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Correlation Coefficients of Extended Hesitant Fuzzy Sets and Their Applications to Decision Making

Na Lu * and Lipin Liang

School of Economics and Administration, Taiyuan University of Technology, Taiyuan 030024, China; liangtyut@126.com
* Correspondence: luna123456@126.com

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Abstract: Extended hesitant fuzzy sets (EHFSs), which allow the membership degree of an element to a set represented by several possible value-groups, can be considered as a powerful tool to express uncertain information in the process of group decision making. Therefore, we derive some correlation coefficients between EHFSs, which contain two cases, the correlation coefficients taking into account the length of extended hesitant fuzzy elements (EHFEs) and the correlation coefficients without taking into account the length of EHFEs, as a new extension of existing correlation coefficients for hesitant fuzzy sets (HFSs) and apply them to decision making under extended hesitant fuzzy environments. A real-world example based on the energy policy problem is employed to illustrate the actual need for dealing with the difference of evaluation information provided by different experts without information loss in decision making processes.

Keywords: EHFSs; correlation; correlation coefficients; decision making

MSC: 03E72; 03E75

1. Introduction

When people make a decision, they are usually hesitant and irresolute for one thing or another, which makes it difficult to reach a final agreement, that is there usually exists a hesitation or uncertainty about the degree of sureness about the final decision. Torra et al. [1,2] proposed the hesitant fuzzy set, which permits the membership to have a set of possible values, and discussed the relationship between hesitant fuzzy sets and Atanassov's intuitionistic fuzzy sets [3]. The hesitant fuzzy set is a very useful tool to deal with uncertainty; more and more decision making theories and methods under the hesitant fuzzy environment have been developed since its appearance. Yi [4] gave some properties of operations and algebraic structures of hesitant fuzzy sets. Xia and Xu [5] proposed hesitant fuzzy information aggregation techniques and their application in decision making. Then, Xu and Xia [6] introduced a variety of distance measures for hesitant fuzzy sets and their corresponding similarity measures. Meanwhile, Xu and Xia [7] defined the distance and correlation measures for hesitant fuzzy information and then discussed their properties in detail. Xu et al. [8] developed some hesitant fuzzy aggregation operators with the aid of quasi-arithmetic means and applied them to group decision making problems. Gu et al. [9] investigated a evaluation model for risk investment with hesitant fuzzy information; they utilized the hesitant fuzzy weighted averaging operator to aggregate the hesitant fuzzy information corresponding to each alternative and then ranked the alternatives and selected the most desirable one(s) according to the score function for hesitant fuzzy sets. Wei [10] developed some prioritized aggregation operators to aggregate hesitant fuzzy information and then applied them to hesitant fuzzy multiple attribute decision making problems, in which the attributes are at different priority levels. Alcantud et al. [11] introduced a novel methodology for ranking hesitant fuzzy sets and built on a recent, theoretically-sound contribution in social choice. Chen et al. [12] proposed some correlation coefficient formulas for hesitant fuzzy sets and applied them to clustering analysis under hesitant fuzzy environments. Additionally, a position and perspective analysis of hesitant fuzzy sets [13] is given to show the important role of hesitant fuzzy sets on information fusion in decision making.

However, hesitant fuzzy sets have some drawbacks. if the two decision makers (DMs) both assign the same value, we can only save one value by the hesitant fuzzy element and lose the other one, which appears to be an information loss problem of HFSs. Further, since generally the DMs have different importance in group decision making [14,15] due to their different social importance, position in the group, previous merits, etc., for example, the loss of information provided by the leading DM may lead to ineffective results.

To resolve the information loss problem, Zhu and Xu [16] introduced the definition of EHFS, which is an extension of the hesitant fuzzy set [1,2]. EHFSs can better deal with the situations that permit the membership of an element to a given set having value-groups, which can avoid giving DMs' preferences anonymously that cause information loss. EHFSs increase the richness of numerical representation based on the value-groups, enhance the modeling abilities of HFSs and can identify different DMs in decision making, which expand the applications of HFSs in practice.

Correlation is one of the most broadly applied indices in many fields and also an important measure in data analysis and classification, pattern recognition, decision making, and so on [17–23]. As many real-world data may be fuzzy, the concept of correlation has been extended to fuzzy environments [21,24–26] and intuitionistic fuzzy environments [27–33]. For instance, Gerstenkorn and Manko [27] introduced the correlation coefficients of intuitionistic fuzzy sets. Hong and Hwang [26] also defined them in probability spaces. Mitchell [31] derived the correlation coefficient of intuitionistic fuzzy sets by interpreting an intuitionistic fuzzy set as an ensemble of ordinary fuzzy sets. Hung proposed a method to calculate the correlation coefficients of intuitionistic fuzzy sets by means of the centroid. Because of the potential applications of correlation coefficients, they have been further extended by Bustince and Burillo [32] and Hong [33] for interval-valued intuitionistic fuzzy sets. Several new methods of deriving the correlation coefficients for both intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets have also been proposed in [18]. In 2013, Chen et al. [12] proposed correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. Thus, we urgently need to put forward the correlation coefficients of EHFSs to deal with these problems. In this paper, we further introduce the correlation of EHFSs, which is a new extension of the correlation of hesitant fuzzy sets and intuitionistic fuzzy sets. Then, we utilize the weighted correlation coefficient to solve extended hesitant fuzzy group decision making problems in which attribute values take the form of extended hesitant fuzzy elements.

The remainder of the paper is organized as follows: In Section 2, we review some basic notions of hesitant fuzzy sets and EHFSs; the correlation coefficients between hesitant fuzzy sets are given as a basis of the main body of the paper in the next section. In Section 3, we propose some correlation coefficients between EHFSs, which contain two cases: the correlation coefficients taking into account the length of EHFEs and the correlation coefficients without taking into account the length of EHFEs. In Section 4, we present methods to deal with group decision making based on extended hesitant fuzzy information, and an example is given to show the actual need for dealing with the difference of evaluation information provided by different experts without information loss in decision making processes. Finally, in Section 5, some conclusions are given.

2. Preliminaries

In this section, we carry out a brief introduction to EHFSs and correlation coefficients of HFSs as a basis of the main body of the paper.

2.1. Several Basic Concepts about HFSs and EHFSs

Torra et al. [1,2] firstly proposed the concept of a hesitant fuzzy set, which is defined as follows:

Definition 1. *Let X be a fixed set; a hesitant fuzzy set A on X is defined in terms of a function* h_A *that when applied to X returns to a finite subset of [0,1], which can be represented as the following mathematical symbol [1,2]:*

$$A = \{< x, h_A(x) > |x \in X|\}, \tag{1}$$

where $h_A(x)$ *is a set of some different values of [0,1], denoting the possible membership degrees of the element* $x \in X$ *to A. For convenience, we call* $h_A(x)$ *a hesitant fuzzy element denoted by h.*

Zhu et al. [16] defined an EHFS, which is an extension of the hesitant fuzzy set, in terms of a function that returns a finite set of membership value-groups.

Definition 2. *Let X be a fixed set,* $h_D(x) = \bigcup_{\gamma_D \in h_D(x)} \{\gamma_D\}$ $(D = 1, \ldots, m)$ *be HFSs on X. Then, an EHFS,that is* H_{h_D}*, is defined as [16]:*

$$H_{h_D}(x) = h_1(x) \times \ldots \times h_m(x) = \bigcup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x), \ldots, \gamma_m \in h_m(x)} \{< x, (\gamma_1(x), \ldots, \gamma_m(x)) > |x \in X\}. \tag{2}$$

For convenience, we call:

$$H = h_1 \times, \ldots, \times h_m = \bigcup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x), \ldots, \gamma_m \in h_m(x)} \{(\gamma_1, \ldots, \gamma_m)\} \tag{3}$$

an extended hesitant fuzzy element (EHFE) and let $u = (\gamma_1, \ldots, \gamma_m)$; then, we call u a membership unit (MU), Based on u, an EHFE H, can also be indicated by:

$$H = \bigcup_{u \in h_m(x)} \{u\} = \bigcup_{\gamma_1 \in h_1(x), \gamma_2 \in h_2(x), \ldots, \gamma_m \in h_m(x)} \{(\gamma_1, \ldots, \gamma_m)\}. \tag{4}$$

From Definition 2, we can see that EHFS increases the richness of numerical representation based on the value-groups, enhances the modeling abilities of hesitant fuzzy sets and can identify different decision makers in decision making processes, which expand the applications of hesitant fuzzy sets in practice. HFSs can be used to construct EHFSs. On the contrary, EHFSs can reduce to HFSs. The existing sets, including fuzzy sets, intuitionistic fuzzy sets, fuzzy multisets, type-2 fuzzy sets, dual hesitant fuzzy sets and especially hesitant fuzzy sets, can handle a more exemplary and flexible access to assign values for each element in the domain.

Example 1. *Let* $X = \{x_1, x_2\}$ *be the reference set,* $H(x_1) = \{(0.2, 0.4), (0.2, 0.5), (0.3, 0.4), (0.3, 0.5)\}$ *and* $H(x_2) = \{(0.1, 0.4), (0.1, 0.5)\}$ *be the EHFEs of* x_i $(i = 1, 2)$ *to a set A, respectively. Then H can be considered as a EHFS, i.e.,*

$A = \{< x_1, \{(0.2, 0.4), (0.2, 0.5), (0.3, 0.4), (0.3, 0.5)\} >, < x_2, \{(0.1, 0.4), (0.1, 0.5)\} >\}.$

To compare the EHFEs, Zhu et al. [16] gave the concepts of score function and deviation function:

Definition 3. *For an MU, $u = (\gamma_1, ..., \gamma_m)$, then we call $s(u) = (1/\sharp u) \sum_{\gamma \in u} \gamma$ the score function of u, where $\sharp u$ is the number of memberships in u. For any two MUs, u_1 and u_2, if $s(u_1) > s(u_2)$, then $u_1 \succ u_2$; if $s(u_1) = s(u_2)$, then $u_1 \sim u_2$, where "$\succ$" denotes "be superior to" and "$\sim$" means "be indifferent to" [16].*

Definition 4. *For an MU, $u = (\gamma_1, ..., \gamma_m)$, let $s(u)$ be the score function of u, then we call $p(u) = [(1/\sharp u) \sum_{\gamma \in u} (\gamma - s(u))^2]^{1/2}$ the deviation function of HFSs, where $\sharp u$ is the number of memberships in u [16].*

Based on the score function and the deviation function, we develop the following comparison law.

Definition 5. *Let u_1 and u_2 be two MUs, $s(u_1)$ and $s(u_2)$ the scores of u_1 and u_2, respectively, and $p(u_1)$ and $p(u_2)$ the deviation degrees of u_1 and u_2, respectively, then [16]:*

(1) if $s(u_1) < s(u_2)$, then $u_1 \prec u_2$;
(2) if $s(u_1) = s(u_2)$, then

   (1) if $p(u_1) = p(u_2)$, then u_1 is equivalent to u_2, denoted by $u_1 \sim u_2$;
   (2) if $p(u_1) < p(u_2)$, then u_1 is superior to u_2, denoted by $u_1 \succ u_2$;
   (3) if $p(u_1) > p(u_2)$, then u_1 is superior to u_2, denoted by $u_1 \prec u_2$.

The comparison laws of fuzzy set theory [1,4,16,34] play an important role in decision making problems, and the score function and accuracy function of EHFEs are the basis of the main body of the next part.

2.2. Correlation Coefficient of Hesitant Fuzzy Sets

Correlation coefficients are an effective tool for addressing the relationship between elements with uncertain information that have been deeply studied [21,24–27]. Chen et al. [12] introduced the informational energy, correlation and correlation coefficients of hesitant fuzzy sets. For a hesitant fuzzy element h, let $\sigma : (1, 2, ..., n) \to (1, 2, ..., n)$ be a permutation satisfying $h_{\sigma(j)} \geq h_{\sigma(j+1)}$ for $j = 1, 2, ..., n-1$ and $h_{\sigma(j)}$ be the j-th largest value in h; the informational energy of hesitant fuzzy sets is given as follows:

Definition 6. *Let A be a hesitant fuzzy set on a universe of discourse $X = \{x_1, x_2, ..., x_n\}$ denoted as $A = \{< x_i, h_A(x_i) > | x_i \in X\}$. Then, the informational energy of A is defined as [12]:*

$$E_{HFS}(A) = \sum_{i=1}^{n} (\frac{1}{l_i} \sum_{j=1}^{l_i} h^2_{A\sigma(j)}(x_i)), \tag{5}$$

where $l_i = l(h_A(x_i))$ represents the number of values in $h_A(x_i)$, $x_i \in X$.

Definition 7. *Let A and B be two hesitant fuzzy sets on a universe of discourse $X = \{x_1, x_2, ..., x_n\}$ denoted as $A = \{< x_i, h_A(x_i) > | x_i \in X\}$ and $B = \{< x_i, h_B(x_i) > | x_i \in X\}$, respectively. Then, the correlation between A and B is defined as [12]:*

$$C_{HFS}(A, B) = \sum_{i=1}^{n} (\frac{1}{l_i} \sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i) h_{B\sigma(j)}(x_i)), \tag{6}$$

here, $l_i = max\{l(h_A(x_i)), l(h_B(x))\}$ for each x_i in X, where $l(h_A(x_i))$ and $l(h_B(x_i))$ represent the number of values in $h_A(x_i)$ and $h_B(x_i)$, respectively. When $l(h_A(x_i)) \neq l(h_B(x_i))$, one can make them have the same number of elements through adding some values to the hesitant fuzzy element, which has less values. According to the pessimistic principle, the smallest element will be added. Therefore, if $l(h_A(x_i)) < l(h_B(x_i))$, $h_A(x_i)$ should be extended by adding the minimum value in it until it has the same length as $h_B(x_i)$.

Definition 8. *Let A and B be two hesitant fuzzy sets on a universe of discourse* $X = \{x_1, x_2, ..., x_n\}$*, denoted as* $A = \{< x_i, h_A(x_i) > | x_i \in X\}$ *and* $B = \{< x_i, h_B(x_i) > | x_i \in X\}$*, respectively. Then, the correlation coefficient between A and B is defined as [12]:*

$$\rho_{HFS}(A,B) = \frac{C_{HFS}(A,B)}{\sqrt{C_{HFS}(A,A)}\sqrt{C_{HFS}(B,B)}} = \frac{\sum_{i=1}^{n}(\frac{1}{l_i}\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i)h_{B\sigma(j)}(x_i))}{\sqrt{\sum_{i=1}^{n}(\frac{1}{l_i}\sum_{j=1}^{l_i} h^2_{A\sigma(j)}(x_i))}\sqrt{\sum_{i=1}^{n}(\frac{1}{l_i}\sum_{j=1}^{l_i} h^2_{B\sigma(j)}(x_i))}}. \quad (7)$$

Theorem 1. *The correlation coefficient between two hesitant fuzzy sets A and B satisfies the following properties [12]:*

(1) $\rho_{HFS}(A,B) = \rho_{HFS}(B,A)$;
(2) $0 \le \rho_{HFS}(A,B) \le 1$;
(3) $\rho_{HFS}(A,B) = 1$, *if* $A = B$.

3. Correlation and Correlation Coefficients of EHFSs

The correlation and correlation coefficients of hesitant fuzzy sets were introduced by Chen et al. [12] to solve practical decision making problems. In this section, we introduce the informational energy, correlation and correlation coefficients of EHFSs as a new extension.

Let $X = \{x_1, x_2, ..., x_n\}$ be a discrete universe of discourse, A be a EHFS on X denoted as $A = \{< x_i, \cup_{u_A \in H_A}\{u_A(x_i)\} > | x_i \in X\}$. The MUs of an EHFE are usually given in disorder, and for convenience, we arrange them in a decreasing order. Based on Definitions 3–5, for a EHFE H, let $\sigma : (1,2,..,n) \to (1,2,...,n)$ be a permutation satisfying $u_{\sigma(i)} \succeq u_{\sigma(i+1)}$, $i = 1,2...,n-1$ and $H_{\sigma(i)}$ be the j-th largest value in H. As is shown in Example 1, $A = \{< x_1, \{(0.2,0.4),(0.2,0.5),(0.3,0.4),(0.3,0.5)\} >, < x_2, \{(0.1,0.4),(0.1,0.5)\} >\}$, so we obtain that EHFEs $H(x_1) = \{(0.2,0.4),(0.2,0.5),(0.3,0.4),(0.3,0.5)\}$ and $H(x_2) = \{(0.1,0.4),(0.1,0.5)\}$, according to Definitions 3–5, as $s((0.2,0.4)) = 0.3 \preceq s((0.2,0.5)) = 0.35 \sim s((0.3,0.4)) = 0.35 \preceq s((0.3,0.5)) = 0.4$ and $p((0.2,0.5)) = 0.2121 \succeq p((0.3,0.4)) = 0.071$, then $H_\sigma(x_1) = \{(0.3,0.5),(0.3,0.4),(0.2,0.5),(0.2,0.4)\}$; similarly, $H_\sigma(x_2) = \{(0.1,0.5),(0.1,0.4)\}$.

It is noted that the number of values in different EHFEs may be different. To compute the correlation coefficients between two EHFSs, let $\sharp H = max\{l(H_A(x_i)), l(H_B(x_i))\}$ for each x_i in X, where $l(H_A(x_i))$ and $l(H_B(x_i))$ represent the number of MUs in $H_A(x_i)$ and $H_B(x_i)$, respectively. When $l(H_A(x_i)) \neq l(H_B(x_i))$, one can make them have the same number of MUs through adding some elements to the EHFE, which has less MUs. Similarly, $\sharp u = max\{l(u(x_i)), u(x_i))\}$ for each x_i in X, where $l(u_A(x_i))$ and $l(u_B(x_i))$ represent the number of values in $u_A(x_i)$ and $u_B(x_i)$, respectively. Motivated by the optimized parameter, Zhu et al. [16] gave the following definitions.

Definition 9. *For a MU,* $u = (\gamma_1, ..., \gamma_m\}$*, let* $u^- = min\{\gamma | \gamma \in u\}$ *and* $u^+ = max\{\gamma | \gamma \in u\}$ *be the minimum and maximum memberships in u, respectively, and* $\varsigma(0 \le \varsigma \le 1)$ *be the optimized parameter, then we call* $\tilde{\gamma} = \varsigma u^+ + (1-\varsigma)u^-$ *an added membership [16].*

For two EFHFEs with different numbers of MUs, we further utilize the optimized parameter to obtain an MU.

Definition 10. *Given an EHFE,* $H_{h_D} = \bigcup_{\gamma_1 \in h_1, ..., \gamma_m \in h_m}\{(\gamma_1, ..., \gamma_m)\}$ $(D = 1, ..., m)$*, let* h_D^- *and* h_D^+ *be the minimum and maximum memberships in* h_D*, respectively, and* $\varsigma(0 \le \varsigma \le 1)$ *be the optimized parameter, then an added MU is defined as* $\tilde{u} = (\tilde{\gamma}_1, ..., \tilde{\gamma}_m)$*, where* $\tilde{\gamma} = \varsigma u^+ + (1-\varsigma)u^-$ $(D = 1, ..., m)$ *[16].*

Similar to the existing works [12], we define the informational energy for EHFSs and the corresponding correlation.

Definition 11. *Let A be an EHFS on a universe of discourse* $X = \{x_1, x_2, ..., x_n\}$*, denoted as* $A = \{< x_i, \cup_{u \in H}\{u(x_i)\} > | x_i \in X\}$*. Then, the informational energy of A is defined as:*

$$E_{EHFS_1}(A) = \sum_{i=1}^{n} \frac{1}{\sharp H} \frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H} \sum_{k=1}^{\sharp u} (\cup_{\gamma^{\sigma(k)}(x_i) \in u^{\sigma(j)}(x_i)} \{(\gamma^{\sigma(k)}(x_i))^2 | u^{\sigma(j)}(x_i) \in H(x_i)\})), \tag{8}$$

where $\sharp H$ *and* $\sharp u$ *are the number of MUs in H and values in MU u, respectively,* S_s *is a function that indicates a summation of all values in the set of* $u^{\sigma(j)}(x_i)$ *in* $H(x_i)$*,* $\gamma^{\sigma(k)}(x_i)$ *is the k-th largest membership in u to* $x_i \in X$ *and* $u^{\sigma(j)}(x_i)$ *is the j-th largest MUs in H.*

Definition 12. *Let A and B be two EHFSs on a universe of discourse* $X = \{x_1, x_2, ..., x_n\}$*, denoted as* $A = \{< x_i, \cup_{u_A \in H_A}\{u_A(x_i)\} > | x_i \in X\}$ *and* $B = \{< x_i, \cup_{u_B \in H_B}\{u_B(x_i)\} > | x_i \in X\}$*, respectively. Then, the correlation between A and B is defined as:*

$$\begin{aligned} C_{EHFS_1}(A, B) = \sum_{i=1}^{n} \frac{1}{\sharp H} \frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H} \sum_{k=1}^{\sharp u} (\cup_{\gamma_A^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}, \gamma_B^{\sigma(k)}(x_i) \in u_B^{\sigma(j)}(x_i)} \{\gamma_A^{\sigma(k)}(x_i) \gamma_B^{\sigma(k)}(x_i) | u_A^{\sigma(j)}(x_i) \in H_A(x_i), \\ u_B^{\sigma(j)}(x_i) \in H_B(x_i)\})), \end{aligned} \tag{9}$$

here, $\sharp H = \sharp H_A = \sharp H_B$*,* $\sharp u = \sharp u_A = \sharp u_B$*,* S_s *is a function that indicates a summation of all values in the set of* $u^{\sigma(j)}(x_i)$ *in* $H(x_i)$*,* $\gamma_A^{\sigma(k)}(x_i)$ *and* $\gamma_B^{\sigma(k)}(x_i)$ *are the k-th largest memberships in* u_A *and* u_B*, respectively, and* $u_A^{\sigma(j)}(x_i)$ *and* $u_B^{\sigma(j)}(x_i)$ *are the j-th largest MUs in* H_A *and* H_B*, respectively.*

It is obvious that the correlation of two EHFSs satisfies the following properties:

(1) $C_{EHFS_1}(A, A) = E_{EHFS_1}(A)$;
(2) $C_{EHFS_1}(A, B) = C_{EHFS_1}(B, A)$.

Definition 13. *Let A and B be two EHFSs on a universe of discourse* $X = \{x_1, x_2, ..., x_n\}$*, denoted as* $A = \{< x_i, \cup_{u_A \in H_A}\{u_A(x_i)\} > | x_i \in X\}$ *and* $B = \{< x_i, \cup_{u_B \in H_B}\{u_B(x_i)\} > | x_i \in X\}$*, respectively. Then, the correlation coefficient between A and B is defined as:*

$$\rho_{EHFS_1}(A, B) = \frac{C_{EHFS_1}(A, B)}{\sqrt{C_{EHFS_1}(A, A)}\sqrt{C_{EHFS_1}(B, B)}}, \tag{10}$$

where:

$$\begin{aligned} C_{EHFS_1}(A, B) = \sum_{i=1}^{n} \frac{1}{\sharp H} \frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H} \sum_{k=1}^{\sharp u} (\cup_{\gamma_A^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}, \gamma_B^{\sigma(k)}(x_i) \in u_B^{\sigma(j)}(x_i)} \{\gamma_A^{\sigma(k)}(x_i) \gamma_B^{\sigma(k)}(x_i) | u_A^{\sigma(j)}(x_i) \in H_A(x_i), \\ u_B^{\sigma(j)}(x_i) \in H_B(x_i)\})), \end{aligned}$$

$$C_{EHFS_1}(A, A) = \sum_{i=1}^{n} \frac{1}{\sharp H} \frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H} \sum_{k=1}^{\sharp u} (\cup_{\gamma_A^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}(x_i)} \{(\gamma_A^{\sigma(k)}(x_i))^2 | u_A^{\sigma(j)}(x_i) \in H_A(x_i)\})),$$

$$C_{EHFS_1}(B, B) = \sum_{i=1}^{n} \frac{1}{\sharp H} \frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H} \sum_{k=1}^{\sharp u} (\cup_{\gamma_B^{\sigma(k)}(x_i) \in u_B^{\sigma(j)}(x_i)} \{(\gamma_B^{\sigma(k)}(x_i))^2 | u_B^{\sigma(j)}(x_i) \in H_B(x_i)\})).$$

Theorem 2. *The correlation coefficient between two EHFSs A and B satisfies the following properties:*

(1) $\rho_{EHFS_1}(A,B) = \rho_{EHFS_1}(B,A)$;
(2) $0 \le \rho_{EHFS_1}(A,B) \le 1$;
(3) $\rho_{EHFS_1}(A,B) = 1$, *if* $A = B$.

Proof.

(1) It is straightforward.
(2) The inequality $\rho_{EHFS_1}(A,B) \ge 0$ is obvious. Below, let us prove $\rho_{EHFS_1}(A,B) \le 1$:

$$
\begin{aligned}
&C_{EHFS_1}(A,B) \\
&= \sum_{i=1}^{n} \frac{1}{\sharp H}\frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{\gamma_A^{\sigma(k)}(x_i)\gamma_B^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i),u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))\\
&= \frac{1}{\sharp H}\frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_1)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_1)\in u_B^{\sigma(j)}(x_1)}\{\gamma_A^{\sigma(k)}(x_1)\gamma_B^{\sigma(k)}(x_1)|u_A^{\sigma(j)}(x_1)\in H_A(x_1),u_B^{\sigma(j)}(x_1)\in H_B(x_1)\}))+\\
&\frac{1}{\sharp H}\frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_2)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_2)\in u_B^{\sigma(j)}(x_2)}\{\gamma_A^{\sigma(k)}(x_2)\gamma_B^{\sigma(k)}(x_2)|u_A^{\sigma(j)}(x_2)\in H_A(x_2),u_B^{\sigma(j)}(x_2)\in H_B(x_2)\}))+...+\\
&\frac{1}{\sharp H}\frac{1}{\sharp u} S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_n)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_n)\in u_B^{\sigma(j)}(x_n)}\{\gamma_A^{\sigma(k)}(x_n)\gamma_B^{\sigma(k)}(x_n)|u_A^{\sigma(j)}(x_n)\in H_A(x_n),u_B^{\sigma(j)}(x_n)\in H_B(x_n)\}))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{\sharp H}\frac{\frac{1}{\sharp u}S_s(\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_1)\in u_A^{\sigma(j)}}\{\gamma_A^{\sigma(k)}(x_1)|u_A^{\sigma(j)}(x_1)\in H_A(x_1)\}))}{\sqrt{\sharp H}}\cdot\\
&\frac{\frac{1}{\sharp u}S_s(\sum_{k=1}^{\sharp u}(\cup_{\gamma_B^{\sigma(k)}(x_1)\in u_B^{\sigma(j)}(x_1)}\{\gamma_B^{\sigma(k)}(x_1)|u_B^{\sigma(j)}(x_1)\in H_B(x_1)\}))}{\sqrt{\sharp H}}+\\
&\sum_{j=1}^{\sharp H}\frac{\frac{1}{\sharp u}S_s(\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_2)\in u_A^{\sigma(j)}}\{\gamma_A^{\sigma(k)}(x_2)|u_A^{\sigma(j)}(x_2)\in H_A(x_2)\}))}{\sqrt{\sharp H}}\cdot\\
&\frac{\frac{1}{\sharp u}S_s(\sum_{k=1}^{\sharp u}(\cup_{\gamma_B^{\sigma(k)}(x_2)\in u_B^{\sigma(j)}(x_2)}\{\gamma_B^{\sigma(k)}(x_2)|u_B^{\sigma(j)}(x_2)\in H_B(x_2)\}))}{\sqrt{\sharp H}}+...+\\
&\sum_{j=1}^{\sharp H}\frac{\frac{1}{\sharp u}S_s(\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_n)\in u_A^{\sigma(j)}}\{\gamma_A^{\sigma(j)}(x_n)|u_A^{\sigma(j)}(x_n)\in H_A(x_n)\}))}{\sqrt{\sharp H}}\cdot\\
&\frac{\frac{1}{\sharp u}S_s(\sum_{k=1}^{\sharp u}(\cup_{\gamma_B^{\sigma(k)}(x_n)\in u_B^{\sigma(j)}(x_n)}\{\gamma_B^{\sigma(k)}(x_n)|u_B^{\sigma(j)}(x_n)\in H_B(x_n)\}))}{\sqrt{\sharp H}},
\end{aligned}
$$

using the Cauchy–Schwarz inequality:

$(x_1y_1 + x_2y_2, ..., x_ny_n)^2 \le (x_1^2 + x_2^2+, ..., x_n^2)(y_1^2 + y_2^2+, ..., y_n^2)$,

where $(x_1, x_2, ..., x_n) \in R^n$, $(y_1, y_2, ..., y^n) \in R^n$; we obtain:

$$
\begin{aligned}
&(C_{EHFS_1}(A,B))^2 \\
&\leq [\frac{1}{\sharp H}\frac{1}{\sharp u}S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_1)\in u_A^{\sigma(j)}(x_1)}\{(\gamma_A^{\sigma(k)}(x_1))^2|u_A^{\sigma(j)}(x_1)\in H_A(x_1)\}))+ \\
&\frac{1}{\sharp H}\frac{1}{\sharp u}S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_2)\in u_A^{\sigma(j)}(x_2)}\{(\gamma_A^{\sigma(k)}(x_2))^2|u_A^{\sigma(j)}(x_2)\in H_A(x_2)\}))+...,+ \\
&\frac{1}{\sharp H}\frac{1}{\sharp u}S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_n)\in u_A^{\sigma(j)}(x_n)}\{(\gamma_A^{\sigma(k)}(x_n))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_n)\}))] \\
&\times [\frac{1}{\sharp H}\frac{1}{\sharp u}S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_B^{\sigma(k)}(x_1)\in u_B^{\sigma(j)}(x_1)}\{(\gamma_B^{\sigma(k)}(x_1))^2|u_B^{\sigma(j)}(x_1)\in H_B(x_1)\}))+ \\
&\frac{1}{\sharp H}\frac{1}{\sharp u}S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_B^{\sigma(k)}(x_2)\in u_B^{\sigma(j)}(x_2)}\{(\gamma_B^{\sigma(k)}(x_2))^2|u_B^{\sigma(j)}(x_2)\in H_B(x_2)\}))+...,+ \\
&\frac{1}{\sharp H}\frac{1}{\sharp u}S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_B^{\sigma(k)}(x_n)\in u_B^{\sigma(j)}(x_n)}\{(\gamma_B^{\sigma(k)}(x_n))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_n)\}))] \\
&= [\sum_{i=1}^{n}\frac{1}{\sharp H}\frac{1}{\sharp u}S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_A^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}))]\times \\
&[\sum_{i=1}^{n}\frac{1}{\sharp H}\frac{1}{\sharp u}S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_B^{\sigma(k)}(x_i))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))] \\
&= C_{EHFS_1}(A,A)\cdot C_{EHFS_1}(B,B).
\end{aligned}
$$

Therefore,

$C_{EHFS_1}(A,B) \leq \sqrt{C_{EHFS_1}(A,A)}\cdot\sqrt{C_{EHFS_1}(B,B)}.$

Therefore, $0 \leq \rho_{EHFS_1}(A,B) \leq 1$.

(3) $A = B \Rightarrow \gamma_A^{\sigma(j)}(x_i) = \gamma_B^{\sigma(j)}(x_i), x_i \in X \Rightarrow \rho_{EHFS_1}(A,B) = 1.$

□

Based on the concepts of HFSs, EHFSs and their informational energies, the correlations and the correlation coefficients, we can easily obtain the following remark:

Remark 1. *If EHFSs reduce to HFSs, the informational energy, the correlation and the correlation coefficient about EHFSs will reduce to the informational energy, the correlation and the correlation coefficient about HFSs, respectively.*

In what follows, we give a new formula of calculating the correlation coefficient, which is similar to that used in HFSs [12]:

Definition 14. *Let A and B be two EHFSs on a universe of discourse $X = \{x_1, x_2, ..., x_n\}$, denoted as $A = \{< x_i, \cup_{u_A \in H_A}\{u_A(x_i)\} > |x_i \in X\}$ and $B = \{< x_i, \cup_{u_B \in H_B}\{u_B(x_i)\} > |x_i \in X\}$, respectively. Then, the correlation coefficient between A and B is defined as:*

$$\rho_{EHFS_2}(A,B) = \frac{C_{EHFS_1}(A,B)}{max\{C_{EHFS_1}(A,A), C_{EHFS_1}(B,B)\}}. \quad (11)$$

Theorem 3. *The correlation coefficient of two EHFSs A and B, $\rho_{EHFS_2}(A,B)$, follows the same properties listed in Theorem 2.*

Proof.

The process to prove Properties (1) and (3) is analogous to that in Theorem 2; we do not repeat it here.

(2) $\rho_{EHFS_2}(A,B) \geq 0$ is obvious. We now only prove $\rho_{EHFS_2}(A,B) \leq 1$.

Based on the proof process of Theorem 2, we have

$C_{EHFS_1}(A,B) \leq \sqrt{C_{EHFS_1}(A,A)} \cdot \sqrt{C_{EHFS_1}(B,B)}$,

and then

$C_{EHFS_1}(A,B) \leq max\{C_{EHFS_1}(A,A), C_{EHFS_1}(B,B)\}$;

thus, $\rho_{EHFS_2}(A,B) \leq 1$. □

Example 2. *Let A and B be two EHFSs in* $X = \{x_1, x_2\}$, *and*

$A = \{< x_1, \{(0.3, 0.4, 0.5), (0.3, 0.4, 0.6)\}, < x_2, \{(0.4, 0.3, 0.2), (0.4, 0.3, 0.1), (0.4, 0.3, 0.5)\} >\}$,

$B = \{< x_1, \{(0.1, 0.2, 0.5)\}, < x_2, \{(0.4, 0.4, 0.2), (0.4, 0.4, 0.1)\} >\}$.

By applying Equation (9), we calculate:

$$\begin{aligned}
C_{EHFS_1}(A,A) = E_{EHFS_1}(A) &= \sum_{i=1}^{n} \frac{1}{\sharp H} \frac{1}{\sharp u} S_s (\sum_{j=1}^{\sharp H} \sum_{k=1}^{\sharp u} (\cup_{\gamma_A^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}(x_i)} \{(\gamma_A^{\sigma(k)}(x_i))^2 | u_A^{\sigma(j)}(x_i) \in H_A(x_i)\})) \\
&= \frac{1}{2}[\frac{1}{3}(0.3^2 + 0.4^2 + 0.5^2) + \frac{1}{3}(0.3^2 + 0.4^2 + 0.6^2)] + \frac{1}{3}[\frac{1}{3}(0.4^2 + 0.3^2 + 0.2^2) \\
&+ \frac{1}{3}(0.4^2 + 0.3^2 + 0.1^2) + \frac{1}{3}(0.5^2 + 0.3^2 + 0.4^2)] \\
&= 0.3017,
\end{aligned}$$

and similarly:

$$\begin{aligned}
C_{EHFS_1}(B,B) = E_{EHFS_1}(B) &= \sum_{i=1}^{n} \frac{1}{\sharp H} \frac{1}{\sharp u} S_s (\sum_{j=1}^{\sharp H} \sum_{k=1}^{\sharp u} (\cup_{\gamma_B^{\sigma(k)}(x_i) \in u_B^{\sigma(j)}(x_i)} \{(\gamma_B^{\sigma(k)}(x_i))^2 | u_B^{\sigma(j)}(x_i) \in H_B(x_i)\})) \\
&= \frac{1}{2}[\frac{1}{3}(0.1^2 + 0.2^2 + 0.5^2) \times 2] + \frac{1}{3}[\frac{1}{3}(0.4^2 + 0.4^2 + 0.2^2) + \frac{1}{3}(0.4^2 + 0.4^2 + 0.1^2) \\
&+ \frac{1}{3}(0.4^2 + 0.4^2 + 0.15^2)] \\
&= 0.2147.
\end{aligned}$$

With $\varsigma = 0.5$, we obtain:

$$\begin{aligned}
&C_{EHFS_1}(A,B) \\
&= \sum_{i=1}^{n} \frac{1}{\sharp H} \frac{1}{\sharp u} S_s (\sum_{j=1}^{\sharp H} \sum_{k=1}^{\sharp u} (\cup_{\gamma_A^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}, \gamma_B^{\sigma(k)}(x_i) \in u_B^{\sigma(j)}(x_i)} \{\gamma_A^{\sigma(k)}(x_i) \gamma_B^{\sigma(k)}(x_i) | u_A^{\sigma(j)}(x_i) \in H_A(x_i), u_B^{\sigma(j)}(x_i) \in H_B(x_i)\})) \\
&= \frac{1}{2}[\frac{1}{3}(0.5 \times 0.5 + 0.4 \times 0.2 + 0.3 \times 0.1) + \frac{1}{3}(0.6 \times 0.5 + 0.4 \times 0.2 + 0.3 \times 0.1)] \\
&+ \frac{1}{3}[\frac{1}{3}(0.4 \times 0.4 + 0.3 \times 0.4 + 0.5 \times 0.2) + \frac{1}{3}(0.4 \times 0.4 + 0.3 \times 0.4 + 0.2 \times 0.15) \\
&+ \frac{1}{3}(0.4 \times 0.4 + 0.3 \times 0.4 + 0.2 \times 0.1)] \\
&= 0.2372.
\end{aligned}$$

Finally, we can calculate the correlation coefficient $\rho_{EHFS_1}(A,B)$ as:

$\rho_{EHFS_1}(A,B) = \frac{C_{EHFS}(A,B)}{\sqrt{C_{EHFS}(A,A)}\sqrt{C_{EHFS}(B,B)}} = \frac{0.2372}{\sqrt{0.3017}\sqrt{0.2147}} = 0.9320$,

and similarly, we can calculate the correlation coefficient $\rho_{EHFS_2}(A,B)$ as:

$\rho_{EHFS_2}(A,B) = \frac{C_{EHFS}(A,B)}{max\{C_{EHFS}(A,A), C_{EHFS}(B,B)\}} = \frac{0.2372}{0.3017} = 0.7862$.

From Example 2, we can find that different results are obtained by extending different values in the short EHFE, so we present several new correlation coefficients of EHFSs, not taking into account the length of EHFEs and the arrangement of their possible value-groups.

Definition 15. *Let H_A and H_B be any two MUs with $u_A \in H_A$, then:*

$$d(u_A, H_B) = min_{u_B \in H_B} \sum_{\gamma_A^{\sigma(i)} \in u_A, \gamma_B^{\sigma(i)} \in u_B} |\gamma_A^{\sigma(i)} - \gamma_B^{\sigma(i)}| \quad (12)$$

is called the distance between the value u_A in H_A and the EHFE H_B; by $u_{B'}$, we denote the value in H_B such that $d(u_A, H_B)$. If there is more that one value in H_B such that $d(u_A, H_B)$, then $u_{B'} = min\{u_B | u_B \in H_B, \sum_{\gamma_A^{\sigma(i)} \in u_A, \gamma_B^{\sigma(i)} \in u_A} |\gamma_A^{\sigma(i)} - \gamma_B^{\sigma(i)}|\} = d(u_A, H_B)$. For convenience, $u_{B'} = \{\gamma_{B'}\}$ and $u_{A'} = \{\gamma_{A'}\}$.

It is obvious that the above distance $d(u_A, H_B)$ satisfies the following properties:

(1) $d(u_A, H_B) = d(H_B, u_A)$;
(2) $0 \leq d(u_A, H_B) \leq 1$;
(3) $d(u_A, H_B) = 0$ if and only if $u_A = u_B$ for any $u_B \in H_B$, where $u_A = u_B$ means $\gamma_A^{\sigma(i)} = \gamma_B^{\sigma(i)}$, $\gamma_A^{\sigma(i)} \in u_A, \gamma_B^{\sigma(i)} \in u_A$.

Definition 16. *Let A and B be two EHFSs on a universe of discourse $X = \{x_1, x_2, ..., x_n\}$, denoted as $A = \{< x_i, \cup_{u_A \in H_A}\{u_A(x_i)\} > | x_i \in X\}$ and $B = \{< x_i, \cup_{u_B \in H_B}\{u_B(x_i)\} > | x_i \in X\}$, respectively. Then, the correlation between A and B is defined as:*

$$C_{EHFS_2}(A,B) = \sum_{i=1}^{n} \Big(\frac{1}{\sharp H_A(x_i)} \frac{1}{\sharp u_A(x_i)} S_s\Big(\sum_{j=1}^{\sharp H_A(x_i)} \sum_{k=1}^{\sharp u_A(x_i)} (\cup_{\gamma_A(x_i)^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}(x_i)} \{\gamma_A^{\sigma(k)}(x_i)\gamma_{B'}^{\sigma(k)}(x_i) | u_A^{\sigma(j)}(x_i)$$
$$\in H_A(x_i)\}) \Big) + \frac{1}{\sharp H_B(x_i)} \frac{1}{\sharp u_B(x_i)} S_s\Big(\sum_{j=1}^{\sharp H_B(x_i)} \sum_{k=1}^{\sharp u_B(x_i)} (\cup_{\gamma_B^{\sigma(k)}(x_i) \in u_B^{\sigma(j)}(x_i)} \{\gamma_B^{\sigma(k)}(x_i)\gamma_{A'}^{\sigma(k)}(x_i) | u_B^{\sigma(j)}(x_i) \in H_B(x_i)\}) \Big)\Big),$$

where $\sharp H_A$ and $\sharp H_B$ are the numbers of extended hesitant fuzzy elements H_A and H_B, respectively. Additionally, $\gamma_A^{\sigma(k)}(x_i), \gamma_{A'}^{\sigma(k)}(x_i), \gamma_B^{\sigma(k)}(x_i)$ and $\gamma_{B'}^{\sigma(k)}(x_i)$ are shown in Definition 15.

It is easy to prove that the above correlation $C_{EHFS_2}(A, B)$ satisfies the following theorem:

Theorem 4. *Let A and B be any two EHFSs in X; the correlation $C_{EHFS_2}(A, B)$ satisfies:*

(1) $C_{EHFS_2}(A, B) = C_{EHFS_2}(B, A)$;
(2) $C_{EHFS_2}(A, A) = 2E_{EHFS_2}(A)$ *with* $E_{EHFS_2}(A) = \sum_{i=1}^{n} \frac{1}{\sharp H_A(x_i)} \frac{1}{\sharp u_A(x_i)} S_s$ $(\sum_{j=1}^{\sharp H_A(x_i)} \sum_{k=1}^{\sharp u_A(x_i)} (\cup_{\gamma_A^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}(x_i)} \{(\gamma_A^{\sigma(k)}(x_i))^2 | u_A^{\sigma(j)}(x_i) \in H_A(x_i)\}))$.

According to the correlation of EHFSs, the correlation coefficient of EHFSs is given as follows:

Definition 17. *Let A and B be any two EHFSs in X; the correlation coefficient between A and B is defined as:*

$$\rho_{EHFS_3}(A,B) = \frac{C_{EHFS_2}(A,B)}{\sqrt{E_{EHFS_2}(A)E_{EHFS_2}(B^A)} + \sqrt{E_{EHFS_2}(B)E_{EHFS_2}(A^B)}}, \quad (13)$$

where:

$$E_{EHFS_2}(A) = \sum_{i=1}^{n} \frac{1}{\sharp H_A(x_i)} \frac{1}{\sharp u_A(x_i)} S_s(\sum_{j=1}^{\sharp H_A(x_i)} \sum_{k=1}^{\sharp u_A(x_i)} (\cup_{\gamma_A^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}(x_i)} \{(\gamma_A^{\sigma(k)}(x_i))^2 | u_A^{\sigma(j)}(x_i) \in H_A(x_i)\})),$$

$$E_{EHFS_2}(B) = \sum_{i=1}^{n} \frac{1}{\sharp H_B(x_i)} \frac{1}{\sharp u_B(x_i)} S_s(\sum_{j=1}^{\sharp H_B(x_i)} \sum_{k=1}^{\sharp u_B(x_i)} (\cup_{\gamma_B^{\sigma(k)}(x_i) \in u_B^{\sigma(j)}(x_i)} \{(\gamma_B^{\sigma(k)}(x_i))^2 | u_B^{\sigma(j)}(x_i) \in H_B(x_i)\})),$$

$$E_{EHFS_2}(A^B) = \sum_{i=1}^{n} \frac{1}{\sharp H_A(x_i)} \frac{1}{\sharp u_A(x_i)} S_s(\sum_{j=1}^{\sharp H_A(x_i)} \sum_{k=1}^{\sharp u_A(x_i)} (\cup_{\gamma_{A'}^{\sigma(k)}(x_i) \in u_A^{\sigma(j)}(x_i)} \{(\gamma_{A'}^{\sigma(k)}(x_i))^2 | u_A^{\sigma(j)}(x_i) \in H_A(x_i)\})),$$

$$E_{EHFS_2}(B^A)=\sum_{i=1}^{n}\frac{1}{\sharp H_B(x_i)}\frac{1}{\sharp u_B(x_i)}S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_{B'}^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{B'}^{\sigma(k)}(x_i))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\})).$$

Theorem 5. *The correlation coefficient $\rho_{EHFS_3}(A,B)$ for any two EHFSs A and B in X satisfies:*

(1) $\rho_{EHFS_3}(A,B)=\rho_{EHFS_3}(B,A)$*;*
(2) $0\le\rho_{EHFS_3}(A,B)\le 1$*;*
(3) $\rho_{EHFS_3}(A,B)=1$*, if* $A=B$*.*

Proof.

(1) It is straightforward.
(2) From Definition 17, it is apparent that $\rho_{EHFS_3}(A,B)\ge 0$. For $\rho_{EHFS_3}(A,B)\le 1$, using the Cauchy–Schwarz inequality:
$(x_1y_1+x_2y_2,...,x_ny_n)^2\le(x_1^2+x_2^2+,...,x_n^2)(y_1^2+y_2^2+,...,y_n^2)$,
where $(x_1,x_2,...,x_n)\in R^n$, $(y_1,y_2,...,y^n)\in R^n$, we drive:

$$(\sum_{i=1}^{n}(\frac{1}{\sharp H_A(x_i)}\frac{1}{\sharp u_A(x_i)}S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{\gamma_A^{\sigma(k)}(x_i)\gamma_{B'}^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}))))^2$$
$$=(\frac{1}{\sharp H_A(x_1)}\frac{1}{\sharp u_A(x_1)}S_s(\sum_{j=1}^{\sharp H_A(x_1)}\sum_{k=1}^{\sharp u_A(x_1)}(\cup_{\gamma_A^{\sigma(k)}(x_1)\in u_A^{\sigma(j)}(x_1)}\{\gamma_A^{\sigma(k)}(x_1)\gamma_{B'}^{\sigma(k)}(x_1)|u_A^{\sigma(j)}(x_1)\in H_A(x_1)\}))+$$
$$(\frac{1}{\sharp H_A(x_2)}\frac{1}{\sharp u_A(x_2)}S_s(\sum_{j=1}^{\sharp H_A(x_2)}\sum_{k=1}^{\sharp u_A(x_2)}(\cup_{\gamma_A^{\sigma(k)}(x_2)\in u_A^{\sigma(j)}(x_2)}\{\gamma_A^{\sigma(k)}(x_2)\gamma_{B'}^{\sigma(k)}(x_2)|u_A^{\sigma(j)}(x_2)\in H_A(x_2)\}))+...+$$
$$(\frac{1}{\sharp H_A(x_n)}\frac{1}{\sharp u_A(x_n)}S_s(\sum_{j=1}^{\sharp H_A(x_n)}\sum_{k=1}^{\sharp u_A(x_n)}(\cup_{\gamma_A^{\sigma(k)}(x_n)\in u_A^{\sigma(j)}(x_n)}\{\gamma_A^{\sigma(k)}(x_n)\gamma_{B'}^{\sigma(k)}(x_n)|u_A^{\sigma(j)}(x_n)\in H_A(x_n)\})))^2$$
$$=(\frac{1}{\sharp u_A(x_1)}S_s(\sum_{j=1}^{\sharp H_A(x_1)}\sum_{k=1}^{\sharp u_A(x_1)}(\cup_{\gamma_A^{\sigma(k)}(x_1)\in u_A^{\sigma(j)}(x_1)}\{\frac{\gamma_A^{\sigma(k)}(x_1)}{\sqrt{\sharp H_A(x_1)}}\frac{\gamma_{B'}^{\sigma(k)}(x_1)}{\sqrt{\sharp H_A(x_1)}}|u_A^{\sigma(j)}(x_1)\in H_A(x_1)\}))+$$
$$(\frac{1}{\sharp u_A(x_2)}S_s(\sum_{j=1}^{\sharp H_A(x_2)}\sum_{k=1}^{\sharp u_A(x_2)}(\cup_{\gamma_A^{\sigma(k)}(x_2)\in u_A^{\sigma(j)}(x_2)}\{\frac{\gamma_A^{\sigma(k)}(x_2)}{\sqrt{\sharp H_A(x_2)}}\frac{\gamma_{B'}^{\sigma(k)}(x_2)}{\sqrt{\sharp H_A(x_2)}}|u_A^{\sigma(j)}(x_2)\in H_A(x_2)\}))+...+$$
$$(\frac{1}{\sharp u_A(x_n)}S_s(\sum_{j=1}^{\sharp H_A(x_n)}\sum_{k=1}^{\sharp u_A(x_n)}(\cup_{\gamma_A^{\sigma(k)}(x_n)\in u_A^{\sigma(j)}(x_n)}\{\frac{\gamma_A^{\sigma(k)}(x_n)}{\sqrt{\sharp H_A(x_n)}}\frac{\gamma_{B'}^{\sigma(k)}(x_n)}{\sqrt{\sharp H_A(x_n)}}|u_A^{\sigma(j)}(x_n)\in H_A(x_n)\})))^2$$
$$\le(\sum_{i=1}^{n}\frac{1}{\sharp H_A(x_i)}\frac{1}{\sharp u_A(x_i)}S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_A^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}))).$$
$$\sum_{i=1}^{n}\frac{1}{\sharp H_A(x_i)}\frac{1}{\sharp u_A(x_i)}S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_{A'}^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_{A'}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}))$$
$$=C_{EHFS_1}(A,A)\cdot C_{EHFS_1}(B,B).$$

Namely,

$$\sum_{i=1}^{n}(\frac{1}{\sharp H_A(x_i)}\frac{1}{\sharp u_A(x_i)}S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{\gamma_A^{\sigma(k)}(x_i)\gamma_{B'}^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\})))\le\sqrt{E_{EHFS_2}(A)E_{EHFS_2}(B^A)}.$$

Similarly, one can have

$$\sum_{i=1}^{n}(\frac{1}{\sharp H_B(x_i)}\frac{1}{\sharp u_B(x_i)}S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{\gamma_B^{\sigma(k)}(x_i)\gamma_{A'}^{\sigma(k)}(x_i)|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\})))\le\sqrt{E_{EHFS_2}(B)E_{EHFS_2}(A^B)}.$$

Thus,

$C_{EHFS_2}(A,B) \le \sqrt{E_{EHFS_2}(A)E_{EHFS_2}(B^A)} + \sqrt{E_{EHFS_2}(B)E_{EHFS_2}(A^B)}$
The result is obtained.

(3) $A = B \Rightarrow C_{EHFS_2}(A,B) = E_{EHFS_2}(A) = E_{EHFS_2}(B^A) = E_{EHFS_2}(B) = E_{EHFS_2}(A^B) \Rightarrow \rho_{WEHFS_3}(A,B) = 1$.

□

Similar to the correlation coefficient of Definition 14, a modified form of the correlation coefficient of EHFSs is defined by:

$$\rho_{EHFS_4}(A,B) = \frac{1}{2}\left(\frac{C_{EHFS_3}(A,B)}{\sqrt{E_{EHFS_2}(A)E_{EHFS_2}(B^A)}} + \frac{C_{EHFS_3}(B,A)}{\sqrt{E_{EHFS_2}(B)E_{EHFS_2}(A^B)}}\right), \tag{14}$$

where:

$$C_{EHFS_3}(A,B) = \sum_{i=1}^{n}\left(\frac{1}{\sharp H_A(x_i)}\frac{1}{\sharp u_A(x_i)}S_s\left(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}\left(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{\gamma_A^{\sigma(k)}(x_i)\gamma_{B'}^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}\right)\right)\right),$$

$$C_{EHFS_3}(B,A) = \sum_{i=1}^{n}\left(\frac{1}{\sharp H_B(x_i)}\frac{1}{\sharp u_B(x_i)}S_s\left(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}\left(\cup_{\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{\gamma_B^{\sigma(k)}(x_i)\gamma_{A'}^{\sigma(k)}(x_i)|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}\right)\right)\right).$$

Theorem 6. *The correlation coefficient $\rho_{EHFS_4}(A,B)$ for any two EHFSs A and B in X satisfies:*

(1) $\rho_{EHFS_4}(A,B) = \rho_{EHFS_4}(B,A)$;
(2) $0 \le \rho_{EHFS_4}(A,B) \le 1$;
(3) $\rho_{EHFS_4}(A,B) = 1$, *if* $A = B$.

Proof. Similar to the proof of Theorem 2, we can easily obtain the conclusions. □

Inspired by Definition 14, the correlation coefficients of EHFSs, for any two EHFSs A and B in X, are defined as follows:

$$\rho_{EHFS_5}(A,B) = \frac{C_{EHFS_2}(A,B)}{max\{E_{EHFS_2}(A),E_{EHFS_2}(B^A)\} + \max\{E_{EHFS_2}(B),E_{EHFS_2}(A^B)\}}, \tag{15}$$

$$\rho_{EHFS_6}(A,B) = \frac{1}{2}\left(\frac{C_{EHFS_3}(A,B)}{max\{E_{EHFS_2}(A),E_{EHFS_2}(B^A)\}} + \frac{C_{EHFS_3}(B,A)}{\max\{E_{EHFS_2}(B),E_{EHFS_2}(A^B)\}}\right), \tag{16}$$

Theorem 7. *The correlation coefficients $\rho_{EHFS_i}(A,B)$ $(i = 5,6)$ for any two EHFSs A and B in X satisfies:*

(1) $\rho_{EHFS_i}(A,B) = \rho_{EHFS_i}(B,A)$;
(2) $0 \le \rho_{EHFS_i}(A,B) \le 1$;
(3) $\rho_{EHFS_i}(A,B) = 1$, *if* $A = B$.

Proof. Similar to the proofs of Theorems 2 and 5, the conclusions obviously hold. □

Example 3. *Now, we calculate Example 2 by the new correlation coefficients $\rho_{EHFS_i}(A,B)$ $(i = 3,4,5,6)$ without taking into account the length of EHFEs.*

The calculation process is given as follows:

$$\begin{aligned}E_{EHFS_2}(A) &= \sum_{i=1}^{n}\frac{1}{\sharp H_A(x_i)}\frac{1}{\sharp u_A(x_i)}S_s\left(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}\left(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_A^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}\right)\right)\\
&= \frac{1}{2}\left[\frac{1}{3}(0.3^2+0.4^2+0.5^2) + \frac{1}{3}(0.3^2+0.4^2+0.6^2)\right] + \frac{1}{3}\left[\frac{1}{3}(0.4^2+0.3^2+0.2^2)\right.\\
&\left.+ \frac{1}{3}(0.4^2+0.3^2+0.1^2) + \frac{1}{3}(0.5^2+0.3^2+0.4^2)\right]\\
&= 0.3017,\end{aligned}$$

$$E_{EHFS_2}(B)=\sum_{i=1}^{n}\frac{1}{\sharp H_B(x_i)}\frac{1}{\sharp u_B(x_i)}S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_B^{\sigma(k)}(x_i))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))$$
$$=\frac{1}{3}(0.1^2+0.2^2+0.5^2)+\frac{1}{2}(\frac{1}{3}(0.4^2+0.4^2+0.2^2)+\frac{1}{3}(0.4^2+0.4^2+0.1^2)]$$
$$=0.2150,$$

$$E_{EHFS_2}(A^B)=\sum_{i=1}^{n}\frac{1}{\sharp H_A(x_i)}\frac{1}{\sharp u_A(x_i)}S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_{A'}^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_{A'}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\})),$$
$$=\frac{1}{3}(0.3^2+0.4^2+0.5^2)+\frac{1}{2}(\frac{1}{3}(0.4^2+0.3^2+0.2^2)+\frac{1}{3}(0.4^2+0.3^2+0.1^2)]$$
$$=0.2583,$$

$$E_{EHFS_2}(B^A)=\sum_{i=1}^{n}\frac{1}{\sharp H_B(x_i)}\frac{1}{\sharp u_B(x_i)}S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_{B'}^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{B'}^{\sigma(k)}(x_i))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))$$
$$=\frac{1}{2}[\frac{1}{3}(0.1^2+0.2^2+0.5^2)+\frac{1}{3}(0.1^2+0.2^2+0.5^2)]+\frac{1}{3}[\frac{1}{3}(0.4^2+0.4^2+0.2^2)$$
$$+\frac{1}{3}(0.4^2+0.4^2+0.2^2)+\frac{1}{3}(0.4^2+0.4^2+0.1^2)]$$
$$=0.2167,$$

$$C_{EHFS_3}(A,B)=\sum_{i=1}^{n}(\frac{1}{\sharp H_A(x_i)}\frac{1}{\sharp u_A(x_i)}S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{\gamma_A^{\sigma(k)}(x_i)\gamma_{B'}^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}))$$
$$=\frac{1}{2}[\frac{1}{3}(0.3\times0.1+0.4\times0.2+0.5\times0.5)+\frac{1}{3}(0.3\times0.1+0.4\times0.2+0.6\times0.5)]+\frac{1}{3}[\frac{1}{3}(0.4\times0.4+$$
$$0.3\times0.4+0.2\times0.2)+\frac{1}{3}(0.4\times0.4+0.3\times0.4+0.1\times0.1)+\frac{1}{3}(0.5\times0.4+0.4\times0.4+0.4\times0.2)]$$
$$=0.2450,$$

$$C_{EHFS_3}(B,A)=\sum_{i=1}^{n}(\frac{1}{\sharp H_B(x_i)}\frac{1}{\sharp u_B(x_i)}S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{\gamma_B^{\sigma(k)}(x_i)\gamma_{A'}^{\sigma(k)}(x_i)|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\})))$$
$$=\frac{1}{3}(0.1\times0.3+0.2\times0.4+0.5\times0.5)+\frac{1}{2}(\frac{1}{3}(0.4\times0.4+0.4\times0.3+0.2\times0.2)+$$
$$\frac{1}{3}(0.4\times0.4+0.4\times0.3+0.1\times0.1)]$$
$$=0.2217,$$

$C_{EHFS_2}(A,B)=C_{EHFS_3}(A,B)+C_{EHFS_3}(B,A)=0.4667.$

Finally, we can calculate the correlation coefficients:

$$\rho_{EHFS_3}(A,B)=\frac{C_{EHFS_2}(A,B)}{\sqrt{E_{EHFS_2}(A)E_{EHFS_2}(B^A)}+\sqrt{E_{EHFS_2}(B)E_{EHFS_2}(A^B)}}=0.9498,$$
$$\rho_{EHFS_4}(A,B)=\frac{1}{2}(\frac{C_{EHFS_3}(A,B)}{\sqrt{E_{EHFS_2}(A)E_{EHFS_2}(B^A)}}+\frac{C_{EHFS_3}(B,A)}{\sqrt{E_{EHFS_2}(B)E_{EHFS_2}(A^B)}})=0.8333,$$
$$\rho_{EHFS_5}(A,B)=\frac{C_{EHFS_2}(A,B)}{max\{E_{EHFS_2}(A),E_{EHFS_2}(B^A)\}+\max\{E_{EHFS_2}(B),E_{EHFS_2}(A^B)\}}=0.9495,$$
$$\rho_{EHFS_6}(A,B)=\frac{1}{2}(\frac{C_{EHFS_3}(A,B)}{max\{E_{EHFS_2}(A),E_{EHFS_2}(B^A)\}}+\frac{C_{EHFS_3}(B,A)}{\max\{E_{EHFS_2}(B),E_{EHFS_2}(A^B)\}})=0.8352.$$

To save all of the information provided by the DMs, distinguish them from each other and consider their different importance in decision making, we now propose the weighted extended hesitant correlation coefficients considering DMs as follows. Assume a decision making problem with m DMs. For any MU, $u=\{\gamma_1,\gamma_2,...,\gamma_m\}$, the weights of DMs are $\omega_D(D=1,2,...,m)$ with $\omega_D\in[0,1]$ and $\sum_{D=1}^{m}=1$. Let $\gamma_{\omega_D}=\omega_D\gamma_D$ be memberships associated with the DMs' weights. On the other hand, in practical applications, the elements x_i $(i=1,2,...,n)$ in the universe X have different weights.

Let $w = (w_i, w_2, ..., w_n)^T$ be the weight vector of x_i $(i = 1, 2, ..., n)$ with $w_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} w_i = 1$, we further extend the correlation coefficient formulas given in Table 1.

Table 1. The correlation coefficients of EHFSs.

ρ of EHFSs	Correlation Coefficient Formulas
$\rho_{WEHFS_1}(A_\omega, B_\omega)$	$\frac{C_{WEHFS_1}(A_\omega,B_\omega)}{\sqrt{C_{WEHFS_1}(A_\omega,A_\omega)}\sqrt{C_{WEHFS_1}(B_\omega,B_\omega)}}$
$\rho_{WEHFS_2}(A_\omega, B_\omega)$	$\frac{C_{WEHFS_1}(A_\omega,B_\omega)}{max\{C_{WEHFS_1}(A_\omega,A_\omega),C_{WEHFS_1}(B_\omega,B_\omega)}$
$\rho_{WEHFS_3}(A_\omega, B_\omega)$	$\frac{C_{WEHFS_2}(A_\omega,B_\omega)}{\sqrt{E_{WEHFS_2}(A_\omega)E_{WEHFS_2}(B_\omega^A)}+\sqrt{E_{WEHFS_2}(B_\omega)E_{WEHFS_2}(A_\omega^B)}}$
$\rho_{WEHFS_4}(A_\omega, B_\omega)$	$\frac{1}{2}(\frac{C_{WEHFS_3}(A_\omega,B_\omega)}{\sqrt{E_{WEHFS_2}(A_\omega)E_{WEHFS_2}(B_\omega^A)}} + \frac{C_{WEHFS_3}(B_\omega,A_\omega)}{\sqrt{E_{WEHFS_2}(B_\omega)E_{WEHFS_2}(A_\omega^B)}})$
$\rho_{WEHFS_5}(A_\omega, B_\omega)$	$\frac{C_{WEHFS_2}(A_\omega,B_\omega)}{max\{E_{WEHFS_2}(A_\omega),E_{WEHFS_2}(B_\omega^A)\}+\max\{E_{WEHFS_2}(B_\omega),E_{WEHFS_2}(A_\omega^B)\}}$
$\rho_{WEHFS_6}(A_\omega, B_\omega)$	$\frac{1}{2}(\frac{C_{WEHFS_3}(A_\omega,B_\omega)}{max\{E_{WEHFS_2}(A_\omega),E_{WEHFS_2}(B_\omega^A)\}} + \frac{C_{WEHFS_3}(B_\omega,A_\omega)}{\max\{E_{WEHFS_2}(B_\omega),E_{WEHFS_2}(A_\omega^B)\}})$

where:

$$C_{WEHFS_1}(A_\omega, B_\omega) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{\gamma_{A_{\omega_D}}^{\sigma(k)}(x_i)\gamma_{B_{\omega_D}}^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i),$$
$$u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))),$$

$$C_{WEHFS_1}(A_\omega, A_\omega) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{A_{\omega_D}}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i),$$
$$u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))),$$

$$C_{WEHFS_1}(B_\omega, B_\omega) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{B_{\omega_D}}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i),$$
$$u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))),$$

$$E_{WEHFS_2}(A_\omega) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_{A_{\omega_D}}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}))),$$

$$E_{WEHFS_2}(B_\omega) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{B_{\omega_D}}^{\sigma(k)}(x_i))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))),$$

$$E_{WEHFS_2}(A_\omega^B) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_{A'}^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_{A'_{\omega_D}}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}))),$$

$$E_{WEHFS_2}(B_\omega^A) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_{B'}^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{B'_{\omega_D}}^{\sigma(k)}(x_i))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))).$$

$$C_{WEHFS_3}(A_\omega, B_\omega) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{\gamma_{A\omega_D}{}^{\sigma(k)}(x_i)\gamma_{B'_{\omega_D}}^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}))),$$

$$C_{WEHFS_3}(B_\omega, A_\omega) = \sum_{i=1}^{n}(w_i S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{\gamma_{B\omega_D}{}^{\sigma(k)}(x_i)\gamma_{A'_{\omega_D}}^{\sigma(k)}(x_i)|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\}))).$$

$C_{WEHFS_2}(A, B) = C_{WEHFS_3}(A, B) + C_{WEHFS_3}(B, A)$.

It can be seen that if $w_i = (1/n, 1/n, ..., 1/n)^T$ and $\omega_i = (1/m, 1/m, ..., 1/m)$, then $\rho_{WEHFS_i}(A_\omega, B_\omega)$ $(i = 1, 2, 3, ..., 6)$ reduce to $\rho_{EHFS_i}(A, B)$ $(i = 1, 2, 3, ..., 6)$. Additionally, it is easy to prove that $\rho_{WEHFS_i}(A_\omega, B_\omega)$ $(i = 1, 2, 3, ..., 6)$ also have the following properties:

Theorem 8. *Let $w = (w_i, w_2, ..., w_n)^T$ be the weight vector of x_i $(i = 1, 2, ..., n)$ with $w_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} w_i = 1$ and $\omega = (\omega_i, \omega_2, ..., \omega_n)^T$ be the weight vector of DMs with $\omega_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} \omega_i = 1$; the correlation coefficients $\rho_{WEHFS_i}(A_\omega, B_\omega)$ $(i = 1, 2, 3, ..., 6)$ between two EHFSs A and B satisfy:*

(1) $\rho_{WEHFS_i}(A_\omega, B_\omega) = \rho_{WEHFS_i}(B_\omega, A_\omega)$;
(2) $0 \leq \rho_{WEHFS_i}(A_\omega, B_\omega) \leq 1$;
(3) $\rho_{WEHFS_i}(A_\omega, B_\omega) = 1$, *if* $A = B$.

However, sometimes, the exact weights w_i of elements x_i are unknown, we present the weighted extended hesitant correlation coefficient of EHFEs as Table 2.

Table 2. The correlation coefficients of EHFEs.

ρ of EHFEs	Correlation Coefficient Formulas
$\rho_{EHFE_1}(A_\omega, B_\omega)$	$\frac{C_{EHFE_1}(A_\omega,B_\omega)}{\sqrt{C_{EHFE_1}(A_\omega,A_\omega)}\sqrt{C_{EHFE_1}(B_\omega,B_\omega)}}$
$\rho_{EHFE_2}(A_\omega, B_\omega)$	$\frac{C_{EHFE_1}(A_\omega,B_\omega)}{max\{C_{EHFE_1}(A_\omega,A_\omega),C_{EHFE_1}(B_\omega,B_\omega)}$
$\rho_{EHFE_3}(A_\omega, B_\omega)$	$\frac{C_{EHFE_2}(A_\omega,B_\omega)}{\sqrt{E_{EHFE_2}(A_\omega)E_{EHFE_2}(B_\omega^A)}+\sqrt{E_{EHFE_2}(B_\omega)E_{EHFE_2}(A_\omega^B)}}$
$\rho_{EHFE_4}(A_\omega, B_\omega)$	$\frac{1}{2}(\frac{C_{EHFS_3}(A_\omega,B_\omega)}{\sqrt{E_{EHFE_2}(A_\omega)E_{EHFE_2}(B_\omega^A)}}+\frac{C_{EHFE_3}(B_\omega,A_\omega)}{\sqrt{E_{EHFE_2}(B_\omega)E_{EHFE_2}(A_\omega^B)}})$
$\rho_{EHFE_5}(A_\omega, B_\omega)$	$\frac{C_{EHFE_2}(A_\omega,B_\omega)}{max\{E_{EHFE_2}(A_\omega),E_{EHFE_2}(B_\omega^A)\}+\max\{E_{EHFE_2}(B_\omega),E_{EHFE_2}(A_\omega^B)\}}$
$\rho_{EHFS_6}(A_\omega, B_\omega)$	$\frac{1}{2}(\frac{C_{EHFE_3}(A_\omega,B_\omega)}{max\{E_{EHFE_2}(A_\omega),E_{EHFE_2}(B_\omega^A)\}}+\frac{C_{EHFE_3}(B_\omega,A_\omega)}{\max\{E_{EHFE_2}(B_\omega),E_{EHFE_2}(A_\omega^B)\}})$

where:

$$C_{EHFE_1}(A_\omega, B_\omega) = \frac{1}{\sharp H} S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{\gamma_{A_{\omega_D}}^{\sigma(k)}(x_i)\gamma_{B_{\omega_D}}^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i),$$
$$u_B^{\sigma(j)}(x_i)\in H_B(x_i)\})),$$

$$C_{EHFE_1}(A_\omega, A_\omega) = \frac{1}{\sharp H} S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{A_{\omega_D}}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i),$$
$$u_B^{\sigma(j)}(x_i)\in H_B(x_i)\})),$$

$$C_{EHFE_1}(B_\omega, B_\omega) = \frac{1}{\sharp H} S_s(\sum_{j=1}^{\sharp H}\sum_{k=1}^{\sharp u}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)},\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{B_{\omega_D}}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i),$$
$$u_B^{\sigma(j)}(x_i)\in H_B(x_i)\})),$$

$$E_{EHFE_2}(A_\omega) = \frac{1}{\sharp H_A(x_i)} S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_{A_{\omega_D}}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\})),$$

$$E_{EHFE_2}(B_\omega) = \frac{1}{\sharp H_B(x_i)} S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_B^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{B_{\omega_D}}^{\sigma(k)}(x_i))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\})),$$

$$E_{EHFE_2}(A_\omega^B) = \frac{1}{\sharp H_A(x_i)} S_s(\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_{A'}^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{(\gamma_{A'_{\omega_D}}^{\sigma(k)}(x_i))^2|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\})),$$

$$E_{EHFE_2}(B_\omega^A) = \frac{1}{\sharp H_B(x_i)} S_s(\sum_{j=1}^{\sharp H_B(x_i)}\sum_{k=1}^{\sharp u_B(x_i)}(\cup_{\gamma_{B'}^{\sigma(k)}(x_i)\in u_B^{\sigma(j)}(x_i)}\{(\gamma_{B'_{\omega_D}}^{\sigma(k)}(x_i))^2|u_B^{\sigma(j)}(x_i)\in H_B(x_i)\})).$$

$$C_{EHFE_3}(A_\omega, B_\omega) = \tfrac{1}{\sharp H_A(x_i)} S_s(\textstyle\sum_{j=1}^{\sharp H_A(x_i)}\sum_{k=1}^{\sharp u_A(x_i)}(\cup_{\gamma_A^{\sigma(k)}(x_i)\in u_A^{\sigma(j)}(x_i)}\{\gamma_{A\omega_D}{}^{\sigma(k)}(x_i)\gamma_{B'_{\omega_D}}^{\sigma(k)}(x_i)|u_A^{\sigma(j)}(x_i)\in H_A(x_i)\}),$$

$$C_{EHFE_3}(B_\omega, A_\omega) = \frac{1}{\sharp H_B(x_i)} S_s\left(\sum_{j=1}^{\sharp H_B(x_i)} \sum_{k=1}^{\sharp u_B(x_i)} \left(\cup_{\gamma_B^{\sigma(k)}(x_i) \in u_B^{\sigma(j)}(x_i)} \{\gamma_{B\omega_D}{}^{\sigma(k)}(x_i) \gamma_{A'_{\omega_D}}^{\sigma(k)}(x_i) | u_B^{\sigma(j)}(x_i) \in H_B(x_i)\}\right)\right),$$

$C_{EHFE_2}(A, B) = C_{EHFE_3}(A, B) + C_{EHFE_3}(B, A)$.

Additionally, it is easy to prove that $\rho_{EHFE_i}(A_\omega, B_\omega)$ $(i = 1, 2, 3, ..., 6)$ also have the following properties:

Theorem 9. *Let $\omega = (\omega_i, \omega_2, ..., \omega_n)^T$ be the weight vector of DMs with $\omega_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} \omega_i = 1$; the correlation coefficients $\rho_{EHFE_i}(A_\omega, B_\omega)$ $(i = 1, 2, 3, ..., 6)$ between two EHFSs A and B satisfy:*

(1) $\rho_{EHFE_i}(A_\omega, B_\omega) = \rho_{EHFE_i}(B_\omega, A_\omega)$;
(2) $0 \leq \rho_{EHFE_i}(A_\omega, B_\omega) \leq 1$;
(3) $\rho_{EHFE_i}(A_\omega, B_\omega) = 1$, *if* $A = B$.

4. Application of the Weighted Correlation Coefficients of the Extend Hesitant Fuzzy Environment

In this section, we shall utilize the weighted correlation coefficients of EHFEs and EHFSs to decision making with extended hesitant fuzzy information.

At first, we utilize the weighted correlation coefficients of EHFSs for decision making problems with extended hesitant fuzzy information. For a decision making problem with extended hesitant fuzzy information, let $A = \{A_1, A_2, ..., A_m\}$ be a discrete set of alternatives and $S = \{S_1, S_2, ..., S_n\}$ be a set of attributes. If the decision makers provide several values for the alternative A_i $(i = 1, 2, ..., m)$ under the attribute C_j $(j = 1, 2, ..., n)$, these values can be considered as an EHFE $H_{ij}(j = 1, 2, .., n; i = 1, 2, ..., m)$. Therefore, we can elicit an extended hesitant fuzzy decision matrix $H = (H_{ij})_{m \times n}$, where $H_{ij}(i = 1, 2, .., m; j = 1, 2, ..., n)$ is in the form of extended hesitant fuzzy elements. In multiple attribute decision making environments, the concept of ideal point has been used to help the identification of the best alternative in the decision set. Although the ideal alternative does not exist in the real world, it does provide a useful theoretical construct to evaluate alternatives. Therefore, we define each ideal EHFE H^* in the ideal alternative $A^* = \{< s_j, H^* > | s_j \in S\}$ $(j = 1, 2, .., n)$.

Method 1:

Step 1. Use the information given by decision makers to establish extended hesitant fuzzy model and construct the extended hesitant fuzzy decision matrix $H = (H_{ij})_{m \times n}$ by EHFEs.

Step 2. Assume that the weights of decision makers ω_D and attributes w and a standard EHFE $H^* = \{(1, 1, 1, 1)\}$ are known; calculate the weighted correlation coefficients between an alternative $A_i(i = 1, 2, .., m)$ and the ideal alternative A^* by using the formulas $\rho_{WEHFS_i}(A_\omega, B_\omega)$ $(i = 1, 2, ..., 6)$ (see Table 1).

Step 3. Rank the alternatives in accordance with the values of $\rho_{WEHFS_i}(A_\omega, B_\omega)$ $(i = 1, 2, ..., 6)$. We may obtain different results and rankings to analyze different weighted correlation coefficients.

Step 4. Select the best alternative according to the maximum values of the weighted correlation coefficients $\rho_{WEHFS_i}(A_\omega, B_\omega)$ $(i = 1, 2, ..., 6)$.

Step 5. End.

However, sometimes, the exact weights w_i of elements x_i are unknown; we present the weighted extended hesitant correlation coefficient of EHFEs with the Dempster–Shafer belief structure [35]. Let C_{ij} be a payoff to the alternative A_i, and the state of nature is S_j, $C = (C_{ij})_{m \times n}$ a payoff matrix and ς the optimized parameter. The DMs knowledge of the states of nature is captured in terms of a belief structure p with the focal elements $B_1, B_2, ..., B_r$, each of which is associated with a weight $p_{(B_k)}$, where $\sum_{k=1}^{r} p_{(B_k)} = 1$. We now develop the following approach to deal with group decision making. The method is similar to Zhu et al.'s (see [16] in detail) as follows.

Method 2:

Step 1. Use the information given by decision makers to establish the extended hesitant fuzzy model and construct the extended hesitant fuzzy decision matrix $H = (H_{ij})_{m\times n}$ by EHFEs.

Step 2. Assume a standard EHFE $H^* = \{(1,1,1,1)\}$ and the optimized parameter, then calculate the correlation coefficients between H^* and H_{ij} by the extended hesitant correlation coefficients of EHFEs in Table 2. Let C_{ij} be equal to the weighted correlation coefficients, and construct the payoff matrix of $\rho_{EHFE_{ij}}(A_\omega, B_\omega)(i = 1,2,...,6, j = 1,2,...,5)$, denoted as C_{ij}.

Step 3. Calculate the belief function p about the states of nature (see [16] in detail).

Step 4. Utilize the optimized parameter to calculate the collection of weights [36,37], which are used in the *OWA* aggregation for each cardinality of focal elements (see [16] in detail).

Step 5. Determine the payoff collection, $M_{ik} = \{C_{ij}|S_j \in B_k\}$, which is a set of payoffs that are possible if we select the alternative A_i and the focal element B_k occurs, and calculate the aggregated payoff, $V_{ik} = OWA(M_{ik})$ (see [16] in detail).

Step 6. Calculate $C_i = \sum_{k=1}^{r} V_{ik} p(B_k)$, and select the alternative that has the best generalized expected value as the optimal alternative (see [16] in detail).

Step 7. End.

Example 4. *[16] Energy is an indispensable factor for the social-economic development of societies. Thus, the correct energy policy affects economic development and the environment; the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) A_i $(i = 1,2,3,4,5)$ to be invested and four criteria to be considered: S_1-technological; S_2-environmental; S_3-socio-political; S_4-economic. Five DMs are invited to evaluate the performances of the five alternatives.*

In order to avoid giving DMs' preferences anonymously given by [6] and deal with this energy policy problem without information loss, the DMs $D_k(k = 1,2,3,4,5)$ provide their preferences over all of the alternatives $A_i(i = 1,2,...,5)$ with respect to the criteria $S_j(j = 1,2,3,4)$ based on hesitant fuzzy sets, then Zhu et al. [16] saved the DMs' preferences by an extended hesitant fuzzy matrix $H = (H_{ij})_{5\times 4}$ shown in Table 3 (see [16] for detail).

Table 3. Extended hesitant fuzzy decision matrix.

	S_1	S_2
A_1	$\{(0.3, 0.4, 0.3, 0.4, 0.5)\}$	$\{(0.7, 0.8, 0.3, 0.8, 0.6), (0.7, 0.8, 0.4, 0.8, 0.6)\}$
A_2	$\{(0.3, 0.4, 0.5, 0.2, 0.5), (0.3, 0.4, 0.5, 0.3, 0.5)\}$	$\{(0.5, 0.6, 0.5, 0.6, 0.6)\}$
A_3	$\{(0.4.0.5.0.5.0.5.0.6)\}$	{(0.5,0.6,0.7,0.6,0.5), (0.6,0.6,0.7,0.6,0.5), (0.5,0.6,0.8,0.6,0.5), (0.6,0.6,0.8,0.6,0.5)}
A_4	$\{(0.3, 0.2, 0.2, 0.3, 0.1)\}$	$\{(0.6, 0.5, 0.7, 0.5, 0.5)\}$
A_5	$\{(0.3, 0.4, 0.6, 0.2, 0.2), (0.3, 0.3, 0.6, 0.2, 0.2)\}$	$\{0.6, 0.8, 0.5, 0.4, 0.6), (0.6, 0.8, 0.5, 0.5, 0.6)\}$
	S_3	S_4
A_1	{(0.3,0.4,0.2,0.3,0.2), (0.4,0.4,0.2,0.3,0.2,), (0.3,0.4,0.3,0.3,0.2), (0.4,0.4,0.3,0.3,0.2)}	{(0.6,0.5,0.5,0.4,0.6)}
A_2	{(0.6,0.4,0.5,0.3,0.5), (0.6,0.4,0.4,0.3,0.5)}	{(0.3,0.4,0.5,0.2,0.2),(0.3,0.4,0.4,0.2,0.2)}
A_3	{(0.7.0.3.0.9.0.8.0.6), (0.7,0.3,0.8,0.8,0.6)}	{(0.7,0.8,0.7,0.8,0.8)}
A_4	{(0.4,0.3,0.2,0.3,0.5)}	{(0.3,0.2,0.7,0.2,0.1)}
A_5	{(0.7,0.5,0.6,0.8,0.6)}	{(0.6,0.4,0.5,0.4,0.6), (0.7,0.4,0.5,0.4,0.6)}

(i) We now use Method 2 to solve the decision making problem first as a comparison to Zhu et al. [16].

Step 1. The decision makers $D_k(k = 1,2,3,4,5)$ provide their preferences over all of the alternatives $A_i(i = 1,2,...,5)$ with respect to the criteria $S_j(j = 1,2,3,4)$ shown in Table 3.

Step 2. Let $A^* = \{(1,1,1,1,1)\}$ be the ideal values of the alternative seen as a standard EHFE H^*, $\varsigma = 0.75$ be the optimized parameter and $\omega = (0.3, 0.1, 0.3, 0.2, 0.1)$ be the weighting vector of the DMs. By Table 2, we can calculate the correlation coefficients between H^* and H_{ij}. Then, construct the payoff matrix shown as Tables 4–8.

Table 4. The payoff matrix of $\rho_{EHFE_1}(A_\omega, B_\omega)$.

	S_1	S_2	S_3	S_4
A_1	0.986883	0.988202	0.97547	0.990684
A_2	0.967077	0.996616	0.974345	0.958094
A_3	0.993058	0.987096	0.986648	0.998221
A_4	0.974355	0.991976	0.955336	0.88474
A_5	0.922226	0.989024	0.992889	0.982754

Table 5. The payoff matrix of $\rho_{EHFE_2}(A_\omega, B_\omega)$.

	S_1	S_2	S_3	S_4
A_1	0.329167	0.608333	0.3	0.525
A_2	0.38125	0.525	0.48125	0.339583
A_3	0.466667	0.633333	0.752083	0.725
A_4	0.25	0.6125	0.955336	0.88474
A_5	0.39375	0.545833	0.666667	0.539583

Table 6. The payoff matrix of $\rho_{EHFE_3}(A_\omega, B_\omega)$.

	S_1	S_2	S_3	S_4
A_1	0.986883	0.989857	0.981194	0.990684
A_2	0.968555	0.996616	0.976718	0.953532
A_3	0.993058	0.987891	0.98561	0.998221
A_4	0.974355	0.991976	0.955336	0.88474
A_5	0.922823	0.990514	0.992889	0.980396

Table 7. The payoff matrix of $\rho_{EHFE_4}(A_\omega, B_\omega)$.

	S_1	S_2	S_3	S_4
A_1	0.986883	0.989843	0.980905	0.990684
A_2	0.968545	0.996616	0.976679	0.953671
A_3	0.993058	0.987869	0.985624	0.998221
A_4	0.974355	0.991976	0.955336	0.88474
A_5	0.922822	0.990505	0.992889	0.980441

Table 8. The payoff matrix of $\rho_{EHFE_5}(A_\omega, B_\omega)$ and $\rho_{EHFE_6}(A_\omega, B_\omega)$.

	S_1	S_2	S_3	S_4
A_1	0.329167	0.614583	0.31875	0.525
A_2	0.384375	0.525	0.490625	0.348958
A_3	0.466667	0.662083	0.761458	0.725
A_4	0.25	0.6125	0.955336	0.88474
A_5	0.394792	0.55	0.666667	0.548958

Step 3. The DMs analyze the energy policy problem so as to obtain the probabilistic information about the states of nature. Assume that the DMs; knowledge of the states of nature consists of the following belief structure, shown in Table 9 (see [16]).

Table 9. Belief structure.

Focal Element	Weights (w)
$B_1 = \{S_1, S_3\}$	0.15
$B_2 = \{S_2, S_4\}$	0.25
$B_3 = \{S_1, S_3, S_4\}$	0.6

Step 4. In order to contrast with the method's results given by [16], we still use the O'Hagan method [36] to obtain weighting vectors associated with the OWA operators for various numbers of arguments. Since $\varsigma = 0.75$, then we can get the weighting vectors shown in Table 10 (see [16]).

Table 10. Weighting vectors for various numbers of arguments.

Number of Arguments	w_1	w_2	w_3
2	0.75	0.25	
2	0.62	0.27	0.11

Step 5. We get V_{ik} for all i and $k(i = 1,2,3,4; k = 1,2,3,4,5)$ as $M_{ik} = \{C_{ij}|S_j \in B_k\}$ and $V_{ik} = OWA(M_{ik})$.

Step 6. Calculate $C_i = \sum_{k=1}^{r} V_{ik}p(B_k)$, the results are given in Table 11. Thus, A_3 is the optimal alternative closest to the ideal values of alternative, which is the same with Zhu and Xu (see the Example in [16]).

Step 7. End.

Table 11. Weighting vectors for various numbers of arguments.

	A_1	A_2	A_3	A_4	A_5	Rankings
Results about ρ_{EHFE_1}	0.985342	0.972911	0.991325	0.962349	0.956626	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
Results about ρ_{EHFE_2}	0.400856	0.422799	0.588053	0.351558	0.495452	$A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$
Results about ρ_{EHFE_3}	0.986794	0.973515	0.991267	0.962349	0.956892	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
Results about ρ_{EHFE_4}	0.986734	0.97352	0.991265	0.962349	0.956895	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$
Results about ρ_{EHFE_5}	0.405769	0.427388	0.593439	0.351558	0.497943	$A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$
Results about ρ_{EHFE_6}	0.405769	0.427388	0.593439	0.351558	0.497943	$A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$

As can be seen from Table 11, A_3 is the optimal alternative closest to the ideal values of alternative, which is the same with Zhu and Xu (see the Example in [16]), but the rankings are not always the same: the rankings of $\rho_{EHFE_2}, \rho_{EHFE_5}, \rho_{EHFE_6}$ and Zhu and Xu [16] are identical, but the rankings of $\rho_{EHFE_1}, \rho_{EHFE_3}$ and ρ_{EHFE_4} are different from [16]. Obviously, the method, which depends on the formulas of $\rho_{EHFE_i}(i = 1,2,3,4,5,6)$, is practical and effective.

(ii) As the weights of the elements x_i $(i = 1,2,...,n)$ in the universe X are easily accessible in practical applications, in other words, the weights of $S_i(i = 1,2,3,4)$ in this example are given by decision makers. Assume that $w_i = (0.3, 0.2, 0.2, 0.3)$, then we can deal with the decision making problem by Method 1 as follows:

Step 1. Construct the extended hesitant fuzzy decision matrix $H = (H_{ik})_{m\times n}$ by EHFEs, $H_{ij}(i = 1,...,q; j = 1,...,n)$, shown in Table 3.

Steps 2 and 3. Calculate the weighted correlation coefficient between an alternative $A_i(i = 1,2,..,m)$ and the ideal alternative A^* by using the formulas $\rho_{WEHFS_i}(A,B)$ $(i = 1,2,...,6)$. The results are given in Table 12.

Step 4. As can be seen from the results, we get that A_3 is the optimal alternative closest to the ideal values of the alternative.

Step 5. End.

Table 12. Weighting vectors for various numbers of arguments.

	A_1	A_2	A_3	A_4	A_5	Rankings
$Results\ of\ \rho_{EHFS_1}$	0.986005	0.971744	0.992133	0.947191	0.967877	$A_3 \succ A_1 \succ A_2 \succ A_5 \succ A_4$
$Results\ of\ \rho_{EHFS_2}$	0.437917	0.4175	0.634583	0.385417	0.5225	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$
$Results\ of\ \rho_{EHFS_3}$	0.987480	0.971293	0.992084	0.947191	0.967646	$A_3 \succ A_1 \succ A_2 \succ A_5 \succ A_4$
$Results\ of\ \rho_{EHFS_4}$	0.987420	0.971324	0.992082	0.947191	0.967658	$A_3 \succ A_1 \succ A_2 \succ A_5 \succ A_4$
$Results\ of\ \rho_{EHFS_5}$	0.442917	0.4231	0.640208	0.385417	0.526458	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$
$Results\ of\ \rho_{EHFS_6}$	0.442917	0.4231	0.640208	0.385417	0.526458	$A_3 \succ A_5 \succ A_1 \succ A_2 \succ A_4$

The example indicates that the proposed decision making methods are simple and effective under extended hesitant fuzzy environments.

5. Conclusions

In this study, we develop some correlation coefficients between EHFSs, which contain two cases: the correlation coefficients taking into account the length of EHFEs and the correlation coefficients without taking into account the length of EHFEs. We also have studied some properties of these correlation coefficients. At last, we give two methods to deal with decision making problems under extended hesitant fuzzy environments, and a real-world example based on the energy policy problem is employed to illustrate the actual need for dealing with the difference of evaluation information provided by different experts without information loss in decision making processes. As EHFSs are a new powerful tool to express uncertain information in the process of group decision making, we will give more studies on the theory and applications in the future.

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