = 0$, $a_m = 2$ and $b_m = 0$. The integration of this ODE is a trivial task, yielding

$$\gamma_m(\tau) = \gamma_m(0) - \frac{\ln\left[1 - g_m \gamma_m'(0)\,\tau\right]}{g_m}, \tag{35b}$$

and the fact that this implies (18) is an immediate consequence of (8) with $r = 0$.

As for the proof of **Remark 2.1**, it is based on the observation that the *decoupled nonlinear* system of N ODEs

$$\ddot{w}_m = i\omega\dot{w}_m + g\left(\dot{w}_m\right)^2 \tag{36a}$$

is Hamiltonian, since it is implied by the standard Hamiltonian equations of motion

$$\dot{w} = \frac{\partial H\left(\vec{v},\,\vec{w};\,t\right)}{\partial v_m},\ \dot{v} = -\frac{\partial H\left(\vec{v},\,\vec{w};\,\right)}{\partial w_m}, \tag{36b}$$

with the (time-dependent) Hamiltonian

$$H\left(\vec{v},\ \vec{w};\ t\right) = \exp\left(i\omega t\right) v^{\alpha} \exp\left(g\alpha w\right), \tag{36c}$$

as the diligent reader will easily verify. Note the *arbitrariness* of the (*nonvanishing*!) parameter α.

However, via the two identities

$$\dot{z}_n = -\left[\prod_{\ell=1,\ \ell\neq n}^{N} \left(z_n - z_\ell\right)^{-1}\right] \sum_{m=1}^{N} \left[\dot{w}\left(z_n\right)^{N-m}\right], \tag{37a}$$

$$\ddot{z}_n = \sum_{\ell=1,\ \ell\neq n}^{N} \left(\frac{2\dot{z}_n \dot{z}_\ell}{z_n - z_\ell}\right) - \left[\prod_{\ell=1,\ \ell\neq n}^{N} \left(z_n - z_\ell\right)^{-1}\right] \sum_{m=1}^{N} \left[\ddot{w}\left(z_n\right)^{N-m}\right], \tag{37b}$$

linking—as proven in [1]—the time evolutions of the N *zeros* $z_n\left(t\right)$ and of the N *coefficients* $w_m\left(t\right)$ of the monic time-dependent polynomial (4b), it is easily seen that the set of ODEs (36a) correspond precisely to the equations of motion (17). This implies that the equations of motion (17) are also Hamiltonian, since the N coordinates z_n are linked to the N *canonical coordinates* w_m by the "point" transformations—*not* involving the *canonical momenta* v_m—which relate the *coefficients* and the *zeros* of a polynomial (see (7) or (11b)).

Let us finally deal with the *third* class of N-body models, obtained by identifying another set of parameters r, a_m, b_m allowing the ODEs (28) to be *explicitly* integrated in terms of *elementary* functions. To this end, we introduce the auxiliary functions

$$\eta_m\left(\tau\right) = \left[\gamma_m\left(\tau\right)\right]^{s_m},\ \gamma_m\left(\tau\right) = \left[\eta_m\left(\tau\right)\right]^{1/s_m}, \tag{38a}$$

with the option to assign the parameters s_m at our convenience (see below). We then see, from (28) with (10b), that these auxiliary functions satisfy the ODEs

$$\eta_m' = s_m \left[A_m\left(\eta^{\alpha_m} + B_m \eta^{\beta_m}\right)\right]^{\frac{1}{2-a_m}}, \tag{38b}$$

$$\alpha_m = \frac{b_m + 1 + \left(s_m - 1\right)\left(2 - a_m\right)}{s_m},\ \beta_m = \frac{\left(s_m - 1\right)\left(2 - a_m\right)}{s_m}, \tag{38c}$$

with A_m, respectively, B_m defined by (28b), respectively, (28c). Thus, we set $\alpha_m = 0$ and $\beta_m = 1$, implying (after a bit of elementary algebra)

$$s_m = -\frac{1}{mr},\ b_m \equiv b_m^{(3)} = -1 + \frac{1}{mr},\quad a_m \equiv a_m^{(3)} = 2 - \frac{1}{1+mr}. \tag{39}$$

The ODE (38b) then reads

$$\eta_m' = -\frac{1}{mr}\left[A_m\left(1 + B_m^{(3)} \eta_m\right)\right]^{1+mr}, \tag{40}$$

with A_m, respectively, $B_m^{(3)}$ defined by (28b) respectively (28c) with (39), while the assignments of the parameters r and g_m are still free (of course $r \neq 0$, $r \neq -1/m$). This ODE, (40), can then be immediately integrated, yielding

$$\eta_m\left(\tau\right) = \left[\frac{1}{B_m^{(3)}} + \eta_m\left(0\right)\right]\left(1 - \frac{\tau}{\check{\tau}_m^{(3)}}\right)^{-\frac{1}{mr}} - \frac{1}{B_m^{(3)}}, \tag{41}$$

$$B_m^{(3)} = \frac{\left[\gamma'\left(0\right)\right]^{\frac{1}{1+mr}}}{A_m} - \left[\gamma\left(0\right)\right]^{\frac{1}{mr}}, \tag{42}$$

$$\check{\tau}_m^{(3)} = -\frac{\left[1 + B_m^{(3)} \eta_m(0)\right]^{-mr}}{(A_m)^{1+mr} B_m^{(3)}}. \quad (43)$$

Hence, finally, via (38a) with (39) and (8), we arrive at the formulas (23), thereby proving **Proposition 2.4**.

4. Outlook

In this Section 4 we outline tersely possible extensions of the results obtained in this paper. A tentative list of such possible developments is provided in the last Section of [6]: the first of those reported there is essentially implemented in the present paper, while the others are still open. Also open is the possibility to investigate a combination of the three classes of models treated above, and of the class of models treated in [6], by assuming that, for different values of the parameter m, some coefficients γ_m evolve according to the three *different* solvable subcases of the ODEs (28) discussed above in Section 3 or to the case discussed in [6]. In addition, of course, also open is the investigation of the *hierarchies* of *solvable* models associated with each of those mentioned herein via the notion of *generations* of monic polynomials such that the *coefficients* of the polynomials of the next generation coincide with the *zeros* of a polynomial of the current generation [2].

Other N-body problems—more general than those treated in [6] and in the present paper—can be investigated by analogous techniques to those employed above, while allowing each $\gamma_m(t)$ to satisfy a *solvable* second-order ODE still belonging to the class (10) but forsaking the restriction on the parameters a_m and b_m necessary and sufficient to allow the solution of these ODEs in terms of *elementary* functions (for instance, a natural extension might include special functions of *elliptic* and *hyperelliptic* type).

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References

1. Calogero, F. New solvable variants of the goldfish many-body problem. *Studies Appl. Math.* **2016**, doi:10.1111/sapm.12096.
2. Bihun, O.; Calogero, F. Generations of monic polynomials such that the coefficients of each polynomial of the next generation coincide with the zeros of a polynomial of the current generation, and new solvable many-body problems. *Lett. Math. Phys.* **2016**, *106*, 1011–1031.
3. Bihun, O.; Calogero, F. A new solvable many-body problem of goldfish type. *J. Nonlinear Math. Phys.* **2016**, *23*, 28–46.
4. Bihun, O.; Calogero, F. Novel solvable many-body problems. *J. Nonlinear Math. Phys.* **2016**, *23*, 190–212.
5. Calogero, F. A solvable N-body problem of goldfish type featuring N^2 arbitrary coupling constants. *J. Nonlinear Math. Phys.* **2016**, *23*, 300–305.
6. Calogero, F. Novel *isochronous* N-body problems featuring N *arbitrary rational* coupling constants. *J. Math. Phys.* **2016**, in press.
7. Calogero, F. Motion of poles and zeros of special solutions of nonlinear and linear partial differential equations, and related "solvable" many-body problems. *Il Nuovo Cimento B* **1978**, *43*, 177–241.
8. Calogero, F. The neatest many-body problem amenable to exact treatments (a "goldfish"?). *Phys. D Nonlinear Phenom.* **2001**, *152–153*, 78–84.
9. Calogero, F. *Classical many-body problems amenable to exact treatments*; Springer: Heidelberg, Germany, 2001; p. 749.
10. Nucci, M.C. Calogero's 'goldfish' is indeed a school of free particles. *J. Phys. A Math. Gen.* **2004**, *37*, 11391–11400.
11. Gómez-Ullate, D.; Sommacal, M. Periods of the goldfish many-body problem. *J. Nonlinear Math. Phys.* **2005**, *12*, 351–362.
12. Suris, Y.B. Time discretization of F. Calogero's "Goldfish" System. *J. Nonlinear Math. Phys.* **2005**, *12*, 633–647.

13. Calogero, F.; Iona, S. Novel solvable extensions of the goldfish many-body model. *J. Math. Phys.* **2005**, *46*, 103515.
14. Guillot, A. The Painlevé property for quasihomogeneous systems and a many-body problem in the plane. *Comm. Math. Phys.* **2005**, *256,* 181–194.
15. Bruschi, M.; Calogero, F. Novel solvable variants of the goldfish many-body model. *J. Math. Phys.* **2006**, *47,* 022703.
16. Calogero, F.; Langmann, E. Goldfishing by gauge theory. *J. Math. Phys.* **2006**, *47*, 082702.
17. Arlind, J.; Bordemann, M.; Hoppe, J.; Lee, C. Goldfish geodesics and Hamiltonian reduction of matrix dynamics. *Lett. Math. Phys.* **2008**, *84*, 89–98.
18. Calogero, F. *Isochronous Systems*; Oxford University Press: Oxford, UK, 2008; p. 264.
19. Bihun, O.; Calogero, F. Solvable many-body models of goldfish type with one-, two- and three-body forces. *SIGMA* **2013**, *9*, 059.
20. Jairuk, U.; Yoo-Kong, S.M.; Tanasittikosol, M. The Lagrangian structure of Calogero's goldfish model. *Theor. Math. Phys*. **2015**, *183*, 665–683.
21. Calogero, F.; Gómez-Ullate, D.; Santini, P.M.; Sommacal, M. The transition from regular to irregular motions, explained as travel on Riemann surfaces. *J. Phys. A Math. Gen.* **2005**, *38*, 8873–8896.
22. Sommacal, M. The Transition from Regular to Irregular Motion, Explained as Travel on Riemann Surfaces. Ph.D. Thesis, SISSA, Trieste, Italy, 2005.
23. Grinevich, P.G.; Santini, P.M. Newtonian dynamics in the plane corresponding to straight and cyclic motions on the hyperelliptic curve: Ergodicity, isochrony and fractals. *Phys. D Nonlinear Phenom.* **2007**, *232*, 22–32.
24. Calogero, F.; Gómez-Ullate, D.; Santini, P.M.; Sommacal, M. Towards a theory of chaos explained as travel on Riemann surfaces. *J. Phys. A. Math. Theor.* **2009**, *42,* 015205.
25. Gómez-Ullate, D.; Santini, P.M.; Sommacal, M.; Calogero, F. Understanding complex dynamics by means of an associated Riemann surface. *Phys. D Nonlinear Phenom.* **2012**, *241*, 1291–1305.