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Bundles over Quantum Real Weighted Projective Spaces

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Abstract: The algebraic approach to bundles in non-commutative geometry and the definition of quantum real weighted projective spaces are reviewed. Principal $U(1)$ -bundles over quantum real weighted projective spaces are constructed. As the spaces in question fall into two separate classes, the *negative* or *odd* class that generalises quantum real projective planes and the *positive* or *even* class that generalises the quantum disc, so do the constructed principal bundles. In the negative case the principal bundle is proven to be non-trivial and associated projective modules are described. In the positive case the principal bundles turn out to be trivial, and so all the associated modules are free. It is also shown that the circle (co)actions on the quantum Seifert manifold that define quantum real weighted projective spaces are almost free.

Keywords: quantum real weighted projective space; principal comodule algebra; noncommutative line bundle

1. Introduction

In an algebraic setup an action of a circle on a quantum space corresponds to a coaction of a Hopf algebra of Laurent polynomials in one variable on the noncommutative coordinate algebra of the quantum space. Such a coaction can equivalently be understood as a $\mathbb{Z}$ -grading of this coordinate algebra. A typical $\mathbb{Z}$ -grading assigns degree ± 1 to every generator of this algebra (different from the identity). The degree zero part forms a subalgebra which in particular cases corresponds to quantum complex or real projective spaces (grading of coordinate algebras of quantum spheres [1] or prolonged quantum

spheres [2]). Often this grading is strong, meaning that the product of i, j -graded parts is equal to the $i + j$ -part of the total algebra. In geometric terms this reflects the freeness of the circle action.

In two recent papers [3,4] circle actions on three-dimensional (and, briefly, higher dimensional) quantum spaces were revisited. Rather than assigning a uniform grade to each generator, separate generators were given degree by pairwise coprime integers. The zero part of such a grading of the coordinate algebra of the quantum odd-dimensional sphere corresponds to the quantum weighted projective space, while the zero part of such a grading of the algebra of the prolonged even dimensional quantum sphere leads to quantum real weighted projective spaces.

In this paper we focus on two classes of algebras $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ (l a positive integer) and $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ (l an odd positive integer) identified in [3] as fixed points of weighted circle actions on the coordinate algebra $\mathcal{O}(\Sigma_q^3)$ of a non-orientable quantum Seifert manifold described in [2]. Our aim is to construct quantum $U(1)$ -principal bundles over the corresponding quantum spaces $\mathbb{RP}_q^2(l; \pm)$ and describe associated line bundles. Recently, the importance of such bundles in non-commutative geometry was once again brought to the fore in [5], where the non-commutative Thom construction was outlined. As a further consequence of the principality of $U(1)$ -coactions we also deduce that $\mathbb{RP}_q^2(l; \pm)$ can be understood as quotients of Σ_q^3 by almost free S^1 -actions.

We begin in Section 2 by reviewing elements of algebraic approach to classical and quantum bundles. We then proceed to describe algebras $\mathcal{O}(\mathbb{RP}_q^2(l; \pm))$ in Section 3. Section 4 contains main results including construction of principal comodule algebras over $\mathcal{O}(\mathbb{RP}_q^2(l; \pm))$. We observe that constructions albeit very similar in each case yield significantly different results. The principal comodule algebra over $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ is non-trivial while that over $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ turns out to be trivial (this means that all associated bundles are trivial, hence we do not mention them in the text). Whether it is a consequence of our particular construction or there is a deeper (topological or geometric) obstruction to constructing non-trivial principal circle bundles over $\mathbb{RP}_q^2(l; +)$ remains an interesting open question.

Throughout we work with involutive algebras over the field of complex numbers (but the algebraic results remain true for all fields of characteristic 0). All algebras are associative and have identity, we use the standard Hopf algebra notation and terminology and we always assume that the antipode of a Hopf algebra is bijective. All topological spaces are assumed to be Hausdorff.

2. Review of Bundles in Non-Commutative Geometry

The aim of this section is to set out the topological concepts in relation to topological bundles, in particular principal bundles. The classical connection is made for interpreting topological concepts in an algebraic setting, providing a manageable methodology for performing calculations. In particular, the connection between principal bundles in topology and the algebraic Hopf–Galois condition is described. The reader familiar with classical theory of bundles can proceed directly to Definition 2.14.

2.1. Topological Aspects of Bundles

As a natural starting point, bundles are defined and topological properties are described. The principal map is defined and shown that injectivity is equivalent to the freeness condition. The image of the

canonical map is deduced and necessary conditions are imposed to ensure the bijectivity of this map. The detailed account of the material presented in this section can be found in [6].

Definition 2.1 A *bundle* is a triple (E, π, M) where E and M are topological spaces and $\pi : E \rightarrow M$ is a continuous surjective map. Here M is called the *base space*, E the *total space* and π the *projection* of the bundle.

For each $m \in M$, the *fibre* over m is the topological space $\pi^{-1}(m)$, i.e., the points on the total space which are projected, under π , onto the point m in the base space. A bundle whose fibres are homeomorphic which satisfies a condition known as local triviality are known as fibre bundles. This is formally expressed in the next definition.

Definition 2.2 A *fibre bundle* is a triple (E, π, M, F) where (E, π, M) is bundle and F is a topological space such that $\pi^{-1}(m)$ are homeomorphic to F for each $m \in M$. Furthermore, π satisfies the local triviality condition.

The local triviality condition is satisfied if for each $x \in E$, there is an open neighbourhood $U \subset B$ such that $\pi^{-1}(U)$ is homeomorphic to the product space $U \times F$, in such a way that π carries over to the projection onto the first factor. That is the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times F \\ \pi \downarrow & \nearrow p_1 & \\ U & & \end{array}$$

The map p_1 is the natural projection $U \times F \rightarrow U$ and $\phi : \pi^{-1}(U) \rightarrow U \times F$ is a homeomorphism.

Example 2.3 An example of a fibre bundle which is non-trivial, i.e., not a global product space, is the Möbius strip. It has a circle that runs lengthwise through the centre of the strip as a base B and a line segment running vertically for the fibre F . The line segments are in fact copies of the real line, hence each $\pi^{-1}(m)$ is homeomorphic to $\mathbb{R}$ and the Möbius strip is a fibre bundle.

Let X be a topological space which is compact and satisfies the Hausdorff property and G a compact topological group. Suppose there is a right action $\triangleleft : X \times G \rightarrow X$ of G on X and write $x \triangleleft g = xg$.

Definition 2.4 An action of G on X is said to be *free* if $xg = x$ for any $x \in X$ implies that $g = e$, the group identity.

With an eye on algebraic formulation of freeness, the *principal map* $F^G : X \times G \rightarrow X \times X$ is defined as $(x, g) \mapsto (x, xg)$.

Proposition 2.5 G acts freely on X if and only if F^G is injective.

Proof. “ $\Leftarrow$ ” Suppose the action is free, hence $xg = x$ implies that $g = e$. If $(x, xg) = (x', x'g')$, then $x = x'$ and $xg = x'g'$. Applying the action of g'^{-1} to both sides of $xg = x'g'$ we get $x(gg'^{-1}) = x$, which implies $gg'^{-1} = e$ by the freeness property, concluding $g = g'$ and F^G is injective as required.

“ $\implies$ ” Suppose F^G is injective, so $F^G(x, g) = F^G(x', g')$ or $(x, xg) = (x', x'g')$ implies $x = x'$ and $g = g'$. Since $x = xe$ from the properties of the action, if $x = xg$ then $g = e$ from the injectivity property. $\square$

Since G acts on X we can define the quotient space X/G ,

$$Y = X/G := \{[x] : x \in X\}, \quad \text{where} \quad [x] = xG = \{xg : g \in G\}$$

The sets xG are called the *orbits* of the points x . They are defined as the set of elements in X to which x can be moved by the action of elements of G . The set of orbits of X under the action of G forms a partition of X , hence we can define the equivalence relation on X as,

$$x \sim y \iff \exists g \in G \text{ such that } xg = y$$

The equivalence relation is the same as saying x and y are in the same orbit, *i.e.*, $xG = yG$. Given any quotient space, then there is a canonical surjective map

$$\pi : X \rightarrow Y = X/G, \quad x \mapsto xG = [x]$$

which maps elements in X to their orbits. We define the pull-back along this map π to be the set

$$X \times_Y X := \{(x, y) \in X \times X : \pi(x) = \pi(y)\}$$

As described above, the image of the principal map F^G contains elements of X in the first leg and the action of $g \in G$ on x in the second leg. To put it another way, the image records elements of $x \in X$ in the first leg and all the elements in the same orbit as this x in the second leg. Hence we can identify the image of the canonical map as the pull back along π , namely $X \times_Y X$. This is formally proved as a part of the following proposition.

Proposition 2.6 G acts freely on X if and only if the map

$$F_X^G : X \times G \rightarrow X \times_Y X, \quad (x, g) \mapsto (x, xg)$$

is bijective.

Proof. First note that the map F_X^G is well-defined since the elements x and xg are in the same orbit and hence map to the same equivalence class under π . Using Proposition 2.5 we can deduce that the injectivity of F_X^G is equivalent to the freeness of the action. Hence if we can show that F_X^G is surjective the proof is complete.

Take $(x, y) \in X \times_Y X$. This means $\pi(x) = \pi(y)$, which implies x and y are in the same equivalence class, which in turn means they are in the same orbit. We can therefore deduce that $y = xg$ for some $g \in G$. So, $(x, y) = (x, xg) = F_X^G(x, g)$ implying $(x, y) \in \text{Im} F_X^G$. Hence $\text{Im} F_X^G = X \times_Y X$ completing the proof. $\square$

Definition 2.7 An action of G on X is said to be *principal* if the map F^G is both injective and continuous (and such that the inverse image of a compact subset is compact in a case of locally compact spaces).

Since the injectivity and freeness condition are equivalent, we can interpret principal actions as both free and continuous actions. We can also deduce that these types of actions give rise to homeomorphisms F_X^G from $X \times G$ onto the space $X \times_{X/G} X$. Principal actions lead to the concept of topological principle bundles.

Definition 2.8 A *principal bundle* is a quadruple (X, π, M, G) such that

- (a) (X, π, M) is a bundle and G is a topological group acting continuously on X with action $\triangleleft : X \times G \rightarrow X, x \triangleleft g = xg$;
- (b) the action $\triangleleft$ is principal;
- (c) $\pi(x) = \pi(y) \iff \exists g \in G$ such that $y = xg$;
- (d) the induced map $X/G \rightarrow M$ is a homeomorphism.

The first two properties tell us that principal bundles are bundles admitting a principal action of a group G on the total space X , *i.e.*, principal bundles correspond to principal actions. By Definition (2.7), principal actions occur when the principal map is both injective and continuous, or equivalently, when the action is free and continuous. The third property ensures that the fibres of the bundle correspond to the orbits coming from the action and the final property implies that the quotient space can topologically be viewed as the base space of the bundle.

Example 2.9 Suppose X is a topological space and G a topological group which acts on X from the right. The triple $(X, \pi, X/G)$ where X/G is the orbit space and π the natural projection is a bundle. A principal action of G on X makes the quadruple $(X, \pi, X/G, G)$ a principal bundle.

We describe a principal bundle (X, π, Y, G) as a G -principal bundle over (X, π, Y) , or X as a G -principal bundle over Y .

Definition 2.10 A *vector bundle* is a bundle (E, π, M) where each fibre $\pi^{-1}(m)$ is endowed with a vector space structure such that addition and scalar multiplication are continuous maps.

Any vector bundle can be understood as a bundle associated to a principal bundle in the following way. Consider a G -principal bundle (X, π, Y, G) and let V be a representation space of G , *i.e.*, a (topological) vector space with a (continuous) left G -action $\triangleright : G \times V \rightarrow V, (g, v) \mapsto g \triangleright v$. Then G acts from the right on $X \times V$ by

$$(x, v) \triangleleft g := (xg, g^{-1} \triangleright v), \quad \text{for all } x \in X, v \in V \text{ and } g \in G$$

We can define $E = (X \times V)/G$ and a surjective (continuous map) $\pi_E : E \rightarrow Y, (x, v) \triangleleft G \mapsto \pi(x)$ and thus have a fibre bundle (E, π_E, Y, V) . In the case where V is a vector space, we assume that G acts linearly on V .

Definition 2.11 A *section* of a bundle (E, π_E, Y) is a continuous map $s : Y \rightarrow E$ such that, for all $y \in Y$,

$$\pi_E(s(y)) = y$$

i.e., a section is simply a section of the morphism π_E . The set of sections of E is denoted by $\Gamma(E)$.

Proposition 2.12 *Sections in a fibre bundle (E, π_E, Y, V) associated to a principal G -bundle X are in bijective correspondence with (continuous) maps $f : X \rightarrow V$ such that*

$$f(xg) = g^{-1} \triangleright f(x)$$

All such G -equivariant maps are denoted by $\text{Hom}_G(X, V)$.

Proof. Remember that $Y = X/G$. Given a map $f \in \text{Hom}_G(X, V)$, define the section $s_f : Y \rightarrow E$, $xG \mapsto (x, f(x)) \triangleleft G$.

Conversely, given $s \in \Gamma(E)$, define $f_s : X \rightarrow V$ by assigning to $x \in X$ a unique $v \in V$ such that $s(xG) = (x, v) \triangleleft G$. Note that v is unique, since if $(x, w) = (x, v) \triangleleft g$, then $xg = x$ and $w = g^{-1} \triangleright v$. Freeness implies that $g = e$, hence $w = v$. The map f_s has the required equivariance property, since the element of $(X \times V)/G$ corresponding to xg is $g^{-1} \triangleright v$. $\square$

2.2. Non-Commutative Principal and Associated Bundles

To make the transition from algebraic formulation of principal and associated bundles to non-commutative setup more transparent, we assume that X is a complex affine variety with an action of an affine algebraic group G and set $Y = X/G$ (all with the usual Euclidean topology). Let $\mathcal{O}(X)$, $\mathcal{O}(Y)$ and $\mathcal{O}(G)$ be the corresponding coordinate rings. Put $A = \mathcal{O}(X)$ and $H = \mathcal{O}(G)$ and note the identification $\mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G)$. Through this identification, $\mathcal{O}(G \times G)$ is a Hopf algebra with comultiplication: $f \mapsto (\Delta f)$, $(\Delta f)(g, h) = f(gh)$, counit $\varepsilon : \mathcal{O}(G) \rightarrow \mathbb{C}$, $\varepsilon(f) = f(e)$, and the antipode $S : H \rightarrow H$, $(Sf)(g) = f(g^{-1})$.

Using the fact that G acts on X we can construct a right coaction of H on A by $\varrho^A : A \rightarrow A \otimes H$, $\varrho^A(a)(x, g) = a(xg)$. This coaction is an algebra map due to the commutativity of the algebras of functions involved.

We have viewed the spaces of polynomial functions on X and G , next we view the space of functions on Y , $B := \mathcal{O}(Y)$, where $Y = X/G$. B is a subalgebra of A by

$$\pi^* : B \rightarrow A, \quad b \mapsto b \circ \pi$$

where π is the canonical surjection defined above. The map π^* is injective, since $b \neq b'$ in $\mathcal{O}(X/G)$ means there exists at least one orbit $xG = [x]$ such that $b([x]) \neq b'([x])$, but $\pi(x) = [x]$, so $b(\pi(x)) \neq b'(\pi(x))$ which implies $\pi^*(b) \neq \pi^*(b')$. Therefore, we can identify B with $\pi^*(B)$. Furthermore, $a \in \pi^*(B)$ if and only if

$$a(xg) = a(x)$$

for all $x \in X, g \in G$. This is the same as

$$\varrho^A(a)(x, g) = (a \otimes 1)(x, g)$$

for all $x \in X, g \in G$, where $1 : G \rightarrow \mathbb{C}$ is the unit function $1(g) = 1$ (the identity element of H). Thus we can identify B with the *coinvariants* of the coaction ϱ^A :

$$B = A^{\text{co}H} := \{a \in A \mid \varrho^A(a) = a \otimes 1\}$$

Since B is a subalgebra of A , it acts on A via the inclusion map $(ab)(x) = a(x)b(\pi(x))$, $(ba)(x) = b(\pi(x))a(x)$. We can identify $\mathcal{O}(X \times_Y X)$ with $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) = A \otimes_B A$ by the map

$$\theta(a \otimes_B a')(x, y) = a(x)a'(y), \quad (\text{with } \pi(x) = \pi(y))$$

Note that θ is well defined because $\pi(x) = \pi(y)$. Proposition 2.6 immediately yields

Proposition 2.13 *The action of G on X is free if and only if $F_X^{G*} : \mathcal{O}(X \times_Y X) \rightarrow \mathcal{O}(X \times G)$, $f \mapsto f \circ F_X^G$ is bijective.*

In view of the definition of the coaction of H on A , we can identify F_X^{G*} with the canonical map

$$\text{can} : a \otimes_B a' \mapsto [(x, g) \mapsto a(x)a'(x.g)] = a\varrho^A(a')$$

Thus the action of G on X is free if and only if this purely algebraic map is bijective. In the classical geometry case we take $A = \mathcal{O}(X)$, $H = \mathcal{O}(G)$ and $B = \mathcal{O}(X/G)$, but in general there is no need to restrict oneself to commutative algebras (of functions on topological spaces). In full generality this leads to the following definition.

Definition 2.14 (Hopf–Galois Extensions) Let H be a Hopf algebra and A a right H -comodule algebra with coaction $\varrho^A : A \rightarrow A \otimes H$. Let $B = A^{\text{co}H} := \{b \in A \mid \varrho^A(b) = b \otimes 1\}$, the coinvariant subalgebra of A . We say that $B \subseteq A$ is a *Hopf–Galois extension* if the left A -module, right H -comodule map

$$\text{can} : A \otimes_B A \rightarrow A \otimes H, \quad a \otimes_B a' \mapsto a\varrho^A(a')$$

is an isomorphism.

Proposition 2.13 tells us that when viewing bundles from an algebraic perspective, the freeness condition is equivalent to the Hopf–Galois extension property. Hence, the Hopf–Galois extension condition is a necessary condition to ensure a bundle is principal. Not all information about a topological space is encoded in a coordinate algebra, so to make a fuller reflection of the richness of the classical notion of a principal bundle we need to require conditions additional to the Hopf–Galois property.

Definition 2.15 Let H be a Hopf algebra with bijective antipode and let A be a right H -comodule algebra with coaction $\varrho^A : A \rightarrow A \otimes H$. Let B denote the coinvariant subalgebra of A . We say that A is a *principal H -comodule algebra* if:

- (a) $B \subseteq A$ is a Hopf–Galois extension;
- (b) the multiplication map $B \otimes A \rightarrow A$, $b \otimes a \mapsto ba$, splits as a left B -module and right H -comodule map (the equivariant projectivity condition).

As indicated already in [7–9], principal comodule algebras should be understood as principal bundles in noncommutative geometry. In particular, if H is the Hopf algebra associated to a C^* -algebra of functions on a quantum group [10], then the existence of the Haar measure together with the results of [8] mean that condition (a) in Definition 2.15 implies condition (b) (i.e., the freeness of the coaction implies its principality).

The following characterisation of principal comodule algebras [11,12] gives an effective method for proving the principality of coaction.

Proposition 2.16 A right H -comodule algebra A with coaction $\varrho^A : A \rightarrow A \otimes H$ is principal if and only if it admits a strong connection form, that is if there exists a map $\omega : H \rightarrow A \otimes A$, such that

$$\omega(1) = 1 \otimes 1 \quad (1a)$$

$$\mu \circ \omega = \eta \circ \varepsilon \quad (1b)$$

$$(\omega \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varrho) \circ \omega \quad (1c)$$

$$(S \otimes \omega) \circ \Delta = (\sigma \otimes \text{id}) \circ (\varrho \otimes \text{id}) \circ \omega \quad (1d)$$

Here $\mu : A \otimes A \rightarrow A$ denotes the multiplication map, $\eta : \mathbb{C} \rightarrow A$ is the unit map, $\Delta : H \rightarrow H \otimes H$ is the comultiplication, $\varepsilon : H \rightarrow \mathbb{C}$ counit and $S : H \rightarrow H$ the (bijective) antipode of the Hopf algebra H , and $\sigma : A \otimes H \rightarrow H \otimes A$ is the flip.

Proof. If a strong connection form ω exists, then the inverse of the canonical map can (see Definition 2.14) is the composite

$$A \otimes H \xrightarrow{\text{id} \otimes \omega} A \otimes A \otimes A \xrightarrow{\mu \otimes \text{id}} A \otimes A \longrightarrow A \otimes_B A$$

while the splitting of the multiplication map (see Definition 2.15 (b)) is given by

$$A \xrightarrow{\varrho^A} A \otimes H \xrightarrow{\text{id} \otimes \omega} A \otimes A \otimes A \xrightarrow{\mu \otimes \text{id}} B \otimes A$$

Conversely, if $B \subseteq A$ is a principal comodule algebra, then ω is the composite

$$H \xrightarrow{\eta \otimes \text{id}} A \otimes H \xrightarrow{\text{can}^{-1}} A \otimes_B A \xrightarrow{\text{id} \otimes s} A \otimes_B B \otimes A \xrightarrow{\cong} A \otimes A$$

where s is the left B -linear right H -colinear splitting of the multiplication $B \otimes A \rightarrow A$. $\square$

Example 2.17 Let A be a right H -comodule algebra. The space of $\mathbb{C}$ -linear maps $\text{Hom}(H, A)$ is an algebra with the convolution product

$$f \otimes g \mapsto \mu \circ (f \otimes g) \circ \Delta$$

and unit $\eta \circ \varepsilon$. A is said to be *cleft* if there exists a right H -colinear map $j : H \rightarrow A$ that has an inverse in the convolution algebra $\text{Hom}(H, A)$ and is normalised so that $j(1) = 1$. Writing j^{-1} for the convolution inverse of j , one easily observes that

$$\omega : H \rightarrow A \otimes A, \quad h \mapsto (j^{-1} \otimes j)(\Delta(h))$$

is a strong connection form. Hence a cleft comodule algebra is an example of a principal comodule algebra. The map j is called a *cleaving map* or a *normalised total integral*.

In particular, if $j : H \rightarrow A$ is an H -colinear algebra map, then it is automatically convolution invertible (as $j^{-1} = j \circ S$) and normalised. A comodule algebra A admitting such a map is termed a *trivial principal comodule algebra*.

Example 2.18 Let H be a Hopf algebra of the compact quantum group. By the Woronowicz theorem [10], H admits an invariant Haar measure, *i.e.*, a linear map $\Lambda : H \rightarrow \mathbb{C}$ such that, for all $h \in H$,

$$\sum h_{(1)} \Lambda(h_{(2)}) = \varepsilon(h), \quad \Lambda(1) = 1$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ is the Sweedler notation for the comultiplication. Next, assume that the lifted canonical map:

$$\overline{\text{can}} : A \otimes A \rightarrow A \otimes H, \quad a \otimes a' \mapsto a \varrho(a') \quad (2)$$

is surjective, and write

$$\ell : H \rightarrow A \otimes A, \quad \ell(h) = \sum \ell(h)^{[1]} \otimes \ell(h)^{[2]}$$

for the $\mathbb{C}$ -linear map such that $\overline{\text{can}}(\ell(h)) = 1 \otimes h$, for all $h \in H$. Then, by the Schneider theorem [8], A is a principal H -comodule algebra. Explicitly, a strong connection form is

$$\omega(h) = \sum \Lambda(h_{(1)} \ell(h_{(2)})^{[1]}_{(1)}) \Lambda(\ell(h_{(2)})^{[2]}_{(1)} S(h_{(3)})) \ell(h_{(2)})^{[1]}_{(0)} \otimes \ell(h_{(2)})^{[2]}_{(0)}$$

where the coaction is denoted by the Sweedler notation $\varrho^A(a) = \sum a_{(0)} \otimes a_{(1)}$; see [13].

Having described non-commutative principal bundles, we can look at the associated vector bundles. First we look at the classical case and try to understand it purely algebraically. Start with a vector bundle (E, π_E, Y, V) associated to a principal G -bundle X . Since V is a vector representation space of G , also the set $\text{Hom}_G(X, V)$ is a vector space. Consequently $\Gamma(E)$ is a vector space. Furthermore, $\text{Hom}_G(X, V)$ is a left module of $B = \mathcal{O}(Y)$ with the action $(bf)(x) = b(\pi_E(x))f(x)$. To understand better the way in which B -module $\Gamma(E)$ is associated to the principal comodule algebra $\mathcal{O}(X)$ we recall the notion of the cotensor product.

Definition 2.19 Given a Hopf algebra H , right H -comodule A with coaction ϱ^A and left H -comodule V with coaction ϱ^V , the *cotensor product* is defined as an equaliser:

$$A \square_H V \longrightarrow A \otimes V \xrightarrow[\text{id} \otimes \varrho^V]{\varrho^A \otimes \text{id}} A \otimes H \otimes V$$

If A is an H -comodule algebra, and $B = A^{coH}$, then $A \square_H V$ is a left B -module with the action $b(a \square v) = ba \square v$. In particular, in the case of a principal G -bundle X over $Y = X/G$, for any left $\mathcal{O}(G)$ -comodule V the cotensor product $\mathcal{O}(X) \square_{\mathcal{O}(G)} V$ is a left $\mathcal{O}(Y)$ -module.

The following proposition indicates the way in which cotensor products enter description of associated bundles.

Proposition 2.20 Assume that the fibre V of a vector bundle (E, π_E, Y, V) associated to a principal G -bundle X is finite dimensional. View V as a left comodule of $\mathcal{O}(G)$ with the coaction $\varrho^V : v \mapsto \sum v_{(-1)} \otimes v_{(0)}$ (summation implicit) determined by $\sum v_{(-1)}(g)v_{(0)} = g^{-1} \triangleright v$. Then the left $\mathcal{O}(Y)$ -module of sections $\Gamma(E)$ is isomorphic to the left $\mathcal{O}(Y)$ -module $\mathcal{O}(X) \square_{\mathcal{O}(G)} V$.

Proof. First identify $\Gamma(E)$ with $\text{Hom}_G(X, V)$. Let $\{v_i \in V^*, v^i \in V\}$ be a (finite) dual basis. Take $f \in \text{Hom}_G(X, V)$, and define $\theta : \text{Hom}_G(X, V) \rightarrow \mathcal{O}(X) \square_{\mathcal{O}(G)} V$ by $\theta(f) = \sum_i v_i \circ f \otimes v^i$.

In the converse direction, define a left $\mathcal{O}(Y)$ -module map

$$\theta^{-1} : \mathcal{O}(X) \square_{\mathcal{O}(G)} V \rightarrow \text{Hom}_G(X, V), \quad a \square v \mapsto a(-)v$$

One easily checks that the constructed map are mutual inverses. $\square$

Moving away from commutative algebras of functions on topological spaces one uses Proposition 2.20 as the motivation for the following definition.

Definition 2.21 Let A be a principal H -comodule algebra. Set $B = A^{coH}$ and let V be a left H -comodule. The left B -module $\Gamma = A \square_H V$ is called a *module associated to the principal comodule algebra* A .

Γ is a projective left B -module, and if V is a finite dimensional vector space, then Γ is a finitely generated projective left B -module. In this case it has the meaning of a module of sections over a non-commutative vector bundle. Furthermore, its class gives an element in the K_0 -group of B . If A is a cleft principal comodule algebra, then every associated module is free, since $A \cong B \otimes H$ as a left B -module and right H -comodule, so that

$$\Gamma = A \square_H V \cong (B \otimes H) \square_H V \cong B \otimes (H \square_H V) \cong B \otimes V$$

3. Weighted Circle Actions on Prolonged Spheres.

In this section we recall the definitions of algebras we study in the sequel.

3.1. Circle Actions and $\mathbb{Z}$ -Gradings.

The coordinate algebra of the circle or the group $U(1)$, $\mathcal{O}(S^1) = \mathcal{O}(U(1))$ can be identified with the $*$ -algebra $\mathbb{C}[u, u^*]$ of Laurent polynomials in a unitary variable u (unitary means $u^{-1} = u^*$). As a Hopf $*$ -algebra $\mathbb{C}[u, u^*]$, is generated by the grouplike element u , i.e.,

$$\Delta(u) = u \otimes u, \quad \varepsilon(u) = 1, \quad S(u) = u^*$$

and thus it can be understood as the group algebra $\mathbb{C}\mathbb{Z}$. As a consequence of this interpretation of $\mathbb{C}[u, u^*]$, an algebra A is a $\mathbb{C}[u, u^*]$ -comodule algebra if and only if A is a $\mathbb{Z}$ -graded algebra,

$$A = \bigoplus_{n \in \mathbb{Z}} A_n, \quad A_n := \{a \in A \mid \varrho^A(a) = a \otimes u^n\}, \quad A_m A_n \subseteq A_{m+n}$$

A_0 is the coinvariant subalgebra of A . Since $\mathbb{C}[u, u^*]$ is spanned by grouplike elements, any convolution invertible map $j : \mathbb{C}[u, u^*] \rightarrow A$ must assign a unit (invertible element) of A to u^n . Furthermore, colinear maps are simply the $\mathbb{Z}$ -degree preserving maps, where $\deg(u) = 1$. Put together, convolution invertible colinear maps $j : \mathbb{C}[u, u^*] \rightarrow A$ are in one-to-one correspondence with sequences

$$(a_n : n \in \mathbb{Z}, a_n \text{ is a unit in } A, \deg(a_n) = n)$$

3.2. The $\mathcal{O}(\Sigma_q^{2n+1})$ and $\mathcal{O}(\mathbb{RP}_q(l_0, \dots, l_n))$ Coordinate Algebras

Let q be a real number, $0 < q < 1$. The coordinate algebra $\mathcal{O}(S_q^{2n})$ of the even-dimensional quantum sphere is the unital complex $*$ -algebra with generators $z_0, z_1, \dots, z_n$, subject to the following relations:

$$z_i z_j = q z_j z_i \quad \text{for } i < j, \quad z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j \quad (3a)$$

$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^n z_m z_m^*, \quad \sum_{m=0}^n z_m z_m^* = 1, \quad z_n^* = z_n \quad (3b)$$

$\mathcal{O}(S_q^{2n})$ is a $\mathbb{Z}_2$ -graded algebra with $\deg(z_i) = 1$ and so is $\mathbb{C}[u, u^*]$ (with $\deg(u) = 1$). In other words, $\mathcal{O}(S_q^{2n})$ is a right $\mathbb{C}\mathbb{Z}_2$ -comodule algebra and $\mathbb{C}[u, u^*]$ is a left $\mathbb{C}\mathbb{Z}_2$ -comodule algebra, hence one can consider the cotensor product algebra $\mathcal{O}(\Sigma_q^{2n+1}) := \mathcal{O}(S_q^{2n}) \square_{\mathbb{C}\mathbb{Z}_2} \mathbb{C}[u, u^*]$. It was shown in [2] that, as a unital $*$ -algebra, $\mathcal{O}(\Sigma_q^{2n+1})$ has generators $\zeta_0, \dots, \zeta_n$ and a central unitary ξ which are related in the following way:

$$\zeta_i \zeta_j = q \zeta_j \zeta_i \quad \text{for } i < j, \quad \zeta_i \zeta_j^* = q \zeta_j^* \zeta_i \quad \text{for } i \neq j \quad (4a)$$

$$\zeta_i \zeta_i^* = \zeta_i^* \zeta_i + (q^{-2} - 1) \sum_{m=i+1}^n \zeta_m \zeta_m^*, \quad \sum_{m=0}^n \zeta_m \zeta_m^* = 1, \quad \zeta_n^* = \zeta_n \xi \quad (4b)$$

For any choice of $n + 1$ pairwise coprime numbers $l_0, \dots, l_n$ one can define the coaction of the Hopf algebra $\mathcal{O}(U(1)) = \mathbb{C}[u, u^*]$ on $\mathcal{O}(\Sigma_q^{2n+1})$ as

$$\varrho_{l_0, \dots, l_n} : \mathcal{O}(\Sigma_q^{2n+1}) \rightarrow \mathcal{O}(\Sigma_q^{2n+1}) \otimes \mathbb{C}[u, u^*], \quad \zeta_i \mapsto \zeta_i \otimes u^{l_i}, \quad \xi \mapsto \xi \otimes u^{-2l_n} \quad (5)$$

for $i = 0, 1, \dots, n$. This coaction is then extended to the whole of $\mathcal{O}(\Sigma_q^{2n+1})$ so that $\mathcal{O}(\Sigma_q^{2n+1})$ is a right $\mathbb{C}[u, u^*]$ -comodule algebra.

The algebra of coordinate functions on the quantum real weighted projective space is now defined as the subalgebra of $\mathcal{O}(\Sigma_q^{2n+1})$ containing all coinvariant elements, *i.e.*,

$$\mathcal{O}(\mathbb{RP}_q(l_0, \dots, l_n)) = \mathcal{O}(\Sigma_q^{2n+1})^{\mathcal{O}(U(1))} := \{x \in \mathcal{O}(\Sigma_q^{2n+1}) : \varrho_{l_0, \dots, l_n}(x) = x \otimes 1\}$$

3.3. The 2D Quantum Real Projective Space $\mathcal{O}(\mathbb{RP}_q(k, l)) \subset \mathcal{O}(\Sigma_q^3)$

In this paper we consider two-dimensional quantum real weighted projective spaces, *i.e.*, the algebras obtained from the coordinate algebra $\mathcal{O}(\Sigma_q^3)$ which is generated by ζ_0, ζ_1 and central unitary ξ such that

$$\zeta_0 \zeta_1 = q \zeta_1 \zeta_0, \quad \zeta_0 \zeta_1^* = q \zeta_1^* \zeta_0 \quad (6a)$$

$$\zeta_0 \zeta_0^* = \zeta_0^* \zeta_0 + (q^{-2} - 1) \zeta_1^2 \xi, \quad \zeta_0 \zeta_0^* + \zeta_1^2 \xi = 1, \quad \zeta_1^* = \zeta_1 \xi \quad (6b)$$

The linear basis of $\mathcal{O}(\Sigma_q^3)$ is

$$\{\zeta_0^r \zeta_1^s \xi^t, \zeta_0^{*r} \zeta_1^{*s} \xi^t, \mid r, s \in \mathbb{N}, t \in \mathbb{Z}\} \quad (7)$$

For a pair k, l of coprime positive integers, the coaction $\varrho_{k,l}$ is given on generators by

$$\zeta_0 \mapsto \zeta_0 \otimes u^k, \quad \zeta_1 \mapsto \zeta_1 \otimes u^l, \quad \xi \mapsto \xi \otimes u^{-2l} \quad (8)$$

and extended to the whole of $\mathcal{O}(\Sigma_q^3)$ so that the coaction is a $*$ -algebra map. We denote the comodule algebra $\mathcal{O}(\Sigma_q^3)$ with coaction $\varrho_{k,l}$ by $\mathcal{O}(\Sigma_q^3(k, l))$.

It turns out that the two dimensional quantum real projective spaces split into two cases depending on not wholly the parameter k but instead whether k is either even or odd, and hence only cases $k = 1$ and $k = 2$ need to be considered [3]. We describe these cases presently.

3.3.1. The Odd or Negative Case

For $k = 1$, $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ is a polynomial $*$ -algebra generated by a, b, c_- which satisfy the relations:

$$a = a^*, \quad ab = q^{-2l}ba, \quad ac_- = q^{-4l}c_-a, \quad b^2 = q^{3l}ac_-, \quad bc_- = q^{-2l}c_-b \quad (9a)$$

$$bb^* = q^{2l}a \prod_{m=0}^{l-1} (1 - q^{2m}a), \quad b^*b = a \prod_{m=1}^l (1 - q^{-2m}a) \quad (9b)$$

$$b^*c_- = q^{-l} \prod_{m=1}^l (1 - q^{-2m}a)b, \quad c_-b^* = q^lb \prod_{m=0}^{l-1} (1 - q^{2m}a) \quad (9c)$$

$$c_-c_-^* = \prod_{m=0}^{2l-1} (1 - q^{2m}a), \quad c_-^*c_- = \prod_{m=1}^{2l} (1 - q^{-2m}a) \quad (9d)$$

The embedding of generators of $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ into $\mathcal{O}(\Sigma_q^3)$ or the isomorphism of $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ with the coinvariants of $\mathcal{O}(\Sigma_q^3(1, l))$ is provided by

$$a \mapsto \zeta_1^2 \xi, \quad b \mapsto \zeta_0^l \zeta_1 \xi, \quad c_- \mapsto \zeta_0^{2l} \xi \quad (10)$$

Up to equivalence $\mathcal{O}(\mathbb{RP}_q^2(l; -))$ has the following irreducible $*$ -representations. There is a family of one-dimensional representations labelled by $\theta \in [0, 1)$ and given by

$$\pi_\theta(a) = 0, \quad \pi_\theta(b) = 0, \quad \pi_\theta(c_-) = e^{2\pi i \theta} \quad (11)$$

All other representations are infinite dimensional, labelled by $r = 1, \dots, l$, and given by

$$\pi_r(a)e_n^r = q^{2(ln+r)}e_n^r, \quad \pi_r(b)e_n^r = q^{ln+r} \prod_{m=1}^l (1 - q^{2(ln+r-m)})^{1/2} e_{n-1}^r, \quad \pi_r(b)e_0^r = 0 \quad (12a)$$

$$\pi_r(c_-)e_n^r = \prod_{m=1}^{2l} (1 - q^{2(ln+r-m)})^{1/2} e_{n-2}^r, \quad \pi_r(c_-)e_0^r = \pi_r(c_-)e_1^r = 0 \quad (12b)$$

where e_n^r , $n \in \mathbb{N}$, is an orthonormal basis for the representation space $\mathcal{H}_r \cong l^2(\mathbb{N})$.

The C^* -algebra of continuous functions on $\mathbb{RP}_q^2(l; -)$, obtained as the completion of these bounded representations, can be identified with the pullback of l -copies of the quantum real projective plane $\mathbb{RP}_q^2$ introduced in [14].

3.3.2. The Even or Positive Case

For $k = 2$ and hence l odd, $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ is a polynomial $*$ -algebra generated by a, c_+ which satisfy the relations:

$$a^* = a, \quad ac_+ = q^{-2l}c_+a \quad (13a)$$

$$c_+c_+^* = \prod_{m=0}^{l-1} (1 - q^{2m}a), \quad c_+^*c_+ = \prod_{m=1}^l (1 - q^{-2m}a) \quad (13b)$$

The embedding of generators of $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ into $\mathcal{O}(\Sigma_q^3)$ or the isomorphism of $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ with the coinvariants of $\mathcal{O}(\Sigma_q^3(2, l))$ is provided by

$$a \mapsto \zeta_1^2 \xi, \quad c_+ \mapsto \zeta_0^l \xi \quad (14)$$

Similarly to the odd k case, there is a family of one-dimensional representations of $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ labelled by $\theta \in [0, 1)$ and given by

$$\pi_\theta(a) = 0, \quad \pi_\theta(c_+) = e^{2\pi i \theta} \quad (15)$$

All other representations are infinite dimensional, labelled by $r = 1, \dots, l$, and given by

$$\pi_r(a)e_n^r = q^{2(ln+r)}e_n^r, \quad \pi_r(c_+)e_n^r = \prod_{m=1}^l (1 - q^{2(ln+r-m)})^{1/2} e_{n-1}^r, \quad \pi_r(c_+)e_0^r = 0 \quad (16)$$

where $e_n^r, n \in \mathbb{N}$ is an orthonormal basis for the representation space $\mathcal{H}_r \cong l^2(\mathbb{N})$.

The C^* -algebra $C(\mathbb{RP}_q^2(l; +))$ of continuous functions on $\mathbb{RP}_q^2(l; +)$, obtained as the completion of these bounded representations, can be identified with the pullback of l -copies of the quantum disk D_q introduced in [15]. Furthermore, $C(\mathbb{RP}_q^2(l; +))$ can also be understood as the quantum double suspension of l points in the sense of [16, Definition 6.1].

4. Quantum Real Weighted Projective Spaces and Quantum Principal Bundles

The general aim of this paper is to construct quantum principal bundles with base spaces given by $\mathcal{O}(\mathbb{RP}_q^2(l; \pm))$ and fibre structures given by the circle Hopf algebra $\mathcal{O}(S^1) \cong \mathbb{C}[u, u^*]$. The question arises as to which quantum space (*i.e.*, a $\mathbb{C}[u, u^*]$ -comodule algebra with coinvariants isomorphic to $\mathcal{O}(\mathbb{RP}_q^2(l; \pm))$) we should consider as the total space within this construction. We look first at the coactions of $\mathbb{C}[u, u^*]$ on $\mathcal{O}(\Sigma_q^3)$ that define $\mathcal{O}(\mathbb{RP}_q^2(k, l))$, *i.e.*, at the comodule algebras $\mathcal{O}(\Sigma_q^3(k, l))$.

4.1. The (Non-)Principality of $\mathcal{O}(\Sigma_q^3(k, l))$

Theorem 4.1 $A = \mathcal{O}(\Sigma_q^3(k, l))$ is a principal comodule algebra if and only if $(k, l) = (1, 1)$.

Proof. As explained in [2] $\mathcal{O}(\Sigma_q^3(1, 1))$ is a prolongation of the $\mathbb{C}\mathbb{Z}_2$ -comodule algebra $\mathcal{O}(S_q^2)$. The latter is a principal comodule algebra (over the quantum real projective plane $\mathcal{O}(\mathbb{RP}_q^2)$ [14]) and since a prolongation of a principal comodule algebra is a principal comodule algebra [8, Remark 3.11], the coaction $\varrho_{1,1}$ is principal as stated.

In the converse direction, we aim to show that the canonical map is not an isomorphism by showing that the image does not contain $1 \otimes u$, *i.e.*, it cannot be surjective since we know $1 \otimes u$ is in the codomain. We begin by identifying a basis for the algebra $\mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(\Sigma_q^3)$; observing the relations in Equations (6a) and (6b) it is clear that a basis for $\mathcal{O}(\Sigma_q^3(k, l))$ is given by elements of the form

$$\begin{aligned} b_1 &= b_1(p_1, p_2, p_3) = \zeta_0^{p_1} \zeta_1^{p_2} \xi^{p_3}, & b_2 &= b_2(\bar{p}_1, \bar{p}_2, \bar{p}_3) = \zeta_0^{\bar{p}_1} \zeta_1^{\bar{p}_2} \xi^{*\bar{p}_3} \\ b_3 &= b_3(q_1, q_2, q_3) = \zeta_0^{*q_1} \zeta_1^{q_2} \xi^{q_3}, & b_4 &= b_4(\bar{q}_1, \bar{q}_2, \bar{q}_3) = \zeta_0^{*\bar{q}_1} \zeta_1^{\bar{q}_2} \xi^{*\bar{q}_3} \end{aligned}$$

noting that all powers are non-negative. Hence a basis for $\mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(\Sigma_q^3)$ is given by elements of the form $b_i \otimes b_j$, where $i, j \in \{1, 2, 3, 4\}$. Applying the canonical map gives

$$\text{can}(b_i \otimes b_j) = b_i \varrho(b_j) = b_i b_j \otimes u^{\deg(b_j)}, \quad \text{where } i, j \in \{1, 2, 3, 4\} \quad (17)$$

where ϱ means $\varrho_{k,l}$ for simplicity of notation. The next stage is to construct all possible elements in $\mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(\Sigma_q^3)$ which map to $1 \otimes u$. To obtain the identity in the first leg we must use one of the following relations:

$$\zeta_0^m \zeta_0^{*n} = \begin{cases} \prod_{p=0}^{m-1} (1 - q^{2p} \zeta_1^2 \xi) & \text{when } m = n \\ \zeta_0^{m-n} \prod_{p=0}^{n-1} (1 - q^{2p} \zeta_1^2 \xi) & \text{when } m > n \\ \prod_{p=0}^{m-1} (1 - q^{2p} \zeta_1^2 \xi) \zeta_0^{*n-m} & \text{when } n > m \end{cases} \quad (18a)$$

$$\zeta_0^{*n} \zeta_0^m = \begin{cases} \prod_{p=1}^m (1 - q^{-2p} \zeta_1^2 \xi) & \text{when } m = n \\ \zeta_0^{*n-m} \prod_{p=1}^m (1 - q^{-2p} \zeta_1^2 \xi) & \text{when } n > m \\ \prod_{p=1}^n (1 - q^{-2p} \zeta_1^2 \xi) \zeta_0^{m-n} & \text{when } n < m \end{cases} \quad (18b)$$

or

$$\xi \xi^* = \xi^* \xi = 1$$

We see that to obtain identity in the first leg we require the powers of ζ_0 and ζ_0^* to be equal. We now construct all possible elements of the domain which map to $1 \otimes u$ after applying the canonical map.

Case 1: use the first relation to obtain $\zeta_0^m \zeta_0^{*m}$ ($m > 0$); this can be done in four ways. First, using $b_1 \varrho(b_3)$, $b_1 \varrho(b_4)$, $b_2 \varrho(b_3)$ and $b_2 \varrho(b_4)$. Now,

$$b_1 \varrho(b_3) \sim \zeta_0^{p_1} \zeta_0^{*q_1} \zeta_1^{p_2+q_2} \xi^{p_3+q_3} \otimes u^{-kq_1+lq_2-2lq_3} \implies p_1 = q_1 = m, p_2 = q_2 = 0, p_3 = q_3 = 0$$

and

$$-kq_1 + lq_2 - 2lq_3 = 1 \implies -mk = 1$$

hence no possible terms. A similar calculation for the three other cases shows that $1 \otimes u$ cannot be obtained as an element of the image of the canonical map in this case.

Case 2: use the second relation to obtain $\zeta_0^{*n} \zeta_0^n$ ($n > 0$); this can be done in four ways $b_3 \varrho(b_1)$, $b_3 \varrho(b_2)$, $b_3 \varrho(b_2)$ and $b_4 \varrho(b_2)$. Now,

$$b_3 \varrho(b_1) \sim \zeta_0^{*q_1} \zeta_0^{p_1} \zeta_1^{p_2+q_2} \xi^{p_3+q_3} \otimes u^{kp_1+lp_2-2lp_3} \implies p_1 = q_1 = n, p_2 = q_2 = 0, p_3 = q_3 = 0$$

and

$$nk = 1 \implies n = 1 \text{ and } k = 1$$

Note that $k = 1$ is not a problem provided l is not equal to 1. This is reviewed at the next stage of the proof. The same conclusion is reached in all four cases.

In all possibilities $\zeta_0^{*n} \zeta_0^n$ appears only when $n = 1$, in which case the relation simplifies to $\zeta_0^* \zeta_0 = 1 - q^{-2} \zeta_1^2 \xi$, so the next stage involves constructing elements in the domain which map to $\zeta_1^2 \xi$. There are eight possibilities altogether to be checked: $b_1 \varrho(b_1)$, $b_1 \varrho(b_2)$, $b_1 \varrho(b_3)$, $b_1 \varrho(b_4)$, $b_3 \varrho(b_1)$, $b_3 \varrho(b_2)$, $b_3 \varrho(b_3)$ and $b_3 \varrho(b_4)$. The first case gives:

$$b_1 \varrho(b_1) \sim \zeta_0^{2p_1} \zeta_1^{2p_2} \xi^{2p_3} \otimes u^{kp_1 + lp_2 - 2lp_3} \implies 2p_1 = 0, 2p_2 = 2, 2p_3 = 1$$

and

$$kp_1 + lp_2 - 2lp_3 = 1 \implies p_1 = 0, p_2 = 1, p_3 \text{ has no possible values and } l = 1.$$

Hence $1 \otimes u$ cannot be obtained as an element in the image in this case. Similar calculations for the remaining possibilities show that either $1 \otimes u$ is not in the image of the canonical map, or that if $1 \otimes u$ is in the image then $k = l = 1$.

Case 3: finally, it seems possible that $1 \otimes u$, using the third relation, could be in the image of the canonical map. All possible elements in the domain which could potentially map to this element are constructed and investigated. There are eight possibilities: $b_1 \varrho(b_2)$, $b_1 \varrho(b_4)$, $b_2 \varrho(b_1)$, $b_2 \varrho(b_3)$, $b_3 \varrho(b_2)$, $b_3 \varrho(b_4)$, $b_4 \varrho(b_1)$ and $b_4 \varrho(b_3)$. The first possibility comes out as

$$b_1 \varrho(b_2) \sim \zeta_0^{p_1 + \bar{p}_1} \zeta_1^{p_2 + \bar{p}_2} \xi^{p_3} \xi^{*\bar{p}_3} \otimes u^{k\bar{p}_1 + l\bar{p}_2 + 2l\bar{p}_3} \implies p_1 = \bar{p}_1 = 0, p_2 = \bar{p}_2 = 0, p_3 = \bar{p}_3 = 1$$

Also

$$k\bar{p}_1 + l\bar{p}_2 + 2l\bar{p}_3 = 1 \implies 2l = 1$$

which implies there are no terms. The same conclusion can be reached for the remaining relations.

This concludes that $1 \otimes u$, which is contained in $\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u, u^*]$, is not in the image of the canonical map, proving that this map is not surjective and ultimately not an isomorphism when k and l are both not simultaneously equal to 1, completing the proof that $\mathcal{O}(\Sigma_q^3(k, l))$ is not a principal comodule algebra in this case. $\square$

Theorem 4.1 tells us that if we use $\mathcal{O}(\Sigma_q^3(k, l))$ as our total space, then we are forced to put $(k, l) = (1, 1)$ to ensure that the required Hopf–Galois condition does not fail. A consequence of this would be the generators ζ_0 and ζ_1 would have $\mathbb{Z}$ -degree 1. This suggests that the comodule algebra $\mathcal{O}(\Sigma_q^3(k, l))$ is too restrictive as there is no freedom with the weights k or l , and that we should in fact consider a subalgebra of $\mathcal{O}(\Sigma_q^3)$ which admits a $\mathcal{O}(S^1)$ -coaction that would offer some choice. Theorem 4.1 indicates that the desired subalgebra should have generators with grades 1 to ensure the Hopf–Galois condition is satisfied. This process is similar to that followed in [4], where the bundles over the quantum teardrops $\mathbb{WP}_q(1, l)$ have the total spaces provided by the quantum lens spaces and structure groups provided by the circle group $U(1)$. We follow a similar approach in the sense that we view $\mathcal{O}(\Sigma_q^3(k, l))$ as a right H -comodule algebra, where H is the Hopf algebra of a suitable cyclic group.

4.2. The Negative Case $\mathcal{O}(\mathbb{RP}_q^2(l; -))$

4.2.1. The Principal $\mathcal{O}(U(1))$ -Comodule Algebra over $\mathcal{O}(\mathbb{RP}_q^2(l; -))$

Take the group Hopf $*$ -algebra $H = \mathbb{C}\mathbb{Z}_l$ which is generated by unitary grouplike element w and satisfies the relation $w^l = 1$. The algebra $\mathcal{O}(\Sigma_q^3)$ is a right $\mathbb{C}\mathbb{Z}_l$ -comodule $*$ -algebra with coaction

$$\mathcal{O}(\Sigma_q^3) \rightarrow \mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}\mathbb{Z}_l, \quad \zeta_0 \mapsto \zeta_0 \otimes w, \quad \zeta_1 \mapsto \zeta_1 \otimes 1, \quad \xi \mapsto \xi \otimes 1 \quad (19)$$

Note that the $\mathbb{Z}_l$ -degree of the generator ξ is determined by the degree of ζ_1 : the relation $\zeta_1^* = \zeta_1 \xi$ and that the coaction must be compatible with all relations imply that $\deg(\zeta_1^*) = \deg(\zeta_1) + \deg(\xi)$. Since ζ_1 has degree zero, ξ must also have degree zero.

The next stage of the process is to find the coinvariant elements of $\mathcal{O}(\Sigma_q^3)$ given the coaction defined above.

Proposition 4.2 *The fixed point subalgebra of the above coaction is isomorphic to the algebra $\mathcal{O}(\Sigma_q^3(l; -))$, generated by x , y and z subject to the following relations*

$$y^* = yz, \quad xy = q^l yx, \quad xx^* = \prod_{p=0}^{l-1} (1 - q^{2p} y^2 z), \quad x^* x = \prod_{p=1}^l (1 - q^{-2p} y^2 z) \quad (20)$$

and z is central unitary. The embedding of $\mathcal{O}(\Sigma_q^3(l; -))$ into $\mathcal{O}(\Sigma_q^3)$ is given by $x \mapsto \zeta_0^l$, $y \mapsto \zeta_1$ and $z \mapsto \xi$

Proof. Clearly ζ_1 , ξ , ζ_0^l and ζ_0^{*l} are coinvariant elements of $\mathcal{O}(\Sigma_q^3)$. Apply the coaction to the basis (7) to obtain

$$\zeta_0^r \zeta_1^s \xi^t \mapsto \zeta_0^r \zeta_1^s \xi^t \otimes w^r, \quad \zeta_0^{*r} \zeta_1^s \xi^t \mapsto \zeta_0^{*r} \zeta_1^s \xi^t \otimes w^{-r}$$

These elements are coinvariant, provided $r = r'l$. Hence every coinvariant element is a polynomial in ζ_1 , ξ , ζ_0^l and ζ_0^{*l} . Equations (20) are now easily derived from Equations (6) and (18). $\square$

The algebra $\mathcal{O}(\Sigma_q^3(l; -))$ is a right $\mathcal{O}(U(1))$ -comodule coalgebra with coaction defined as

$$\varphi : \mathcal{O}(\Sigma_q^3(l; -)) \rightarrow \mathcal{O}(\Sigma_q^3(l; -)) \otimes \mathcal{O}(U(1)), \quad x \mapsto x \otimes u, \quad y \mapsto y \otimes u, \quad z \mapsto z \otimes u^{-2} \quad (21)$$

Note in passing that the second and third relations in Equations (20) tell us that the grade of z must be double the grade of y^* since xx^* and x^*x have degree zero, and so

$$\deg(y^2 z) = \deg(y^2) + \deg(z) = 2 \deg(y) + \deg(z) = 0 \implies \deg(z) = -2 \deg(y) = 2 \deg(y^*)$$

Proposition 4.3 *The algebra $\mathcal{O}(\Sigma_q^3(l; -))^{co\mathcal{O}(U(1))}$ of invariant elements under the coaction φ is isomorphic to the $\mathcal{O}(\mathbb{RP}_q(l; -))$.*

Proof. We aim to show that the $*$ -subalgebra of $\mathcal{O}(\Sigma_q^3(l; -))$ of elements which are invariant under the coaction is generated by $x^2 z$, xyz and $y^2 z$. The isomorphism of $\mathcal{O}(\Sigma_q^3(l; -))^{co\mathcal{O}(U(1))}$ with $\mathcal{O}(\mathbb{RP}_q(l; -))$ is then obtained by using the embedding of $\mathcal{O}(\Sigma_q^3(l; -))$ in $\mathcal{O}(\Sigma_q^3)$ described in Proposition 4.2, i.e., $y^2 z \mapsto \zeta_1 \xi \mapsto a$, $xyz \mapsto \zeta_0^l \zeta_1 \xi \mapsto b$ and $x^2 z \mapsto \zeta_0^{2l} \xi \mapsto c$.

The algebra $\mathcal{O}(\Sigma_q^3(l; -))$ is spanned by elements of the type $x^r y^s z^t$, $x^{*r} y^s z^t$, where $r, s \in \mathbb{N}$ and $t \in \mathbb{Z}$. Applying the coaction φ to these basis elements gives $x^r y^s z^t \mapsto x^r y^s z^t \otimes u^{r+s-2t}$. Hence $x^r y^s z^t$ is φ -invariant if and only if $2t = r + s$. If r is even, then s is even and

$$x^r y^s z^t = x^r y^s z^{(r+s)/2} = (x^2 z)^{r/2} (y^2 z)^{s/2}$$

If r is odd, then so is s and

$$x^r y^s z^t = x^r y^s z^{(r+s)/2} \sim (x^2 z)^{(r-1)/2} (y^2 z)^{(s-1)/2} (xyz)$$

The case of $x^{*r} y^s z^t$ is dealt with similarly, thus proving that all coinvariants of φ are polynomials in $x^2 z$, xyz , $y^2 z$ and their $*$ -conjugates. $\square$

The main result of this section is contained in the following theorem.

Theorem 4.4 $\mathcal{O}(\Sigma_q^3(l; -))$ is a non-cleft principal $\mathcal{O}(U(1))$ -comodule algebra over $\mathcal{O}(\mathbb{RP}_q(l; +))$ via the coaction φ .

Proof. To prove that $\mathcal{O}(\Sigma_q^3(l; -))$ is a principal $\mathcal{O}(U(1))$ -comodule algebra over $\mathcal{O}(\mathbb{RP}_q(l; +))$ we employ Proposition 2.16 and construct a strong connection form as follows.

Define $\omega : \mathcal{O}(U(1)) \rightarrow \mathcal{O}(\Sigma_q^3(l; -)) \otimes \mathcal{O}(\Sigma_q^3(l; -))$ recursively as follows.

$$\omega(1) = 1 \otimes 1 \quad (22a)$$

$$\omega(u^n) = x^* \omega(u^{n-1}) x - \sum_{m=1}^l (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} y^{2m-1} z^m \omega(u^{n-1}) y \quad (22b)$$

$$\omega(u^{-n}) = x \omega(u^{-n+1}) x^* - \sum_{m=1}^l (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} y^{2m-1} z^{m-1} \omega(u^{-n+1}) y z \quad (22c)$$

where $n \in \mathbb{N}$ and, for all $s \in \mathbb{R}$, the deformed or q -binomial coefficients $\binom{l}{m}_s$ are defined by the following polynomial equality in indeterminate t

$$\prod_{m=1}^l (1 + s^{m-1} t) = \sum_{m=0}^l s^{m(m-1)/2} \binom{l}{m}_s t^m \quad (23)$$

The map ω has been designed such that normalisation property, Equation (1a), is automatically satisfied. To check Equation (1b) for ω given by Equation (22b) and (22c) takes a bit more work. We use proof by induction, but first have to derive an identity to assist with the calculation. Set $s = q^{-2}$, $t = -q^{-2} y^* y$ in Equation (23) to arrive at

$$\sum_{m=1}^l (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} y^{*m} y^m = \prod_{m=1}^l (1 + q^{-2(m-1)} (-q^{-2} y^* y)) - 1$$

which, using Equations (20), simplifies to

$$\sum_{m=1}^l (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} y^{2m} z^m = \prod_{m=1}^l (1 - q^{-2m} y^2 z) - 1 = x^* x - 1 \quad (24)$$

Now to start the induction process we consider the case $n = 1$. By Equation (24) $(\mu \circ \omega)(u) = 1$ providing the basis. Next, we assume that the relation holds for $n = N$, that is $(\mu \circ \omega)(u^N) = 1$, and consider the case $n = N + 1$,

$$\omega(u^{N+1}) = x^* \omega(u^N) x - \sum_{m=1}^l (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} y^{2m-1} z^m \omega(u^N) y$$

applying the multiplication map to both sides and using the induction hypothesis,

$$(m \circ \omega)(u^{N+1}) = x^* x - \sum_{m=1}^l (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} y^{2m} z^m = x^* x - (x^* x - 1) = 1$$

showing Equation (1b) holds for all $u^n \in \mathcal{O}(U(1))$, where $n \in \mathbb{N}$. To show this property holds for each $u^{*n} = u^{-n}$ we adopt the same strategy; this is omitted from the proof as it does not provide further insight, instead repetition of similar arguments.

Equation (1c): this is again proven by induction. Applying $(\text{id} \otimes \varphi)$ to $\omega(u)$ gives

$$\begin{aligned} x^* \otimes x \otimes u &= \sum_{m=1}^l (-1)^m q^{-m(m-1)} \binom{l}{m}_{q^2} y^{2m-1} z^m \otimes y \otimes u \\ &= (x^* \otimes x - \sum_{m=1}^l (-1)^m q^{-m(m-1)} \binom{l}{m}_{q^2} y^{2m-1} z^m \otimes y) \otimes u \\ &= \omega(u) \otimes u = (\omega \otimes \text{id}) \circ \Delta(u) \end{aligned}$$

This shows that Equation (1c) holds for ω given by Equation (22b) when $n = 1$. We now assume the property holds for $n = N - 1$, hence $(\text{id} \otimes \varphi) \circ w(u^{N-1}) = (\omega \otimes \text{id}) \circ \Delta(u^{N-1}) = \omega(u^{N-1}) \otimes u^{N-1}$, and consider the case $n = N$.

$$\begin{aligned} (\text{id} \otimes \varphi)(w(u^N)) &= (\text{id} \otimes \varphi)(x^* \omega(u^{N-1}) x - \sum_{m=1}^l (-1)^m q^{-m(m-1)} \binom{l}{m}_{q^{-2}} y^{2m-1} z^m \omega(u^{N-1}) y) \\ &= x^* ((\text{id} \otimes \varphi)(\omega(u^{N-1}) x)) \\ &\quad - \sum_{m=1}^l (-1)^m q^{-m(m-1)} \binom{l}{m}_{q^{-2}} y^{2m-1} z^m ((\text{id} \otimes \varphi)(\omega(u^{N-1}) y)) \\ &= x^* \omega(u^{N-1}) x \otimes u^N - \sum_{m=1}^l (-1)^m q^{-m(m-1)} \binom{l}{m}_{q^{-2}} y^{2m-1} z^m \omega(u^{N-1}) y \otimes u^N \\ &= \omega(u^N) \otimes u^N = (\omega \otimes \text{id}) \circ \Delta(u^N) \end{aligned}$$

hence Equation (1c) is satisfied for all $u^n \in \mathcal{O}(U(1))$ where $n \in \mathbb{N}$. The case for u^{*n} is proved in a similar manner, as is Equation (1d). Again, the details are omitted as the process is identical. This completes the proof that ω is a strong connection form, hence $\mathcal{O}(\Sigma_q^3(l, -))$ is a principal comodule algebra.

Following the discussion of Section 3.1, to determine whether the constructed comodule algebra is cleft we need to identify invertible elements in $\mathcal{O}(\Sigma_q^3(l, -))$. Since

$$\mathcal{O}(\Sigma_q^3(l, -)) \subset \mathcal{O}(\Sigma_q^3) \cong \mathcal{O}(S_q^2) \square_{\mathbb{C}Z_2} \mathcal{O}(U(1)) \subset \mathcal{O}(S_q^2) \otimes \mathcal{O}(U(1))$$

and the only invertible elements in the algebraic tensor $\mathcal{O}(S_q^2) \otimes \mathcal{O}(U(1))$ are scalar multiples of $1 \otimes u^n$ for $n \in \mathbb{N}$, we can conclude that the only invertible elements in $\mathcal{O}(S_q^2) \square_{\mathbb{C}\mathbb{Z}_2} \mathcal{O}(U(1))$ are the elements of the form $1 \otimes u^n$. These elements correspond to the elements ξ^n in $\mathcal{O}(\Sigma_q^3)$, which in turn correspond to z^n in $\mathcal{O}(\Sigma_q^3(l, -))$.

Suppose $j : H \rightarrow A$ is the cleaving map; to ensure the map is convolution invertible we are forced to put $u \mapsto z^n$. Since u has degree 1 in $H = \mathcal{O}(U(1))$ and z has degree -2 in $\mathcal{O}(\Sigma_q^3(l, -))$, the map j fails to preserve the degrees, hence it is not colinear. Therefore, $\mathcal{O}(\Sigma_q^3(l, -))$ is a non-cleft principal comodule algebra. $\square$

4.2.2. Almost Freeness of the Coaction $\varrho_{1,l}$

At the classical limit, $q \rightarrow 1$, the algebras $\mathcal{O}(\mathbb{RP}_q(l; -))$ represent singular manifolds or orbifolds. It is known that every orbifold can be obtained as a quotient of a manifold by an *almost free* action. The latter means that the action has finite (rather than trivial as in the free case) stabiliser groups. As explained in Section 2, on the algebraic level, freeness is encoded in the bijectivity of the canonical map can , or, more precisely, in the surjectivity of the lifted canonical map $\overline{\text{can}}$ (Equation (2)). The surjectivity of $\overline{\text{can}}$ means the triviality of the cokernel of $\overline{\text{can}}$, thus the size of the cokernel of $\overline{\text{can}}$ can be treated as a measure of the size of the stabiliser groups. This leads to the following notion proposed in [4].

Definition 4.5 Let H be a Hopf algebra and let A be a right H -comodule algebra with coaction $\varrho^A : A \rightarrow A \otimes H$. We say that the coaction is almost free if the cokernel of the (lifted) canonical map

$$\overline{\text{can}} : A \otimes A \rightarrow A \otimes H, \quad a \otimes a' \mapsto a\varrho^A(a')$$

is finitely generated as a left A -module.

Although the coaction φ defined in the preceding section is free, at the classical limit $q \rightarrow 1$ $\mathcal{O}(\Sigma_q^3(l, -))$ represents a singular manifold or an orbifold. On the other hand, at the same limit, $\mathcal{O}(\Sigma_q^3)$ corresponds to a genuine manifold, one of the Seifert three-dimensional non-orientable manifolds; see [17]. It is therefore natural to ask, whether the coaction $\varrho_{1,l}$ of $\mathcal{O}(U(1))$ on $\mathcal{O}(\Sigma_q^3)$ which has $\mathcal{O}(\mathbb{RP}_q(l; -))$ as fixed points is almost free in the sense of Definition 4.5.

Proposition 4.6 The coaction $\varrho_{1,l}$ is almost free.

Proof. Denote by $\iota_- : \mathcal{O}(\Sigma_q^3(l, -)) \hookrightarrow \mathcal{O}(\Sigma_q^3)$, the $*$ -algebra embedding described in Proposition 4.2. One easily checks that the following diagram

$$\begin{array}{ccc} \mathcal{O}(\Sigma_q^3(l, -)) & \xrightarrow{\iota_-} & \mathcal{O}(\Sigma_q^3) \\ \varphi \downarrow & & \downarrow \varrho_{1,l} \\ \mathcal{O}(\Sigma_q^3(l, -)) \otimes \mathcal{O}(U(1)) & \xrightarrow{\iota_- \otimes (-)^l} & \mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(U(1)) \end{array}$$

where $(-)^l : u \rightarrow u^l$, is commutative. The principality or freeness of φ proven in Theorem 4.4 implies that $1 \otimes u^{ml} \in \text{Im}(\overline{\text{can}})$, $m \in \mathbb{Z}$, where $\overline{\text{can}}$ is the (lifted) canonical map corresponding to coaction

$\varrho_{1,l}$. This means that $\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^l, u^{-l}] \subseteq \text{Im}(\overline{\text{can}})$. Therefore, there is a short exact sequence of left $\mathcal{O}(\Sigma_q^3)$ -modules

$$(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u, u^{-1}]) / (\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^l, u^{-l}]) \longrightarrow \text{coker}(\overline{\text{can}}) \longrightarrow 0$$

The left $\mathcal{O}(\Sigma_q^3)$ -module $(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u, u^{-1}]) / (\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^l, u^{-l}])$ is finitely generated, hence so is $\text{coker}(\overline{\text{can}})$. $\square$

4.2.3. Associated Modules or Sections of Line Bundles

One can construct modules associated to the principal comodule algebra $\mathcal{O}(\Sigma_q^3(l, -))$ following the procedure outlined at the end of Section 2.2; see Definition 2.21.

Every one-dimensional comodule of $\mathcal{O}(U(1)) = \mathbb{C}[u, u^*]$ is determined by the grading of a basis element of $\mathbb{C}$, say 1. More precisely, for any integer n , $\mathbb{C}$ is a left $\mathcal{O}(U(1))$ -comodule with the coaction

$$\varrho_n : \mathbb{C} \rightarrow \mathbb{C}[u, u^*] \otimes \mathbb{C}, \quad 1 \mapsto u^n \otimes 1$$

Identifying $\mathcal{O}(\Sigma_q^3(l, -)) \otimes \mathbb{C}$ with $\mathcal{O}(\Sigma_q^3(l, -))$ we thus obtain, for each coaction ϱ_n

$$\Gamma[n] := \mathcal{O}(\Sigma_q^3(l, -)) \square_{\mathcal{O}(U(1))} \mathbb{C} \cong \{f \in \Sigma_q^3(l, -) \mid \varphi(f) = f \otimes u^n\} \subset \mathcal{O}(\Sigma_q^3(l, -))$$

In other words, $\Gamma[n]$ consists of all elements of $\mathcal{O}(\Sigma_q^3(l, -))$ of $\mathbb{Z}$ -degree n . In particular $\Gamma[0] = \mathcal{O}(\mathbb{RP}_q(l; -))$. Each of the $\Gamma[n]$ is a finitely generated projective left $\mathcal{O}(\mathbb{RP}_q(l; -))$ -module, *i.e.*, it represents the module of sections of the non-commutative line bundle over $\mathbb{RP}_q(l; -)$. The idempotent matrix $E[n]$ defining $\Gamma[n]$ can be computed explicitly from a strong connection form ω (see Equations (22) in the proof of Theorem 4.4) following the procedure described in [11]. Write $\omega(u^n) = \sum_i \omega(u^n)^{[1]}_i \otimes \omega(u^n)^{[2]}_i$. Then

$$E[n]_{ij} = \omega(u^n)^{[2]}_i \omega(u^n)^{[1]}_j \in \mathcal{O}(\mathbb{RP}_q^2(l; -)) \quad (25)$$

For example, for $l = 2$ and $n = 1$, using Equations (22b) and (22a) as well as redistributing numerical coefficients we obtain

$$E[1] = \begin{pmatrix} (1-a)(1-q^2a) & q^{-1}\sqrt{1+q^{-2}}b & iq^{-3}ba \\ q^{-1}\sqrt{1+q^{-2}}b^* & q^{-2}(1+q^{-2})a & iq^{-4}\sqrt{1+q^{-2}}a^2 \\ iq^{-3}b^* & iq^{-4}\sqrt{1+q^{-2}}a & -q^{-6}a^2 \end{pmatrix} \quad (26)$$

Although the matrix $E[1]$ is not hermitian, the left-upper 2×2 block is hermitian. On the other hand, once $\mathcal{O}(\mathbb{RP}_q(2; -))$ is completed to the C^* -algebra $C(\mathbb{RP}_q(2; -))$ of continuous functions on $\mathbb{RP}_q(2; -)$ (and then identified with the suitable pullback of two algebras of continuous functions over the quantum real projective space; see [3]), then a hermitian projector can be produced out of $E[1]$ by using the Kaplansky formula; see [18, page 88].

The traces of tensor powers of each of the $E[n]$ make up a cycle in the cyclic complex of $\mathcal{O}(\mathbb{RP}_q(l; -))$, whose corresponding class in the cyclic homology $HC_\bullet(\mathcal{O}(\mathbb{RP}_q(l; -)))$ is known as the *Chern character* of $\Gamma[n]$. Again, as an illustration of the usage of an explicit form of a strong connection form, we compute the traces of $E[n]$ for general l .

Lemma 4.7 *The zero-component of the Chern character of $\Gamma[n]$ is the class of the polynomial c_n in generator a of $\mathcal{O}(\mathbb{RP}_q(l; -))$, given by the following recursive formula. First, $c_0(a) = 1$, and then, for all positive n ,*

$$c_n(a) = c_{n-1}(q^{2l}a) \prod_{p=0}^{l-1} (1 - q^{2p}a) + c_{n-1}(a) \left(1 - \prod_{p=1}^l (1 - q^{-2p}a) \right) \quad (27a)$$

$$c_{-n}(a) = c_{-n+1}(q^{-2l}a) \prod_{p=1}^l (1 - q^{-2p}a) + c_{-n+1}(a) \left(1 - \prod_{p=0}^{l-1} (1 - q^{2p}a) \right) \quad (27b)$$

Proof. We will prove the formula (27a) as (27b) is proven by similar arguments. Recall that $c_n = \text{Tr } E[n]$. By normalisation (22a) of the strong connection ω , obviously $c_0 = 1$. In view of Equation (22b) we obtain the following recursive formula

$$c_n = xc_{n-1}x^* - \sum_{m=1}^l (-1)^m q^{-m(m+1)} \binom{l}{m}_{q^{-2}} y c_{n-1} y^{2m-1} z^m \quad (28)$$

In principle, c_n could be a polynomial in a, b and c_- . However, the third of Equations (20) together with Equation (24) and identification of a as $y^2 z$ yield

$$c_1 = \prod_{p=0}^{l-1} (1 - q^{2p}a) + \left(1 - \prod_{p=1}^l (1 - q^{-2p}a) \right) \quad (29)$$

that is a polynomial in a only. As commuting x and y through a polynomial in a in Equation (28) will produce a polynomial in a again, we conclude that each of the c_n is a polynomial in a . The second of Equations (20), the centrality of z and the identification of a as $y^2 z$ imply that

$$xc_{n-1}(a) = c_{n-1}(q^{2l}a), \quad yc_{n-1}(a) = c_{n-1}(a)y$$

and in view of Equations (28) and (29) yield Equation (27a). $\square$

4.3. The Positive Case $\mathcal{O}(\mathbb{RP}_q(l; +))$

4.3.1. The Principal $\mathcal{O}(U(1))$ -Comodule Algebra over $\mathcal{O}(\mathbb{RP}_q^2(l; +))$

In the same light as the negative case we aim to construct quantum principal bundles with base spaces $\mathcal{O}(\mathbb{RP}_q(l; +))$, and proceed by viewing $\mathcal{O}(\Sigma_q^3)$ as a right H' -comodule algebra, where H' is a Hopf-algebra of a finite cyclic group. The aim is to construct the total space $\mathcal{O}(\Sigma_q^3(l, +))$ of the bundle over $\mathcal{O}(\mathbb{RP}_q(l; +))$ as the coinvariant subalgebra of $\mathcal{O}(\Sigma_q^3)$. $\mathcal{O}(\Sigma_q^3(l, +))$ must contain generators $\zeta_1^2 \xi$ and $\zeta_0^l \xi$ of $\mathcal{O}(\mathbb{RP}_q(l; +))$. Suppose $H' = \mathbb{C}\mathbb{Z}_m$ and $\Phi : \mathcal{O}(\Sigma_q^3) \rightarrow \mathcal{O}(\Sigma_q^3) \otimes H'$ is a coaction. We require Φ to be compatible with the algebraic relations and to give zero $\mathbb{Z}_m$ -degree to $\zeta_1^2 \xi$ and $\zeta_0^l \xi$ are zero. These requirements yield

$$2 \deg(\zeta_1) + \deg(\xi) = 0 \pmod{m}, \quad l \deg(\zeta_0) + \deg(\xi) = 0 \pmod{m}$$

Bearing in mind that l is odd, the simplest solution to these requirements is provided by $m = 2l$, $\deg(\xi) = 0$, $\deg(\zeta_0) = 2$, $\deg(\zeta_1) = l$. This yields the coaction

$$\Phi : \mathcal{O}(\Sigma_q^3) \rightarrow \mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}\mathbb{Z}_{2l} \quad \zeta_0 \mapsto \zeta_0 \otimes v^2, \quad \zeta_1 \mapsto \zeta_1 \otimes v^l, \quad \xi \mapsto \xi \otimes 1$$

where v ($v^{2l} = 1$) is the unitary generator of $\mathbb{C}\mathbb{Z}_{2l}$. Φ is extended to the whole of $\mathcal{O}(\Sigma_q^3)$ so that Φ is an algebra map, making $\mathcal{O}(\Sigma_q^3)$ a right $\mathbb{C}\mathbb{Z}_{2l}$ -comodule algebra.

Proposition 4.8 *The fixed point subalgebra of the coaction Φ is isomorphic to the $*$ -algebra $\mathcal{O}(\Sigma_q^3(l, +))$ generated by x', y' and central unitary z' subject to the following relations:*

$$x'y' = q^{2l}y'x', \quad y'^* = y'z'^2 \quad (30a)$$

$$x'x'^* = \prod_{p=0}^{l-1} (1 - q^{2p}y'z'), \quad x'^*x' = \prod_{p=1}^l (1 - q^{-2p}y'z') \quad (30b)$$

The isomorphism between $\mathcal{O}(\Sigma_q^3(l, +))$ and the coinvariant subalgebra of $\mathcal{O}(\Sigma_q^3)$ is given by $x' \mapsto \zeta_0^l$, $y' \mapsto \zeta_1^2$ and $z' \mapsto \xi$.

Proof. Clearly $\zeta_1^2, \xi, \zeta_0^l$ and ζ_0^{*l} are coinvariant elements of $\mathcal{O}(\Sigma_q^3)$. Apply the coaction Φ to the basis (7) to obtain

$$\zeta_0^r \zeta_1^s \xi^t \mapsto \zeta_0^r \zeta_1^s \xi^t \otimes v^{2r+ls}, \quad \zeta_0^{*r} \zeta_1^s \xi^t \mapsto \zeta_0^{*r} \zeta_1^s \xi^t \otimes v^{-2r+ls}$$

These elements are coinvariant, provided $2r + ls = 2ml$ in the first case or $-2r + ls = 2ml$ in the second. Since l is odd, s must be even and then $r = r'l$, hence the invariant elements must be of the form

$$(\zeta_0^l)^{r'} (\zeta_1^2)^{s/2} \xi^t, \quad (\zeta_0^{*l})^{r'} (\zeta_1^2)^{s/2} \xi^t$$

as required. Equations (30) are now easily derived from Equations (6) and (18). $\square$

The algebra $\mathcal{O}(\Sigma_q^3(l, +))$ is a right $\mathcal{O}(U(1))$ -comodule with coaction defined as,

$$\Omega : \mathcal{O}(\Sigma_q^3(l, +)) \rightarrow \mathcal{O}(\Sigma_q^3(l, +)) \otimes \mathcal{O}(U(1)), \quad x' \mapsto x' \otimes u, \quad y' \mapsto y' \otimes u, \quad z' \mapsto z' \otimes u^{-1} \quad (31)$$

The first relation in Equations (30a) bears no information on the possible gradings of the generators of $\mathcal{O}(\Sigma_q^3(l, +))$, however the second relation in Equations (30a) tells us that the grade of y'^* must be the same as that of z' since,

$$\deg(y'^*) = -\deg(y') = \deg(y') + 2\deg(z')$$

hence,

$$2\deg(y'^*) = 2\deg(z'), \text{ or, } \deg(y'^*) = \deg(z')$$

This is consistent with Equations (30b) since the left hand sides, $x'x'^*$ and x'^*x' , have degree zero, as do the right hand sides,

$$\deg(y'z') = \deg(y') + \deg(y'^*) = \deg(y') + (-\deg(y')) = 0$$

The coaction Ω is defined setting the grades of x' and y' as 1, and putting the grade of z' as -1 to ensure the coaction is compatible with the relations of the algebra $\mathcal{O}(\Sigma_q^3(l, +))$.

Proposition 4.9 *The right $\mathcal{O}(U(1))$ -comodule algebra $\mathcal{O}(\Sigma_q^3(l, +))$ has $\mathcal{O}(\mathbb{RP}_q(l; +))$ as its subalgebra of coinvariant elements under the coaction Ω .*

Proof. The fixed points of the algebra $\mathcal{O}(\Sigma_q^3(l, +))$ under the coaction Ω are found using the same method as in the odd k case. A basis for the algebra $\mathcal{O}(\Sigma_q^3(l, +))$ is given by $x'^r y'^s z'^t$, $x'^{r*} y'^s z'^t$, where $r, s \in \mathbb{N}$ and $t \in \mathbb{Z}$.

Applying the coaction Ω to the first of these basis elements gives,

$$x'^r y'^s z'^t \mapsto x'^r y'^s z'^t \otimes u^{r+s-t}$$

Hence the invariance of $x'^r y'^s z'^t$ is equivalent to $t = r + s$. Simple substitution and re-arranging gives,

$$x'^r y'^s z'^t = x'^r y'^s z'^{r+s} = (x' z')^r (y' z')^s$$

i.e., $x'^r y'^s z'^t$ is a polynomial in $x' z'$ and $y' z'$. Repeating the process for the second type of basis element gives the $*$ -conjugates of $x' z'$ and $y' z'$. Using Proposition 4.8 we can see that $a = \zeta_1^2 \xi = y' z'$ and $c_+ = \zeta_0^l \xi = x' z'$. $\square$

In contrast to the odd k case, although $\mathcal{O}(\Sigma_q^3(l, +))$ is a principal comodule algebra it yields trivial principal bundle over $\mathcal{O}(\mathbb{RP}_q(l; +))$.

Proposition 4.10 *The right $\mathcal{O}(U(1))$ -comodule algebra $\mathcal{O}(\Sigma_q^3(l, +))$ is trivial.*

Proof. The cleaving map is given by,

$$j : \mathcal{O}(U(1)) \rightarrow \mathcal{O}(\Sigma_q^3(l, +)), \quad j(u) = z'^*$$

which is an algebra map since z'^* is central unitary in $\mathcal{O}(\Sigma_q^3(l, +))$, hence must be convolution invertible. Also, j is a right $\mathcal{O}(U(1))$ -comodule map since,

$$(\Omega \circ j)(u) = \Omega(z'^*) = z'^* \otimes u = j(u) \otimes u = (j \otimes \text{id}) \circ \Delta(u)$$

completing the proof. $\square$

Since $\mathcal{O}(\Sigma_q^3(l, +))$ is a trivial principal comodule algebra, all associated $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ -modules are free.

4.3.2. Almost Freeness of the Coaction $\varrho_{2,l}$

As was the case for $\mathcal{O}(\Sigma_q^3(l, -))$, the principality of $\mathcal{O}(\Sigma_q^3(l, +))$ can be used to determine that the $\mathcal{O}(U(1))$ -coaction $\varrho_{2,l}$ on $\mathcal{O}(\Sigma_q^3)$ that defines $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ is almost free.

Proposition 4.11 *The coaction $\varrho_{2,l}$ is almost free.*

Proof. Denote by $\iota_+ : \mathcal{O}(\Sigma_q^3(l, +)) \hookrightarrow \mathcal{O}(\Sigma_q^3)$, the $*$ -algebra embedding described in Proposition 4.8. One easily checks that the following diagram

$$\begin{array}{ccc} \mathcal{O}(\Sigma_q^3(l, +)) & \xrightarrow{\iota_+} & \mathcal{O}(\Sigma_q^3) \\ \Omega \downarrow & & \downarrow \varrho_{2,l} \\ \mathcal{O}(\Sigma_q^3(l, +)) \otimes \mathcal{O}(U(1)) & \xrightarrow{\iota_+ \otimes (-)^{2l}} & \mathcal{O}(\Sigma_q^3) \otimes \mathcal{O}(U(1)) \end{array}$$

where $(-)^{2l} : u \rightarrow u^{2l}$ is commutative. By the arguments analogous to those in the proof of Proposition 4.6 one concludes that there is a short exact sequence of left $\mathcal{O}(\Sigma_q^3)$ -modules

$$(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u, u^{-1}]) / (\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^{2l}, u^{-2l}]) \longrightarrow \text{coker}(\overline{\text{can}}) \longrightarrow 0$$

where $\overline{\text{can}}$ is the lifted canonical map corresponding to coaction $\varrho_{2,l}$. The left $\mathcal{O}(\Sigma_q^3)$ -module $(\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u, u^{-1}]) / (\mathcal{O}(\Sigma_q^3) \otimes \mathbb{C}[u^{2l}, u^{-2l}])$ is finitely generated, hence so is $\text{coker}(\overline{\text{can}})$. $\square$

5. Conclusions

In this paper we discussed the principality of the $\mathcal{O}(U(1))$ -coactions on the coordinate algebra of the quantum Seifert manifold $\mathcal{O}(\Sigma_q^3)$ weighted by coprime integers k and l . We concluded that the coaction is principal if and only if $k = l = 1$, which corresponds to the case of a $U(1)$ -bundle over the quantum real projective plane. In all other cases the coactions are almost free. We identified subalgebras of $\mathcal{O}(\Sigma_q^3)$ which admit principal $\mathcal{O}(U(1))$ -coactions, whose invariants are isomorphic to coordinate algebras $\mathcal{O}(\mathbb{RP}_q^2(l; \pm))$ of quantum real weighted projective spaces. The structure of these subalgebras depends on the parity of k . For the odd k case, the constructed principal comodule algebra $\mathcal{O}(\Sigma_q^3(l, -))$ is non-trivial, while for the even case, the corresponding principal comodule algebra $\mathcal{O}(\Sigma_q^3(l, +))$ turns out to be trivial. The triviality of $\mathcal{O}(\Sigma_q^3(l, +))$ is a disappointment. Whether a different nontrivial principal $\mathcal{O}(U(1))$ -comodule algebra over $\mathcal{O}(\mathbb{RP}_q^2(l; +))$ can be constructed or whether such a possibility is ruled out by deeper geometric, topological or algebraic reasons remains to be seen.

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