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Multi-objective Fuzzy Bi-matrix Game Model: A Multicriteria Non-Linear Programming Approach

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Abstract: A multi-objective bi-matrix game model based on fuzzy goals is established in this paper. It is shown that the equilibrium solution of such a game model problem can be translated into the optimal solution of a multi-objective, non-linear programming problem. Finally, the results of this paper are demonstrated through a numerical example.

Keywords: multi-objective fuzzy bi-matrix game; equilibrium solution; multi-objective nonlinear programming

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1. Introduction

When the bet is a small amount of money, the multi-objective bi-matrix game model is accurate. In real life, however, the interests of the relationship are more complex, particularly in some areas of the economy where the interests of the two players is precisely opposite. It is well known that these games are two person non-zero-sum games, called multi-objective bi-matrix games. Therefore, the research of multi-objective bi-matrix game problems has become more and more widespread in recent years.

The fuzzy set theory was introduced initially in 1965 by Zadeh [1]. The fuzziness occurring in game problems is categorized as fuzzy game problems. Single objective fuzzy game problems and related problems attached a wide range of research [2–9]. Tan et al. [5] presented a concept of the potential function for solving fuzzy games problems. They also reached a conclusion that the solution of fuzzy games and the marginal value of potential functions are equivalent. Chakeri et al. [10] used fuzzy logic to determine the priority of the pay-off based on the linguistic preference relation and proposed the notion of linguistic Nash equilibriums. Fuzzy preference relation has been widely used in fuzzy game theory [7–9,11]. At the same time, they [11] utilize the same method [10] to determine the priority of the pay-off based on fuzzy preference relation. In order to deal with this game model, a new approach was put forward. Moreover, Sharifian et al. [6] also applied fuzzy linguistic preference relation to fuzzy game theory.

The notions of max–min and min–max values were the earliest applied to solve the multi-objective game model in [12]. Roy et al. [13] presented solution procedures in view of the multi-objective bi-matrix game model. Besides, they [14] applied fuzzy optimization means to solve the fuzzy multicriteria bi-matrix game model. Nishizaki et al. [15–17] solved the multi-objective bi-matrix game via the resolution approach. Chen et al. [18,19] proposed an alternative technique for solving fuzzy multi-objective bi-matrix game problems through genetic algorithms in [20]. Angelov [21] proposed a new concept of the optimization problem based on degrees of satisfaction. Precup [22] introduced a new optimisation criteria in the development of fuzzy controllers with dynamics based on an attractive development method. In order to solve numerical optimization problems, a new algorithm was introduced in [23]. Ghosn et al. [24] investigated the use of parallel genetic algorithms in order to discuss the open-shop scheduling problem. Roy et al. [25] provided a mathematical optimization model for solving the multiple objective bi-matrix goal game problem on account of the entropy circumstance. Additionally, to solve the formulated mathematical model, they proposed a solution procedure of the fuzzy optimization method.

Since Wierzbicki [26] proposed equilibrium solutions for game problems, he analysed multi-objective game models based on pay-offs related to scalarising functions. There is a debate about the existence of equilibrium solutions of multicriteria bi-matrix games put forward by Borm et al. [27]. Nishizaki et al. [17] studied an equilibrium solution of multi-objective bi-matrix games. Qiu et al. [28] discussed the relationship of two fuzzy numbers via the lower limit$-\frac{1}{2}$ of the possibility degree. They also concluded that the equilibrium solution of multiple objective fuzzy games and the optimal solution of multi-objective linear optimization problems are of equal value. Bector et al. [2] only considered a single objective bi-matrix game based on fuzzy goals. Having gained enlightenment from [2,29–31], we will consider a multiple objective bi-matrix game based on fuzzy goals, so as to obtain better results.

The outline of this paper is as follows. Section 2 is about basic definitions and recalls results with regard to a crisp multi-objective bi-matrix game. In Section 3, a multi-objective bi-matrix game model based on fuzzy goals is established. Section 4 presents a kind of multicriteria, non-linear programming problem in some special cases. The results of this paper are demonstrated through a numerical example in Section 5.

2. Preliminaries

In this section, we recall some basic definitions and preliminaries. Further, we shall describe a crisp multi-objective bi-matrix game model in [29].

Definition 1. *[32] The set of mixed strategies for Player* I *is denoted by:*

$$S^m = \{x = (x_1, x_2, \cdots, x_m)^T \in \mathbb{R}^m | \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, 2, \cdots, m.\} \tag{1}$$

Similarly, the set of mixed strategies for Player II *is denoted by:*

$$S^n = \{y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n | \sum_{j=1}^{n} y_j = 1, y_j \geq 0, j = 1, 2, \cdots, n.\} \tag{2}$$

where x^T is the transposition of x, $\mathbb{R}^m$ and $\mathbb{R}^n$ are m- and n-dimensional Euclidean spaces.

The multiple pay-off matrices of Player I and Player II in multi-objective bi-matrix games are denoted by [29]:

$$A^1 = \begin{pmatrix} a^1_{11} & \cdots & a^1_{1n} \\ \vdots & \ddots & \vdots \\ a^1_{m1} & \cdots & a^1_{mn} \end{pmatrix}, \cdots, A^r = \begin{pmatrix} a^r_{11} & \cdots & a^r_{1n} \\ \vdots & \ddots & \vdots \\ a^r_{m1} & \cdots & a^r_{mn} \end{pmatrix} \tag{3}$$

and

$$B^1 = \begin{pmatrix} b^1_{11} & \cdots & b^1_{1n} \\ \vdots & \ddots & \vdots \\ b^1_{m1} & \cdots & b^1_{mn} \end{pmatrix}, \cdots, B^s = \begin{pmatrix} b^s_{11} & \cdots & b^s_{1n} \\ \vdots & \ddots & \vdots \\ b^s_{m1} & \cdots & b^s_{mn} \end{pmatrix} \tag{4}$$

respectively. Here, Player I and Player II have r and s objectives, respectively. Without any loss of generality, we assume that the Player I and Player II are both maximized players.

A multi-objective bi-matrix game ($MOBG$) model is taken as:

$$MOBG = (S^m, S^n, A^k(1,2,\cdots,r), B^l(l=1,2,\cdots,s)).$$

Definition 2. *[25] Let $A = (A^1, A^2, \cdots, A^r)$. When Player I chooses a mixed strategy $x \in S^m$, the expected pay-off of Player I is denoted by:*

$$\begin{aligned} E(x,y) &= x^T A y = [E_1(x,y), E_2(x,y), \cdots, E_r(x,y)] \\ &= \left[x^T A^1 y, x^T A^2 y, \cdots, x^T A^r y\right] \\ &= \left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^1 x_i y_j, \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2 x_i y_j, \cdots, \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^r x_i y_j\right] \end{aligned} \tag{5}$$

Similarly, let $B = (B^1, B^2, \cdots, B^s)$, when Player II chooses a mixed strategy $y \in S^n$, the expected pay-off of Player II is denoted by:

$$\begin{aligned} E(x,y) &= x^T B y = [E_1(x,y), E_2(x,y), \cdots, E_s(x,y)] \\ &= \left[x^T B^1 y, x^T B^2 y, \cdots, x^T B^s y\right] \\ &= \left[\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}^1 x_i y_j, \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}^2 x_i y_j, \cdots, \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}^s x_i y_j\right] \end{aligned} \tag{6}$$

Definition 3. *[29] Suppose $D^k = \{x^T A^k y : (x,y) \in S^m \times S^n\} \subseteq \mathbb{R}$ be the domain of k^{th} pay-offs of Player I. Then a fuzzy goal $\widetilde{g}_I^k$ of Player I corresponding to the k^{th} pay-offs is a fuzzy set on D^k, whose the membership function is defined by:*

$$u_{\widetilde{g}_I^k} : D^k \to [0,1], \tag{7}$$

Similarly, suppose $D^l = \{x^T B^l y : (x,y) \in S^m \times S^n\} \subseteq \mathbb{R}$ be the domain of lth payoff of Player II. Then a fuzzy goal $\widetilde{g}_{II}^l$ of Player II corresponding to the l^{th} pay-offs is a fuzzy set on D^l, whose the membership function is defined by:

$$v_{\widetilde{g}_{II}^l} : D^l \to [0,1]. \tag{8}$$

3. A Multi-objective Bi-matrix Game with Fuzzy Goals

In this section, we first introduce the concepts of fuzzy sets and fuzzy numbers.

A fuzzy set $\widetilde{F}$ of $\mathbb{R}$ is characterized by a membership function $u_{\widetilde{F}} : \mathbb{R} \to [0,1]$ [1]. An α-level set of $\widetilde{F}$ is given as $[\widetilde{F}]_\alpha = \{x \in \mathbb{R} : u_{\widetilde{F}}(x) \geq \alpha\}$ for each $\alpha \in (0,1]$. A strict α-level set of $\widetilde{F}$ is given by $(\widetilde{F})_\alpha = \{x \in \mathbb{R} : u_{\widetilde{F}}(x) > \alpha\}$ for each $\alpha \in (0,1]$. We define the set $[\widetilde{F}]_0$ by $[\widetilde{F}]_0 = \overline{\{x \in \mathbb{R} : u_{\widetilde{F}}(x) > 0\}}$, where $\overline{F}$ denotes the closure of a crisp set F. A fuzzy set $\widetilde{F}$ is said to be a fuzzy number if it satisfies the following conditions [33]:

(1) $\widetilde{F}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u_{\widetilde{F}}(x_0) = 1$;
(2) $\widetilde{F}$ is convex, i.e., $u_{\widetilde{F}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u_{\widetilde{F}}(x_1), u_{\widetilde{F}}(x_2)\}$, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$;
(3) $\widetilde{F}$ is upper semi-continuous;
(4) $[\widetilde{F}]_0$ is compact.

In the following, we establish a multi-objective bi-matrix game model in the fuzzy environment. Suppose S^m, S^n, A^k $(k = 1,2,\cdots,r)$, and B^l $(l = 1,2,\cdots,s)$ be as introduced in Section 2.

Definition 4. *Let $A = (A^1, A^2, \cdots, A^r)$. When Player I chooses a mixed strategy $x \in S^m$, an aspiration level of Player I with respect to the k^{th} pay-offs is denoted by:*

$$V_0^k = \max_{y \in S^n} E_k(x,y) = \max_{y \in S^n} x^T A^k y, \quad (k = 1,2,\cdots,r).$$

Similarly, let $B = (B^1, B^2, \cdots, B^s)$*, when Player* II *chooses a mixed strategy* $y \in S^n$*, an aspiration level of Player* II *with respect to the* l^{th} *pay-offs is denoted by:*

$$W_0^l = \max_{x \in S^m} E_l(x, y) = \max_{x \in S^m} x^T B^l y, \quad (l = 1, 2, \cdots, s).$$

Therefore, we obtain that the multi-objective bi-matrix game based on fuzzy goals, denoted by $MOBGFG$, can be presented as:

$$MOBGFG = (S^m, S^n, A^k, B^l, V_0^k, \gtrsim, W_0^l, \lesssim, (k = 1, 2, \cdots, r; l = 1, 2, \cdots, s)) \tag{9}$$

where $\gtrsim$ and $\lesssim$ are the fuzzified versions of symbols $\geq$ and $\leq$, respectively in [34].

Let t, a, and $p \in \mathbb{R}$ $(p > 0)$, then the membership function of the fuzzy set $\tilde{F}$ defining the fuzzy inequality $t \gtrsim_p a$, where this fuzzy inequality $t \gtrsim_p a$ can be interpreted as "t essentially greater than or equal to a with tolerance error p", can be defined by [2]:

$$u_{\tilde{F}}(t) = \begin{cases} 1, & t \geq a, \\ 1 - (\frac{a-t}{p}), & a - p \leq t \leq a, \\ 0, & t < a - p. \end{cases} \tag{10}$$

Based on the above discussion, let p_0^k and $p_0^{'k}$ $(k = 1, 2, \cdots, r)$ (respectively, q_0^l and $q_0^{'l}$ $(l = 1, 2 \cdots, s)$) be the positive tolerance errors of Player I (respectively, Player II) about the fuzzy inequalities, with respect to k^{th} pay-offs (respectively, l^{th} pay-offs). Thus the game $MOBGFG$ model becomes:

$$MOBGFG = (S^m, S^n, A^k, B^l, V_0^k, p_0^k, p_0^{'k}, W_0^l, q_0^l, q_0^{'l}, \gtrsim, \lesssim, (k = 1, 2, \cdots, r; l = 1, 2, \cdots, s)) \tag{11}$$

Definition 5. $(\bar{x}, \bar{y}) \in S^m \times S^n$ *is called a pair of equilibrium solution of the game* $(MOBGFG)$ *model if:*

$$\begin{aligned} & x^T A^k \bar{y} \lesssim_{p_0^k} V_0^k, k = 1, 2, \cdots, r; \forall x \in S^m, \\ & \bar{x}^T B^l y \lesssim_{q_0^l} W_0^l, l = 1, 2, \cdots, s; \forall y \in S^n, \\ & \bar{x}^T A^k \bar{y} \gtrsim_{p_0^{'k}} V_0^k, k = 1, 2, \cdots, r, \\ & \bar{x}^T B^l \bar{y} \gtrsim_{q_0^{'l}} W_0^l, l = 1, 2, \cdots, s. \end{aligned} \tag{12}$$

In order to deal with the above game $(MOBGFG)$ model, we can get the following theorem.

Theorem 1. *Suppose* $(\bar{x}, \bar{y}, \bar{\lambda})$ *is an optimal solution of the problem* $(MONLP2)$ *if and only if we have that* $(\bar{x}, \bar{y})$ *is a pair of equilibrium solution of the game* $(MOBGFG)$ *model. Additionally,* $\bar{\lambda}$ *is the security level of satisfaction of Player I and Player II.* V_0^k $(k = 1, 2, \cdots, r)$ *and* W_0^l $(l = 1, 2, \cdots, s)$ *are the aspiration levels of Player I and II, respectively.*

$$(MONLP2) \qquad max \quad \lambda \tag{13}$$

$$\begin{aligned} subject\ to \quad & A_i^k y + (\lambda - 1) p_0^k \leq V_0^k, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\ & B_j^{l^T} x + (\lambda - 1) q_0^l \leq W_0^l, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\ & x^T A^k y + (1 - \lambda) p_0^{'k} \geq V_0^k, (k = 1, 2, \cdots, r), \\ & x^T B^l y + (1 - \lambda) q_0^{'l} \geq W_0^l, (l = 1, 2, \cdots, s), \\ & 0 \leq \lambda \leq 1, \\ & x \in S^m, y \in S^n. \end{aligned}$$

Proof. Since $(\bar{x}, \bar{y})$ is a pair of equilibrium solutions of the game $(MOBGFG)$ model. By using Definition 5, we can get that the equilibrium solution of the game $(MOBGFG)$ model and the following multiple objective fuzzy optimization problem $(MOFOP)$ are of equal value.

$$(MOFOP) \qquad \text{Find} \quad (x,y) \in S^m \times \in S^n \quad \text{subject to:}$$

$$\begin{array}{l} A_i^k y \lesssim_{p_0^k} V_0^k, (k = 1,2,\cdots,r; i = 1,2,\cdots m), \\ B_j^{T^l} x \lesssim_{q_0^l} W_0^l, (l = 1,2,\cdots,s; j = 1,2,\cdots n), \\ x^T A^k y \gtrsim_{p_0'^k} V_0^k, (k = 1,2,\cdots,r), \\ x^T B^l y \gtrsim_{q_0'^l} W_0^l, (l = 1,2,\cdots,s), \end{array} \tag{14}$$

where A_i^k $(i = 1,2,\cdots,m)$ is the i^{th} row of the matrix A^k and B_j^l $(j = 1,2,\cdots,n)$ is the j^{th} column of the matrix B^l.

By using (9), we obtain that membership functions $u_i^k(A_i^k y)$, $(i = 1,2,\cdots,m)$ (respectively, $v_j^l(B_j^{l^T} x)$, $(j = 1,2,\cdots,n)$) of fuzzy inequalities $A_i^k y \lesssim_{p_0^k} V_0^k$ $(\forall y \in S^n)$ (respectively, $B_j^{l^T} x \lesssim_{q_0^l} W_0^l$ $(\forall x \in S^m)$) can be presented as:

$$u_i^k(A_i^k y) = \begin{cases} 1, & A_i^k y \leq V_0^k, \\ 1 - \frac{A_i^k y - V_0^k}{p_0^k}, & V_0^k \leq A_i^k y \leq V_0^k + p_0^k, \\ 0, & A_i^k y \geq V_0^k + p_0^k, \end{cases} \tag{15}$$

and

$$v_j^l(B_j^{l^T} x) = \begin{cases} 1, & B_j^{l^T} x \leq W_0^l, \\ 1 - \frac{B_j^{l^T} x - W_0^l}{q_0^l}, & W_0^l \leq B_j^{l^T} x \leq W_0^l + q_0^l, \\ 0, & B_j^{l^T} x \geq W_0^l + q_0^l. \end{cases} \tag{16}$$

respectively.

Similarly, we have that the non-linear membership functions of the fuzzy inequalities $x^T A^k y \gtrsim_{p_0'^k} V_0^k$ (respectively, $x^T B^l y \gtrsim_{q_0'^l} W_0^l$) can be expressed as:

$$u_{\tilde{g}_I^k}(x^T A^k y) = \begin{cases} 1, & x^T A^k y \geq V_0^k, \\ 1 - \frac{V_0^k - x^T A^k y}{p_0'^k}, & V_0^k \geq x^T A^k y \geq V_0^k - p_0'^k, \\ 0, & x^T A^k y \leq V_0^k - p_0'^k, \end{cases} \tag{17}$$

and

$$v_{\tilde{g}_{II}^l}(x^T B^l y) = \begin{cases} 1, & x^T B^l y \geq W_0^l, \\ 1 - \frac{W_0^l - x^T B^l y}{q_0'^l}, & W_0^l \geq x^T B^l y \geq W_0^l - q_0'^l, \\ 0, & x^T B^l y \leq W_0^l - q_0'^l. \end{cases} \tag{18}$$

respectively.

Inspired by [35], by combining (10)–(13) we obtain that the problem $(MOFOP)$ model is equivalent to the multicriteria non-linear programming $(MONLP1)$ problem.

$$
(MONLP1) \qquad \max \quad \lambda
$$

$$
\begin{array}{ll}
\textit{Subject to} & \lambda \leq 1 - \frac{A_i^k y - V_0^k}{p_0^k}, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\
& \lambda \leq 1 - \frac{B_j^{l^T} x - W_0^l}{q_0^l}, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\
& \lambda \leq 1 + \frac{x^T A^k y - V_0^k}{p_0^{'k}}, (k = 1, 2, \cdots, r), \\
& \lambda \leq 1 + \frac{x^T B^l y - W_0^l}{q_0^{'l}}, (l = 1, 2, \cdots, s), \\
& 0 \leq \lambda \leq 1, \\
& x \in S^m, y \in S^n.
\end{array} \tag{19}
$$

That is, by simplifying the above problem, that is equal to:

$$
(MONLP2) \qquad \max \quad \lambda
$$

$$
\begin{array}{ll}
\textit{subject to} & A_i^k y + (\lambda - 1) p_0^k \leq V_0^k, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\
& B_j^{l^T} x + (\lambda - 1) q_0^l \leq W_0^l, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\
& x^T A^k y + (1 - \lambda) p_0^{'k} \geq V_0^k, (k = 1, 2, \cdots, r), \\
& x^T B^l y + (1 - \lambda) q_0^{'l} \geq W_0^l, (l = 1, 2, \cdots, s), \\
& 0 \leq \lambda \leq 1, \\
& x \in S^m, y \in S^n.
\end{array} \tag{20}
$$

Then, we have that $(\bar{x}, \bar{y})$ is a pair of equilibrium solutions of the game $(MOBGFG)$ model if and only if $(\bar{x}, \bar{y}, \bar{\lambda})$ is an optimal solution of the problem $(MONLP2)$.

$$
(MONLP2) \qquad \max \quad \lambda
$$

$$
\begin{array}{ll}
\textit{subject to} & A_i^k y + (\lambda - 1) p_0^k \leq V_0^k, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\
& B_j^{l^T} x + (\lambda - 1) q_0^l \leq W_0^l, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\
& x^T A^k y + (1 - \lambda) p_0^{'k} \geq V_0^k, (k = 1, 2, \cdots, r), \\
& x^T B^l y + (1 - \lambda) q_0^{'l} \geq W_0^l, (l = 1, 2, \cdots, s), \\
& 0 \leq \lambda \leq 1, \\
& x \in S^m, y \in S^n.
\end{array} \tag{21}
$$

□

Remark 1. *Let* $\bar{\lambda} = 1$ *and suppose* $(\bar{x}, \bar{y}, \bar{\lambda})$ *is an optimal solution of the problem* $(MONLP2)$*. Then, we obtain that the game* $(MOBG)$ *model is a special case of the game* $(MOBGFG)$ *model.*

Remark 2. *Let* $\bar{\lambda} = 1$ *and suppose* $(\bar{x}, \bar{y}, \bar{\lambda})$ *is an optimal solution of the problem* $(MONLP2)$*. Then the problem* $(MONLP2)$ *model changes into:*

$$
(MONLP3) \qquad \max \quad \lambda
$$

$$
\begin{array}{ll}
\textit{subject to} & A_i^k y \leq V_0^k, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\
& B_j^{l^T} x \leq W_0^l, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\
& x^T A^k y \geq V_0^k, (k = 1, 2, \cdots, r), \\
& x^T B^l y \geq W_0^l, (l = 1, 2, \cdots, s), \\
& 0 \leq \lambda \leq 1, \\
& x \in S^m, y \in S^n.
\end{array} \tag{22}
$$

4. Special Case:

In this section, we present a multicriteria non-linear programming problem in some special cases.

Theorem 2. *Let $V_0^k = \overline{a}^k$, $p_0^k = p_0^{'k} = \overline{a}^k - \underline{a}^k$, $W_0^l = \overline{b}^l$, and $q_0^l = q_0^{'l} = \overline{b}^l - \underline{b}^l$. Suppose $(\bar{x}, \bar{y})$ are a pair of equilibrium solutions of the game $(MOBGFG)$ model if and only if $(\bar{x}, \bar{y}, \bar{\lambda})$ is an optimal solution of the problem $(MONLP4)$.*

$$(MONLP4) \qquad \max \quad \lambda$$

$$\text{subject to} \quad \begin{aligned} & A_i^k y + (\lambda - 1)(\overline{a}^k - \underline{a}^k) \le \overline{a}^k, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\ & B_j^{l^T} x + (\lambda - 1)(\overline{b}^l - \underline{b}^l) \le \overline{b}^l, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\ & x^T A^k y + (1 - \lambda)(\overline{a}^k - \underline{a}^k) \ge \overline{a}^k, (k = 1, 2, \cdots, r), \\ & x^T B^l y + (1 - \lambda)(\overline{b}^l - \underline{b}^l) \ge \overline{b}^l, (l = 1, 2, \cdots, s), \\ & x \in S^m, y \in S^n, \end{aligned} \tag{23}$$

where

$$\underline{a}^k = \min_{x \in S^m} \min_{y \in S^n} x^T A^k y = \min_{x \in S^m} \min_{y \in S^n} a_{ij}^k, \quad \overline{a}^k = \max_{x \in S^m} \max_{y \in S^n} x^T A^k y = \max_{x \in S^m} \max_{y \in S^n} a_{ij}^k,$$

$$\underline{b}^l = \min_{x \in S^m} \min_{y \in S^n} x^T B^l y = \min_{x \in S^m} \min_{y \in S^n} b_{ij}^l, \quad \overline{b}^l = \max_{x \in S^m} \max_{y \in S^n} x^T B^l y = \max_{x \in S^m} \max_{y \in S^n} b_{ij}^l.$$

Proof. Since $(\bar{x}, \bar{y})$ is a pair of equilibrium solutions of the game $(MOBGFG)$ model and $V_0^k = \overline{a}^k$, and $p_0^k = p_0^{'k} = \overline{a}^k - \underline{a}^k$, $W_0^l = \overline{b}^l$, $q_0^l = q_0^{'l} = \overline{b}^l - \underline{b}^l$. By using Definition 5 and Theorem 1, we can get that the equilibrium solutions of the game $(MOBGFG)$ model and the following multiple objective fuzzy optimization problem $(MOFOP1)$ are of equal value.

$$(MOFOP1) \qquad \text{Find} \quad (x, y) \in S^m \times \in S^n \quad \text{subject to}$$

$$\begin{aligned} & A_i^k y \lesssim_{\overline{a}^k - \underline{a}^k} \overline{a}^k, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\ & B_j^{T^l} x \lesssim_{\overline{b}^l - \underline{b}^l} \overline{b}^l, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\ & x^T A^k y \gtrsim_{\overline{a}^k - \underline{a}^k} \overline{a}^k, (k = 1, 2, \cdots, r), \\ & x^T B^l y \gtrsim_{\overline{b}^l - \underline{b}^l} \overline{b}^l, (l = 1, 2, \cdots, s), \end{aligned} \tag{24}$$

Inspired by [2,29], now combining (10), (11), (12) and (13), we take membership functions $u_i^k(A_i^k y)$ $(i = 1, 2, \cdots, m)$, $v_j^l(B_j^{l^T} x)$ $(j = 1, 2, \cdots, n)$, $u_{\tilde{g}_I^k}(x^T A^k y)$ and $v_{\tilde{g}_{II}^l}(x^T B^l y)$ $(k = 1, 2, \cdots, r; l = 1, 2, \cdots, s)$ as:

$$u_i^k(A_i^k y) = \begin{cases} 1, & A_i^k y \le \overline{a}^k, \\ 1 - \frac{A_i^k y - \overline{a}^k}{\overline{a}^k - \underline{a}^k}, & \overline{a}^k \le A_i^k y \le 2\overline{a}^k - \underline{a}^k, \\ 0, & A_i^k y \ge 2\overline{a}^k - \underline{a}^k, \end{cases} \tag{25}$$

$$v_j^l(B_j^{l^T} x) = \begin{cases} 1, & B_j^{l^T} x \le \overline{b}^l, \\ 1 - \frac{B_j^{l^T} x - \overline{b}^l}{\overline{b}^l - \underline{b}^l}, & \overline{b}^l \le B_j^{l^T} x \le 2\overline{b}^l - \underline{b}^l, \\ 0, & B_j^{l^T} x \ge 2\overline{b}^l - \underline{b}^l. \end{cases} \tag{26}$$

$$u_{\tilde{g}_I^k}(x^T A^k y) = \begin{cases} 1, & x^T A^k y \ge \overline{a}^k, \\ 1 - \frac{\overline{a}^k - x^T A^k y}{\overline{a}^k - \underline{a}^k}, & \overline{a}^k \ge x^T A^k y \ge \underline{a}^k, \\ 0, & x^T A^k y \le \underline{a}^k, \end{cases} \tag{27}$$

and

$$v_{\tilde{g}_{II}^{l}}(x^T B^l y) = \begin{cases} 1, & x^T B^l y \geq \overline{b}^l, \\ 1 - \frac{\overline{b}^l - x^T B^l y}{q_0^{\prime l}}, & \overline{b}^l \geq x^T B^l y \geq \underline{b}^l, \\ 0, & x^T B^l y \leq \underline{b}^l. \end{cases} \tag{28}$$

Similarly, we obtain that the problem $(MOFOP1)$ model changes into:

$$(MONLP5) \quad \max \quad \lambda$$

$$\begin{aligned} \text{subject to} \quad & \lambda \leq 1 - \frac{A_i^k y - \overline{a}^k}{\overline{a}^k - \underline{a}^k}, (k = 1,2,\cdots,r; i = 1,2,\cdots m), \\ & \lambda \leq 1 - \frac{B_j^{l^T} x - \overline{b}^l}{\overline{b}^l - \underline{b}^l}, (l = 1,2,\cdots,s; j = 1,2,\cdots n), \\ & \lambda \leq 1 + \frac{x^T A^k y - \overline{a}^k}{\overline{a}^k - \underline{a}^k}, (k = 1,2,\cdots,r), \\ & \lambda \leq 1 + \frac{x^T B^l y - \overline{b}^l}{\overline{b}^l - \underline{b}^l}, (l = 1,2,\cdots,s), \\ & 0 \leq \lambda \leq 1, \\ & x \in S^m, y \in S^n. \end{aligned} \tag{29}$$

That is, the problem $(MONLP5)$ model is equal to:

$$(MONLP4) \quad \max \quad \lambda$$

$$\begin{aligned} \text{subject to} \quad & A_i^k y + (\lambda - 1)(\overline{a}^k - \underline{a}^k) \leq \overline{a}^k, (k = 1,2,\cdots,r; i = 1,2,\cdots m), \\ & B_j^{l^T} x + (\lambda - 1)(\overline{b}^l - \underline{b}^l) \leq \overline{b}^l, (l = 1,2,\cdots,s; j = 1,2,\cdots n), \\ & x^T A^k y + (1 - \lambda)(\overline{a}^k - \underline{a}^k) \geq \overline{a}^k, (k = 1,2,\cdots,r), \\ & x^T B^l y + (1 - \lambda)(\overline{b}^l - \underline{b}^l) \geq \overline{b}^l, (l = 1,2,\cdots,s), \\ & x \in S^m, y \in S^n. \end{aligned} \tag{30}$$

Then, we have that $(\bar{x}, \bar{y})$ is a pair of equilibrium solutions of the game $(MOBGFG)$ model if and only if $(\bar{x}, \bar{y}, \bar{\lambda})$ is an optimal solution of the problem $(MONLP4)$.

$$(MONLP4) \quad \max \quad \lambda$$

$$\begin{aligned} \text{subject to} \quad & A_i^k y + (\lambda - 1)(\overline{a}^k - \underline{a}^k) \leq \overline{a}^k, (k = 1,2,\cdots,r; i = 1,2,\cdots m), \\ & B_j^{l^T} x + (\lambda - 1)(\overline{b}^l - \underline{b}^l) \leq \overline{b}^l, (l = 1,2,\cdots,s; j = 1,2,\cdots n), \\ & x^T A^k y + (1 - \lambda)(\overline{a}^k - \underline{a}^k) \geq \overline{a}^k, (k = 1,2,\cdots,r), \\ & x^T B^l y + (1 - \lambda)(\overline{b}^l - \underline{b}^l) \geq \overline{b}^l, (l = 1,2,\cdots,s), \\ & x \in S^m, y \in S^n. \end{aligned} \tag{31}$$

□

Theorem 3. *Suppose $(\bar{x}, \bar{y}, \bar{\lambda})$ are an optimal solution of the problem $(MONLP2)$. Let $V_0^k = \overline{a}^k$, and $p_0^k = p_0^{\prime k} = \overline{a}^k - \underline{a}^k$, $W_0^l = \overline{b}^l$, $q_0^l = q_0^{\prime l} = \overline{b}^l - \underline{b}^l$. Then the problem $(MONLP2)$ model changes into the following problem $(MONLP4)$.*

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda})$ is an optimal solution of the problem $(MONLP2)$, then we can get:

$$(MONLP2) \qquad max \quad \lambda$$

$$\begin{aligned} subject\ to \quad & A_i^k y + (\lambda - 1)p_0^k \leq V_0^k, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\ & B_j^{l^T} x + (\lambda - 1)q_0^l \leq W_0^l, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\ & x^T A^k y + (1 - \lambda)p_0^{'k} \geq V_0^k, (k = 1, 2, \cdots, r), \\ & x^T B^l y + (1 - \lambda)q_0^{'l} \geq W_0^l, (l = 1, 2, \cdots, s), \\ & 0 \leq \lambda \leq 1, \\ & x \in S^m, y \in S^n. \end{aligned} \tag{32}$$

Now, let $V_0^k = \overline{a}^k$, $p_0^k = p_0^{'k} = \overline{a}^k - \underline{a}^k$, $W_0^l = \overline{b}^l$, and $q_0^l = q_0^{'l} = \overline{b}^l - \underline{b}^l$. Hence, we obtain that the problem $(MONLP2)$ model changes into:

$$(MONLP4) \qquad max \quad \lambda$$

$$\begin{aligned} subject\ to \quad & A_i^k y + (\lambda - 1)(\overline{a}^k - \underline{a}^k) \leq \overline{a}^k, (k = 1, 2, \cdots, r; i = 1, 2, \cdots m), \\ & B_j^{l^T} x + (\lambda - 1)(\overline{b}^l - \underline{b}^l) \leq \overline{b}^l, (l = 1, 2, \cdots, s; j = 1, 2, \cdots n), \\ & x^T A^k y + (1 - \lambda)(\overline{a}^k - \underline{a}^k) \geq \overline{a}^k, (k = 1, 2, \cdots, r), \\ & x^T B^l y + (1 - \lambda)(\overline{b}^l - \underline{b}^l) \geq \overline{b}^l, (l = 1, 2, \cdots, s), \\ & x \in S^m, y \in S^n. \end{aligned} \tag{33}$$

Then, we have that $(\bar{x}, \bar{y}, \bar{\lambda})$ is an optimal solution of the problem $(MONLP4)$. □

5. Example

Now, we consider the following multi-objective fuzzy bi-matrix game ($MOBGFG$) model.

Example 1. *A the multi-objective bi-matrix game is considered. The multiple pay-off matrices of the Player I and Player II are taken as:*

$$A^1 = \begin{pmatrix} 6 & 3 & 4 \\ 3 & 6 & 8 \\ 7 & 3 & 4 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 9 & 2 & 7 \\ 4 & 5 & 8 \\ 2 & 7 & 3 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 5 & 1 & 2 \\ 3 & 4 & 8 \\ 1 & 8 & 1 \end{pmatrix},$$

and

$$B^1 = \begin{pmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{pmatrix}, \qquad B^2 = \begin{pmatrix} 1 & 6 & 7 \\ 8 & 2 & 3 \\ 4 & 9 & 3 \end{pmatrix}, \qquad B^3 = \begin{pmatrix} 8 & 2 & 3 \\ -5 & 6 & 0 \\ -3 & 1 & 6 \end{pmatrix}$$

respectively.

We now solve this problem with the above model. Thus, by Theorem 2, we have:

$$\underline{a}^1 = \min_{x \in S^3} \min_{y \in S^3} a_{ij}^1 = 3,\ V_0^1 = \overline{a}^1 = \max_{x \in S^3} \max_{y \in S^3} a_{ij}^1 = 8,\ p_0^1 = p_0^{'1} = \overline{a}^1 - \underline{a}^1 = 5; \tag{34}$$

$$\underline{a}^2 = \min_{x \in S^3} \min_{y \in S^3} a_{ij}^2 = 2,\ V_0^2 = \overline{a}^2 = \max_{x \in S^3} \max_{y \in S^3} a_{ij}^2 = 9, p_0^2 = p_0^{'2} = \overline{a}^2 - \underline{a}^2 = 7; \tag{35}$$

$$\underline{a}^3 = \min_{x \in S^3} \min_{y \in S^3} a_{ij}^3 = 1,\ V_0^3 = \overline{a}^3 = \max_{x \in S^3} \max_{y \in S^3} a_{ij}^3 = 8, p_0^3 = p_0^{'3} = \overline{a}^3 - \underline{a}^3 = 7; \tag{36}$$

$$\underline{b}^1 = \min_{x \in S^3} \min_{y \in S^3} b_{ij}^1 = 0,\ W_0^1 = \overline{b}^1 = \max_{x \in S^3} \max_{y \in S^3} b_{ij}^1 = 9,\ q_0^1 = q_0^{'1} = \overline{b}^1 - \underline{b}^1 = 9; \tag{37}$$

$$\underline{b}^2 = \min_{x \in S^3} \min_{y \in S^3} b_{ij}^2 = 1,\ W_0^2 = \overline{b}^2 = \max_{x \in S^3} \max_{y \in S^3} b_{ij}^2 = 9,\ q_0^2 = q_0^{'2} = \overline{b}^2 - \underline{b}^2 = 8; \tag{38}$$

$$\underline{b}^3 = \min_{x \in S^3} \min_{y \in S^3} b_{ij}^3 = -5,\ W_0^3 = \overline{b}^3 = \max_{x \in S^3} \max_{y \in S^3} b_{ij}^3 = 8,\ q_0^3 = q_0^{'3} = \overline{b}^3 - \underline{b}^3 = 13. \tag{39}$$

By the above numerical values, we can get that the equilibrium solutions of the above model and the following multiple objective fuzzy optimization problem $(MOFOP2)$ are of equal value.

$(MOFOP2)$ Find $(x, y) \in S^3 \times \in S^3$

$$\begin{array}{lll}
\textit{subject to} & 6y_1 + 3y_2 + 4y_3 \lesssim_5 8, & 9x_1 + 0x_2 + 5x_3 \lesssim_9 9, \\
& 3y_1 + 6y_2 + 8y_3 \lesssim_5 8, & 1x_1 + 6x_2 + 2x_3 \lesssim_9 9, \\
& 7y_1 + 3y_2 + 4y_3 \lesssim_5 8, & 4x_1 + 3x_2 + 8x_3 \lesssim_9 9, \\
& 9y_1 + 2y_2 + 7y_3 \lesssim_7 9, & 1x_1 + 8x_2 + 4x_3 \lesssim_8 9, \\
& 4y_1 + 5y_2 + 8y_3 \lesssim_7 9, & 6x_1 + 2x_2 + 9x_3 \lesssim_8 9, \\
& 2y_1 + 7y_2 + 3y_3 \lesssim_7 9, & 7x_1 + 3x_2 + 3x_3 \lesssim_8 9, \\
& 5y_1 + 1y_2 + 2y_3 \lesssim_7 8, & 8x_1 - 5x_2 - 3x_3 \lesssim_{13} 8, \\
& 3y_1 + 4y_2 + 8y_3 \lesssim_7 8, & 2x_1 + 6x_2 + 1x_3 \lesssim_{13} 8, \\
& 1y_1 + 8y_2 + 1y_3 \lesssim_7 8, & 3x_1 + 0x_2 + 6x_3 \lesssim_{13} 8,
\end{array}$$
$$\begin{array}{l}
6x_1y_1 + 3x_2y_1 + 7x_3y_1 + 3x_1y_2 + 6x_2y_2 + 3x_3y_2 + 4x_1y_3 + 8x_2y_3 + 4x_3y_3 \gtrsim_5 8, \\
9x_1y_1 + 4x_2y_1 + 2x_3y_1 + 2x_1y_2 + 5x_2y_2 + 7x_3y_2 + 7x_1y_3 + 8x_2y_3 + 3x_3y_3 \gtrsim_7 9, \\
5x_1y_1 + 3x_2y_1 + 1x_3y_1 + 1x_1y_2 + 4x_2y_2 + 8x_3y_2 + 2x_1y_3 + 8x_2y_3 + 1x_3y_3 \gtrsim_7 8, \\
9x_1y_1 + 0x_2y_1 + 5x_3y_1 + 1x_1y_2 + 6x_2y_2 + 2x_3y_2 + 4x_1y_3 + 3x_2y_3 + 8x_3y_3 \gtrsim_9 9, \\
1x_1y_1 + 8x_2y_1 + 4x_3y_1 + 6x_1y_2 + 2x_2y_2 + 9x_3y_2 + 7x_1y_3 + 3x_2y_3 + 3x_3y_3 \gtrsim_8 9, \\
8x_1y_1 - 5x_2y_1 - 3x_3y_1 + 2x_1y_2 + 6x_2y_2 + 1x_3y_2 + 3x_1y_3 + 0x_2y_3 + 6x_3y_3 \gtrsim_{13} 8.
\end{array} \tag{40}$$

Now we get the following membership functions based on the above fuzzy inequalities.

$$u_1^1(6y_1 + 3y_2 + 4y_3) = \begin{cases} 1, & 6y_1 + 3y_2 + 4y_3 \le 8, \\ 1 - \frac{6y_1 + 3y_2 + 4y_3 - 8}{5}, & 8 \le 6y_1 + 3y_2 + 4y_3 \le 13, \\ 0, & 6y_1 + 3y_2 + 4y_3 \ge 13, \end{cases} \tag{41}$$

$$u_2^1(3y_1 + 6y_2 + 8y_3) = \begin{cases} 1, & 3y_1 + 6y_2 + 8y_3 \le 8, \\ 1 - \frac{3y_1 + 6y_2 + 8y_3 - 8}{5}, & 8 \le 3y_1 + 6y_2 + 8y_3 \le 13, \\ 0, & 3y_1 + 6y_2 + 8y_3 \ge 13, \end{cases} \tag{42}$$

$$u_3^1(7y_1 + 3y_2 + 4y_3) = \begin{cases} 1, & 7y_1 + 3y_2 + 4y_3 \le 8, \\ 1 - \frac{7y_1 + 3y_2 + 4y_3 - 8}{5}, & 8 \le 7y_1 + 3y_2 + 4y_3 \le 13, \\ 0, & 7y_1 + 3y_2 + 4y_3 \ge 13, \end{cases} \tag{43}$$

$$u_1^2(9y_1 + 2y_2 + 7y_3) = \begin{cases} 1, & 9y_1 + 2y_2 + 7y_3 \le 9, \\ 1 - \frac{9y_1 + 2y_2 + 7y_3 - 9}{7}, & 9 \le 9y_1 + 2y_2 + 7y_3 \le 16, \\ 0, & 9y_1 + 2y_2 + 7y_3 \ge 16, \end{cases} \tag{44}$$

$$u_2^2(4y_1 + 5y_2 + 8y_3) = \begin{cases} 1, & 4y_1 + 5y_2 + 8y_3 \le 9, \\ 1 - \frac{4y_1 + 5y_2 + 8y_3 - 9}{7}, & 9 \le 4y_1 + 5y_2 + 8y_3 \le 16, \\ 0, & 4y_1 + 5y_2 + 8y_3 \ge 16, \end{cases} \tag{45}$$

$$u_3^2(2y_1 + 7y_2 + 3y_3) = \begin{cases} 1, & 2y_1 + 7y_2 + 3y_3 \le 9, \\ 1 - \frac{2y_1 + 7y_2 + 3y_3 - 9}{7}, & 9 \le 2y_1 + 7y_2 + 3y_3 \le 16, \\ 0, & 2y_1 + 7y_2 + 3y_3 \ge 16, \end{cases} \tag{46}$$

$$u_1^3(5y_1+1y_2+2y_3)=\begin{cases}1, & 5y_1+1y_2+2y_3\le 8,\\ 1-\frac{5y_1+1y_2+2y_3-8}{7}, & 8\le 5y_1+1y_2+2y_3\le 15,\\ 0, & 5y_1+1y_2+2y_3\ge 15,\end{cases}\quad(47)$$

$$u_2^3(3y_1+4y_2+8y_3)=\begin{cases}1, & 3y_1+4y_2+8y_3\le 8,\\ 1-\frac{3y_1+4y_2+8y_3-8}{7}, & 8\le 3y_1+4y_2+8y_3\le 15,\\ 0, & 3y_1+4y_2+8y_3\ge 15,\end{cases}\quad(48)$$

$$u_3^3(1y_1+8y_2+1y_3)=\begin{cases}1, & 1y_1+8y_2+1y_3\le 8,\\ 1-\frac{1y_1+8y_2+1y_3-8}{7}, & 8\le 1y_1+8y_2+1y_3\le 15,\\ 0, & 1y_1+8y_2+1y_3\ge 15,\end{cases}\quad(49)$$

$$v_1^1(9x_1+0x_2+5x_3)=\begin{cases}1, & 9x_1+0x_2+5x_3\le 9,\\ 1-\frac{9x_1+0x_2+5x_3-9}{9}, & 9\le 9x_1+0x_2+5x_3\le 18,\\ 0, & 9x_1+0x_2+5x_3\ge 18.\end{cases}\quad(50)$$

$$v_2^1(1x_1+6x_2+2x_3)=\begin{cases}1, & 1x_1+6x_2+2x_3\le 9,\\ 1-\frac{1x_1+6x_2+2x_3-9}{9}, & 9\le 1x_1+6x_2+2x_3\le 18,\\ 0, & 1x_1+6x_2+2x_3\ge 18.\end{cases}\quad(51)$$

$$v_3^1(4x_1+3x_2+8x_3)=\begin{cases}1, & 4x_1+3x_2+8x_3\le 9,\\ 1-\frac{4x_1+3x_2+8x_3-9}{9}, & 9\le 4x_1+3x_2+8x_3\le 18,\\ 0, & 4x_1+3x_2+8x_3\ge 18.\end{cases}\quad(52)$$

$$v_1^2(1x_1+8x_2+4x_3)=\begin{cases}1, & 1x_1+8x_2+4x_3\le 9,\\ 1-\frac{1x_1+8x_2+4x_3-9}{8}, & 9\le 1x_1+8x_2+4x_3\le 17,\\ 0, & 1x_1+8x_2+4x_3\ge 17.\end{cases}\quad(53)$$

$$v_2^2(6x_1+2x_2+9x_3)=\begin{cases}1, & 6x_1+2x_2+9x_3\le 9,\\ 1-\frac{6x_1+2x_2+9x_3-9}{8}, & 9\le 6x_1+2x_2+9x_3\le 17,\\ 0, & 6x_1+2x_2+9x_3\ge 17.\end{cases}\quad(54)$$

$$v_3^2(7x_1+3x_2+3x_3)=\begin{cases}1, & 7x_1+3x_2+3x_3\le 9,\\ 1-\frac{7x_1+3x_2+3x_3-9}{8}, & 9\le 7x_1+3x_2+3x_3\le 17,\\ 0, & 7x_1+3x_2+3x_3\ge 17.\end{cases}\quad(55)$$

$$v_1^3(8x_1-5x_2-3x_3)=\begin{cases}1, & 8x_1-5x_2-3x_3\le 8,\\ 1-\frac{8x_1-5x_2-3x_3-8}{13}, & 8\le 8x_1-5x_2-3x_3\le 21,\\ 0, & 8x_1-5x_2-3x_3\ge 21.\end{cases}\quad(56)$$

$$v_2^3(2x_1+6x_2+1x_3)=\begin{cases}1, & 2x_1+6x_2+1x_3\le 8,\\ 1-\frac{2x_1+6x_2+1x_3-8}{13}, & 8\le 2x_1+6x_2+1x_3\le 21,\\ 0, & 2x_1+6x_2+1x_3\ge 21.\end{cases}\quad(57)$$

$$v_3^3(3x_1+0x_2+6x_3)=\begin{cases}1, & 3x_1+0x_2+6x_3\le 8,\\ 1-\frac{3x_1+0x_2+6x_3-8}{13}, & 8\le 3x_1+0x_2+6x_3\le 21,\\ 0, & 3x_1+0x_2+6x_3\ge 21.\end{cases}\quad(58)$$

$$u_{\tilde{g}_l^1}(x^TA^1y)=\begin{cases}1, & x^TA^1y\ge 8,\\ 1-\frac{8-x^TA^1y}{5}, & 8\ge x^TA^1y\ge 3,\\ 0, & x^TA^1y\le 3,\end{cases}\quad(59)$$

$$u_{\tilde{g}_I^2}(x^TA^2y) = \begin{cases} 1, & x^TA^2y \geq 9, \\ 1 - \frac{9 - x^TA^2y}{7}, & 9 \geq x^TA^2y \geq 2, \\ 0, & x^TA^2y \leq 2, \end{cases} \tag{60}$$

$$u_{\tilde{g}_I^3}(x^TA^3y) = \begin{cases} 1, & x^TA^3y \geq 8, \\ 1 - \frac{8 - x^TA^3y}{7}, & 8 \geq x^TA^3y \geq 1, \\ 0, & x^TA^3y \leq 1, \end{cases} \tag{61}$$

$$v_{\tilde{g}_{II}^1}(x^TB^1y) = \begin{cases} 1, & x^TB^1y \geq 9, \\ 1 - \frac{9 - x^TB^1y}{9}, & 9 \geq x^TB^1y \geq 0, \\ 0, & x^TB^1y \leq 0, \end{cases} \tag{62}$$

$$v_{\tilde{g}_{II}^2}(x^TB^2y) = \begin{cases} 1, & x^TB^2y \geq 9, \\ 1 - \frac{9 - x^TB^2y}{8}, & 9 \geq x^TB^2y \geq 1, \\ 0, & x^TB^2y \leq 1, \end{cases} \tag{63}$$

and

$$v_{\tilde{g}_{II}^3}(x^TB^3y) = \begin{cases} 1, & x^TB^3y \geq 8, \\ 1 - \frac{8 - x^TB^3y}{13}, & 8 \geq x^TB^3y \geq -5, \\ 0, & x^TB^3y \leq -5, \end{cases} \tag{64}$$

where

$$\begin{aligned} x^TA^1y &= 6x_1y_1 + 3x_2y_1 + 7x_3y_1 + 3x_1y_2 + 6x_2y_2 + 3x_3y_2 + 4x_1y_3 + 8x_2y_3 + 4x_3y_3, \\ x^TA^2y &= 9x_1y_1 + 4x_2y_1 + 2x_3y_1 + 2x_1y_2 + 5x_2y_2 + 7x_3y_2 + 7x_1y_3 + 8x_2y_3 + 3x_3y_3, \\ x^TA^3y &= 5x_1y_1 + 3x_2y_1 + 1x_3y_1 + 1x_1y_2 + 4x_2y_2 + 8x_3y_2 + 2x_1y_3 + 8x_2y_3 + 1x_3y_3, \\ x^TB^1y &= 9x_1y_1 + 0x_2y_1 + 5x_3y_1 + 1x_1y_2 + 6x_2y_2 + 2x_3y_2 + 4x_1y_3 + 3x_2y_3 + 8x_3y_3, \\ x^TB^2y &= 1x_1y_1 + 8x_2y_1 + 4x_3y_1 + 6x_1y_2 + 2x_2y_2 + 9x_3y_2 + 7x_1y_3 + 3x_2y_3 + 3x_3y_3, \\ x^TB^3y &= 8x_1y_1 - 5x_2y_1 - 3x_3y_1 + 2x_1y_2 + 6x_2y_2 + 1x_3y_2 + 3x_1y_3 + 0x_2y_3 + 6x_3y_3. \end{aligned} \tag{65}$$

Now using (21) and (22), we have the following multiple objective non-liner programming problem ($MONLP6$).

$$(MONLP6) \qquad max \qquad \lambda$$

$$\begin{aligned} subject\ to\quad & 6y_1 + 3y_2 + 4y_3 + (\lambda - 1)5 \leq 8,\ 9x_1 + 0x_2 + 5x_3 + (\lambda - 1)9 \leq 9, \\ & 3y_1 + 6y_2 + 8y_3 + (\lambda - 1)5 \leq 8,\ 1x_1 + 6x_2 + 2x_3 + (\lambda - 1)9 \leq 9, \\ & 7y_1 + 3y_2 + 4y_3 + (\lambda - 1)5 \leq 8,\ 4x_1 + 3x_2 + 8x_3 + (\lambda - 1)9 \leq 9, \\ & 9y_1 + 2y_2 + 7y_3 + (\lambda - 1)7 \leq 9,\ 1x_1 + 8x_2 + 4x_3 + (\lambda - 1)8 \leq 9, \\ & 4y_1 + 5y_2 + 8y_3 + (\lambda - 1)7 \leq 9,\ 6x_1 + 2x_2 + 9x_3 + (\lambda - 1)8 \leq 9, \\ & 2y_1 + 7y_2 + 3y_3 + (\lambda - 1)7 \leq 9,\ 7x_1 + 3x_2 + 3x_3 + (\lambda - 1)8 \leq 9, \\ & 5y_1 + 1y_2 + 2y_3 + (\lambda - 1)7 \leq 8,\ 8x_1 - 5x_2 - 3x_3 + (\lambda - 1)13 \leq 8, \\ & 3y_1 + 4y_2 + 8y_3 + (\lambda - 1)7 \leq 8,\ 2x_1 + 6x_2 + 1x_3 + (\lambda - 1)13 \leq 8, \\ & 1y_1 + 8y_2 + 1y_3 + (\lambda - 1)7 \leq 8,\ 3x_1 + 0x_2 + 6x_3 + (\lambda - 1)13 \leq 8, \\ & 6x_1y_1 + 3x_2y_1 + 7x_3y_1 + 3x_1y_2 + 6x_2y_2 + 3x_3y_2 + 4x_1y_3 + 8x_2y_3 + 4x_3y_3 + (1 - \lambda)5 \geq 8, \\ & 9x_1y_1 + 4x_2y_1 + 2x_3y_1 + 2x_1y_2 + 5x_2y_2 + 7x_3y_2 + 7x_1y_3 + 8x_2y_3 + 3x_3y_3 + (1 - \lambda)7 \geq 9, \\ & 5x_1y_1 + 3x_2y_1 + 1x_3y_1 + 1x_1y_2 + 4x_2y_2 + 8x_3y_2 + 2x_1y_3 + 8x_2y_3 + 1x_3y_3 + (1 - \lambda)7 \geq 8, \\ & 9x_1y_1 + 0x_2y_1 + 5x_3y_1 + 1x_1y_2 + 6x_2y_2 + 2x_3y_2 + 4x_1y_3 + 3x_2y_3 + 8x_3y_3 + (1 - \lambda)9 \geq 9, \\ & 1x_1y_1 + 8x_2y_1 + 4x_3y_1 + 6x_1y_2 + 2x_2y_2 + 9x_3y_2 + 7x_1y_3 + 3x_2y_3 + 3x_3y_3 + (1 - \lambda)8 \geq 9, \\ & 8x_1y_1 - 5x_2y_1 - 3x_3y_1 + 2x_1y_2 + 6x_2y_2 + 1x_3y_2 + 3x_1y_3 + 0x_2y_3 + 6x_3y_3 + (1 - \lambda)13 \geq 8, \\ & 0 \leq \lambda \leq 1,\ x \in S^3, y \in S^3. \end{aligned} \tag{66}$$

For some sample values of λ, we obtain the optimal solutions of the problem $(MONLP6)$ for Player I and Player II in Table 1. Similarly, for other values of $\lambda \in [0, 1]$, we can obtain the optimal solutions of the problem $(MONLP6)$ model through the same approach.

In particular, let $\bar{\lambda} = 0.2885$, then we can have that $(\bar{x}_1 = 0.1, \bar{x}_2 = 0.1, \bar{x}_3 = 0.8)$ and $(\bar{y}_1 = 0.6, \bar{y}_2 = 0.3, \bar{y}_3 = 0.1)$ are the mixed strategies of Player I and Player II, respectively.

Table 1. Strategies of Example 1.

Strategies	$\bar{\lambda}$	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{y}_1$	$\bar{y}_2$	$\bar{y}_3$
1	0.2285	0.1	0.2	0.7	0.4	0.2	0.4
2	0.2685	0.2	0.3	0.5	0.7	0.1	0.2
3	0.2720	0.6	0.2	0.2	0.2	0.5	0.3
4	0.2885	0.1	0.1	0.8	0.6	0.3	0.1
5	0.2971	0.5	0.1	0.4	0.5	0.2	0.3
6	0.3000	0.4	0.1	0.5	0.3	0.4	0.3

6. Conclusions

In this paper, we have presented a multi-objective bi-matrix game with a fuzzy goals $(MOBGFG)$ model. The inspiration of the model is from [2,29,30,36] and we have solved the game $(MOBGFG)$ model via a multi-objective non-linear programming method. We will discuss a situation where the elements of matrices $A^k(l = 1, 2, \cdots, r)$ and $B^l(l = 1, 2, \cdots, s)$ of the game $(MOBGFG)$ model become fuzzy numbers in our future research. We have also concluded that the game model with entropy is becoming more and more significant and it is related to practical problems of our real life [13,14,37]. Inspired by [37], we will extend the some results of this paper to the game $(MOBGFG)$ model in an entropy or fuzzy entropy environment.

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