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First and Second Zagreb Eccentricity Indices of Thorny Graphs

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Abstract: The Zagreb eccentricity indices are the eccentricity reformulation of the Zagreb indices. Let H be a simple graph. The first Zagreb eccentricity index ($E_1(H)$) is defined to be the summation of squares of the eccentricity of vertices, i.e., $E_1(H) = \sum_{u \in V(H)} \varepsilon_H^2(u)$. The second Zagreb eccentricity index ($E_2(H)$) is the summation of product of the eccentricities of the adjacent vertices, i.e., $E_2(H) = \sum_{uv \in E(H)} \varepsilon_H(u)\varepsilon_H(v)$. We obtain the thorny graph of a graph H by attaching thorns i.e., vertices of degree one to every vertex of H. In this paper, we will find closed formulation for the first Zagreb eccentricity index and second Zagreb eccentricity index of different well known classes of thorny graphs.

Keywords: graphs; vertices; complete graph; path; star; cycle

1. Introduction

In theoretical chemistry, molecular descriptors or topological indices are utilized to configure properties of chemical compounds. A topological index is a real number connected with chemical structure indicating relationships of chemical configuration with different physical properties, chemical reactivity or biological activity, which is utilized to understand properties of chemical compounds in theoretical chemistry. Topological indices have been observed to be helpful in chemical documentation, isomer discrimination, structure-property relations, structure-activity (SAR) relations and pharmaceutical medication plans. All through the paper, all graphs are considered to be simple and connected.

Let $H = (V, E)$ be a simple graph with $m = |V|$ vertices and $n = |E|$ edges. For $u \in V$, degree of u, denoted by $d(u)$, is number of vertices attached to u in the graph. The maximum distance from a vertex to any other vertex in the graph H is called eccentricity of the vertex and is denoted by $\varepsilon_H(u)$ i.e., $\varepsilon_H(u) = \max\{d(u,v) | v \in V\}$, where $d(u,v)$ denotes the distance between u and v in H. The first Zagreb index (M_1) and second Zagreb index (M_2) are the oldest known indices introduced by Gutman and Trinajstić [1] defined as

$$M_1 = M_1(H) = \sum_{u \in V(H)} d_u^2,$$

$$M_2 = M_2(H) = \sum_{uv \in E(H)} d_u d_v.$$

Several topological indices depend upon the eccentricity of the vertices and are very effective in drug design. Sharma, Goswami and Madan [2] proposed the eccentric connectivity index of the graph H, which is defined as

$$C^{\xi}(H) = \sum_{w \in V(H)} \frac{d_H(w)}{\varepsilon_H(w)}.$$

In 2000, Gupta, Singh and Madan [3] introduced another distance-cum-degree based topological descriptor termed the connective eccentricity index:

$$\xi^{C}(H) = \sum_{w \in V(H)} d_H(w)\varepsilon_H(w).$$

Other eccentricity related indices include the eccentric distance sum [4], augmented and super augmented eccentric connectivity indices [5–7], and adjacent eccentric distance sum index [8,9].

Recently, the first Zagreb eccentricity index and second Zagreb eccentricity index E_1 and E_2 have been proposed as the revised versions of the Zagreb indices M_1 and M_2, respectively, by Ghorbani and Hosseinzadeh [10]. The first Zagreb eccentricity index (E_1) and the second Zagreb eccentricity index (E_2) of a graph H are defined as

$$E_1 = E_1(H) = \sum_{u \in V(H)} \varepsilon_H^2(u),$$

$$E_2 = E_2(H) = \sum_{uv \in E(H)} \varepsilon_H(u)\varepsilon_H(v),$$

respectively. Das et al. [11] gave a few lower and upper bounds on the first Zagreb eccentricity index and the second Zagreb eccentricity index of trees and graphs, and also characterized the extremal graphs. Nilanjan [12] computed a few new lower and upper bounds on the first Zagreb eccentricity index and the second Zagreb eccentricity index. Zhaoyang and Jianliang [13] computed Zagreb eccentricity indices under different graph operations. Farahani [14] computed precise equations for the First Zagreb Eccentricity index of Polycyclic Aromatic Hydrocarbons. Evidently, Zagreb indices and the family of all connectivity indices express mathematically attractive invariants. In this manner, we expect numerous more studies on these indices and anticipate further development of this area of mathematical chemistry.

2. Results and Discussion

Consider a graph H with vertex set $\{u_1, u_2, \ldots, u_m\}$ and a set of positive integers $\{p_1, p_2, \ldots, p_m\}$. The thorn graph of H, denoted by $H^*(p_1, p_2, \ldots, p_n)$, is obtained by attaching p_j pendant vertices to u_j for each j. The idea of a thorn graph was presented by Gutman [15], and various studies on thorn graphs and different topological indices have been conducted by some researchers in the recent past [16–19]. In this paper, we will derive explicit expressions for computing the first Zagreb eccentricity index and the second Zagreb eccentricity index of thorny graphs of some well-known classes of graphs like complete graphs, complete bipartite graphs, star graphs, cycles and paths.

2.1. The Thorny Complete Graph

Suppose that we take the complete graph K_m with m vertices. Obviously, $E_1(K_m) = m$ and $E_2(K_m) = \frac{m(m-1)}{2}$. The thorny complete graph K_m^* is obtained from K_m by attaching p_j thorns at each vertex of K_m, $j = 1, 2, \ldots, m$. Suppose that the total number of thorns attached to K_m are denoted by T.

Theorem 1. *The first Zagreb eccentricity index and the second Zagreb eccentricity index of K_m^* are given by:*

$$E_1(K_m^*) = 4E_1(K_m) + 9T \text{ and } E_2(K_m^*) = 4E_2(K_m) + 6T, \text{ respectively.}$$

Proof. Let K_m be a compete graph. Suppose that v_j, $j = 1, 2, \ldots, m$ are the vertices of K_m, and v_{jk}, $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p_j$ are the newly attached pendant vertices. Then, $\varepsilon_{K_m^*}(v_j) = 2$, $\varepsilon_{K_m^*}\left(v_{jk}\right) = 3$ for $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p_j$ are the eccentricities of the vertices of K_m^*. Thus, the first Zagreb eccentricity index and the second Zagreb eccentricity index of K_m^* are given by

$$
\begin{aligned}
&E_1(K_m^*) = \sum_{j=1}^{m} \varepsilon^2_{K_m^*}(v_j) + \sum_{j=1}^{m} \sum_{k=1}^{p_j} \varepsilon^2_{K_m^*}(v_{jk}) \\
&= \sum_{j=1}^{m} (2)^2 + \sum_{j=1}^{m} \sum_{k=1}^{p_j} (3)^2 \\
&= 4m + 9T = 4E_1(K_m) + 9T, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
&M_2^*(K_m^*) = \sum_{v_j v_k \in E(K_m^*)} \varepsilon_{K_m^*}(v_j) \varepsilon_{K_m^*}(v_k) + \sum_{k=1}^{p_j} \sum_{v_j v_{jk} \in E(K_m^*)} \varepsilon_{K_m^*}(v_j) \varepsilon_{K_m^*}(v_{jk}) \\
&= \sum_{v_j v_k \in E(K_m^*)} 4 + \sum_{k=1}^{p_j} \sum_{v_j v_{jk} \in E(K_m^*)} 6 \\
&= 4 \tfrac{m(m-1)}{2} + 6T \\
&= 4E_2(K_m^*) + 6T.
\end{aligned}
$$

2.2. The Thorny Complete Bipartite Graph

Assume that we take the complete bipartite graph $K_{n,m}$ having $(n+m)$ vertices. Obviously, the eccentricities are equal to two for all the vertices of $K_{n,m}$. Then, $E_1(K_{n,m}) = 4(n+m)$ and $E_2(K_{n,m}) = 4nm$. The thorny complete bipartite graph $K_{n,m}^*$ is attained by attaching pendant vertices to each vertex of $K_{n,m}$. Let T be the total number of pendent vertices.

Theorem 2. *The first Zagreb eccentricity index and the second Zagreb eccentricity index of $K_{n,m}^*$ are given by:*

$$E_1\left(K_{n,m}^*\right) = 9(n+m) + 16T \text{ and } E_2\left(K_{n,m}^*\right) = 9nm + 12T, \text{ respectively.}$$

Proof. Suppose that $\{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$ is the vertex set of $K_{n,m}$, and let v_{ik} be the newly attached pendant vertices to v_i, $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, p_i$ and u_{jl} be the pendant vertices of u_j, $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p_l$. Then, the eccentricity of the vertices of $K_{n,m}^*$ is given by $\varepsilon_{K_{n,m}^*}(v_i) = 3$, $\varepsilon_{K_{n,m}^*}(v_{ik}) = 4$, for $i = 1, 2, \ldots n$; $k = 1, 2, \ldots, p_i$ and $\varepsilon_{K_{n,m}^*}(u_j) = 3$, $\varepsilon_{K_{n,m}^*}\left(u_{jl}\right) = 4$, for $j = 1, 2, \ldots, m$; $l = 1, 2, \ldots, p_j'$. Thus, the Zagreb eccentricity indices of $K_{n,m}^*$ are given by:

$$
\begin{aligned}
&E_1\left(K_{n,m}^*\right) = \sum_{i=1}^{n} \varepsilon^2_{K_{n,m}^*}(v_i) + \sum_{j=1}^{m} \varepsilon^2_{K_{n,m}^*}(u_j) + \sum_{i=1}^{n} \sum_{k=1}^{p_i} \varepsilon^2_{K_{n,m}^*}(v_{ik}) + \sum_{j=1}^{m} \sum_{l=1}^{p_j} \varepsilon^2_{K_{n,m}^*}(u_{jl}) \\
&= \sum_{i=1}^{n} (3)^2 + \sum_{j=1}^{m} (3)^2 + \sum_{i=1}^{n} \sum_{k=1}^{p_i} (4)^2 + \sum_{j=1}^{m} \sum_{l=1}^{p_j} (4)^2 \\
&= 9n + 9m + 16 \sum_{i=1}^{n} p_i + 16 \sum_{j=1}^{m} p_j \\
&= 9(n+m) + 16\left(\sum_{i=1}^{n} p_i + \sum_{j=1}^{m} p_j\right).
\end{aligned}
$$

The second Zagreb eccentricity index is computed as:

$$E_2(K^*_{n,m}) = \sum_{u_iv_j\in E(K^*_{n,m})} \varepsilon_{K^*_{n,m}}(u_i)\varepsilon_{K^*_{n,m}}(v_j) + \sum_{i=1}^{n}\sum_{k=1}^{p_i} \varepsilon_{K^*_{n,m}}(v_i)\varepsilon_{K^*_{n,m}}(v_{ik}) + \sum_{j=1}^{m}\sum_{l=1}^{p_j} \varepsilon_{K^*_{n,m}}(u_j)\varepsilon_{K^*_{n,m}}(u_{jl})$$
$$= \sum_{u_iv_j\in E(K^*_{n,m})} 9 + \sum_{i=1}^{n}\sum_{k=1}^{p_i} 12 + \sum_{j=1}^{m}\sum_{l=1}^{p_j} 12$$
$$= 9|E(K_{n,m})| + 12\sum_{i=1}^{n} p_i + 12\sum_{j=1}^{m} p_j.$$

2.3. The Thorny Star Graph

Suppose that we have the star graph $S_m = K_{1,(m-1)}$ of m vertices. Obviously, $E_1(S_m) = 4m-3$ and $E_2(S_m) = 2(m-1)$. Let the thorny star graph S^*_m be obtained by joining p_j pendant vertices to every vertex v_j, $j = 2,3,\ldots,m$ and p_1 pendant vertices to the central vertex v_1 of S_m.

Theorem 3. *The first Zagreb eccentricity index and the second Zagreb eccentricity index of S^*_m are given by:*

$$E_1(S^*_m) = 9m - 5 + 16T - 7p_1 \text{ and } E_2(S^*_m) = 6(m-1) - 6p_1 + 12T, \text{ respectively.}$$

Proof. Assume v_{1k}, $k = 1,2,\ldots,p_1$ and v_{jk}, for $j = 2,3,\ldots,m;\ k = 1,2,\ldots,p_j$ are the newly attached pendant vertices. Then, the eccentricities of the vertices of S^*_m are given by $\varepsilon_{S^*_m}(v_1) = 2$, $\varepsilon_{S^*_m}(v_j) = 3$, for $j = 2,3,\ldots,m$, $\varepsilon_{S^*_m}\left(v_{jk}\right) = 4$, for $j = 2,3,\ldots,m;\ k = 1,2,\ldots,p_j$, $\varepsilon_{S^*_m}(v_{1k}) = 3$, for $k = 1,2,\ldots,p_1$. Thus, the Zagreb eccentricity indices of S^*_m are

$$E_1(S^*_m) = \sum_{j=1}^{m} \varepsilon^2_{S^*_m}(v_j) + \sum_{j=1}^{m}\sum_{k=1}^{p_j} \varepsilon^2_{S^*_m}(v_{jk})$$
$$= \varepsilon^2_{S^*_m}(v_1) + \sum_{j=2}^{m} \varepsilon^2_{S^*_m}(v_j) + \sum_{k=1}^{p_1} \varepsilon^2_{S^*_m}(v_{1k}) + \sum_{j=2}^{m}\sum_{k=1}^{p_j} \varepsilon^2_{S^*_m}(v_{jk})$$
$$= (2)^2 + \sum_{j=2}^{m} (3)^2 + \sum_{k=1}^{p_1} (3)^2 + \sum_{j=2}^{m}\sum_{k=1}^{p_j} \varepsilon^2_{S^*_m}(4)^2$$
$$= 4 + 9(m-1) + 9p_1 + 16\sum_{j=2}^{m} p_j$$
$$= 9m - 5 + 9p_1 + 16\sum_{j=1}^{m} p_j - 16p_1,$$

from which we get the desired result. Now,

$$E_2(S^*_m) = \sum_{j=2}^{m} \varepsilon_{S^*_m}(v_1)\varepsilon_{S^*_m}(v_j) + \sum_{j=1}^{m}\sum_{k=1}^{p_j} \varepsilon_{S^*_m}(v_j)\varepsilon_{S^*_m}(v_{jk})$$
$$= \sum_{j=2}^{m} 6 + \sum_{k=1}^{p_1} 6 + \sum_{j=2}^{m}\sum_{k=1}^{p_j} 12$$
$$= 6(m-1) + 6p_1 + 12\sum_{j=2}^{m} p_j$$
$$= 6(m-1) + 6p_1 + 12\sum_{j=1}^{m} p_j - 12p_1,$$

and the result follows.

2.4. The Thorny Cycle

Let C_m be a cycle having m vertices and m edges. Clearly, $E_1(C_m) = E_2(C_m) = \frac{m(m-1)^2}{4}$, if m is odd and $E_1(C_m) = E_2(C_m) = \frac{m^3}{4}$, if m is even. Let C_m^* be the thorny cycle of C_m obtained by joining p_j thorns v_{jk} to each vertex v_j, $j = 1, 2, \ldots, m$ of C_m.

Theorem 4. *The first Zagreb eccentricity index and the second Zagreb eccentricity index of C_m^* are given by*

$$E_1(C_m^*) = \begin{cases} \frac{m(m+1)^2+T(m+3)^2}{4}, \; if\ m\ is\ odd \\ \frac{m(m+2)^2+T(m+4)^2}{4}, \; if\ m\ is\ even \end{cases}$$

and

$$E_2(C_m^*) = \begin{cases} \frac{(m+1)[m(m+1)+(m+3)T]}{4}, \; if\ m\ is\ odd \\ \frac{(m+2)[m(m+2)+(m+4)T]}{4}, \; if\ m\ is\ even \end{cases},$$

respectively.

Proof. The vertex eccentricities of C_m^* are given as $\varepsilon_{C_m^*}(v_j) = \frac{m+1}{2}$ and $\varepsilon_{C_m^*}(v_{jk}) = \frac{m+3}{2}$, if m is odd; $\varepsilon_{C_m^*}(v_j) = \frac{m+2}{2}$ and $\varepsilon_{C_m^*}(v_{jk}) = \frac{m+4}{2}$, if m is even; for $j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, p_j$.

Thus, when m is an odd number, the first Zagreb eccentricity index of C_m^* is

$$\begin{aligned} E_1(C_m^*) &= \sum_{j=1}^{m} \varepsilon^2_{C_m^*}(v_j) + \sum_{j=1}^{m}\sum_{k=1}^{p_j} \varepsilon^2_{C_m^*}(v_{jk}) \\ &= \sum_{j=1}^{m}\left(\frac{m+1}{2}\right)^2 + \sum_{j=1}^{m}\sum_{k=1}^{p_j}\left(\frac{m+3}{2}\right)^2 \\ &= \frac{m(m+1)^2}{4} + \frac{(m+3)^2}{4}\sum_{j=1}^{m} p_j, \end{aligned}$$

and the second Zagreb eccentricity index of C_m^* is

$$\begin{aligned} E_2(C_m^*) &= \sum_{v_jv_k \in E(C_m^*)} \varepsilon_{C_m^*}(v_j)\varepsilon_{C_m^*}(v_k) + \sum_{j=1}^{m}\sum_{k=1}^{p_j} \varepsilon_{C_m^*}(v_j)\varepsilon_{C_m^*}(v_{jk}) \\ &= \sum_{v_jv_k \in E(C_m^*)}\left(\frac{m+1}{2}\right)^2 + \sum_{j=1}^{m}\sum_{k=1}^{p_j}\left(\frac{m+1}{2}\right)\left(\frac{m+3}{2}\right) \\ &= \left(\frac{m+1}{2}\right)^2|E(C_m)| + \left(\frac{m+1}{2}\right)\left(\frac{m+3}{2}\right)\sum_{j=1}^{m} p_j \\ &= \frac{m(m+1)^2}{4} + \frac{(m+1)(m+3)}{4}T. \end{aligned}$$

Now, when m is an even number, the first Zagreb eccentricity index of C_m^* is given by

$$\begin{aligned} E_1(C_m^*) &= \sum_{j=1}^{m} \varepsilon^2_{C_m^*}(v_j) + \sum_{j=1}^{m}\sum_{k=1}^{p_j} \varepsilon^2_{C_m^*}(v_{jk}) \\ &= \sum_{j=1}^{m}\left(\frac{m+2}{2}\right)^2 + \sum_{j=1}^{m}\sum_{k=1}^{p_j}\left(\frac{m+4}{2}\right)^2 \\ &= \frac{m(m+2)^2}{4} + \frac{(m+4)^2}{4}\sum_{j=1}^{m} p_j. \end{aligned}$$

Next, we proceed for the second Zagreb eccentricity index as

$$\begin{aligned}
E_2(C_m^*) &= \sum_{v_jv_k\in E(C_m^*)} \varepsilon_{C_m^*}(v_j)\varepsilon_{C_m^*}(v_k) + \sum_{j=1}^{m}\sum_{k=1}^{p_j} \varepsilon_{C_m^*}(v_j)\varepsilon_{C_m^*}(v_{jk}) \\
&= \sum_{v_jv_k\in E(C_m^*)} \left(\tfrac{m+2}{2}\right)^2 + \sum_{j=1}^{m}\sum_{k=1}^{p_j} \left(\tfrac{m+2}{2}\right)\left(\tfrac{m+4}{2}\right) \\
&= \left(\tfrac{m+2}{2}\right)^2 |E(C_m)| + \left(\tfrac{m+2}{2}\right)\left(\tfrac{m+4}{2}\right)\sum_{j=1}^{m} p_j \\
&= \tfrac{m(m+2)^2}{4} + \tfrac{(m+2)(m+4)}{4}T.
\end{aligned}$$

2.5. The Thorny Path Graph

Consider the path graph P_m with m vertices. If m is even, then we write $m = 2n+2$, and suppose that the vertices of P_m are serially indicated by $v'_n, v'_{n-1}, \ldots, v'_2, v'_1, v'_0, v_0, v_1, v_2, \ldots, v_{n-1}, v_n$, where the centers of the path P_{2n+2} are v'_0 and v_0 having eccentricity $n+1$. If m is odd, then we write $m = 2n+1$, and we suppose that we have $v'_n, v'_{n-1}, \ldots, v'_2, v'_1, v_0, v_1, v_2, \ldots, v_{n-1}, v_n$ as the consecutive vertices of P_m, where the center of the path P_{2n+1} is v_0 having the eccentricity n. Then, the thorny path graph P_m^* is obtained from P_m by attaching p_j and p'_j pendant vertices to each v_j and v'_j $(j = 1, 2, \ldots, n)$, respectively. We define $p'_0 = 0$. Now, we will find the first Zagreb eccentricity index and the second Zagreb eccentricity index of P_m^*.

Theorem 5. *The first Zagreb eccentricity index and the second Zagreb eccentricity index of P_m^* are given by*

$$E_1(P_m^*) = \begin{cases} 2\sum_{j=0}^{n}(n+j+2)^2 + \sum_{j=0}^{n}(p_j+p'_j)(n+j+3)^2, \text{ if } m \text{ is even} \\ 2\sum_{j=0}^{n}(n+j+1)^2 + \sum_{j=0}^{n}(p_j+p'_j)(n+j+2)^2, \text{ if } m \text{ is odd} \end{cases},$$

and

$$E_2(P_m^*) = \begin{cases} 2\sum_{j=0}^{n-1}(n+j+2)(n+j+3) + \sum_{j=0}^{n}(p_j+p'_j)(n+j+2)(n+j+3)(n+2)^2, \text{ if } m \text{ is even} \\ 2\sum_{j=0}^{n-1}(n+j+1)(n+j+2) + \sum_{j=0}^{n}(p_j+p'_j)(n+j+1)(n+j+2), \text{ if } m \text{ is odd} \end{cases},$$

respectively.

Proof. If $m = 2n+2$, then all the vertices of P_m^* have eccentricities $\varepsilon_{P_m^*}(v_j) = n+j+2 = \varepsilon_{P_m^*}(v'_j)$, for $j = 0, 1, \ldots, n$; $\varepsilon_{P_m^*}(v_{jk}) = n+j+3 = \varepsilon_{P_m^*}(v'_{jk})$, for $j = 0, 1, \ldots, n$; $k = 1, 2, \ldots, p_j$. Thus, the Zagreb eccentricity indices of P_m^* are given by

$$\begin{aligned}
E_1(P_m^*) &= \sum_{j=0}^{n} \varepsilon^2_{P_m^*}(v_j) + \sum_{j=0}^{n} \varepsilon^2_{P_m^*}(v'_j) + \sum_{j=0}^{n}\sum_{k=1}^{p_j} \varepsilon^2_{P_m^*}(v_{jk}) + \sum_{j=0}^{n}\sum_{k=1}^{p'_j} \varepsilon^2_{P_m^*}(v'_{jk}) \\
&= \sum_{j=0}^{n} 2(n+j+2)^2 + \sum_{j=0}^{n} p_j(n+j+3)^2 + \sum_{j=0}^{n} p'_j(n+j+3)^2 \\
&= 2\sum_{j=0}^{n} (n+j+2)^2 + \sum_{j=0}^{n} (p_j+p'_j)(n+j+3)^2,
\end{aligned}$$

and

$$\begin{aligned}
&E_2(P_m^*) = E_2'(P_m^*) + E_2''(P_m^*)\\
&E_2'(P_m^*) = \sum_{j=0}^{n-1} \varepsilon_{P_m^*}(v_j)\varepsilon_{P_m^*}(v_{j+1}) + \varepsilon_{P_m^*}(v_0)\varepsilon_{P_m^*}(v_0') + \sum_{j=0}^{n-1} \varepsilon_{P_m^*}(v_j')\varepsilon_{P_m^*}(v_{j+1}')\\
&= \sum_{j=0}^{n-1} (n+j+2)(n+j+3) + (n+2)^2 + \sum_{j=0}^{n-1} (n+j+2)(n+j+3)\\
&= 2\sum_{j=0}^{n-1} (n+j+2)(n+j+3) + (n+2)^2.
\end{aligned}$$

In addition,

$$\begin{aligned}
&E_2''(P_m^*) = \sum_{j=0}^{n}\sum_{k=1}^{p_j} \varepsilon_{P_m^*}(v_j)\varepsilon_{P_m^*}(v_{jk}) + \sum_{j=0}^{n}\sum_{k=1}^{p_j'} \varepsilon_{P_m^*}(v_j')\varepsilon_{P_m^*}(v_{jk}')\\
&= \sum_{j=0}^{n}\sum_{k=1}^{p_j} (n+j+2)(n+j+3) + \sum_{j=0}^{n}\sum_{k=1}^{p_j'} (n+j+2)(n+j+3)\\
&= \sum_{j=0}^{n} p_j(n+j+2)(n+j+3) + \sum_{j=0}^{n} p_j'(n+j+2)(n+j+3)\\
&= \sum_{j=0}^{n} (p_j+p_j')(n+j+2)(n+j+3),
\end{aligned}$$

and the result follows.

If m is odd, then the vertices of P_m^* have the eccentricities, $\varepsilon_{P_m^*}(v_j) = n+j+1 = \varepsilon_{P_m^*}(v_j')$, for $j = 0,1,\ldots,n$; $\varepsilon_{P_m^*}(v_0) = n+1$, $\varepsilon_{P_m^*}(v_{jk}) = n+j+2 = \varepsilon_{P_m^*}(v_{jk}')$, for $j = 0,1,\ldots,n$; $k = 1,2,\ldots,p_j$ (the equalities do not apply for v_0' and v_{0j}'). Now, the Zagreb eccentricity indices of P_{2n+1}^* are given as

$$\begin{aligned}
&E_1(P_m^*) = E_1'(P_m^*) + E_1''(P_m^*)\\
&E_1'(P_m^*) = \sum_{j=1}^{n-1} \varepsilon^2_{P_m^*}(v_j) + \varepsilon^2_{P_m^*}(v_n) + \varepsilon^2_{P_m^*}(v_0) + \sum_{j=1}^{n-1} \varepsilon^2_{P_m^*}(v_j') + \varepsilon^2_{P_m^*}(v_n')\\
&= \sum_{j=1}^{n-1} (n+j+1)^2 + (2n+1)^2 + (n+1)^2 + \sum_{j=1}^{n-1} (n+j+1)^2 + (2n+1)^2\\
&= 2\sum_{j=1}^{n} (n+j+1)^2 + (n+1)^2.
\end{aligned}$$

In addition,

$$\begin{aligned}
&E_1''(P_m^*) = \sum_{j=1}^{n}\sum_{k=1}^{p_j} \varepsilon^2_{P_m^*}(v_{jk}) + \sum_{j=1}^{n}\sum_{k=1}^{p_j'} \varepsilon^2_{P_m^*}(v_{jk}') + \sum_{k=1}^{p_0} \varepsilon^2_{P_m^*}(v_{0k})\\
&= \sum_{j=1}^{n}\sum_{k=1}^{p_j} (n+j+2)^2 + \sum_{j=1}^{n}\sum_{k=1}^{p_j'} (n+j+2)^2 + \sum_{k=1}^{p_0} (n+2)^2\\
&= \sum_{j=1}^{n} (p_j+p_j')(n+j+2)^2 + p_0(n+2)^2\\
&= \sum_{j=0}^{n} (p_j+p_j')(n+j+2)^2,
\end{aligned}$$

and we get the desired result.

Now, $E_2(P_m^*) = E_2'(P_m^*) + E_2''(P_m^*)$

$$\begin{aligned}
E_2'(P_m^*) &= \sum_{j=0}^{n-1} \varepsilon_{P_m^*}(v_j)\varepsilon_{P_m^*}(v_{j+1}) + \varepsilon_{P_m^*}(v_0)\varepsilon_{P_m^*}(v_1') + \sum_{j=1}^{n-1} \varepsilon_{P_m^*}(v_j')\varepsilon_{P_m^*}(v_{j+1}') \\
&= \sum_{j=1}^{n-1} (n+j+1)(n+j+2) + (n+1)(n+2) + \sum_{j=1}^{n-1} (n+j+1)(n+j+2) \\
&= 2\sum_{j=0}^{n-1} (n+j+1)(n+j+2).
\end{aligned}$$

In addition,

$$\begin{aligned}
E_2''(P_m^*) &= \sum_{j=0}^{n}\sum_{k=1}^{p_j} \varepsilon_{P_m^*}(v_j)\varepsilon_{P_m^*}(v_{jk}) + \sum_{j=1}^{n}\sum_{k=1}^{p_j'} \varepsilon_{P_m^*}(v_j')\varepsilon_{P_m^*}(v_{jk}') \\
&= \sum_{k=1}^{p_0} \varepsilon_{P_m^*}(v_0)\varepsilon_{P_m^*}(v_{0k}) + \sum_{j=1}^{n}\sum_{k=1}^{p_j} \varepsilon_{P_m^*}(v_j)\varepsilon_{P_m^*}(v_{jk}) + \sum_{j=1}^{n}\sum_{k=1}^{p_j'} \varepsilon_{P_m^*}(v_j')\varepsilon_{P_m^*}(v_{jk}') \\
&= \sum_{k=1}^{p_0} (n+1)(n+2) + \sum_{j=1}^{n}\sum_{k=1}^{p_j} (n+j+1)(n+j+2) + \sum_{j=1}^{n}\sum_{k=1}^{p_j'} (n+j+1)(n+j+2) \\
&= p_0(n+1)(n+2) + \sum_{j=1}^{n} (p_j+p_j')(n+j+1)(n+j+2) \\
&= \sum_{j=0}^{n} (p_j+p_j')(n+j+1)(n+j+2),
\end{aligned}$$

and we obtain the equality.

3. Conclusions

In this article we computed closed formulas for computing first Zagreb eccentricity index as well as second Zagreb eccentricity index for thorny graphs of important families of graphs like complete graph, complete bipartite graph, cycle, star and path. These relations are given in Theorems 1–4. Moreover, it can be observed from these formulas that values of these indices increase by increasing the number of vertices and number of thorns attached to graphs. These invariants have applications in computational chemistry.

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