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Communication

An Itô Formula for an Accretive Operator

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Abstract: We give an Itô formula associated to a non-linear semi-group associated to a m-accretive operator.

Keywords: non-linear semi-group; Itô formula

1. Introduction

Let us recall the Itô formula in the Stratonovich Calculus [1]. Let B_t be a one dimensional Brownian motion and f be a smooth function on R. Then

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s \tag{1}$$

where we consider the Stratonovich differential.

In [2,3], we have remarked that the couple $(B_t, f(B_t))$ is a diffusion on $R \times R$ whose generator can be easily computed. This leads to an interpretation inside the semi-group theory of the Itô formula. Various Itô formulas were stated by ourself for various partial differential equations where there is no stochastic process [4–9]. See [9] for a review. For an Itô formula associated to a bilaplacian viewed inside the Fock space, we refer to [10].

There is roughly speaking following Hunt theory a stochastic process associated to a linear semi-group when the infinitesimal generator of the semi-group satisfied the maximum principle.

For nonlinear semi-group, the role of maximum principle is played by the notion of accretive operator. The goal of this paper is to state an Itô formula for a nonlinear semi-group associated to a m-accretive operator on $C_b(T^d)$, the space of continuous functions on the d-dimensional torus T^d endowed with the uniform metric $\|.\|_\infty$.

2. Statement of the Theorems

Let $(E, \|.\|)$ be a Banach space. Let L be a non-linear operator densely defined on E. We suppose $L0 = 0$. We recall that L is said to be accretive if for $\lambda \geq 0$

$$\|e_1 - e_2 + \lambda(L(e_1) - L(e_2))\| \geq \|e_1 - e_2\| \tag{2}$$

It is said to be m-accretive if for $\lambda > 0$

$$Im(I + \lambda L) = E \tag{3}$$

Let us recall what is a mild solution of the non-linear parabolic equation

$$\frac{\partial}{\partial t}u_t + Lu_t = 0; \quad u_0 = e \tag{4}$$

We consider a subdivision $0 \leq t_1 < \cdots < t_N = 1$. We say that u_{t_i} is an ϵ-discretization of Equation (4) if:

$$t_{i+1} - t_i < \epsilon \tag{5}$$

$$\frac{u_{t_i} - u_{t_{i-1}}}{t_{i+1} - t_i} + Lu_i = 0 \tag{6}$$

Definition 1. *v is said to be a mild solution of Equation (4) if for all ϵ there exist an ϵ-discretization u of Equation (6) such that $\|u_t - v_t\| \leq \epsilon$.*

Let us recall the main theorem of [11,12]:

Theorem 1. *If L is m-accretive, there exists for all e in E a unique mild-solution of Equation (4). This generates therefore a non-linear semi-group* $\exp[-tL]$.

We consider the d-dimensional torus. We consider $E = C_b(T^d)$ and let L be an m-accretive operator whose domain contains $C_b^\infty(T^d)$, the space of smooth functions on T^d with bounded derivatives at each order which is continuous from $C_b^\infty(T^d)$ into $C_b(T^d)$.

Let $f \in C_b^\infty(T^d)$. We consider $g \in C_b(T^d \times R)$.

We consider the diffeomorphism ψ^f of $T^d \times R$:

$$\psi^f(x, y) = (x, y + f(x)) \tag{7}$$

It defines a continuous linear isometry Ψ^f of $C_b(T^d \times R)$

$$\Psi^f[g](x, y) = g \circ \psi^f(x, y) \tag{8}$$

Definition 2. *The Itô transform L^f of L is the operator densely defined on $C_b(T^d \times R)$*

$$L^f = (\Psi^f)^{-1} \circ (L \otimes I_1) \circ \Psi^f \tag{9}$$

Let us give the domain of $L \otimes I_1$. $C_b(T^d \times R)$ is constituted of function $g(x, y)$.

$$L \otimes I_1[g](x, y) = L_x g(x, y) \tag{10}$$

where we apply the operator L on the continuous function $x \to g(x, y)$ supposed in the domain of L for all y. We suppose moreover that $(x, y) \to L_x g(x, y)$ is bounded continuous. The domain contains clearly $C_b^\infty(T^d \times R)$.

Theorem 2. *If L is m-accretive on $C_b(T^d)$, its Itô-transform is m-accretive on $C_b(T^d \times R)$.*

We deduce therefore two non-linear semi-groups if L is m-accretive:

- $\exp[-tL]$ acting on $C_b(T^d)$.
- $\exp[-tL^f]$ acting on $C_b(T^d \times R)$.

Let g be an element of $C_b(T^d \times R)$. We consider $g^f(x) = g(x, f(x))$. We get:

Theorem 3. *(Itô formula) We have the relation*

$$\exp[-tL][g^f](x) = \exp[-tL^f][g](x, f(x)) \tag{11}$$

This formula is an extension in the non-linear case of the classical Itô formula for the Brownian motion. If we take $L = -1/2\frac{\partial^2}{\partial x^2}$ acting densely on $C_b(R)$, we have

$$\exp[-tL][g](x) = E[g(B_t + x)] \tag{12}$$

where $t \to B_t$ is a Brownian motion on R starting from 0. $(B_t + x, f(B_t + x) + y)$ is a diffusion on $R \times R$ whose generator is L^f.

3. Proof of the Theorems

Proof of Theorem 2. $L \otimes I_1$ is clearly m-accretive on $C_b(T^d \times R)$. Let us show this result.

- $L \otimes I_1$ is densely defined. Let g be a bounded continuous function on $T^d \times R$. By using a suitable partition of unity on R, we can write

$$g(x, y) = \sum g^n(x, y) \tag{13}$$

  where $g^n(x, y) = 0$ if y does not belong to $[-n-1, n+1]$. By an approximation by convolution we can find a smooth function $g^{n,\epsilon}(x, y)$ close from $g(x, y)$ for the supremum norm and with bounded derivative of each order. $x \to L_x g^{n,\epsilon}$ is continuous in x and the joint function $(x, y) \to L_x g^{n,\epsilon}(x, y)$ is bounded continuous in (x, y) by the hypothesis on L.
- Clearly Equation (2) is satisfied.
- It remains to show Equation (3). If g belong to $C_b(T^d \times R)$ we can find $x \to h(x, y)$ such that

$$h(x, y) + \lambda L_x h(x, y) = g(x, y) \tag{14}$$

$$\|g(, y) - g(., y')\|_\infty \quad \geq \|h(., y) - h(., y')\|_\infty \tag{15}$$

  Therefore $(x, y) \to h(x, y)$ is jointly bounded continuous.

Since Ψ^f is a linear isometry of $C_b(T^d \times R)$ which transform a smooth function into a smooth function,

$$L^f = (\Psi^f)^{-1} \circ (L \otimes I_1) \circ \Psi^f \tag{16}$$

is clearly still m-accretive. □

Proof of Theorem 3. Let us consider $t_i = i/N$ to simplify the exposition. Let us consider an ϵ-discretization $u_.$ of the parabolic equation associated to L^f. This means that

$$u_{t_i} \in (\Psi^f)^{-1}(I_{d+1} + 1/N(L \otimes I_1))^{-i}\Psi^f g \tag{17}$$

I_{d+1} is the identity on $C_b(T^d \times R)$. But

$$(I_{d+1} + 1/N(L \otimes I_1)) = (I_d + 1/NL) \otimes I_1 \tag{18}$$

such that

$$((I_d + 1/NL)^i \otimes I_1)\Psi^f u_{t_i} = \Psi^f g \tag{19}$$

By doing $y = 0$ in the previous equality, we deduce that

$$(1 + L/N)^i u^f_{t_i} = g^f \tag{20}$$

Therefore $u^f_{t_i}$ is an ϵ-discretization to the parabolic equation associated to L. □

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