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On Submanifolds in a Riemannian Manifold with a Semi-Symmetric Non-Metric Connection

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Academic Editor: Angel Garrido
Received: 26 May 2017; Accepted: 28 June 2017; Published: 8 July 2017

Abstract: In this paper, we study submanifolds in a Riemannian manifold with a semi-symmetric non-metric connection. We prove that the induced connection on a submanifold is also semi-symmetric non-metric connection. We consider the total geodesicness and minimality of a submanifold with respect to the semi-symmetric non-metric connection. We obtain the Gauss, Cadazzi, and Ricci equations for submanifolds with respect to the semi-symmetric non-metric connection.

Keywords: semi-symmetric non-metric connection; submanifold

1. Introduction

In 1924, Friedmann and Schouten [1] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\widehat{\nabla}$ on a differentiable manifold $\widetilde{M}$ is said to be a semi-symmetric connection if the torsion $\widehat{T}$ of the connection $\widehat{\nabla}$ satisfies

$$\widehat{T}(\widetilde{X},\widetilde{Y}) = \pi(\widetilde{Y})\widetilde{X} - \pi(\widetilde{X})\widetilde{Y}, \tag{1}$$

where π is a 1-form.

In 1932, Hayden [2] introduced the notion of a semi-symmetric metric connection on a Riemannian manifold $(\widetilde{M}, g)$. A semi-symmetric connection $\widehat{\nabla}$ is said to be a semi-symmetric metric connection if

$$\widehat{\nabla} g = 0. \tag{2}$$

Yano [3] studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. Submanifolds of a Riemannian manifold with a semi-symmetric metric connection were studied by Nakao [4].

After a long gap, the study of a semi-symmetric connection $\widehat{\nabla}$ satisfying

$$\widehat{\nabla} g \neq 0 \tag{3}$$

was initiated by Prvanovic [5] with the name pseudo-metric semi-symmetric connection, and was just followed by Smaranda and Andonie [6].

A semi-symmetric connection $\widehat{\nabla}$ is said to be a semi-symmetric non-metric connection if it satisfies the condition Equation (3).

In 1992, Agashe and Chafle [7] introduced a semi-symmetric non-metric connection on a Riemannian manifold $(\widetilde{M}, g)$ given by

$$\check{\check{\nabla}}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} + \pi(\widetilde{X})\widetilde{Y},$$

where $\widetilde{\nabla}$ is the Levi-Civita connection of $(\widetilde{M}, g)$ and π is a 1-form. Agashe and Chafle [8] studied submanifolds of a Riemannian manifold with this semi-symmetric non-metric connection. In 2000, Sengupta, De, and Binh [9] gave another type of semi-symmetric non-metric connection. Özgür [10] studied properties of submanifolds of a Riemannian manifold with this semi-symmetric non-metric connection. Recently, De, Han, and Zhao [11] introduced a new type of semi-symmetric non-metric connection which is given by

$$\check{\check{\nabla}}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} + a\omega(\widetilde{X})\widetilde{Y} + b\omega(\widetilde{Y})\widetilde{X}, \tag{4}$$

where a and b are two non-zero real numbers and ω is a 1-form. They proved the existence of this new type of linear connection and studied a Riemannian manifold admitting this type of semi-symmetric non-metric connection in [11].

Motivated by [8] and [10], we have studied submanifolds of a Riemannian manifold endowed with the semi-symmetric non-metric connection defined by Equation (4) in this paper. The paper has been organized as follows: In Section 2, we give some properties of the semi-symmetric non-metric connection; In Section 3, we consider a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection and show that the induced connection on the submanifold is also a semi-symmetric non-metric connection. We also consider the total geodesicness and minimality of a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection; In section 4, we deduce the Gauss, Codazzi, and Ricci equations with respect to the semi-symmetric non-metric connection. Using this Gauss equation, we give the relation between the sectional curvatures with respect to the semi-symmetric non-metric connection of a Riemannian manifold and a submanifold, which is analogous to Synger's inequality [12]. Finally, we consider these fundamental equations of a submanifold in a space form with constant curvature with the semi-symmetric non-metric connection.

2. Preliminaries

Let $\widetilde{M}$ be an $(n+d)$-dimensional Riemannian manifold with a Riemannian metric g and $\widetilde{\nabla}$ be the Levi-Civita connection of $(\widetilde{M}, g)$. De, Han, and Zhao [11] defined a special type of linear connection on $\widetilde{M}$ by

$$\check{\check{\nabla}}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} + a\omega(\widetilde{X})\widetilde{Y} + b\omega(\widetilde{Y})\widetilde{X}, \tag{5}$$

where a and b are two non-zero real numbers and ω is a 1-form on $\widetilde{M}$. Denote by $\widetilde{U} = \omega^\sharp$, i.e., the vector field $\widetilde{U}$ is defined by $\omega(\widetilde{X}) = g(\widetilde{X}, \widetilde{U})$ for all $\widetilde{X} \in \mathcal{X}(\widetilde{M})$, $\mathcal{X}(\widetilde{M})$ is the set of all differentiable vector fields on $\widetilde{M}$.

By Equation (5), the torsion tensor $\check{\check{T}}$ with respect to the connection $\check{\check{\nabla}}$ is given by

$$\check{\check{T}}(\widetilde{X}, \widetilde{Y}) = (b-a)\omega(\widetilde{Y})\widetilde{X} - (b-a)\omega(\widetilde{X})\widetilde{Y} = \pi(\widetilde{Y})\widetilde{X} - \pi(\widetilde{X})\widetilde{Y},$$

where $\pi(\widetilde{X}) = (b-a)\omega(\widetilde{X})$ is a 1-form.

Therefore, the connection $\check{\check{\nabla}}$ is a semi-symmetric connection. Additionally,

$$(\check{\check{\nabla}}_{\widetilde{X}}g)(\widetilde{Y}, \widetilde{Z}) = -2a\omega(\widetilde{X})g(\widetilde{Y}, \widetilde{Z}) - b\omega(\widetilde{Y})g(\widetilde{X}, \widetilde{Z}) - b\omega(\widetilde{Z})g(\widetilde{X}, \widetilde{Y}) \neq 0.$$

Hence, the semi-symmetric connection $\check{\check{\nabla}}$ defined by Equation (5) is a semi-symmetric non-metric connection.

Analogous to the definition of the curvature tensor $\widetilde{R}$ of $\widetilde{M}$ with respect to the Levi-Civita connection $\widetilde{\nabla}$, we define the curvature tensor $\check{\widetilde{R}}$ of $\widetilde{M}$ with respect to the semi-symmetric non-metric connection $\check{\widetilde{\nabla}}$ given by

$$\check{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \check{\widetilde{\nabla}}_{\widetilde{X}}\check{\widetilde{\nabla}}_{\widetilde{Y}}\widetilde{Z} - \check{\widetilde{\nabla}}_{\widetilde{Y}}\check{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Z} - \check{\widetilde{\nabla}}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, \tag{6}$$

where $\widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathcal{X}(\widetilde{M})$.

Using Equations (5) and (6), we have

$$\begin{aligned}\check{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z} &= \widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} - a(\widetilde{\nabla}_{\widetilde{Y}}\omega)(\widetilde{X})\widetilde{Z} + a(\widetilde{\nabla}_{\widetilde{X}}\omega)(\widetilde{Y})\widetilde{Z} - b(\widetilde{\nabla}_{\widetilde{Y}}\omega)(\widetilde{Z})\widetilde{X} \\ &\quad + b(\widetilde{\nabla}_{\widetilde{X}}\omega)(\widetilde{Z})\widetilde{Y} + b^2\omega(\widetilde{Y})\omega(\widetilde{Z})\widetilde{X} - b^2\omega(\widetilde{X})\omega(\widetilde{Z})\widetilde{Y}.\end{aligned} \tag{7}$$

The Riemannian Christoffel tensors of the connections $\widetilde{\nabla}$ and $\check{\widetilde{\nabla}}$ are defined by

$$\widetilde{R}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}) = g(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z},\widetilde{W})$$

and

$$\check{\widetilde{R}}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}) = g(\check{\widetilde{R}}(\widetilde{X},\widetilde{Y})\widetilde{Z},\widetilde{W}),$$

respectively.

3. Submanifolds of a Riemannian Manifold with the Semi-Symmetric Non-Metric Connection $\check{\widetilde{\nabla}}$

Let M be an n-dimensional submanifold of an $(n+d)$-dimensional Riemannian manifold with the semi-symmetric non-metric connection $\check{\widetilde{\nabla}}$. We decompose the vector field $\widetilde{U}$ on M uniquely into their tangent and normal components $U^{\top}, U^{\perp}$.

The Gauss formula for the submanifold M with respect to the Levi-Civita connection $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall X,Y \in \mathcal{X}(M), \tag{8}$$

where h is the second fundamental form of M in $\widetilde{M}$.

For the second fundament form h, the covariant of h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \quad \forall X,Y,Z \in \mathcal{X}(M). \tag{9}$$

Then, $\overline{\nabla} h$ is a normal bundle valued tensor of type $(0,3)$ and is called the third fundamental form of M. $\overline{\nabla}$ is called the van der Waerden–Bortolotti connection of M; i.e., $\overline{\nabla}$ is the connection in $TM \oplus T^{\perp}M$ built with ∇ and $\nabla^{\perp}$.

Let $\check{\nabla}$ be the induced connection from the semi-symmetric non-metric connection $\check{\widetilde{\nabla}}$. We define

$$\check{\widetilde{\nabla}}_X Y = \check{\nabla}_X Y + \check{h}(X,Y), \quad \forall X,Y \in \mathcal{X}(M), \tag{10}$$

where $\check{h}$ is a $(1,2)$-tensor field in $T^{\perp}M$, the normal part of M. The Equation (10) may be called the Gauss formula for M with respect to the semi-symmetric non-metric connection $\check{\widetilde{\nabla}}$.

Using Equations (5), (8), and (10), we have

$$\check{\nabla}_X Y + \check{h}(X,Y) = \nabla_X Y + h(X,Y) + a\omega(X)Y + b\omega(Y)X. \tag{11}$$

Comparing the tangential and normal parts of Equation (11), we obtain

$$\check{\nabla}_X Y = \nabla_X Y + a\omega(X)Y + b\omega(Y)X \tag{12}$$

and

$$\check{h}(X,Y) = h(X,Y). \tag{13}$$

From Equation (12), we have

$$\check{T}(X,Y) = \check{\nabla}_X Y - \check{\nabla}_Y X - [X,Y] = (b-a)\omega(Y)X - (b-a)\omega(X)Y \tag{14}$$

where $\check{T}$ is the torsion tensor of the connection $\check{\nabla}$ on M. Moreover, using Equation (12), we have

$$\begin{aligned}(\check{\nabla}_X g)(Y,Z) &= \check{\nabla}_X(g(Y,Z)) - g(\check{\nabla}_X Y, Z) - g(Y, \check{\nabla}_X Z) \\ &= -2a\omega(X)g(Y,Z) - b\omega(Y)g(X,Z) - b\omega(Z)g(X,Y) \\ &\neq 0.\end{aligned} \tag{15}$$

In view of Equations (12), (14), and (15), we can state the following theorem:

Theorem 1. *The induced connection $\check{\nabla}$ on a submanifold of a Riemannian manifold endowed with the semi-symmetric non-metric connection $\check{\check{\nabla}}$ is also a semi-symmetric non-metric connection.*

If $\check{h}(X,Y) = 0$ for all $X, Y \in \mathcal{X}(M)$, then M is called totally geodesic with respect to the semi-symmetric non-metric connection. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of the tangent space of M. We define the mean curvature vector $\check{H}$ of M with respect to the semi-symmetric non-metric connection by

$$\check{H} = \frac{1}{n}\sum_{i=1}^{n} \check{h}(e_i, e_i). \tag{16}$$

From Equation (13) we know that

$$\check{H} = H, \tag{17}$$

where H is the mean curvature vector of the submanifold M. If $\check{H} = 0$, then M is called minimal with respect to the semi-symmetric non-metric connection.

From Equations (13) and (17), we have the following result:

Theorem 2. *Let M be an n-dimensional submanifold of an $(n+d)$-dimensional Riemannian manifold $\check{M}$ with the semi-symmetric non-metric connection $\check{\check{\nabla}}$. Then,*
(1) M is totally geodesic with respect to the semi-symmetric non-metric connection if and only if M is totally geodesic with respect to the Levi-Civita connection.
(2) M is minimal with respect to the semi-symmetric non-metric connection if and only if M is minimal with respect to the Levi-Civita connection.

Let ξ be a normal vector field on M. From Equation (5), we have

$$\check{\check{\nabla}}_X \xi = \widetilde{\nabla}_X \xi + a\omega(X)\xi + b\omega(\xi)X. \tag{18}$$

It is well known that the Weingarten formula for a submanifold of a Riemannian manifold is given by

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{19}$$

where A_ξ is the shape operator of M in the direction of ξ.

Using Equation (19), we can write Equation (18) as

$$\check{\check{\nabla}}_X \xi = -A_\xi X + b\omega(\xi)X + \nabla^\perp_X \xi + a\omega(X)\xi. \tag{20}$$

Now we define a $(1, 1)$-tensor field on M by

$$\check{A}_\xi = (A_\xi - b\omega(\xi))I. \tag{21}$$

Then, Equation (20) turns into

$$\check{\check{\nabla}}_X \xi = -\check{A}_\xi X + \nabla^\perp_X \xi + a\omega(X)\xi. \tag{22}$$

Equation (22) is called the Weingarten formula for M with respect to the semi-symmetric non-metric connection.

Since A_ξ is symmetric, it is easy to verify that

$$g(\check{A}_\xi X, Y) = g(X, \check{A}_\xi Y)$$

and

$$g([\check{A}_\xi, \check{A}_\eta]X, Y) = g([A_\xi, A_\eta]X, Y), \tag{23}$$

where $[\check{A}_\xi, \check{A}_\eta] = \check{A}_\xi \check{A}_\eta - \check{A}_\eta \check{A}_\xi$, $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$ and ξ, η are normal vector fields on M.

From Equations (21) and (23), we can also obtain the following theorems:

Theorem 3. *Principal directions of the unit normal vector ξ with respect to the Levi-Civita connection $\widetilde{\nabla}$ and the semi-symmetric non-metric connection $\check{\check{\nabla}}$, and the principle curvatures are equal if and only if ξ is orthogonal to $U^\perp$.*

Theorem 4. *Let M be a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection $\check{\check{\nabla}}$. Then, the shape operators with respect to the semi-symmetric non-metric connection are simultaneously diagonalizable if and only if the shape operators with respect to the Levi-Civita connection are simultaneously diagonalizable.*

4. Gauss, Codazzi, and Ricci Equations with Respect to the Semi-Symmetric Non-Metric Connection

We denote the curvature tensor of a submanifold M of a Riemannian manifold $\widetilde{M}$ with respect to the induced semi-symmetric non-metric connection $\check{\nabla}$ and the induced Levi-Civita connection ∇ by

$$\check{R}(X, Y)Z = \check{\nabla}_X \check{\nabla}_Y Z - \check{\nabla}_Y \check{\nabla}_X Z - \check{\nabla}_{[X,Y]} Z \tag{24}$$

and

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

respectively, where $X, Y, Z \in \mathcal{X}(M)$.

Theorem 5. *Let M be a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection $\check{\nabla}$. Then, for all $X, Y, Z, W \in \mathcal{X}(M)$, we have*

$$\begin{aligned} \check{\check{R}}(X, Y, Z, W) = \check{R}(X, Y, Z, W) - g(\check{h}(Y, Z), \check{h}(X, W)) + g(\check{h}(X, Z), \check{h}(Y, W)) \\ + b\omega(\check{h}(Y, Z))g(X, W) - b\omega(\check{h}(X, Z))g(Y, W). \end{aligned} \tag{25}$$

Here Equation (25) is called the Gauss equation for the submanifold M with respect to the semi-symmetric non-metric connection.

Proof. From Equations (10) and (20), we have

$$\check{\check{\nabla}}_X \check{\check{\nabla}}_Y Z = \check{\nabla}_X \check{\nabla}_Y Z + \check{h}(X, \check{\nabla}_Y Z) - A_{\check{h}(Y,Z)} X \\ + b\omega(\check{h}(Y,Z))X + \nabla^{\perp}_X \check{h}(Y,Z) + a\omega(X)\check{h}(Y,Z), \tag{26}$$

$$\check{\check{\nabla}}_Y \check{\check{\nabla}}_X Z = \check{\nabla}_Y \check{\nabla}_X Z + \check{h}(Y, \check{\nabla}_X Z) - A_{\check{h}(X,Z)} Y \\ + b\omega(\check{h}(X,Z))Y + \nabla^{\perp}_Y \check{h}(X,Z) + a\omega(Y)\check{h}(X,Z), \tag{27}$$

and

$$\check{\check{\nabla}}_{[X,Y]} Z = \check{\nabla}_{[X,Y]} Z + \check{h}([X,Y], Z). \tag{28}$$

Using Equations (24), (26)–(28), we obtain

$$\begin{aligned} \check{\check{R}}(X,Y)Z = \check{R}(X,Y)Z &+ \check{h}(X, \check{\nabla}_Y Z) - \check{h}(Y, \check{\nabla}_X Z) - \check{h}([X,Y], Z) \\ &- A_{\check{h}(Y,Z)} X + A_{\check{h}(X,Z)} Y + b\omega(\check{h}(Y,Z))X - b\omega(\check{h}(X,Z))Y \\ &+ \nabla^{\perp}_X \check{h}(Y,Z) - \nabla^{\perp}_Y \check{h}(X,Z) + a\omega(X)\check{h}(Y,Z) - a\omega(Y)\check{h}(X,Z). \end{aligned} \tag{29}$$

Since $g(A_\xi X, Y) = g(h(X,Y), \xi)$ and $h = \check{h}$, from Equation (29) we find

$$\begin{aligned} \check{\check{R}}(X,Y,Z,W) &= \check{R}(X,Y,Z,W) - g(A_{\check{h}(Y,Z)} X, W) + g(A_{\check{h}(X,Z)} Y, W) \\ &\quad + b\omega(\check{h}(Y,Z))g(X,W) - b\omega(\check{h}(X,Z))g(Y,W) \\ &= \check{R}(X,Y,Z,W) - g(\check{h}(Y,Z), \check{h}(X,W)) + g(\check{h}(X,Z), \check{h}(Y,W)) \\ &\quad + b\omega(\check{h}(Y,Z))g(X,W) - b\omega(\check{h}(X,Z))g(Y,W). \end{aligned}$$

□

Recalling that if $\pi \in T_pM$ is a 2-dimensional subspace of T_pM spanned by an orthonormal base $\{X, Y\}$, we define the sectional curvature $\check{K}(\pi)$ with respect to the semi-symmetric non-metric connection as $\check{R}(X,Y,Y,X)$. Let $\check{\check{K}}(\pi)$ denote the corresponding sectional curvature in $\widetilde{M}$. As an application of the Gauss Equation (25), we can obtain the following Synger's inequality with respect to the semi-symmetric non-metric connection.

Corollary 1. *Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with the semi-symmetric non-metric connection $\check{\check{\nabla}}$ and γ be a geodesic in $\widetilde{M}$ which lies in M, and T be a unit tangent vector field of γ. π is a subspace of the tangent space T_pM spanned by $\{X, T\}$. Then,*
(1) $\check{\check{K}}(\pi) \geq \check{K}(\pi)$ along γ.
(2) if X is a unit tangent vector field on M which is parallel along γ and orthogonal to T, then the equality of (1) holds if and only if X is parallel along γ in $\widetilde{M}$.

Proof. (1) Let γ be a geodesic in $\widetilde{M}$ which lies in M and T be a unit tangent vector field of γ. Then, we have

$$h(T, T) = 0. \tag{30}$$

Let π be a subspace of the tangent space T_pM spanned by an orthonormal base $\{X, T\}$. Applying the Gauss Equation (25) and $h = \check{h}$, we obtain

$$\begin{aligned} \check{\check{K}}(\pi) &= \check{\check{R}}(X,T,T,X) \\ &= \check{R}(X,T,T,X) - g(h(X,X), h(T,T)) + g(h(X,T), (X,T)) + b\omega(h(T,T)) \\ &= \check{K}(\pi) + g(h(X,T), (X,T)) \\ &\geq \check{K}(\pi). \end{aligned} \tag{31}$$

(2) If X be parallel along γ, we have $\nabla_T X = 0$. Thus, we have

$$\widetilde{\nabla}_T X = h(T, X).$$

Then, the equality of Equation (31) holds if and only if $h(X, T) = 0$; i.e., $\widetilde{\nabla}_T X = 0$. □

Theorem 6. *Let M be a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection $\check{\nabla}$. Then, for all $X, Y, Z \in \mathcal{X}(M)$, we have*

$$(\check{\check{R}}(X,Y)Z)^{\perp} = \check{\check{\nabla}}_X \check{h}(Y,Z) - \check{\check{\nabla}}_Y \check{h}(X,Z) - (b-a)\omega(X)\check{h}(Y,Z) + (b-a)\omega(Y)\check{h}(X,Z), \tag{32}$$

where $\check{\check{\nabla}}_X \check{h}(Y,Z) = \nabla^{\perp}_X \check{h}(Y,Z) - \check{h}(\check{\nabla}_X Y, Z) - \check{h}(Y, \check{\nabla}_X Z)$. Equation (32) is called the Codazzi equation with respect to the semi-symmetric non-metric connection.

Proof. From Equation (29), the normal component of $\check{\check{R}}(X,Y)Z$ is given by

$$\begin{aligned}
(\check{\check{R}}(X,Y)Z)^{\perp} &= \check{h}(X, \check{\nabla}_Y Z) - \check{h}(Y, \check{\nabla}_X Z) - \check{h}([X,Y], Z) + \nabla^{\perp}_X \check{h}(Y,Z) \\
&\quad - \nabla^{\perp}_Y \check{h}(X,Z) + a\omega(X)\check{h}(Y,Z) - a\omega(Y)\check{h}(X,Z) \\
&= \nabla^{\perp}_X \check{h}(Y,Z) - \nabla^{\perp}_Y \check{h}(X,Z) - \check{h}(Y, \check{\nabla}_X Z) + \check{h}(X, \check{\nabla}_Y Z) \\
&\quad - \check{h}(\check{\nabla}_X Y - \check{\nabla}_Y X + (b-a)\omega(X)Y - (b-a)\omega(Y)X, Z) \\
&\quad + a\omega(X)\check{h}(Y,Z) - a\omega(Y)\check{h}(X,Z) \\
&= \check{\check{\nabla}}_X \check{h}(Y,Z) - \check{\check{\nabla}}_Y \check{h}(X,Z) \\
&\quad - (b-2a)\omega(X)\check{h}(Y,Z) + (b-2a)\omega(Y)\check{h}(X,Z),
\end{aligned}$$

where $\check{\check{\nabla}}_X \check{h}(Y,Z) = \nabla^{\perp}_X \check{h}(Y,Z) - \check{h}(\check{\nabla}_X Y, Z) - \check{h}(Y, \check{\nabla}_X Z)$. □

Remark 1. *$\check{\check{\nabla}}$ is the connection in $TM \oplus T^{\perp}M$ built with $\check{\nabla}$ and $\nabla^{\perp}$. It may be called the van der Waerden–Bortolotti connection with respect to the semi-symmetric non-metric connection.*

Theorem 7. *Let M be a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection $\check{\nabla}$. Then, for all $X, Y \in \mathcal{X}(M)$ and normal vector fields ξ, μ on M, we have*

$$\check{\check{R}}(X,Y,\xi,\mu) = R^{\perp}(X,Y,\xi,\mu) - g([A_{\xi}, A_{\mu}]X, Y) + a[g(Y, \nabla_X U^{\top}) - g(X, \nabla_Y U^{\top})]g(\xi,\mu). \tag{33}$$

Equation (33) is called the Ricci equation for the submanifold M with respect to the semi-symmetric non-metric connection.

Proof. From Equations (10) and (22), we get

$$\begin{aligned}
\check{\check{\nabla}}_X \check{\check{\nabla}}_Y \xi &= -\check{\nabla}_X(\check{A}_{\xi} Y) - \check{h}(X, \check{A}_{\xi} Y) - \check{A}_{\nabla^{\perp}_Y \xi} X + \nabla^{\perp}_X \nabla^{\perp}_Y \xi \\
&\quad + a\omega(X)\nabla^{\perp}_Y \xi + ag(\widetilde{\nabla}_X Y, U^{\top})\xi + ag(Y, \widetilde{\nabla}_X U^{\top})\xi \\
&\quad + a\omega(Y)\widetilde{\nabla}_X \xi + a^2\omega(X)\omega(Y)\xi + ab\omega(Y)\omega(\xi)X,
\end{aligned} \tag{34}$$

$$\begin{aligned}
\check{\check{\nabla}}_Y \check{\check{\nabla}}_X \xi &= -\check{\nabla}_Y(\check{A}_{\xi} X) - \check{h}(Y, \check{A}_{\xi} X) - \check{A}_{\nabla^{\perp}_X \xi} Y + \nabla^{\perp}_Y \nabla^{\perp}_X \xi \\
&\quad + a\omega(Y)\nabla^{\perp}_X \xi + ag(\widetilde{\nabla}_Y X, U^{\top})\xi + ag(X, \widetilde{\nabla}_Y U^{\top})\xi \\
&\quad + a\omega(X)\widetilde{\nabla}_Y \xi + a^2\omega(Y)\omega(X)\xi + ab\omega(X)\omega(\xi)Y
\end{aligned} \tag{35}$$

and
$$\check{\check{\nabla}}_{[X,Y]}\xi = -\check{A}_{\xi}[X,Y] + \nabla^{\perp}_{[X,Y]}\xi + ag([X,Y],U^{\top})\xi. \tag{36}$$

Using Equations (34)–(36), we have

$$\begin{aligned}\check{\check{R}}(X,Y,\xi,\mu) &= g(\check{\check{R}}(X,Y)\xi),\mu)\\ &= R^{\perp}(X,Y,\xi,\mu) - g(\check{h}(X,\check{A}_{\xi}Y),\mu) + g(\check{h}(Y,\check{A}_{\xi}X),\mu)\\ &\quad + a[g(Y,\widetilde{\nabla}_X U^{\top}) - g(X,\widetilde{\nabla}_Y U^{\top})]g(\xi,\mu).\end{aligned}$$

In view of Equations (10), (13), and (21), the above equation turns into

$$\begin{aligned}\check{\check{R}}(X,Y,\xi,\mu) &= R^{\perp}(X,Y,\xi,\mu) - g(h(X,A_{\xi}Y),\mu) + g(h(Y,A_{\xi}X),\mu)\\ &\quad + a[g(Y,\nabla_X U^{\top}) - g(X,\nabla_Y U^{\top})]g(\xi,\mu)\\ &= R^{\perp}(X,Y,\xi,\mu) - g((A_{\xi}A_{\mu} - A_{\mu}A_{\xi})X,Y)\\ &\quad + a[g(Y,\nabla_X U^{\top}) - g(X,\nabla_Y U^{\top})]g(\xi,\mu)\\ &= R^{\perp}(X,Y,\xi,\mu) - g([A_{\xi},A_{\mu}]X,Y)\\ &\quad + a[g(Y,\nabla_X U^{\top}) - g(X,\nabla_Y U^{\top})]g(\xi,\mu)\end{aligned}$$

□

It will be useful to examine the form of our fundamental equations with respect to the semi-symmetric non-metric connection when the ambient space $\widetilde{M}$ has constant curvature. Now, assume that $\widetilde{M}$ is an $(n+d)$-dimensional space form of constant curvature C with the semi-symmetric non-metric connection $\check{\check{\nabla}}$. Let M be a submanifold of $\widetilde{M}$. Then, from Equation (7) we have

$$\begin{aligned}\check{\check{R}}(X,Y)Z &= C[g(Y,Z)X - g(X,Z)Y] - a(\widetilde{\nabla}_Y\omega)(X)Z + a(\widetilde{\nabla}_X\omega)(Y)Z\\ &\quad - b(\widetilde{\nabla}_Y\omega)(Z)X + b(\widetilde{\nabla}_X\omega)(Z)Y + b^2\omega(Y)\omega(Z)X - b^2\omega(X)\omega(Z)Y,\end{aligned} \tag{37}$$

where $X,Y,Z \in \mathcal{X}(M)$.

Hence from Equation (25) we know that the Gauss equation becomes

$$\begin{aligned}\check{R}(X,Y)Z &= C[g(Y,Z)X - g(X,Z)Y] - a(\widetilde{\nabla}_Y\omega)(X)Z + a(\widetilde{\nabla}_X\omega)(Y)Z\\ &\quad - b(\widetilde{\nabla}_Y\omega)(Z)X + b(\widetilde{\nabla}_X\omega)(Z)Y + b^2\omega(Y)\omega(Z)X - b^2\omega(X)\omega(Z)Y\\ &\quad + g(\check{h}(Y,Z),\check{h}(X,W)) - g(\check{h}(X,Z),\check{h}(Y,W))\\ &\quad - b\omega(\check{h}(Y,Z))g(X,W) + b\omega(\check{h}(X,Z))g(Y,W).\end{aligned}$$

From Equation (37) we know
$$(\check{\check{R}}(X,Y)Z)^{\perp} = 0$$

So from Equation (32) we know that the Codazzi equation becomes

$$\check{\check{\nabla}}_X\check{h}(Y,Z) - \check{\check{\nabla}}_Y\check{h}(X,Z) = (b-2a)\omega(X)\check{h}(Y,Z) - (b-2a)\omega(Y)\check{h}(X,Z).$$

Since $\widetilde{M}$ is a space form of constant C, it follows that $\widetilde{R}(X,Y,\xi,\mu) = 0$. On the other hand, from Equation (37) we have

$$\begin{aligned}\check{\check{R}}(X,Y,\xi,\mu) &= a[(\check{\nabla}_X\omega)Y - (\check{\nabla}_Y\omega)X]g(\xi,\mu) \\ &= a[X(g(U^\top,Y)) - g(\nabla_X Y,U^\top) - Y(g(U^\top,X)) - g(\nabla_Y X,U^\top)]g(\xi,\mu) \\ &= a[g(\nabla_X U^\top,Y) - g(\nabla_Y U^\top,X)g(\xi,\mu).\end{aligned} \tag{38}$$

Then, using Equations (33) and (38), we obtain that the Ricci equation becomes

$$R^\perp(X,Y,\xi,\mu) = g([A_\xi,A_\mu]X,Y) \tag{39}$$

Using Equations (23) and (39), we can state the following result:

Corollary 2. *Let M be a submanifold of a space form of constant curvature with the semi-symmetric non-metric connection $\check{\check{\nabla}}$. Then, the normal connection $\nabla^\perp$ is flat if and only if all second fundamental tensors with respect to the Levi-Civita connection and the semi-symmetric non-metric connection are simultaneously diagonalizable.*

Example. Let $\mathbb{T}^2 : S^1(1) \times S^1(1) \in \mathbb{R}^4$ be a torus embedded in $\mathbb{R}^4$ defined by

$$\mathbb{T}^2 = \{(\cos u, \sin u, \cos v, \sin v) : u, v \in \mathbb{R}\}.$$

For $p = (\cos u, \sin u, \cos v, \sin v)$, $T_P(\mathbb{T}^2)$ is spanned by

$$e_1 = (-\sin u, \cos u, 0, 0),$$

$$e_2 = (0, 0 - \sin v, \cos v)$$

and $T_P^\perp(\mathbb{T}^2)$ is spanned by

$$e_3 = (\cos u, \sin u, 0, 0),$$

$$e_4 = (0, 0, \cos v, \sin v).$$

Differentiating these, we get

$$\begin{aligned}&\widetilde{\nabla}_{e_1}e_1 = -e_3, \quad \widetilde{\nabla}_{e_1}e_2 = 0, \quad \widetilde{\nabla}_{e_1}e_3 = e_1, \quad \widetilde{\nabla}_{e_1}e_4 = 0, \\ &\widetilde{\nabla}_{e_2}e_1 = 0, \quad \widetilde{\nabla}_{e_2}e_2 = -e_4, \quad \widetilde{\nabla}_{e_2}e_3 = 0, \quad \widetilde{\nabla}_{e_2}e_4 = e_2.\end{aligned} \tag{40}$$

Let ω be a 1-form on $\mathbb{R}^4$. A semi-symmetric non-metric connection $\check{\check{\nabla}}$ on $\mathbb{R}^4$ is given by

$$\widetilde{\check{\check{\nabla}}}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} + a\pi(\widetilde{X})\widetilde{Y} + b\pi(\widetilde{Y})\widetilde{X}. \tag{41}$$

From Equations (40) and (41), we have

$$\begin{aligned}&\check{\check{\nabla}}_{e_1}e_1 = -e_3 + (a+b)\omega(e_1)e_1, \ \ \check{\check{\nabla}}_{e_1}e_2 = a\omega(e_1)e_2 + b\omega(e_2)e_1, \\ &\check{\check{\nabla}}_{e_2}e_1 = a\omega(e_2)e_1 + b\omega(e_1)e_2, \ \ \check{\check{\nabla}}_{e_2}e_2 = -e_4 + (a+b)\omega(e_2)e_2.\end{aligned} \tag{42}$$

Using Equation (42), we obtain

$$\begin{aligned}&\check{\nabla}_{e_1}e_1 = (a+b)\omega(e_1)e_1, \ \ \check{\nabla}_{e_1}e_2 = a\omega(e_1)e_2 + b\omega(e_2)e_1, \\ &\check{\nabla}_{e_2}e_1 = a\omega(e_2)e_1 + b\omega(e_1)e_2, \ \ \check{\nabla}_{e_2}e_2 = (a+b)\omega(e_2)e_2\end{aligned} \tag{43}$$

and

$$\check{h}(e_1,e_1) = -e_3, \ \ \check{h}(e_1,e_2) = \check{h}(e_2,e_1) = 0, \ \ \check{h}(e_2,e_2) = -e_4. \tag{44}$$

From Equation (43), we have

$$\begin{aligned}\check{T}(e_1,e_2) &= \check{\nabla}_{e_1}e_2 - \check{\nabla}_{e_2}e_1 - [e_1,e_2]\\ &= (a-b)\omega(e_2)e_1 - (a-b)\omega(e_1)e_2.\end{aligned} \tag{45}$$

and

$$\begin{aligned}&(\check{\nabla}_{e_1}g)(e_1,e_1) = -2(a+b)\omega(e_1),\ \ (\check{\nabla}_{e_1}g)(e_1,e_2) = -b\omega(e_2),\\ &(\check{\nabla}_{e_1}g)(e_2,e_2) = -2a\omega(e_1),\ \ (\check{\nabla}_{e_2}g)(e_1,e_1) = -2a\omega(e_2),\\ &(\check{\nabla}_{e_2}g)(e_1,e_2) = -b\omega(e_2),\ \ (\check{\nabla}_{e_2}g)(e_2,e_2) = -2(a+b)\omega(e_2).\end{aligned} \tag{46}$$

Equations (45) and (46) show that the induced connection $\check{\nabla}$ is also a semi-symmetric non-metric connection.

Using Equation (44), we know that the mean curvature vector of $\mathbb{T}^2$ with respect to the semi-symmetric non-metric connection is

$$\check{H} = \frac{1}{2}[\check{h}(e_1,e_1) + \check{h}(e_2,e_2)] = -\frac{1}{2}(e_3+e_4).$$

Acknowledgments: This work is supported by National Natural Science Foundation of China (No. 11371194, No. 11571172) and Natural Science Foundation of the Anhui Higher Education Institutions of China (No. KJ2017A324).

Author Contributions: All authors contributed equally and significantly this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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