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A Hybrid Fuzzy DEA/AHP Methodology for Ranking Units in a Fuzzy Environment

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Abstract: In this paper, a novel approach combining fuzzy data envelopment analysis (DEA) and the analytical hierarchical process (AHP) is proposed to rank units with multiple fuzzy criteria. The hybrid fuzzy DEA/AHP approach derives the AHP pairwise comparisons by fuzzy DEA and utilizes AHP to fully rank units. It shows that the proposed approach generates a logical ranking of units that has perfect compatibility with fuzzy DEA ranking and there is no any form of subjective analysis engaged within the methodology. A study on the facility layout design in manufacturing systems is provided to illustrate the superiority of the proposed approach and show the compatibility between the proposed approach and fuzzy DEA ranking.

Keywords: data envelopment analysis (DEA); fuzzy optimization; analytical hierarchical process (AHP); ranking

1. Introduction

Ranking organizational units is a subject of tremendous interest in multi-criteria decision-making (MCDM) which has been developed rapidly for dealing with complex decision-making problems. Of the MCDM approaches, the analytic hierarchy process (AHP) [1] is particularly suitable for modeling qualitative criteria and has found extensive applications in selection, evaluation, planning and development, decision-making, and so on [2,3]. AHP utilizes pairwise comparisons between criteria and between units, assessed subjectively by the decision maker, to rank the units overall [4]. In spite of completely ranking units in AHP, the process of making pairwise comparison matrix is based on experts' choices, and it causes error and inconsistency in the resulting matrix [5].

The relationship between MCDM and data envelopment analysis (DEA) was highlighted by many researchers [4,6,7]. It has been recognized for more than a decade that the MCDM and DEA formulations coincide if inputs and outputs can be viewed as criteria for performance evaluation, with minimization of inputs and/or maximization of outputs as associated objectives [8,9]. DEA is a linear programming methodology to measure the efficiency of all decision-making units (DMUs). Since the pioneering work by Charnes et al. [10], DEA has developed in many directions and in numerous applications [11]. All the basic models in DEA divide DMUs into two groups: efficient DMUs and inefficient DMUs, and lack of discrimination of efficient units is a serious problem. There have been attempts to fully rank units in the context of DEA during the last decade. As highlighted by Friedman and Sinuany-Stern [12], each ranking method in the context of DEA has its limitations. Some are based on subjective data and others are limited to part of the units, yet none provides an ultimately good model for fully ranking units in the DEA context [4].

Another attempt to fully rank DMUs in DEA utilizes AHP. Shang and Sueyoshi [13] used the subjective AHP results in DEA for selection of a flexible manufacturing system. However, this approach has the limitations of both methods, the subjectivity of AHP and the Pareto solutions of DEA. Sinuany-Stern et al. [4] proposed a combined method of AHP and DEA for ranking units. The AHP pairwise comparisons are generated by running pairwise DEA. Thus, there is no subjective evaluation [4]. This method was a new idea in ranking but has lots of problems. The most principal problem is that its ranking is incompatible with the traditional model in DEA when there are multiple inputs and outputs, and this incompatibility causes some efficient units to be ranked lower than inefficient units. To cover the incompatibility in [4], newly developed AHP/DEA approaches were studied by some researchers [5,14]. Table 1 summaries the integrated AHP/DEA approaches proposed in literature.

Table 1. Literature on the integrated AHP/DEA approaches.

Overview	Advantage	Disadvantage
Shang and Sueyoshi [13]	This work attempts to fully rank DMUs in DEA utlilizing AHP.	It includes the subjectivity of AHP and the Pareto solutions of DEA.
Sinuany-Stern et al. [4]	The AHP pairwise comparisons are generated by running pairwise DEA. Thus, there is no subjective evaluation.	Its ranking is incompatible with traditional model in DEA when there are multiple inputs and outputs.
Alirezaee and Sani [14]	This approach overcomes the draw-backs of the AHP/DEA method developed in [4].	The integrated AHP/DEA models can not reflect the vagueness of human thought while ranking units with multiple fuzzy criteria.
Rakhshan et al. [5]	The proposed approach generates the ranking of units which is compatible with traditional DEA ranking.	It has the limitation on dealing with human thoughts with uncertainty in the real-world applications.

Due to the vagueness involved in the real-world decision-making problems, different fuzzy modeling approaches are introduced to deal with human thoughts with uncertainty in various fields [15–18]. The integrated fuzzy AHP and DEA methodologies are developed to reflect human thoughts with uncertainty for performance evaluation and assessment of units in a fuzzy environment [19–21]. These approaches employ fuzzy AHP to reflect the vagueness of human thought for allocating the relative importance of criteria and using DEA to measure the relative efficiency of DMUs, which leads to weak discrimination of efficient units. Inspired by the recent research, this work considers combining fuzzy DEA and AHP for fully ranking units with multiple fuzzy criteria. The proposed approach takes the best of both fuzzy DEA and AHP methods by avoiding the pitfalls of each. The AHP pairwise comparisons are derived mathematically from the results of the average of efficiencies by running fuzzy DEA models, and DMUs are fully ranked by AHP. It shows that the proposed approach presents a logical ranking of DMUs that is compatible with the efficient/inefficient classification derived from fuzzy DEA. The rest of the paper is organized as follows. In Section 2, a hybrid fuzzy DEA/AHP approach for fully ranking DMUs with multiple fuzzy criteria is presented. An algorithm of the proposed approach and its validation are provided in Section 3. A study on the facility layout design application is included to illustrate the superiority of the proposed approach in Section 4. The paper is concluded in Section 5.

2. The Methodology

The hybrid fuzzy DEA/AHP approach consists of two stages for ranking DMUs with multiple fuzzy criteria. In the first stage of the ranking method, fuzzy DEA is employed to construct the pairwise comparisons of AHP based on results of the average of efficiencies. In the second stage of the ranking method, a single level AHP is utilized to fully rank units according to the pairwise evaluation matrix generated in the first stage. Section 2.1 presents the construction of pairwise comparisons of AHP by fuzzy DEA and Section 2.2 presents the ranking by AHP.

2.1. Construction of Pairwise Comparisons of AHP

In the proposed approach, the AHP pairwise comparisons are derived from the results of average of efficiencies, which is the average of any DMUs before and after removing other DMUs [14]. To obtain the efficiency of DMUs, the input-orientated fuzzy CCR model named after Charnes, Cooper, and Rhodes [10] is employed. Suppose there are m inputs, s outputs and n DMUs being evaluated. Denote $\tilde{x}_{i,j}$ as the i-th fuzzy input and $\tilde{y}_{r,j}$ as the r-th fuzzy output of the j-th DMU. The input-oriented CCR model with multiple fuzzy criteria for measuring the efficiency of DMU_p can be formulated as:

$$\begin{aligned} FE^p = \min \quad & \theta_p \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j \tilde{x}_{i,j} \leq \theta_p \tilde{x}_{i,p}, \forall\, i = 1, 2, \cdots, m, \\ & \sum_{j=1}^{n} \lambda_j \tilde{y}_{r,j} \geq \tilde{y}_{r,p}, \forall\, r = 1, 2, \cdots, s \\ & \lambda_j \geq 0,\ \forall\, j = 1, 2, \cdots, n, \end{aligned} \tag{1}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ is a column vector of a linear combination of n DMUs and θ_p is the efficiency score of DMU_p. The interpretation of the input-oriented fuzzy CCR model can be found in many references [22]. In the input-oriented fuzzy CCR model, an efficiency score is generated for a DMU by minimizing inputs with fixed outputs and for each observed DMU_p an imaginary composite unit is constructed that outperforms DMU_p. In addition, λ_j represents the proportion to which $DMU_j, j = 1, 2, \cdots, n$, is used to construct the composite unit for DMU_p. In problem (1), the composite unit produces inputs that are at most equal to a proportion θ_p of the inputs of DMU_p with $0 < \theta_p \leq 1$ and consumes at least the same levels of outputs as DMU_p [23]. If $\theta_p < 1, DMU_p$ is not efficient and the parameter θ_p indicates the extent by which DMU_p has to decrease its inputs to become efficient. Moreover, it could be easily checked that problem (1) has a feasible solution when $\theta_p = 1, \lambda_j = 0$ for $j \neq p$, and $\lambda_p = 1$.

Since the input and output variables in problem (1) are not known precisely, problem (1) cannot be solved by a standard linear programming solver. The decision maker may define the risk-free and impossible bounds for each fuzzy input and output variables for transforming the fuzzy optimization problem (1) to a traditional linear program. Risk-free bounds are interpreted as the conservative values that are most realistically found, whereas the impossible bounds are associated with the values that are the least realistic. For each fuzzy input and output variable, the change from its risk-free to impossible bounds is represented by its membership function. It is assumed that membership functions of each fuzzy input and output variable are monotonically linear, and are equal to zero, if the input or output bounds are impossible, and are equal to one if they are risk free. Suppose that $x_{i,j}^L$ and $x_{i,j}^U$ represent the impossible and risk-free bounds of the i-th fuzzy input of the j-th DMU, respectively. A possible linear membership function associated with the i-th fuzzy input for the j-th DMU is given by:

$$\mu_{\tilde{x}_{i,j}}(x) = \frac{x_{i,j}^L - x}{x_{i,j}^L - x_{i,j}^U}, \text{ where } x_{i,j}^L \leq x \leq x_{i,j}^U,\ i = 1, 2, \cdots, m, j = 1, 2, \cdots, n. \tag{2}$$

Suppose also that $y^L_{r,j}$ and $y^U_{r,j}$ represent the risk-free and impossible bounds of the r-th fuzzy output of the j-th DMU, respectively. A possible linear membership function associated with the r-th fuzzy output for the j-th DMU is given by:

$$\mu_{\tilde{y}_{r,j}}(y) = \frac{y - y^U_{r,j}}{y^L_{r,j} - y^U_{r,j}}, \text{ where } y^L_{r,j} \leq y \leq y^U_{r,j},\ r = 1, 2, \cdots, s, j = 1, 2, \cdots, n. \tag{3}$$

According to (2) and (3), $\tilde{x}_{i,j}$ and $\tilde{y}_{r,j}$ can be replaced by the new variables $\mu_{\tilde{x}_{i,j}}$ and $\mu_{\tilde{y}_{r,j}}$, which locate the levels of inputs and outputs within the impossible and risk-free bounds, as follows:

$$\tilde{x}_{i,j} = x^L_{i,j} + \mu_{\tilde{x}_{i,j}}(x^U_{i,j} - x^L_{i,j}), i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \text{ with } 0 \leq \mu_{\tilde{x}_{i,j}} \leq 1,$$

$$\tilde{y}_{r,j} = y^U_{r,j} + \mu_{\tilde{y}_{r,j}}(y^L_{r,j} - y^U_{r,j}), r = 1, 2, \cdots, s, j = 1, 2, \cdots, n \text{ with } 0 \leq \mu_{\tilde{y}_{r,j}} \leq 1.$$

With these transformations, we have:

$$\begin{aligned} FE^p = \min \quad & \theta_p \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j(x^L_{i,j} + (x^U_{i,j} - x^L_{i,j})\mu_{\tilde{x}_{i,j}}) \leq \theta_p(x^L_{i,j} + (x^U_{i,j} - x^L_{i,j})\mu_{\tilde{x}_{i,j}}), \forall\, i = 1, 2, \cdots, m, \\ & \sum_{j=1}^{n} \lambda_j(y^U_{r,j} + \mu_{\tilde{y}_{r,j}}(y^L_{r,j} - y^U_{r,j})) \geq y^U_{r,p} + \mu_{\tilde{y}_{r,p}}(y^L_{r,p} - y^U_{r,p}), \forall\, r = 1, 2, \cdots, s, \\ & \lambda_j \geq 0,\ \forall\, j = 1, 2, \cdots, n. \end{aligned} \tag{4}$$

According to [24], the fuzzy decision of (4) can be reached with the membership degree:

$$\mu = \min_{\substack{i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, n \\ r = 1, 2, \cdots, s}} \{\mu_{\tilde{x}_{i,j}}, \mu_{\tilde{y}_{r,j}}\}.$$

For a specific $\mu \in [0, 1]$, problem (4) is rewritten as the linear programming problem:

$$\begin{aligned} FE^p = \min \quad & \theta_p \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j(x^L_{i,j} + (x^U_{i,j} - x^L_{i,j})\mu) \leq \theta_p(x^L_{i,j} + (x^U_{i,j} - x^L_{i,j})\mu), \forall\, i = 1, 2, \cdots, m, \\ & \sum_{j=1}^{n} \lambda_j(y^U_{r,j} + \mu(y^L_{r,j} - y^U_{r,j})) \geq y^U_{r,p} + \mu(y^L_{r,p} - y^U_{r,p}), \forall\, r = 1, 2, \cdots, s \\ & \lambda_j \geq 0,\ \forall\, j = 1, 2, \cdots, n. \end{aligned} \tag{5}$$

The efficiency of $DMU_p, p = 1, 2, \cdots, n$, can then be obtained by solving the linear programming problem (5).

Moreover, define $FE^p_k, p = 1, 2, \cdots, n, k = 1, 2, \cdots, n$, the efficiency of DMU_p after removing the k-th DMU from the observation set of DMUs, which can be described by the fuzzy DEA model:

$$\begin{aligned} FE^p_k = \min \quad & \theta_p \\ \text{s.t.} \quad & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j \tilde{x}_{i,j} \leq \theta_p \tilde{x}_{i,p}, \forall\, i = 1, 2, \cdots, m, \\ & \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j \tilde{y}_{r,j} \geq \tilde{y}_{r,p}, r = 1, 2, \cdots, s, \\ & \lambda_j \geq 0,\ \forall\, j = 1, 2, \cdots, n. \end{aligned} \tag{6}$$

As similar to the discussion on solving problem (1), for $\mu \in [0,1]$, the efficiency of DMU_p after removing the k-th DMU defined in (6) can be obtained by solving the linear programming problem:

$$\begin{aligned} FE_k^p = \min \quad & \theta_p \\ \text{s.t.} \quad & \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j (x_{i,j}^L + (x_{i,j}^U - x_{i,j}^L)\mu) \leq \theta_p (x_{i,j}^L + (x_{i,j}^U - x_{i,j}^L)\mu), \forall\, i = 1,2,\cdots,m, \\ & \sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j (y_{r,j}^U + \mu(y_{r,j}^L - y_{r,j}^U)) \geq y_{r,p}^U + \mu(y_{r,p}^L - y_{r,p}^U), \forall\, r = 1,2,\cdots,s, \\ & \lambda_j \geq 0,\ \forall\, j = 1,2,\cdots,n, \end{aligned} \tag{7}$$

where $x_{i,j}^L$ and $x_{i,j}^U$ represent the impossible and risk-free bounds of the i-th fuzzy input of the j-th DMU, respectively, and $y_{r,j}^L$ and $y_{r,j}^U$ represent the risk-free and impossible bounds of the r-th fuzzy output of the j-th DMU, respectively. For each $p = 1,2,\cdots,n$, we solve problem (7) and obtain the efficiency of unit after removing the k-th DMU, $k = 1,2,\cdots,n$. Table 2 lists the results.

Table 2. Efficiency of units after removing the k-th DMU.

DMUs	**1**	**2**	**3**	$\cdots$	n
Remove 1	$*$	FE_1^2	FE_1^3	$\cdots$	FE_1^n
Remove 2	FE_2^1	$*$	FE_2^3	$\cdots$	FE_2^n
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Remove n	FE_n^1	FE_n^2	FE_n^3	$\cdots$	$*$

"$*$" denotes that FE_k^k, $k = 1, 2, \cdots, n$, does not exist.

Using the results in Table 2, for each pair of DMU i and j, the priority of DMU i to DMU j is defined as [14]:

$$a_{ij} = \frac{\sum\limits_{\substack{s=1 \\ s\neq i,j}}^{n} FE_s^i + FE^i}{\sum\limits_{\substack{s=1 \\ s\neq i,j}}^{n} FE_s^j + FE^j},\ i,j = 1,2,\cdots,n;\ a_{ii} = 1;\ a_{ji} = \frac{1}{a_{ij}}, \tag{8}$$

where FE_s^i and FE_s^j are the efficiencies of the i-th and j-th DMU after removing the s-th DMU, respectively; FE^i and FE^j are efficiencies of the i-th and j-th DMU obtained by solving problem (5), respectively. It should be noticed that the priority of the i-th DMU to the j-th DMU, a_{ij}, defined in (8) is the ratio of two averages. The first one is the average of efficiencies of the i-th DMU after removing the s-th DMU $(s = 1,2,\cdots,n; k \neq i,j)$ and the efficiency of the i-th DMU. The second average is a similar average of efficiencies of the j-th DMU after removing the s-th DMU $(s = 1,2,\cdots,n; k \neq i,j)$ and the efficiency of the j-th DMU.

The pairwise comparison matrix $A = (a_{ij})$ generated by (8) is utilized to fully rank DMUs in the second stage of the proposed method via a single level AHP.

2.2. Ranking with AHP

The AHP is a method developed for subjective evaluation of a set of alternatives based on multiple criteria. It provides a structured framework for setting priorities on each level of the hierarchy using pairwise comparisons that are subjective created by decision makers. Let A be a pairwise comparison matrix of AHP:

$$A = (a_{ij})_{n\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix},$$

where a_{ij} denotes pairwise comparison between criteria/units, $a_{ii} = 1$ and $a_{ij} = \dfrac{1}{a_{ji}}$ for every $i, j = 1, 2, \cdots, n$. If $a_{ij} = a_{ik} \cdot a_{kj}$ for $i, j, k = 1, 2, \cdots, n$ in a pairwise comparison matrix of $A = (a_{ij})_{n\times n}$, then $A = (a_{ij})_{n\times n}$ is completely compatible; otherwise, it is said to be incompatible [5].

Theorem 1. *[25] Suppose that $A = (a_{ij})_{n\times n}$ is a pairwise comparison matrix, if $A = (a_{ij})_{n\times n}$ is compatible, then its maximal eigenvalue $\lambda_{max} = n$. Otherwise $\lambda_{max} > n$.*

According to Theorem 1, a pairwise comparison matrix, for which its maximum eigenvalue is closer to the matrix dimension, has less incompatibility [5].

With the pairwise comparison matrix A constructed in the first stage of the proposed, a single hierarchical level AHP is run to determine the weight vector $\mathbf{w} = (w_1, \cdots, w_n)^T$ by the following characteristic equation:

$$A\mathbf{w} = \lambda_{max}\mathbf{w}, \tag{9}$$

where λ_{max} is the maximal eigenvalue of A. Such a method for determining the weight vector of a pairwise comparison matrix is referred to as the principal right eigenvector method [1]. The j-th component of $\mathbf{w}$ reflects the relative importance given to the j-th DMU. The DMU that has higher corresponding value of w_j has higher ranking. We assign the rank 1 to the DMU with the maximal value of w_j.

3. An Algorithm and the Validation of the Hybrid Fuzzy DEA/AHP Method

Based on the discussion in the previous section, an algorithm of the ranking method by the hybrid fuzzy DEA/AHP can be organized as below (Algorithm 1).

Algorithm 1: The hybrid fuzzy DEA/AHP ranking method

Step 1. Construct the pairwise comparison matrix by fuzzy DEA.

- **Step 1.1** Decision makers provide the risk-free and impossible bounds for each fuzzy criterion and assign the value of $\mu \in [0, 1]$.
- **Step 1.2** Solve problem (5) and obtain the efficiency of DMU_p, FE^p, $p = 1, 2, \cdots, n$.
- **Step 1.3** Solve problem (7) and obtain the efficiency of DMU_p after removing the k-th DMU, FE^p_k, $k = 1, 2, \cdots, n$, $p = 1, 2, \cdots, n$.
- **Step 1.4** Construct the pairwise comparison matrix $A = (a_{ij})_{n\times n}$ by Equation (8) using the results obtained in Steps 1.2 and 1.3.

Step 2. Rank units by AHP

- **Step 2.1** Solve Equation (9) based on the pairwise comparison matrix $A = (a_{ij})_{n\times n}$ generated in Step 1 and obtain the weight vector $\mathbf{w} = (w_1, \cdots, w_n)^T$.
- **Step 2.2** Assign the rank 1 to the DMU with the maximal value of w_j and stop. (The DMU which has higher corresponded value of w_j has higher ranking.)

The flow chart with the steps of the proposed algorithm is presented in Figure 1.

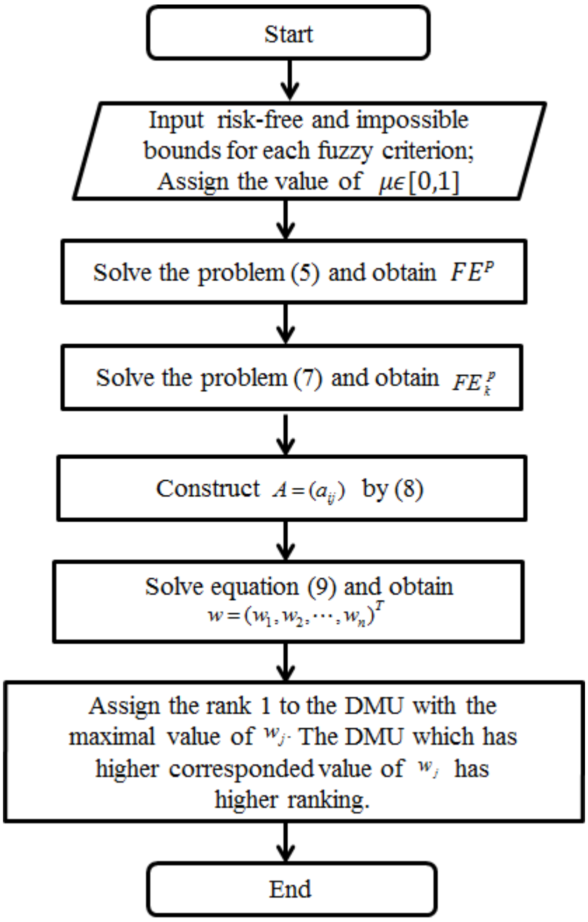

Figure 1. The flow chart with the steps of the proposed algorithm.

To show that there is perfect compatibility between the rank derived from the proposed method and efficient/inefficient classification derived from fuzzy DEA, we have the following result.

Theorem 2. *If DMU_i is efficient and DMU_j is inefficient according to the results obtained by the fuzzy DEA model (1), and w_i and w_j are corresponding weights obtained by the hybrid fuzzy DEA/AHP method, then $w_i > w_j$.*

Proof. To show that the weights $w_i > w_j$ with DMU_i are efficient and DMU_j is inefficient, we have to prove that $a_{ik} \geq a_{jk}, k = 1, 2, \cdots, n$, in the pairwise comparison matrix and for at least one $k, k = 1, 2, \cdots, n$, it is a restrict inequality [26]. □

For $\mu \in [0, 1]$, problems (5) and (7) can be regarded as traditional input-oriented CCR models with $x^L_{i,j} + (x^U_{i,j} - x^L_{i,j})\mu$ as the i-th input and $y^U_{r,j} + \mu(y^L_{r,j} - y^U_{r,j})$ as the r-th fuzzy output of the j-th DMU.

Denote $FE^i, i = 1, 2, \cdots, n$, the efficiency of the i-th DMU obtained by solving the problem (5) and FE^i_s the efficiency of the i-th DMU after removing the s-th DMU obtained by solving problem (7). Since an efficient DMU is efficient after removing other DMUs, for every efficient DMU_i, $FE^i_s = FE^i, s = 1, 2, \cdots, n$, and $s \neq i$ [14]. Therefore, for an efficient DMU_i and an inefficient DMU_j, we have:

$$\sum_{\substack{s=1 \\ s \neq j,k}}^{n} FE^j_s + FE^j \leq \sum_{\substack{s=1 \\ s \neq i,k}}^{n} FE^i_s + FE^i. \tag{10}$$

Moreover, since an efficient frontier does not change after eliminating an inefficient DMU_j, for an efficient DMU_i and an inefficient DMU_j, we have:

$$FE_j^k = FE^k \leq FE_i^k,$$

where $k \neq j$. It implies that:

$$\frac{1}{\sum\limits_{\substack{s=1 \\ s\neq i,k}}^{n} FE_s^k + FE^k} = \frac{1}{(FE_1^k+\cdots+FE_j^k+\cdots+FE_n^k)+FE^k} \geq \frac{1}{(FE_1^k+\cdots+FE_i^k+\cdots+FE_n^k)+FE^k} = \frac{1}{\sum\limits_{\substack{s=1 \\ s\neq j,k}}^{n} FE_s^k + FE^k}. \tag{11}$$

Combining (10) and (11), we have:

$$a_{ik} = \frac{\sum\limits_{\substack{s=1 \\ s\neq i,k}}^{n} FE_s^i + FE^i}{\sum\limits_{\substack{s=1 \\ s\neq i,k}}^{n} FE_s^k + FE^k} \geq \frac{\sum\limits_{\substack{s=1 \\ s\neq j,k}}^{n} FE_s^j + FE^j}{\sum\limits_{\substack{s=1 \\ s\neq j,k}}^{n} FE_s^k + FE^k} = a_{jk}, k = 1, 2, \cdots, n. \tag{12}$$

Moreover, since DMU_i is efficient and DMU_j is inefficient,

$$\sum_{\substack{s=1 \\ s\neq i,j}}^{n} FE_s^i + FE^i > \sum_{\substack{s=1 \\ s\neq i,j}}^{n} FE_s^j + FE^j.$$

Consequently, we have:

$$a_{ij} = \frac{\sum\limits_{\substack{s=1 \\ s\neq i,j}}^{n} FE_s^i + FE^i}{\sum\limits_{\substack{s=1 \\ s\neq i,j}}^{n} FE_s^j + FE^j} > 1 = \frac{\sum\limits_{\substack{s=1 \\ s\neq i,j}}^{n} FE_s^j + FE^j}{\sum\limits_{\substack{s=1 \\ s\neq i,j}}^{n} FE_s^j + FE^j} = a_{jj}. \tag{13}$$

Equations (12) and (13) imply that $w_i > w_j$.

According to Theorem 2, the proposed fuzzy DEA/AHP method ranks the efficient DMUs in a better position than the inefficient DMUs. Therefore, there is perfect compatibility between the rank derived from the proposed ranking method and efficient/inefficient classification derived from fuzzy DEA. In other words, the integrated fuzzy DEA/AHP method ranks efficient DMUs, which are not ranked by fuzzy DEA, and also ranks inefficient DMUs, assuring at the same time that efficient DMUs have the better position than the inefficient DMUs.

4. An Illustrated Example on the Facility Layout Design Application

To illustrate the idea of the proposed approach, the modified real data set of a case study for evaluating the facility layout designs of the plastic profile production system from the study of [27] is utilized. The effective facility layout evaluation procedure considers fuzzy criteria, e.g., flexibility in volume and variety and quality related to the product and production, as well as quantitative criteria such as material handling cost and adjacency score, shape ratio, and material handling vehicle utilization in the decision process.

Eighteen facility layout alternatives are evaluated using four output variables and two input variables in the illustrated example. To take care of the multiple criteria by fuzzy DEA, the criteria that are to be minimized are viewed as inputs and the criteria to be maximized are considered as outputs. The two inputs are material handling cost (x_1) and adjacency scores (x_2). The four outputs are shape ratio (y_1), flexibility ($\tilde{y}_2$), quality ($\tilde{y}_3$) and utilization of material-handling vehicle (y_4). Table 3 provides inputs and outputs data for the example.

Table 3. Input and output data.

DMU	Inputs		Outputs			
j	x_{1j}	x_{2j}	y_{1j}	$\tilde{y}_{2j}$	$\tilde{y}_{3j}$	y_{4j}
1	20,309.56	6405	0.4697	[113, 123]	[410, 425]	30.89
2	20,411.22	5393	0.4380	[337, 360]	[484, 510]	31.34
3	20,280.28	5294	0.4392	[308, 330]	[653, 680]	30.26
4	20,053.20	4450	0.3776	[245, 265]	[638, 660]	28.03
5	19,998.75	4370	0.3526	[856, 880]	[484, 510]	25.43
6	20,193.68	4393	0.3674	[717, 760]	[361, 380]	29.11
7	19,779.73	2862	0.2854	[245, 260]	[846, 880]	25.29
8	19,831.00	5473	0.4398	[113, 130]	[129, 140]	24.80
9	19,608.43	5161	0.2868	[674, 690]	[724, 750]	24.45
10	20,038.10	6078	0.6624	[856, 880]	[653, 675]	26.45
11	20,330.68	4516	0.3437	[856, 870]	[638, 675]	29.46
12	20,155.09	3702	0.3526	[856, 880]	[846, 860]	28.07
13	19,641.86	5726	0.2690	[337, 360]	[361, 380]	24.58
14	20,575.67	4639	0.3441	[856, 885]	[638, 650]	32.20
15	20,687.50	5646	0.4326	[337, 360]	[452, 470]	33.21
16	20,779.75	5507	0.3312	[856, 880]	[653, 675]	33.60
17	19,853.38	3912	0.2847	[245, 260]	[638, 660]	31.29
18	19,853.38	5974	0.4398	[337, 360]	[179, 190]	25.12

To evaluate the facility layout alternatives by the proposed fuzzy DEA/AHP approach, pairwise comparisons of AHP are derived from the results of average of efficiencies by running fuzzy DEA. In Step 1 of the proposed algorithm, the fuzzy DEA model (5) is applied to obtain the efficiency of $DMU_p, FE^p, p = 1, 2, \cdots, 18$. For instance, to measure the efficiency of DMU_1, we have

$$\begin{aligned} FE^1 = \min \quad & \theta_1 \\ \text{s.t.} \quad & 20309.56\lambda_1 + 20411.22\lambda_2 + \cdots + 19853.38\lambda_{18} \leq 20309.56\theta_1 \\ & 6405\lambda_1 + 5393\lambda_2 + \cdots + 5974\lambda_{18} \leq 6405\theta_1 \\ & 0.4697\lambda_1 + 0.4380\lambda_2 + \cdots + 0.4398\lambda_{18} \geq 0.4697 \\ & (10\mu - 123)\lambda_1 + (23\mu - 360)\lambda_2 + \cdots + (23\mu - 360)\lambda_{18} \leq 10\mu - 123 \\ & (15\mu - 425)\lambda_1 + (26\mu - 510)\lambda_2 + \cdots + (11\mu - 190)\lambda_{18} \leq 15\mu - 425 \\ & 30.89\lambda_1 + 31.34\lambda_2 + \cdots + 25.12\lambda_{18} \geq 30.89 \\ & \lambda_1, \lambda_2, \cdots, \lambda_{18} \geq 0. \end{aligned} \tag{14}$$

The efficiencies of $DMU_p, FE^p, p = 1, 2, \cdots, 18$ are listed in the second column of Table 4.

To obtain the efficiency of DMU_p after removing the k-th DMU, $p = 1, 2, \cdots, 18, k = 1, 2, \cdots, 18$, the DEA model (7) is applied in Step 1.3 of the proposed algorithm. For instance, to measure the efficiency of DMU_2 after removing DMU_1, the following problem is considered:

$$\begin{aligned} FE_1^2 = \min \quad & \theta_2 \\ \text{s.t.} \quad & 20411.22\lambda_+ 20280.28\lambda_3 + \cdots + 19853.38\lambda_{18} \leq 20411.22\theta_2 \\ & 5393\lambda_2 + 5294\lambda_3 + \cdots + 5974\lambda_{18} \leq 5393\theta_2 \\ & 0.4380\lambda_2 + 0.4392\lambda_2 + \cdots + 0.4398\lambda_{18} \geq 0.4380 \\ & (23\mu - 360)\lambda_2 + (22\mu - 330)\lambda_3 + \cdots + (23\mu - 360)\lambda_{18} \leq 23\mu - 360 \\ & (26\mu - 510)\lambda_2 + (27\mu - 680)\lambda_3 + \cdots + (11\mu - 190)\lambda_{18} \leq 26\mu - 510 \\ & 31.34\lambda_2 + 30.26\lambda_3 + \cdots + 25.12\lambda_{18} \geq 31.34 \\ & \lambda_2, \cdots, \lambda_{18} \geq 0. \end{aligned} \tag{15}$$

Table 5 shows the efficiency of DMU_p after removing the k-th DMU, $FE_k^p, p = 1, 2, \cdots, 18$, $k = 1, 2, \cdots, 18$.

Using the results in Table 3 and the second column of Table 4, the pairwise comparison matrix $A = (a_{ij})_{18\times18}$ can be constructed by (8) as shown in Table 6.

In Step 2 of the proposed algorithm, the pairwise comparison matrix $A = (a_{ij})_{18\times18}$ is utilized to fully rank DMUs by a single level AHP. The weight vector obtained by solving (9) is listed in the last column of Table 4 with the maximum eigenvalue $\lambda_{max} = 18.0000$. According to Theorem 1, the pairwise comparison matrix in Table 6 is almost compatible.

Table 4 shows the results of ranking by the fuzzy DEA model (1), the method in [27] and the proposed ranking method by the hybrid fuzzy DEA/AHP. The numbers in parentheses are rankings of the corresponding DMUs. The second column of Table 4 shows the efficiency scores, $FE^i, i = 1, 2, \cdots, n$, calculated from the fuzzy DEA model (1). There are 12 efficient DMUs which cannot be differentiated. The third column of Table 4 shows efficiency scores calculated from the model (3) in [27]. There are eight efficient DMUs that cannot be differentiated. The last column of Table 4 shows the weight vector obtained by the proposed hybrid fuzzy DEA/AHP ranking method. According to Table 4, the proposed ranking method leads to more discrimination and the DMU_5 has the best rank of all DMUs by the hybrid fuzzy DEA/AHP approach. If we examine Table 5, we can see that DMU_5 has more effect on DMU_4, DMU_8, DMU_9 and DMU_{18}, so it has the best rank in comparison to other DMUs. In addition, from Table 4, DMUs that are efficient in the fuzzy DEA rank higher than inefficient DMUs by using the proposed ranking method.

Table 4. Ranking by different methods.

DMUs	Fuzzy DEA	Method in [27]	Weight Vector in Fuzzy DEA/AHP
1	0.9989(13)	0.985(12)	0.2419804 (13)
2	1.0000(1)	0.988(11)	0.2422704 (4)
3	1.0000(1)	0.997(10)	0.2422452 (5)
4	0.9595(14)	0.949(14)	0.2350762 (14)
5	1.0000(1)	1.000(1)	0.2424019 (1)
6	1.0000(1)	0.973(13)	0.2422439 (6)
7	1.0000(1)	1.000(1)	0.2421172 (10)
8	0.8617(16)	0.857(16)	0.2138516 (17)
9	0.9098(15)	0.889(15)	0.2250112 (15)
10	1.0000(1)	1.000(1)	0.2423731 (2)
11	1.0000(1)	0.998(9)	0.2421616 (7)
12	1.0000(1)	1.000(1)	0.2421571 (8)
13	0.7901(18)	0.776(18)	0.1945052 (18)
14	1.0000(1)	1.000(1)	0.2420896 (12)
15	1.0000(1)	1.000(1)	0.2422820 (3)
16	1.0000(1)	1.000(1)	0.2421566 (9)
17	1.0000(1)	1.000(1)	0.2420933 (11)
18	0.8611(17)	0.852(17)	0.2190128 (16)

From the experimental results, we see that the proposed method leads to a more discrimination ranking of units with multiple fuzzy criteria. Moreover, it also shows the perfect compatibility between the proposed ranking method by hybrid fuzzy DEA/AHP and the fuzzy DEA.

Table 5. The efficiencies of DMUs after removing the k-th DMU, $k = 1, 2, \cdots, 18$.

DMUs	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Remove 1	$*$	1	1	0.9595	1	1	1	0.8617	0.9098	1	1	1	0.7901	1	1	1	1	0.8611
Remove 2	1.0000	$*$	1	1.0000	1	1	1	0.8715	0.9265	1	1	1	0.8130	1	1	1	1	1.0000
Remove 3	1.0000	1	$*$	0.9703	1	1	1	0.8889	0.9138	1	1	1	0.8072	1	1	1	1	1.0000
Remove 4	0.9989	1	1	$*$	1	1	1	0.8617	0.9098	1	1	1	0.7901	1	1	1	1	0.8611
Remove 5	1.0000	1	1	0.9983	$*$	1	1	1.0000	0.9513	1	1	1	0.8088	1	1	1	1	1.0000
Remove 6	1.0000	1	1	0.9769	1	$*$	1	0.8637	0.9322	1	1	1	0.8074	1	1	1	1	1.0000
Remove 7	1.0000	1	1	0.9761	1	1	$*$	0.8642	0.9207	1	1	1	0.8081	1	1	1	1	0.8662
Remove 8	0.9989	1	1	0.9595	1	1	1	$*$	0.9098	1	1	1	0.7901	1	1	1	1	0.8611
Remove 9	0.9989	1	1	0.9595	1	1	1	0.8617	$*$	1	1	1	0.7901	1	1	1	1	0.8611
Remove 10	1.0000	1	1	0.9771	1	1	1	1.0000	0.9396	$*$	1	1	0.8073	1	1	1	1	1.0000
Remove 11	1.0000	1	1	0.9598	1	1	1	0.8621	0.9268	1	$*$	1	0.8650	1	1	1	1	0.8640
Remove 12	1.0000	1	1	0.9697	1	1	1	0.8622	1	1	1	$*$	0.7906	1	1	1	1	0.8620
Remove 13	1.0000	1	1	0.9595	1	1	1	0.8617	0.9098	1	1	1	$*$	1	1	1	1	0.8611
Remove 14	1.0000	1	1	0.9624	1	1	1	0.8627	0.9255	1	1	1	0.7912	$*$	1	1	1	0.8633
Remove 15	1.0000	1	1	0.9889	1	1	1	0.9002	0.9388	1	1	1	0.8155	1	$*$	1	1	0.9360
Remove 16	1.0000	1	1	0.9774	1	1	1	0.8924	0.9269	1	1	1	0.8167	1	1	$*$	1	0.8654
Remove 17	1.0000	1	1	0.9661	1	1	1	0.8637	0.9257	1	1	1	0.7918	1	1	1	$*$	0.8621
Remove 18	0.9989	1	1	0.9595	1	1	1	0.8617	0.9098	1	1	1	0.7901	1	1	1	1	$*$

"$*$" denotes that efficiencies of DMU_k after removing the k-th DMU, $k = 1, 2, \cdots, 18$, does not exist.

Table 6. The pairwise comparison matrix in the example.

1.000	0.999	0.999	1.028	0.999	0.999	0.999	1.129	1.074	0.999	0.999	0.999	1.242	0.999	0.999	0.999	0.999	1.101
1.004	1.000	1.000	1.0316	1.000	1.000	1.000	1.1310	1.0756	1.000	1.000	1.000	1.242	0.999	0.999	0.999	0.999	1.112
1.000	1.000	1.000	1.031	1.000	1.000	1.000	1.140	1.077	1.000	1.000	1.000	1.2449	1.000	1.000	1.000	1.000	1.112
0.972	0.969	0.971	1.000	0.969	0.970	0.970	1.098	1.044	0.970	0.971	0.971	1.208	0.971	0.970	0.970	0.971	1.071
1.000	1.000	1.000	1.031	1.000	1.000	1.000	1.140	1.0773	1.000	1.000	1.000	1.245	1.000	1.000	1.000	1.000	1.112
1.000	1.000	1.000	1.030	1.000	1.000	1.000	1.130	1.076	1.000	1.000	1.000	1.244	1.000	1.000	1.000	1.000	1.112
1.000	1.000	1.000	1.030	1.000	1.000	1.000	1.130	1.075	1.000	1.000	1.000	1.245	1.000	1.000	1.000	1.000	1.102
0.885	0.884	0.883	0.910	0.876	0.884	0.884	1.000	0.950	0.876	0.884	0.884	1.100	0.884	0.882	0.882	0.884	0.975
0.931	0.929	0.930	0.957	0.928	0.929	0.930	1.052	1.000	0.928	0.929	0.925	1.157	0.929	0.926	0.929	0.929	1.025
1.000	1.000	1.000	1.030	1.000	1.000	1.000	1.140	1.076	1.000	1.000	1.000	1.244	1.000	1.000	1.000	1.000	1.112
1.000	1.000	1.000	1.029	1.000	1.000	1.000	1.130	1.075	1.000	1.000	1.000	1.250	1.000	1.000	1.000	1.000	1.102
1.000	1.000	1.000	1.029	1.000	1.000	1.000	1.130	1.080	1.000	1.000	1.000	1.243	1.000	1.000	1.000	1.000	1.102
0.804	0.803	0.803	0.827	0.803	0.803	0.803	0.909	0.864	0.803	0.799	0.804	1.000	0.804	0.802	0.802	0.804	0.886
1.000	1.000	1.000	1.029	1.000	1.000	1.000	1.130	1.075	1.000	1.000	1.000	1.243	1.000	1.000	1.000	1.000	1.102
1.000	1.000	1.000	1.030	1.000	1.000	1.000	1.133	1.079	1.000	1.000	1.000	1.245	1.000	1.000	1.000	1.000	1.107
1.000	1.000	1.000	1.030	1.000	1.000	1.000	1.132	1.075	1.000	1.000	1.000	1.245	1.000	1.000	1.000	1.000	1.102
1.000	1.000	1.000	1.029	1.000	1.000	1.000	1.130	1.075	1.000	1.000	1.000	1.243	1.000	1.000	1.000	1.000	1.102
0.907	0.899	0.899	0.933	0.899	0.899	0.907	1.025	0.974	0.899	0.907	0.907	1.128	0.907	0.902	0.907	0.907	1.000

5. Conclusions

In this work, a ranking method by the hybrid fuzzy DEA/AHP is proposed for fully ranking units in a fuzzy environment. The proposed approach takes care of the multiple fuzzy criteria by fuzzy DEA and ranks units by AHP. It employs benefits of both fuzzy DEA and AHP methods to present a logical ranking of DMUs that is compatible with the efficient/inefficient classification derived from fuzzy DEA. Compared with other methods from the literature, the ranking by the hybrid fuzzy DEA/AHP approach leads to more discrimination. Moreover, since the AHP pairwise comparisons are derived mathematically from the fuzzy multiple criteria by fuzzy DEA, there is no subjective assessment of a decision-maker evaluation involved. A study on the facility layout design in manufacturing systems is provided to illustrate the superiority of the proposed approach. It shows that the ranking by the proposed approach is compatible with the results by fuzzy DEA and it furthers the analysis by providing full ranking for all units with multiple fuzzy criteria.

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