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Group Decision-Making for Hesitant Fuzzy Sets Based on Characteristic Objects Method

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Abstract: There are many real-life problems that, because of the need to involve a wide domain of knowledge, are beyond a single expert. This is especially true for complex problems. Therefore, it is usually necessary to allocate more than one expert to a decision process. In such situations, we can observe an increasing importance of uncertainty. In this paper, the Multi-Criteria Decision-Making (MCDM) method called the Characteristic Objects Method (COMET) is extended to solve problems for Multi-Criteria Group Decision-Making (MCGDM) in a hesitant fuzzy environment. It is a completely new idea for solving problems of group decision-making under uncertainty. In this approach, we use L-R-type Generalized Fuzzy Numbers (GFNs) to get the degree of hesitancy for an alternative under a certain criterion. Therefore, the classical COMET method was adapted to work with GFNs in group decision-making problems. The proposed extension is presented in detail, along with the necessary background information. Finally, an illustrative numerical example is provided to elaborate the proposed method with respect to the support of a decision process. The presented extension of the COMET method, as opposed to others' group decision-making methods, is completely free of the rank reversal phenomenon, which is identified as one of the most important MCDM challenges.

Keywords: hesitant fuzzy sets; L-R-type generalized fuzzy numbers; Multi-Criteria Group Decision-Making (MCGDM); Characteristic Objects Method (COMET)

1. Introduction

For human activities and their problems, the Multi-Criteria Group Decision-Making (MCGDM) is an important tool [1,2]. In complex real-world conditions, it is not possible for a single Decision-Maker (DM) to recognize all of the relevant aspects of a decision-making problem [3]. Thus, the decision-making procedure requires considering many DMs or experts from different fields. In many group decision-making problems, a group is established by various DMs from different fields, including work experience, education backgrounds and knowledge structure [4]. It could be implemented to select the most suitable alternative from a given set of decision variants or their subset [5,6]. The essential prerequisite of the MCGDM is the combination of experts' preferences and judgments about the candidate alternatives versus the conflicting criteria [7], which is a popular trend of present research to develop new group MCDM methods [8–11].

In the decision-making, the problems of uncertainty and hesitancy usually turn out to be unavoidable. To express the DMs' evaluation information more objectively, several tools have been developed, such as fuzzy set [1,12], interval-valued fuzzy set [13,14], linguistic fuzzy set [15–17], which allow one to present an element's membership function as a set denoted by a fuzzy number, an interval fuzzy number, a linguistic variable and a fuzzy set, respectively. Intuitionistic fuzzy set [18] and fuzzy multiset [19,20] are another two generalizations of the fuzzy set. Whilst the former contains three types of information (the membership, the non-membership and the hesitancy), the latter permits the elements to repeat more than once.

In many practical problems, sometimes, it is difficult to define the membership grade of an element, because of a set of possible membership values [21]. This issue is very important in MCGDM problems, when the DMs do not support the same membership grade for an element [22,23]. In this case, the difficulty of establishing a common membership grade is caused not by the margin of error (as happens in Intuitionistic Fuzzy Set (IFS)) or some possible distribution values (as happens in Type-2 Fuzzy Sets), but by the fact that several membership values are possible [10]. To deal with these cases, the Hesitant Fuzzy Set (HFS) was introduced [24] as a new generalization of fuzzy sets. Many MCDM methods have been extended by using the HFS theory, e.g., the ELECTRE family methods [25], Viekriterijumsko Kompromisno Rangiranje (VIKOR) [26] or prospect theory [27]. There was also established a number of new methods [13,28–30] or aggregation operators [31,32], which are based on the HFS concept. Presently, group decision-making problems are solved for hesitant fuzzy sets and with aggregation operators in [33–36]. Interval-valued hesitant fuzzy sets have been used in the applications of group decision-making in [28,37–40]. MCGDM with hesitant two-tuple linguistic information and by using trapezoidal valued HFSs is discussed in [41,42]. Yu [43] gave the concept of triangular hesitant fuzzy sets and used it for the solution of decision-making problems. Unfortunately, all of the mentioned group decision-making methods are susceptible to the occurrence of the rank reversal phenomenon paradox, which lies at the heart of the main MCDM challenges.

The Characteristic Objects Method (COMET) is a useful technique in dealing with Multi-Criteria Decision-Making (MCDM) problems [44–48]. It helps a DM to organize the structure of the problems to be solved and carry out the analysis, comparisons and ranking of the alternatives, where the complexity of the algorithm is completely independent of the alternatives' number [49,50]. Additionally, comparisons between the Characteristic Objects (COs) are easier than comparisons between alternatives. However, the most important merit of the COMET method is the fact that this method is completely free of the rank reversals phenomenon [51] because the final ranking is constructed based on COs and fuzzy rules.

In this study, we extend the COMET concept to develop a methodology for solving multi-criteria group decision-making problems under uncertainty. The proposed method allows a group of DMs to make their opinion independent of linguistic terms by using HFS. The proposed method is designed for modeling uncertainty from different sources, which are related to expert knowledge. The main motivation of this research is the fact that the presented extension is also completely free of the rank reversals paradox as the classical version.

The group version of the HFS COMET method can be used in various research fields and disciplines such as economics [29,30,32], resource management [51], production [52], transport [53], game theory (Nash equilibrium) [54–63], medical problems [48,64], sustainability manufacturing [65] or web systems [66]; especially in decision situations requiring the involvement of many experts [67].

The rest of this paper is organized as follows. In Section 2, we introduced some basic concepts related to the hesitant fuzzy sets, L-R-type Generalized Fuzzy Numbers (GFNs), the fuzzy rule, the rule base and the t-norm. In Section 3, we established a group decision-making method based on COMET to deal with the uncertainty environment. In Section 4, an illustrative example is given to demonstrate the practicality and effectiveness of the proposed approach. Finally, we conclude the paper and give some remarks in Section 5.

2. Preliminaries

The HFS [24], as a generalization of the fuzzy set, maps the membership degree of an element to a set presented as several possible values between zero and one, which can better describe the situations where people have hesitancy in providing their preferences over objects in the process of decision-making.

In this section, we recall some important concepts that are necessary to understand our proposed decision-making method.

Definition 1. *A hesitant fuzzy set A on X is a function h^A that when applied to X returns a finite subset of $[0,1]$, which can be represented as the following mathematical symbol [24]:*

$$A = \{(x, h^A(x)) | x \in X\},$$

where $h^A(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set A. For convenience, Xia and Xu [68] named $h^A(x)$ a Hesitant Fuzzy Element (HFE).

Definition 2. *For an HFS represented by its membership function h, we define its complement as follows [24]:*

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}.$$

Definition 3. *In reference [68], for an HFE h, $Sc(h) = \frac{1}{l_h} \sum_{\gamma \in h} \gamma$, is called the score function of h, where l_h is the number of elements in h and $Sc(h) \in [0,1]$. For two HFEs h_1 and h_2, if $Sc(h_1) > Sc(h_2)$, then $h_2 \prec h_1$, if TODO$Sc(h_1) = Sc(h_2)$, then $h_1 \approx h_2$.*

Xia and Xu [68] define some operations on the HFEs (h, h_1 and h_2) and the scalar number k :

1. $kh = \bigcup_{\gamma \in h} \{1 - (1-\gamma)^k\}$;
2. $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$;
3. $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}$.

Definition 4. *Let L and R both be decreasing, shape functions from $\Re^+ = [0, \infty)$ to $[0,1]$ with $L(0) = \omega$; $L(x) < \omega$ for all $x < 1$; $L(1) = 0$ or ($L(x) > 0$ for all x and $L(+\infty) = 0$) (and the same for R). A GFN is called the L-Rtype if there are real numbers m, $\alpha > 0, \beta > 0$ and ω $(0 \leq \omega \leq 1)$ with [69]:*

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega L(\frac{m-x}{\alpha}), & x \leq m \\ \omega R(\frac{x-m}{\beta}), & x \geq m \end{cases}$$

where m is called the mean value of $\tilde{A}$ and α and β are called the left and right spreads, respectively. The L-R-type GFN $\tilde{A}$ is symbolically denoted by $\tilde{A} = (m, \alpha, \beta; \omega)_{LR}$. If $\omega = 1$, then $\tilde{A}$ is called the L-R-type fuzzy number and simply denoted by $\tilde{A} = (m, \alpha, \beta)_{LR}$.

For an L-R-type GFN $\tilde{A} = (m, \alpha, \beta; \omega)_{LR}$, if L and R are of the form:

$$T(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then, $\tilde{A}$ is called a generalized triangular fuzzy number denoted by $\tilde{A} = (m, \alpha, \beta; \omega)_T$. Similarly, for $\omega = 1$, $\tilde{A}$ is simply called a triangular fuzzy number denoted by $\tilde{A} = (m, \alpha, \beta)_T$.

A fuzzy number $\tilde{A}$ is called an L-R-type generalized trapezoidal fuzzy number if there are real numbers $m_1, m_2, \alpha > 0$ and $\beta > 0$ with the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega L(\frac{m_1 - x}{\alpha}), & x \le m_1 \\ \omega, & m_1 \le x \le m_2 \\ \omega R(\frac{x - m_2}{\beta}), & x \ge m_2 \end{cases}$$

where m_1 and m_2 are called the mean values of $\tilde{A}$ and α, β are called the left and right spreads, respectively. Symbolically, $\tilde{A}$ is denoted by $(m_1, m_2, \alpha, \beta; \omega)_{LR}$. The L-R-type generalized trapezoidal fuzzy number $\tilde{A} = (m_1, m_2, \alpha, \beta; \omega)_{LR}$ divides into three parts: left part, middle part and right part. The left, middle and right parts include the intervals $[m_1 - \alpha, m_1], [m_1, m_2]$ and $[m_2, m_2 + \beta]$, respectively.

If we take L and R to be of the form as mentioned in Equation (4), then $\tilde{A}$ is called the generalized trapezoidal fuzzy number denoted by $(m_1, m_2, \alpha, \beta; \omega)_T$. A generalized trapezoidal fuzzy number $\tilde{A}(m_1, m_2, \alpha, \beta; \omega)_T$ is simply called a trapezoidal fuzzy number denoted by $\tilde{A}(m_1, m_2, \alpha, \beta)_T$ when $\omega = 1$.

We know that the L-R-type fuzzy numbers are used to present real numbers in a fuzzy environment, and trapezoidal fuzzy numbers are used to present fuzzy intervals that are widely applied in linguistics, knowledge representation, control systems, database, and so forth [21,70–72]. Similarly, the L-R-type GFNs are very general and allow one to represent the different types of information. For example, the L-R-type GFN $\tilde{B} = (m, m, 0, 0; \omega)_{LR}$ with $m \in \Re = (-\infty, \infty)$ is used to denote a real number $\tilde{B}$, and the L-R-type GFN $\tilde{C} = (m_1, m_2, 0, 0; \omega)_{LR}$ with $m_1, m_2 \in \Re$ and $m_1 < m_2$ is used to denote an interval $\tilde{C}$.

Definition 5. *For a triangular fuzzy number $\tilde{A}$, we define:*

1. *The support of $\tilde{A}$ is $S(\tilde{A}) = \{x : \mu_{\tilde{A}}(x) > 0\}$.*
2. *The core of $\tilde{A}$ is $C(\tilde{A}) = \{x : \mu_{\tilde{A}}(x) = 1\}$.*

Definition 6. *The fuzzy rule [73,74]:*

The single fuzzy rule can be based on the modus ponens tautology [73,74]. The reasoning process uses logical connectives IF-THEN, OR and AND.

Definition 7. *The rule base [75]:*

The rule base consists of logical rules determining causal relationships existing in the system between the fuzzy sets of its inputs and outputs [75].

Definition 8. *In reference [76], a triangular norm (t-norm) is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ satisfying $\forall x, y, z \in [0,1]$:*

1. *$T(x, y) = T(y, x)$ (commutativity),*
2. *$T(x, y) \le T(x, z)$, if $y \le z$ (monotonicity),*
3. *$T(x, T(y, z)) = T(T(x, y), z)$ (associativity),*
4. *$T(x, 1) = x$ (neutrality of one).*

Throughout this paper, only the product is used as a t-norm operator, i.e.,
$P(\mu_{\alpha_1}(x), \mu_{\alpha_2}(y)) = \mu_{\alpha_1}(x).\mu_{\alpha_2}(y)$.

3. COMET for MCGDM Using HFS

Consider an MCGDM problem in which the ratings of the alternative evaluations are expressed as HFSs. The solution procedure for the proposed MCGDM approach is described below.

Let A_j $(j = 1, 2, ..., m)$ be the set of alternatives and suppose a group of DMs $D = \{d_1, d_2, ..., d_k\}$ is asked to evaluate the given alternatives with respect to several criteria C_i $(i = 1, 2, ..., n)$. The ranking algorithm of the COMET has the following five steps:

Step 1: Define the space of the problem as follows:

Let $\mathcal{F}$ be the collection of all L-R-type GFNs and $F_i^{1\delta}, F_i^{2\delta}, ..., F_i^{q\delta}$ be different families of subsets of $\mathcal{F}$ selected by a DM d_δ $(\delta = 1, 2, ..., k)$ for each criterion C_i $(i = 1, 2, ..., n)$ where

$F_i^{1\delta} = \{F_{i1}^{1\delta}, F_{i2}^{1\delta}, ..., F_{ic_i}^{1\delta}\}$;
$F_i^{2\delta} = \{F_{i1}^{2\delta}, F_{i2}^{2\delta}, ..., F_{ic_i}^{2\delta}\}$;
$\vdots$
$F_i^{q\delta} = \{F_{i1}^{q\delta}, F_{i2}^{q\delta}, ..., F_{ic_i}^{q\delta}\}$.

In this way, the following result is obtained:

$C_1 = \left\{F_{11}^{b\delta}, F_{12}^{b\delta}, ..., F_{1c_1}^{b\delta}\right\}$;
$C_2 = \left\{F_{21}^{b\delta}, F_{22}^{b\delta}, ..., F_{2c_2}^{b\delta}\right\}$;
$\vdots$
$C_n = \left\{F_{n1}^{b\delta}, F_{n2}^{b\delta}, ..., F_{nc_n}^{b\delta}\right\}$;

where $1 \leq b \leq q$ and $c_1, c_2, ..., c_n$ are the numbers of fuzzy numbers in each family $F_i^{b\delta}$ $(1 \leq b \leq q, 1 \leq i \leq n)$ for all criteria.

Initially, suppose each alternative is assessed by all DMs by means of n criteria in the form of a single family of TFNs F_i^t $(1 \leq i \leq n)$ with their fuzzy semantics as shown in Figures 1–6. Suppose each DM further provides the hesitant information of an alternative for each criterion in the form of L-R-type GFNs. Note that, in this method, the observations already provided by all of the DMs for each criterion in the form of the single family of TFNs set F_i^t $(1 \leq i \leq n)$ is a necessary part of all of the family of the remaining L-R-type GFNs set during the computation. The core of each criterion is defined as the core of each F_i^t $(1 \leq i \leq n)$, i.e.,

$C(C_1) = \left\{C(F_{11}^t), C(F_{12}^t), ..., C(F_{1c_1}^t)\right\}$;
$C(C_2) = \left\{C(F_{21}^t), C(F_{22}^t), ..., C(F_{2c_2}^t)\right\}$;
$\vdots$
$C(C_n) = \left\{C(F_{n1}^t), C(F_{n2}^t), ..., C(F_{nc_n}^t)\right\}$.

Step 2: Generate the characteristic objects:

By using the Cartesian product of all TFNs cores, the COs can be obtained as follows:

$CO = C(C_1) \times C(C_2) \times ... \times C(C_n)$

As the result of this, the ordered set of all COs is obtained:

$CO_1 = \left\{C(F_{11}^t), C(F_{21}^t), ..., C(F_{n1}^t)\right\}$;
$CO_2 = \left\{C(F_{11}^t), C(F_{21}^t), ..., C(F_{n2}^t)\right\}$;
$\vdots$
$CO_s = \left\{C(F_{1c_1}^t), C(F_{2c_2}^t), ..., C(F_{nc_n}^t)\right\}$;

where $s = \prod_{i=1}^{n} c_i$ is a number of COs.

Step 3: Rank and evaluate the characteristic objects:

A comparison of COs is obtained by adding the opinion of DMs. After this, determine the Matrix of Expert Judgment (MEJ) as follows:

$$\mathbf{ME}J = \begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \cdots & \tilde{h}_{1s} \\ \tilde{h}_{21} & \tilde{h}_{22} & \cdots & \tilde{h}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{s1} & \tilde{h}_{s2} & \cdots & \tilde{h}_{ss} \end{bmatrix}$$

where $\tilde{h}_{\alpha\beta} = \{\tilde{h}_{\alpha\beta}^{\omega}, \omega = 1, 2, ..., l_{\tilde{h}_{\alpha\beta}}\}$ is the HFE containing preferences of all DMs and is obtained as a result of comparing CO_α and CO_β. The more preferred CO obtains a stronger preference degree, and the second object obtains a weaker one. If the preferences are balanced, then both objects obtain a preference degree denoted by HFE $\tilde{h}_f = \{0.5\}$. The selection of $\tilde{h}_{\alpha\beta}$ depends solely on the knowledge and opinion of the experts. Mathematically, $\tilde{h}_{\alpha\beta}$ should satisfy the following conditions:

1. $\tilde{h}_{\alpha\beta}^{\sigma(\omega)} + \tilde{h}_{\beta\alpha}^{\sigma(l_{\tilde{h}_{\alpha\beta}}-\omega+1)} = 1, \alpha, \beta = l, 2, ..., s;$
2. $\tilde{h}_{\alpha\alpha} = \{0.5\}, \alpha = l, 2, ..., s;$
3. $l_{\tilde{h}_{\alpha\beta}} = l_{\tilde{h}_{\beta\alpha}}, \alpha, \beta = l, 2, ..., s.$

where the values in $\tilde{h}_{\alpha\beta}$ are assumed to be arranged in increasing order for convenience, and let $\tilde{h}_{\alpha\beta}^{\sigma(\omega)} (\omega = 1, 2, ..., l_{\tilde{h}_{\alpha\beta}})$ denote the ωth smallest value in $\tilde{h}_{\alpha\beta}$ and $l_{\tilde{h}_{\alpha\beta}}$ the number of the values in $\tilde{h}_{\alpha\beta}$.

The last equation indicates that the sum of the ωth smallest value in $\tilde{h}_{\alpha\beta}$ and the ωth largest value in $\tilde{h}_{\beta\alpha}$ should be equivalent to one, which is the complement condition as introduced by Torra in [24] (see Definition 2). In other words, if $\tilde{h}_{\alpha\beta} = \{\tilde{h}_{\alpha\beta}^{\omega}, \omega = 1, 2, ..., l_{\tilde{h}_{\alpha\beta}}\}$ is known, then we can obtain $\tilde{h}_{\beta\alpha}$, which is given by $\tilde{h}_{\beta\alpha} = \{1 - \tilde{h}_{\alpha\beta}^{\omega}, \omega = 1, 2, ..., l_{\tilde{h}_{\alpha\beta}}\}$. The second equation indicates that the diagonal elements in **MEJ** should be equivalent to $\{0.5\}$, which implies the balanced preference degrees of CO_α and CO_β. The third equation indicates that the number of elements in $\tilde{h}_{\alpha\beta}$ and $\tilde{h}_{\beta\alpha}$ should be the same.

Suppose $\tilde{H}_\alpha = \oplus_{\beta=1}^{s} \tilde{h}_{\alpha\beta}$, where each $\tilde{H}_\alpha$ is an HFE. Afterward, we get a vertical vector SJ of the summed judgments where $SJ_\alpha = Sc(\tilde{H}_\alpha) = \frac{1}{l_{\tilde{H}_\alpha}} \sum_{\gamma \in \tilde{H}_\alpha} \gamma$ (see Definition 3). To assign the approximate value of preference to each CO, we use the same MATLAB code as used by Salabun in [45]. As a result, we get a vertical vector P, where the α-th component of P represents the approximate value of preference for CO_α.

Step 4: The rule base:
Each CO and value of preference is converted to a fuzzy rule as follows:
$IF \quad CO_\alpha \quad THEN \quad P_\alpha$
$IF \quad C(F_{1\alpha}^t) \quad AND \quad C(F_{2\alpha}^t) \quad AND \ ... \ THEN \quad P_\alpha$
In this way, the complete fuzzy rule base is obtained, which can be presented as follows:
$IF \quad CO_1 \quad THEN \quad P_1$
$IF \quad CO_2 \quad THEN \quad P_2$
$\vdots$
$IF \quad CO_s \quad THEN \quad P_s$

Step 5: Inference in a fuzzy model and final ranking:

Each alternative activates the specified number of fuzzy rules, where for each one, the fulfillment degree of the conjunctive complex premise is determined. The fulfillment degrees of each activated rule corresponding to each element of F_i^b $(1 \leq b \leq q, 1 \leq i \leq n)$ always sum to one. Each alternative is a set of crisp numbers, corresponding to criteria $C_1, C_2, ..., C_n$. It can be presented as follows:
$A_j = \{a_{1j}, a_{2j}, ..., a_{nj}\}$, where the following conditions must be satisfied:
$a_{1j} \in [C(F_{11}^t), C(F_{1c_1}^t)];$
$a_{2j} \in [C(F_{21}^t), C(F_{2c_2}^t)];$
$\vdots$
$a_{nj} \in [C(F_{n1}^t), C(F_{nc_n}^t)].$
To infer the final ranking of the alternatives corresponding to each criterion, we proceed as follows:
For each $j = 1, 2, ..., m$,

$a_{1j} \in [C(F_{1k_1}^t), C(F_{1(k_1+1)}^t)];$
$a_{2j} \in [C(F_{2k_2}^t), C(F_{2(k_2+1)}^t)];$
$\vdots$

$a_{nj} \in [C(F^t_{nk_n}), C(F^t_{n(k_n+1)})]$;
where $k_i = 1, 2, ..., (c_i - 1)$, $(1 \le i \le n)$. The activated rules (COs), i.e., the group of those COs where the membership function of each alternative A_j $(1 \le j \le m)$ is non-zero, are:
$\left(C(F^t_{1k_1}), C(F^t_{2k_2}), ..., C(F^t_{nk_n})\right)$;
$\left(C(F^t_{1k_1}), C(F^t_{2k_2}), ..., C(F^t_{n(k_n+1)})\right)$;
$\vdots$
$\left(C(F^t_{1(k_1+1)}), C(F^t_{2(k_2+1)}), ..., C(F^t_{n(k_n+1)})\right)$.
The number of COs are obviously 2^n and $1 \le 2^n \le s$.

Let $p_1, p_2, ..., p_{2^n}$ be the approximate values of the preference of the activated rules (COs), which were already calculated in Step 3, where p_η's $(1 \le \eta \le 2^n)$ are some values in P_α's $(1 \le \alpha \le s)$. We denote the HFE at the point $x \in A_j$ $(1 \le j \le m)$ provided by a DM d_δ $(\delta = 1, 2, ..., k)$ as:

$$h^\delta_{ij}(x) = \{F^{1\delta}_{ij}(x), F^{2\delta}_{ij}(x), ..., F^{q^\delta}_{ij}(x)\}$$

for each criterion C_i $(i = 1, 2, .., n)$.

To aggregate the information in the form of HFEs from every DM, in order to achieve a single HFE, which summarizes all of the information provided by the different DMs, there are several aggregation operators that are available in the literature. However, in this paper, we simply use the average operator to get the average of the membership values obtained from LR-type GFNs provided by the DMs in the form of HFE corresponding to each $a_{ij} \in A_j$ $(1 \le i \le n, 1 \le j \le m)$. Suppose $h_{ij}(x)$ is an HFE obtained as a result of aggregating the HFEs $h^\delta_{ij}(x)$, $(\delta = 1, 2, ..., k)$ where:

$$h_{ij}(x) = \{F^1_{ij}(x), F^2_{ij}(x), ..., F^q_{ij}(x)\}$$

Let $\mathbf{A}_j$ be HFE, which is computed as the sum of the products of all activated rules, as their fulfillment degrees and their values of the preference, i.e.,
$\mathbf{A}_j = p_1\left(h_{1k_1}(a_{1j}) \otimes h_{2k_2}(a_{2j}) \otimes \ldots h_{nk_n}(a_{nj})\right) \oplus p_2\left(h_{1k_1}(a_{1j}) \otimes h_{2k_2}(a_{2j}) \otimes \ldots h_{n(k_n+1)}(a_{nj})\right)$
$\oplus \ldots p_{2^n}\left(h_{1(k_1+1)}(a_{1j}) \otimes h_{2(k_2+1)}(a_{2j}) \otimes \ldots h_{n(k_n+1)}(a_{nj})\right)$
The preference of each alternative A_j $(1 \le j \le m)$ can be found by finding the score of the corresponding HFE $\mathbf{A}_j$ $(1 \le j \le m)$ as follows:

$$Sc(\mathbf{A}_j) = \frac{1}{l_{\mathbf{A}_j}} \sum_{y \in \mathbf{A}_j} y$$

The final ranking of alternatives is obtained by sorting the preference of alternatives. The greater the preference value, the better the alternative A_j $(1 \le j \le m)$.

As the summary of this section, Figure 1 presents the stepwise procedure of the proposed extension of the COMET method. After initiating the decision process, the procedure starts by modeling the structure of a considered decision problem. At this point, each expert determine generalized fuzzy numbers for each criterion. This is followed by generating characteristic objects in Step 2, evaluating the preferences of the characteristic objects in Step 3 and generating the fuzzy rule base in Step 4. The procedure ends by computing the assessment for each alternative from the considered set. The set of alternatives can be ranked according to the descending order of the computed assessments.

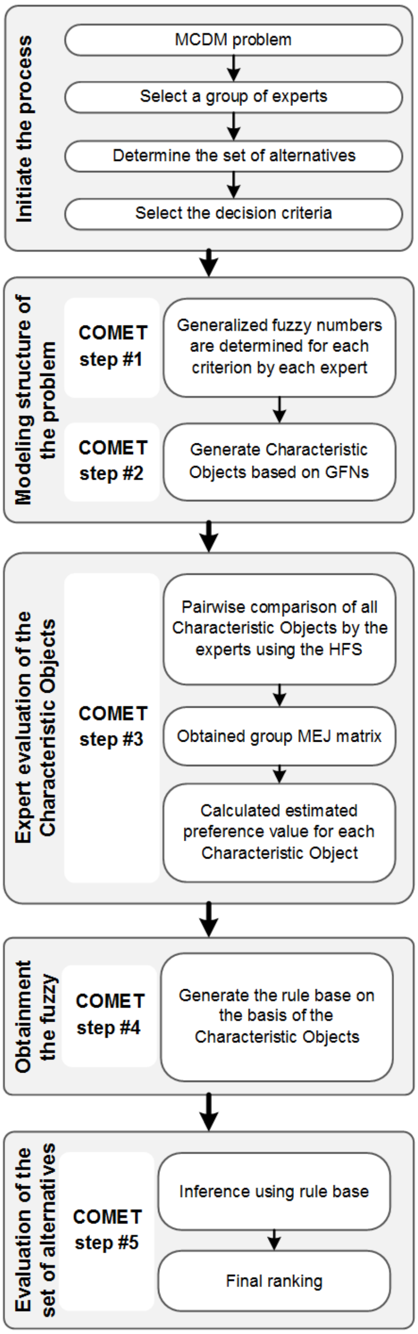

Figure 1. The procedure of the proposed extension of COMET to group decision-making.

4. An Illustrative Example

In this section, an example is given to understand our approach. We used the method proposed in Section 3 to get the most desirable alternative, as well as to rank the alternatives from the best to the worst or vice versa.

Let us consider a factory, whose maximum capacity of using mobile units is a total of 1000 per month, which intends to select a new mobile company. Four companies A_1, A_2, A_3 and A_4 are available, and three DMs are asked to consider two criteria C_1 (fixed line rent) and C_2 (rates per unit) to decide which mobile company to choose. The fixed line rent, rates per unit and the original ranking order of the feasible mobile companies are shown in Table 1.

Table 1. Original ranking of the alternatives, where LR - fixed line rent and R/U - rates per unit.

Alternatives	C_1 (LR)	C_2 (R/U)	Bill Amount	Original Rank
A_1	150	1500	1650	2
A_2	50	2000	2050	3
A_3	250	1250	1500	1
A_4	30	2150	2180	4

A set of TFNs and trapezoidal fuzzy numbers for both criteria C_1 and C_2 set by three DMs are shown in Tables 2 and 3. The average of the membership values obtained from LR-type GFNs for both the criteria are shown in Table 4.

Table 2. LR-type Group Fuzzy Numbers (GFNs) selected by the Decision-Makers (DMs) for criteria C_1.

DM1	$\{(30,30,200),(30,200,300),(200,300,300)\}$ $\{(30,30,30,170),(30,170,220,300),(220,300,300,300)\}$
DM2	$\{(30,30,200),(30,200,300),(200,300,300)\}$ $\{(30,30,30,180),(30,180,230,300),(230,300,300,300)\}$
DM3	$\{(30,30,200),(30,200,300),(200,300,300)\}$ $\{(30,30,30,160),(30,160,215,300),(215,300,300,300)\}$

Table 3. LR-type GFNs selected by the DMs for criteria C_2.

DM1	$\{(1200,1200,1800),(1200,1800,2500),(1800,2500,2500)\}$ $\{(1200,1200,1200,1600),(1200,1600,1900,2500),(1900,2500,2500,2500)\}$
DM2	$\{(1200,1200,1800),(1200,1800,2500),(1800,2500,2500)\}$ $\{(1200,1200,1200,1700),(1200,1700,1900,2500),(1900,2500,2500,2500)\}$
DM3	$\{(1200,1200,1800),(1200,1800,2500),(1800,2500,2500)\}$ $\{(1200,1200,1200,1650),(1200,1650,1950,2500),(1950,2500,2500,2500)\}$

Table 4. Average of the membership values obtained from LR-type GFNs for criteria C_1.

Average of the Membership Values Obtained from LR-Type GFNs for Criterion C_2			
30	50	150	250
$(1,0,0)$	$(0.8824,0.1176,0)$	$(0.2941,0.7059,0)$	$(0,0.5000,0.5000)$
$(1,0,0)$	$(0.8567,0.1433,0)$	$(0.8567,0.1433,0)$	$(0,0.6425,0.3575)$
1250	1500	2000	2150
$(0.9167,0.0833,0)$	$(0.5000,0.5000,0)$	$(0,0.7143,0.2857)$	$(0,0.5000,0.5000)$
$(0.8880,0.1120,0)$	$(0.3278,0.6722,0)$	$(0,0.8586,0.1414)$	$(0,0.6010,0.3990)$

The graphical representation of L-R-type GFNs selected by the DMs for both the criteria C_1 and C_2 are shown in Figures 2–7, respectively.

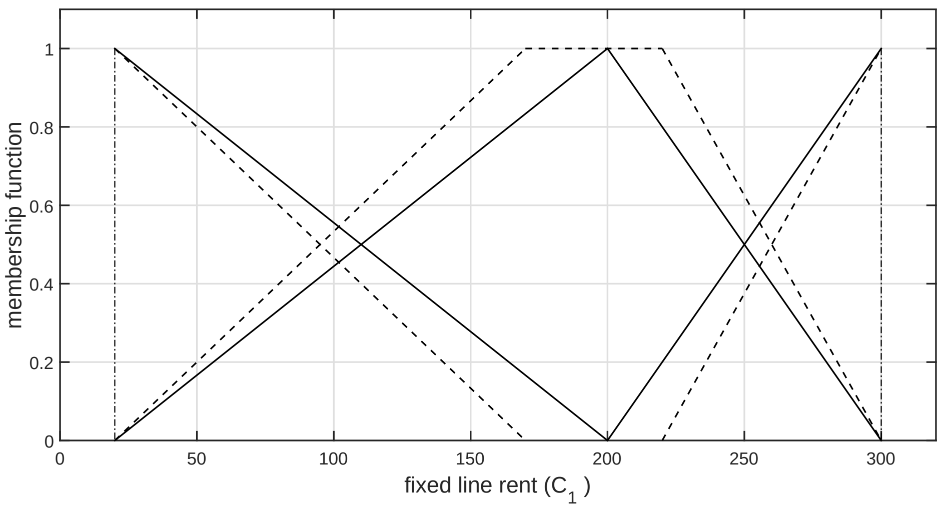

Figure 2. Graphical representation of *LR*-type GFNs selected by DM1 for the criterion C_1.

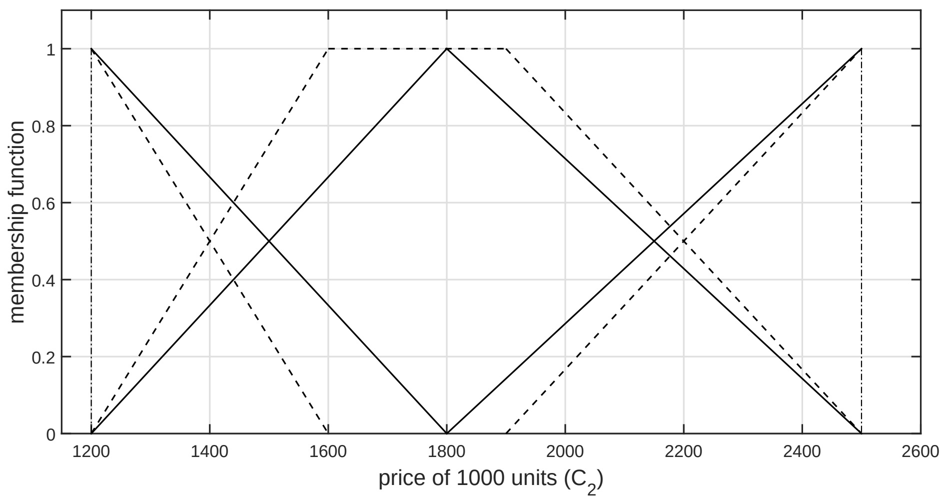

Figure 3. Graphical representation of *LR*-type GFNs selected by DM1 for the criterion C_2.

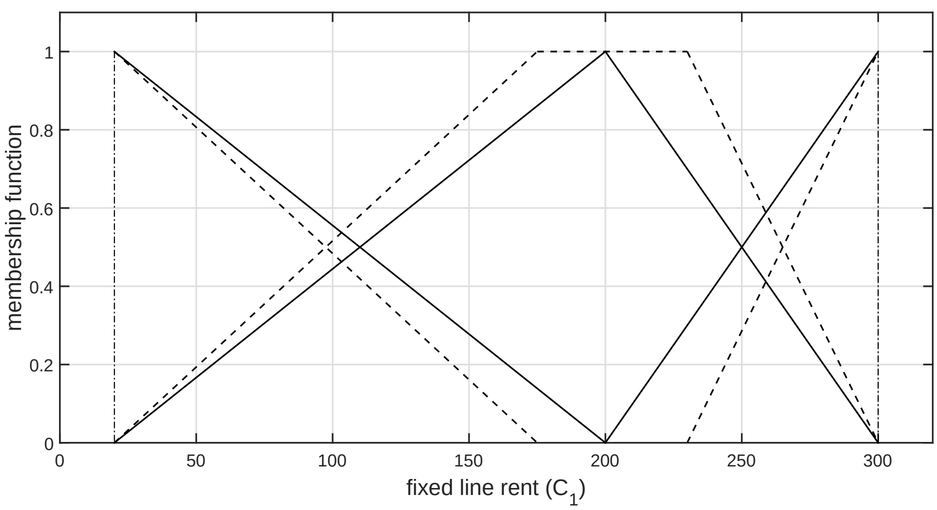

Figure 4. Graphical representation of *LR*-type GFNs selected by DM2 for the criterion C_1.

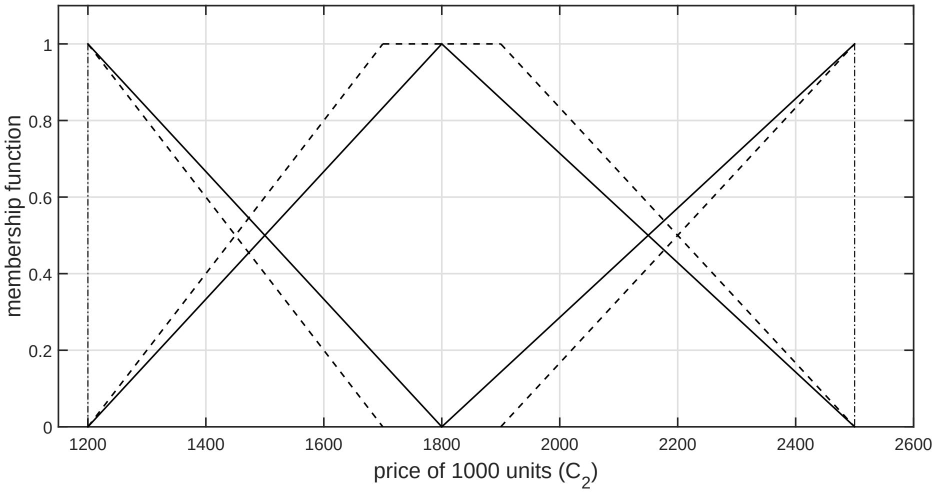

Figure 5. Graphical representation of *LR*-type GFNs selected by DM2 for the criterion C_2.

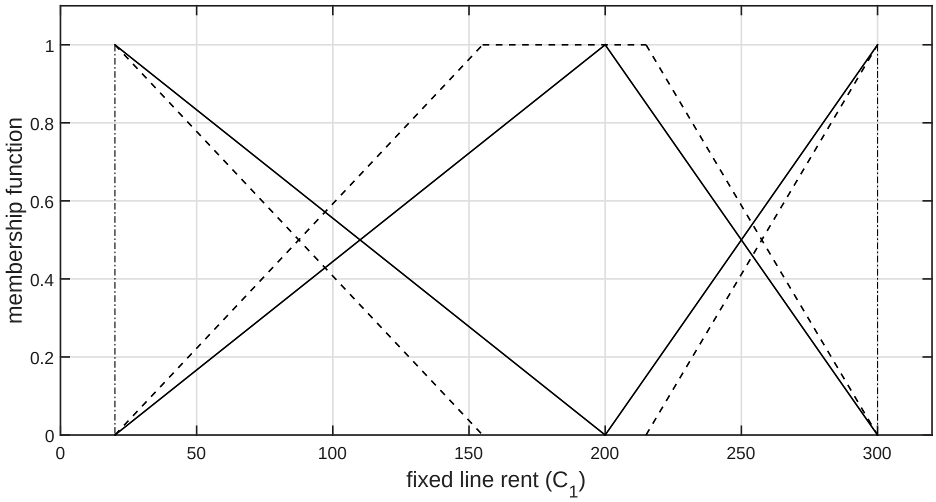

Figure 6. Graphical representation of *LR*-type GFNs selected by DM3 for the criterion C_1.

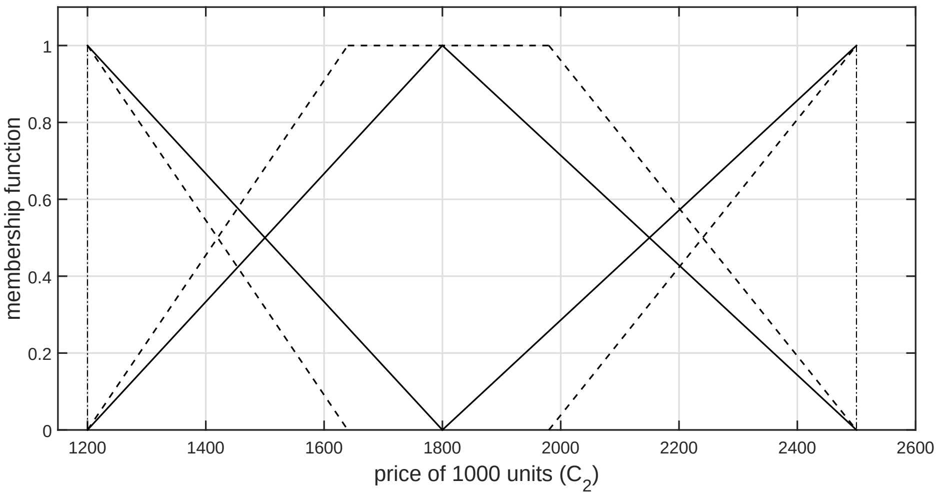

Figure 7. Graphical representation of *LR*-type GFNs selected by DM3 for the criterion C_2.

The cores of the family of TFNs for both the criteria C_1 and C_2 are respectively $\{30, 200, 300\}$ and $\{1200, 1800, 2500\}$. The solution of the COMET is obtained for different number of COs. The simplest solution involves the use of nine COs, which are presented as follows:

$$\begin{array}{l} CO_1 = \{30, 1200\},\ CO_2 = \{30, 1800\},\ CO_3 = \{30, 2500\} \\ CO_4 = \{200, 1200\},\ CO_5 = \{200, 1800\},\ CO_6 = \{200, 2500\} \\ CO_7 = \{300, 1200\},\ CO_8 = \{300, 1800\},\ CO_9 = \{300, 2500\} \end{array}$$

To rank and evaluate the COs, suppose the three DMs give their assessments by providing the HFEs as shown in Tables 5 and 6, and therefore, the Matrix of Expert Judgment (MEJ) is as follows:

Table 5. Matrix of Expert Judgment (MEJ).

	CO_1	CO_2	CO_3	CO_4	CO_5
CO_1	$\{0.5\}$	$\{0.8, 1\}$	$\{0.8, 0.9\}$	$\{0.7, 0.8\}$	$\{0.8, 0.9, 1\}$
CO_2	$\{0, 0.2\}$	$\{0.5\}$	$\{0.8, 1\}$	$\{0, 0.1, 0.2\}$	$\{0.9, 1\}$
CO_3	$\{0.1, 0.2\}$	$\{0, 0.2\}$	$\{0.5\}$	$\{0, 0.2, 0.3\}$	$\{0.1, 0.2\}$
CO_4	$\{0.2, 0.3\}$	$\{0.8, 0.9, 1\}$	$\{0.7, 0.8, 1\}$	$\{0.5\}$	$\{0.8, 0.9, 1\}$
CO_5	$\{0, 0.1, 0.2\}$	$\{0, 0.1\}$	$\{0.8, 0.9\}$	$\{0, 0.1, 0.2\}$	$\{0.5\}$
CO_6	$\{0, 0.2\}$	$\{0, 0.2\}$	$\{0, 0.1\}$	$\{0, 0.2\}$	$\{0, 0.1, 0.2\}$
CO_7	$\{0, 0.2\}$	$\{0.8, 1\}$	$\{0, 0.8\}$	$\{0, 0.2\}$	$\{0.8, 0.9, 1\}$
CO_8	$\{0, 0.1, 0.2\}$	$\{0.1, 0.2\}$	$\{0.7, 0.8, 1\}$	$\{0.1, 0.2\}$	$\{0, 0.1, 0.2\}$
CO_9	$\{0, 0.2\}$	$\{0, 0.1\}$	$\{0.2, 0.3\}$	$\{0, 0.1, 0.2\}$	$\{0, 0.2\}$

Table 6. Matrix of Expert Judgment (MEJ).

	CO_6	CO_7	CO_8	CO_9	SJ
CO_1	$\{0.8, 1\}$	$\{0.8, 1\}$	$\{0.8, 0.9, 1\}$	$\{0.8, 1\}$	0.999999
CO_2	$\{0.8, 1\}$	$\{0, 0.2\}$	$\{0.8, 0.9\}$	$\{0.9, 1\}$	0.999980
CO_3	$\{0.9, 1\}$	$\{0, 0.2\}$	$\{0, 0.2, 0.3\}$	$\{0.7, 0.8\}$	0.995033
CO_4	$\{0.8, 1\}$	$\{0.8, 1\}$	$\{0.8, 0.9\}$	$\{0.8, 0.9, 1\}$	0.999998
CO_5	$\{0.8, 0.9, 1\}$	$\{0, 0.1, 0.2\}$	$\{0.8, 0.9, 1\}$	$\{0.8, 1\}$	0.999751
CO_6	$\{0.5\}$	$\{0, 0.1, 0.2\}$	$\{0.2, 0.3\}$	$\{0.8, 0.9, 1\}$	0.968745
CO_7	$\{0.8, 0.9, 1\}$	$\{0.5\}$	$\{0.8, 1\}$	$\{0.8, 0.9\}$	0.999970
CO_8	$\{0.7, 0.8\}$	$\{0, 0.2\}$	$\{0.5\}$	$\{0.7, 0.8, 0.9\}$	0.996636
CO_9	$\{0, 0.1, 0.2\}$	$\{0.1, 0.2\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5\}$	0.841614

The vector SJ on the basis of MEJ is obtained as follows:

$$\begin{aligned} SJ = {} & [0.999999, 0.999980, 0.995033, 0.999998, 0.999751, 0.968745, 0.999970, \\ & 0.996636, 0.841614]^T \end{aligned}$$

A vertical vector P is obtained by using a MATLAB code (see [45]) as follows:

$$P = [1.0000, 0.7500, 0.2500, 0.8750, 0.5000, 0.1250, 0.6250, 0.3750, 0]^T$$

Each component of the vector P represents the approximate values of the preference for the generated COs. Each CO and the value of preference p_i is converted to a fuzzy rule, as follows:

$$
\begin{array}{llllllll}
IF & LR \sim 30 & AND & R/U \sim 1200 & THEN & P_1 \sim 1.0000 \\
IF & LR \sim 30 & AND & R/U \sim 1800 & THEN & P_2 \sim 0.7500 \\
IF & LR \sim 30 & AND & R/U \sim 2500 & THEN & P_3 \sim 0.2500 \\
IF & LR \sim 200 & AND & R/U \sim 1200 & THEN & P_4 \sim 0.8750 \\
IF & LR \sim 200 & AND & R/U \sim 1800 & THEN & P_5 \sim 0.5000 \\
IF & LR \sim 200 & AND & R/U \sim 2500 & THEN & P_6 \sim 0.1250 \\
IF & LR \sim 300 & AND & R/U \sim 1200 & THEN & P_7 \sim 0.6250 \\
IF & LR \sim 300 & AND & R/U \sim 1800 & THEN & P_8 \sim 0.3750 \\
IF & LR \sim 300 & AND & R/U \sim 2500 & THEN & P_9 \sim 0.0000
\end{array}
$$

With respect to Model 4, for the alternative $A_1 = \{150, 1500\}$, we have nine rules (COs), but the activated rules are $CO_1,\ CO_2,\ CO_4,\ CO_5$. The approximate values of the preference of corresponding COs are $p_1 \sim 1,\ p_2 \sim 0.7500,\ p_3 \sim 0.8750,\ p_4 \sim 0.5000$. The HFE $\mathbf{A}_1$ and the preference value of the corresponding alternative A_1 are computed respectively as follows:

$$
\begin{aligned}
\mathbf{A}_1 = {} & p_1 h_{11}(150) \otimes h_{21}(1500) \oplus p_2 h_{11}(150) \otimes h_{22}(1500) \oplus p_3 h_{12}(150) \otimes h_{21}(1500) \oplus \\
& p_4 h_{12}(150) \otimes h_{22}(1500)
\end{aligned}
$$

$$
Sc(\mathbf{A}_1) = \frac{1}{l_{\mathbf{A}_1}} \sum_{y \in \mathbf{A}_1} y = 0.5725
$$

Similarly, we can find the preference values for the rest of the alternatives and their ranking, which are shown in Table 7.

Table 7. Comparison of the original ranking with the ranking obtained using the proposed method.

Alternatives	C_1 (LR)	C_2 **(R/U)**	**Original Ranking**	**Preference Values**	**New Ranking**
A_1	150	1500	2	0.5725	3
A_2	50	2000	3	0.6236	2
A_3	250	1250	1	0.6272	1
A_4	30	2150	4	0.5281	4

The best choice is the alternative A_3 followed by A_2, A_1 and A_4. The worst choice is the alternative A_4. The extrema elements are consistent with the original ranking. However, the ranking obtained by the COMET method is not perfect. The main reason is that this problem was solved under an uncertain environment by a group of decision-makers. In other words, it is extremely difficult to make a reliable decision using uncertain data, but we believe that it is possible. This example also illustrates how hard it is to make a group decision under uncertainty. Notwithstanding, the COMET method shows the best and the worst decision.

The main contribution of the proposed approach can be expressed by the most important properties of this extension, i.e., the proposed approach is completely free of the rank reversal phenomenon and obtains not only a discrete value of priority, but the mathematical function, which can be used to calculate the priority for all alternatives from the space of the problem. Quantitative expression of efficiency is a very difficult task because a large number of assumptions is needed. Additionally, the reference ranking of the alternatives set is needed in this task, but the reference rank is almost always unknown. However, the problem of quantitative effectiveness assessment is a very important and interesting direction for further research.

5. Conclusions

The hesitant fuzzy sets theory is a useful tool to deal with uncertainty in multi-criteria group decision-making problems. Various sources of uncertainty can be a challenge to make a reliable decision. The paper presented the extension of the COMET method, which was proposed for solving real-life problems under the opinions of experts in a hesitant fuzzy environment. Therefore, the proposed approach successfully helps to deal with group decision-making under uncertainty. The basic concept of the proposed method is based on the distance of alternatives from the nearest characteristic objects and their values of preference. The characteristic objects are obtained from the crisp values of all of the considered fuzzy numbers for each criterion. The proposed method is different from all of the previous techniques for MCGDM due to the fact that it uses hesitant fuzzy sets theory and the modification of the COMET method. The prominent feature of the proposed method is that it could provide a useful and flexible way to efficiently facilitate DMs under a hesitant fuzzy environment. The related calculations are simple and have a low computational complexity. Hence, it enriched and developed the theories and methods of MCGDM problems and also provided a new idea for solving MCGDM problems. Finally, a practical example was given to verify the developed approach and to demonstrate its practicality and effectiveness.

During the research, some possible areas of improvement of the proposed approach were identified. From a formal way, the COMET method can be extended over intuitionistic fuzzy sets, hesitant intuitionistic fuzzy sets, hesitant intuitionistic fuzzy linguistic term sets or other uncertain forms. Additionally, analysis and improvement of the accuracy of the presented extension of the COMET method should be performed. The future works may cover the practical usage of the proposed approach in the different decision-making domains.

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