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B-Fredholm Spectra of Drazin Invertible Operators and Applications

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Abstract: In this article, we consider Drazin invertible operators for study of the relationship between their B-Fredholm spectra and the transfer between some of the spectral properties defined through B-Fredholm spectra of this class of operators. Among other results, we investigate the transfer of generalized a-Weyl's theorem from T to their Drazin inverse S, if it exists.

Keywords: B-Fredholm spectra; Drazin invertible operator; Drazin inverse; generalized a-Weyl's theorem

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1. Introduction

In this paper, $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. By $\mathcal{H}(\sigma(T))$, we denote the set of all analytic functions defined on an open neighborhood of $\sigma(T)$, where $\sigma(T)$ is the spectrum of $T \in L(X)$, and also for $f \in \mathcal{H}(\sigma(T))$, we define $f(T)$ by means of the classical functional calculus. The notion of invertibility for an operator $T \in L(X)$ admits several generalizations and has some significance in investigating the relationships between the spectral properties of T and the spectral properties of a "generalized inverse" of T, if this exists. For instance, the relationship of "reciprocity" mentioned above between the points of the approximate point spectrum has been recently observed in the case that the "generalized inverse" is given in the sense of left m-invertible operators [1]. Another generalization of the notion of invertibility, which satisfies some relationships of "reciprocity" observed above, is provided by the concept of Drazin invertibility. Recall that an operator $T \in L(X)$ is said to be *Drazin invertible* if there exists an operator $S \in L(X)$ (called the Drazin inverse of T) and an integer $n \geq 0$ such that

$$TS = ST,\ STS = S,\ T^nST = T^n. \tag{1}$$

The operator S described in (1) is unique and also is Drazin invertible (see [2]). In this case, T^* (the dual operator of T) is Drazin invertible with Drazin inverse S^*, because $(T^*)^n = T^*S^*(T^n)^* = T^*T^*S^*(T^{n-1})^* = (T^*)^2S^*(T^{n-1})^* = \ldots = (T^*)^nS^*T^*$. In addition, if $T \in L(X)$ is a Drazin invertible operator, then T and S satisfy the equation $T^nST^n = T^nSTT^{n-1} = T^nT^{n-1} = T^j$ for the same integers $j = 2n-1 > n \geq 0$ (see [3]). The transfer of some spectral properties from T to S was studied in [4], but none of these properties involve the B-Fredholm theory. Moreover, in the literature, the relationship among the B-Fredholm spectra of a Drazin invertible operator with those of its Drazin inverse has not been studied. In this work, we study the transfer of the polaroid condition and the single-valued extension property, from a Drazin invertible operator T to its Drazin inverse S. Furthermore, we show that the classical Weyl type theorems and other spectral properties are equivalent for $f(S^*)$, where $f \in \mathcal{H}(\sigma(T))$. Next, we show that the nonzero points among the B-Fredholm spectra of T and S satisfy a reciprocal relationship. Finally, we establish that the forty-four spectral properties in [5] (Table 1) are transferred from T to S; in particular, the properties defined with B-Fredholm spectra. The importance of this study is that it enables an extension of the theoretical framework of the transmission of Weyl and Browder type theorems (generalized or not) from a Drazin invertible operator to its Drazin inverse.

2. Preliminaries and Basic Results

In this section, we present some basic definitions and results that will be useful throughout this manuscript. For $T \in L(X)$, we denote by $\alpha(T)$ the dimension of ker (T) (the kernel of T), by $\beta(T)$ the co-dimension of $T(X)$ (the range of T), by $p(T)$ and $q(T)$ the ascent and descent of T, respectively. We refer to [6] for more details on notations and terminologies. However, we give the following notations for some spectra:

- Fredholm spectrum: $\sigma_e(T)$,
- Upper semi-Fredholm spectrum: $\sigma_{usf}(T)$,
- Lower semi-Fredholm spectrum: $\sigma_{lsf}(T)$,
- B-Fredholm spectrum: $\sigma_{bf}(T)$,
- Upper semi B-Fredholm spectrum: $\sigma_{ubf}(T)$,
- Lower semi B-Fredholm spectrum: $\sigma_{lbf}(T)$,
- Approximate point spectrum: $\sigma_a(T)$,
- Surjective spectrum: $\sigma_s(T)$,
- Weyl spectrum: $\sigma_w(T)$,
- Upper semi-Weyl spectrum: $\sigma_{uw}(T)$,
- Lower semi-Weyl spectrum: $\sigma_{lw}(T)$,
- B-Weyl spectrum: $\sigma_{bw}(T)$,
- Upper semi B-Weyl spectrum: $\sigma_{ubw}(T)$,
- Lower semi B-Weyl spectrum: $\sigma_{lbw}(T)$,
- Browder spectrum: $\sigma_b(T)$,
- Upper semi-Browder spectrum: $\sigma_{ub}(T)$,
- Drazin invertible spectrum: $\sigma_d(T)$,
- Left Drazin invertible spectrum: $\sigma_{ld}(T)$,
- Right Drazin invertible spectrum: $\sigma_{rd}(T)$.

The single-valued extension property introduced by Finch in [7] plays a relevant role in local spectral theory. An operator $T \in L(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbf{D}$ with $\lambda_0 \in \mathbf{D}$, the only analytic function $f : \mathbf{D} \to X$ that satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbf{D}$ is the function $f \equiv 0$. The operator T is said to have SVEP, if it has SVEP at every point $\lambda \in \mathbb{C}$. It is easy to prove that $T \in L(X)$ has SVEP at every isolated point of $\sigma(T)$ and at each point of the resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover,

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{2}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{3}$$

see [8] (Theorem 3.8). From the definition of the localized SVEP, it is easily seen that

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ and has SVEP at } \lambda, \tag{4}$$

and dually

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ and has SVEP at } \lambda. \tag{5}$$

Note that $H_0(T) := \{x \in X : \|T^n(x)\|^{1/n} \to 0,\ n \to \infty\}$, the quasi-nilpotent part of $T \in L(X)$, generally is not closed and by [8] (Theorem 2.31), we have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{6}$$

Remark 1. *The converse of the implications (2)–(6) holds, whenever $\lambda I - T$ is a quasi-Fredholm operator; in particular, whenever $\lambda I - T$ is left Drazin invertible or right Drazin invertible (see [9]).*

Denote by $\operatorname{iso} A$, the set of all isolated points of $A \subseteq \mathbb{C}$. For $T \in L(X)$, define the following sets:

$$\begin{aligned}
\pi_{00}(T) &:= \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},\\
E(T) &:= \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T)\},\\
E^a(T) &:= \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(\lambda I - T)\}.
\end{aligned}$$

The previous sets allow defining some spectral properties that will be treated in this article.

Definition 1. *An operator $T \in L(X)$ is said to satisfy:*

1. Property (R) [10] *if* $\sigma_a(T) \setminus \sigma_{ub}(T) = \pi_{00}(T)$.
2. Property (gR) [11] *if* $\sigma_a(T) \setminus \sigma_{ld}(T) = E(T)$.
3. Property (w) [12] *if* $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T)$.
4. Property (gw) [13] *if* $\sigma_a(T) \setminus \sigma_{ubw}(T) = E(T)$.
5. Property (Sb) [14] *if* $\sigma_{ubw}(T) = \sigma_b(T)$.
6. Property (V_E) [5] *if* $\sigma(T) \setminus \sigma_{uw}(T) = E(T)$.
7. Property (gz) [15] *if* $\sigma(T) \setminus \sigma_{ubw}(T) = E^a(T)$.

Theorem 1 ([5])**.** *An operator $T \in L(X)$ satisfies property (V_E) if and only if T satisfies property (gz) and $\sigma_{ubw}(T) = \sigma_{uw}(T)$.*

Theorem 2 ([16])**.** *If $T \in L(X)$ is Drazin invertible with Drazin inverse S, then $0 \in \sigma(S) \setminus \sigma_{uw}(S)$ if and only if $0 \in \sigma(T) \setminus \sigma_{uw}(T)$.*

Next, we consider five results that were proved in [4], which are interesting since they present some basic relationships for Drazin invertible operators.

Lemma 1. *If $T \in L(X)$ is Drazin invertible with Drazin inverse S, then the following statements hold:*

1. $0 \in \sigma(T)$ *if and only if* $0 \in \sigma(S)$.
2. $0 \in \operatorname{iso}\sigma(T)$ *if and only if* $0 \in \operatorname{iso}\sigma(S)$.
3. $0 < \alpha(T) < \infty$ *if and only if* $0 < \alpha(S) < \infty$.

Theorem 3. *If $T \in L(X)$ is Drazin invertible with Drazin inverse S, then T has SVEP at $\lambda \neq 0$ if and only if S has SVEP at λ^{-1}.*

Theorem 4. *If $T \in L(X)$ is Drazin invertible with Drazin inverse S and $\lambda \neq 0$, then for all $k \in \mathbb{N}$, we have:*

1. $\ker(\lambda I - S)^k = \ker(\lambda^{-1} I - T)^k$.
2. $(\lambda I - S)^k(X) = (\lambda^{-1} I - T)^k(X)$.

Theorem 5. *If $T \in L(X)$ is Drazin invertible with Drazin inverse S and $\lambda \neq 0$, then the following statements hold:*

1. $p(\lambda I - T) = p(\lambda^{-1} I - S)$.
2. $q(\lambda I - T) = q(\lambda^{-1} I - S)$.

Theorem 6. *If $T \in L(X)$ is Drazin invertible with Drazin inverse S, then the following statements hold:*

1. $\sigma_w(S) \setminus \{0\} = \{\lambda^{-1} : \lambda \in \sigma_w(T) \setminus \{0\}\}$.
2. $\sigma_{uw}(S) \setminus \{0\} = \{\lambda^{-1} : \lambda \in \sigma_{uw}(T) \setminus \{0\}\}$.

The following remark will be useful in obtaining some of our results.

Remark 2. *Note that the implication $p(T) < \infty \Rightarrow p(T_{[n]}) < \infty$ always holds, since* $\ker(T^p|_{T^n(X)}) = \ker(T^p) \cap T^n(X)$.

3. Weyl and Browder Type Theorems and Related Properties

In this section, we study the connection among some Weyl and Browder type theorems. This will enable obtaining some additional results of this paper. In order to established some relations among property (gR) (resp. property (R)) and other Weyl type theorems, we require the following two theorems.

Theorem 7. *An operator $T \in L(X)$ satisfies property (gw) if and only if T satisfies property (gR) and T has SVEP at each $\lambda \notin \sigma_{ubw}(T)$.*

Proof. Assume that T satisfies property (gw). Let $\lambda \in E(T)$. Then $\lambda \in \operatorname{iso}\sigma(T)$ and $0 < \alpha(\lambda I - T)$, it follows that T has SVEP in λ. By hypothesis, we have $\lambda \in \sigma_a(T) \setminus \sigma_{ubw}(T)$, and so exists an integer $n \geq 0$ such that $(\lambda I - T)_{[n]}$ is an upper semi-Weyl operator. By Remark 1, $p(\lambda I - T) < \infty$, which implies by Remark 2 that $p((\lambda I - T)_{[n]}) < \infty$. Hence, $\lambda \in \sigma_a(T) \setminus \sigma_{ld}(T)$ and so $E(T) \subseteq \sigma_a(T) \setminus \sigma_{ld}(T)$. As we have always $\sigma_a(T) \setminus \sigma_{ld}(T) \subseteq \sigma_a(T) \setminus \sigma_{ubw}(T)$, and as T satisfies property (gw), it follows that $\sigma_a(T) \setminus \sigma_{ld}(T) \subseteq E(T)$. Therefore, $\sigma_a(T) \setminus \sigma_{ld}(T) = E(T)$ and T satisfies property (gR). On the other hand, let $\lambda \notin \sigma_{ubw}(T)$. We consider two cases:

Case 1. $\lambda \notin \sigma_a(T)$.

Case 2. $\lambda \in \sigma_a(T)$.

In Case 1, obviously T has SVEP at λ. In the Case 2, $\lambda \in \sigma_a(T) \setminus \sigma_{ubw}(T) = E(T)$ and so $\lambda \in \operatorname{iso}\sigma(T)$, hence T has SVEP at λ again.

Conversely, let $\lambda \in \sigma_a(T) \setminus \sigma_{ubw}(T)$. Since T has SVEP at λ, by Remark 1, we have $p(\lambda I - T) < \infty$ and so $\lambda \in \sigma_a(T) \setminus \sigma_{ld}(T) = E(T)$. Furthermore, $E(T) = \sigma_a(T) \setminus \sigma_{ld}(T) \subseteq \sigma_a(T) \setminus \sigma_{ubw}(T)$. Therefore, $\sigma_a(T) \setminus \sigma_{ubw}(T) = E(T)$ and T satisfies property (gw). □

Remark 3. *In [13] (Theorem 2.6), we can see another proof of Theorem 7 using different methods.*

Considering Remark 1 and proceeding analogously as in the proof of Theorem 7, we obtain the following result.

Theorem 8. *An operator $T \in L(X)$ satisfies property (w) if and only if T satisfies property (R) and T has SVEP at each $\lambda \notin \sigma_{uw}(T)$.*

We end this section by giving some results in connection with property (V_E), which we will use in Section 4.

Theorem 9. *Let $T \in L(X)$. The following statements are equivalent:*

1. *T satisfies properties (Sb) and (gz).*
2. *T satisfies property (V_E).*

Proof. (1) $\Rightarrow$ (2). If T satisfies both properties (Sb) and (gz), then $\sigma_{ubw}(T) = \sigma_b(T)$ and $\sigma(T) \setminus \sigma_{ubw}(T) = E_a(T)$. Thus, $\sigma_{ubw}(T) = \sigma_{uw}(T) = \sigma_{ub}(T) = \sigma_b(T)$ and by [16] (Lemma 2.1), it follows that $\sigma(T) = \sigma_a(T)$. Therefore, $\sigma(T) \setminus \sigma_{uw}(T) = \sigma(T) \setminus \sigma_{ubw}(T) = E_a(T) = E(T)$ and so T satisfies property (V_E).

(2) $\Rightarrow$ (1). It follows from [5] (Theorem 2.27). $\square$

Theorem 10. *If $T \in L(X)$ satisfies property (V_E) and T has SVEP at each $\lambda \notin \sigma_{lw}(T)$, then $\sigma_w(T) = \sigma_{uw}(T) = \sigma_{lw}(T)$.*

Proof. The first equality $\sigma_{uw}(T) = \sigma_w(T)$ was given in [5] (Theorem 2.27). It only remains to prove that $\sigma_w(T) \subseteq \sigma_{lw}(T)$. Indeed, let $\lambda \notin \sigma_{lw}(T)$. Then $\lambda I - T$ is a lower semi-Fredholm operator and as T has SVEP at λ, by Remark 1, we have $p(\lambda I - T) < \infty$, which implies that $\lambda \notin \sigma_{uw}(T)$. Hence, $\lambda \notin \sigma_w(T)$ and so $\sigma_w(T) = \sigma_{lw}(T)$. $\square$

Recall that an operator $T \in L(X)$ is polaroid if the isolated points of the spectrum of T, points belonging $\operatorname{iso}\sigma(T)$, are poles of the resolvent of T. It is well-known, from Reference [17] (Theorem 2.11), that T^* is polaroid if and only if T is polaroid.

Theorem 11. *Let $T \in L(X)$ be a polaroid operator that has SVEP at each $\lambda \notin \sigma_{lw}(T)$. If T satisfies property (V_E), then T^* satisfies property (V_E).*

Proof. Since T is polaroid, T^* is also polaroid, it follows that $E(T) = \Pi(T) = \Pi(T^*) = E(T^*)$. On the other hand, as T has SVEP at each $\lambda \notin \sigma_{lw}(T)$, then by Theorem 10, we have $\sigma(T^*) \setminus \sigma_{uw}(T^*) = \sigma(T) \setminus \sigma_{lw}(T) = \sigma(T) \setminus \sigma_{uw}(T)$ and therefore, T^* satisfies property (V_E). $\square$

4. *B*-Fredholm Spectra and Drazin Invertible Operators

In this section, we consider the transfers of the SVEP and the polaroid condition from $T \in L(X)$ to their Drazin inverse S. Consequently, we establish some spectral properties for $f(S^*)$, where $f \in \mathcal{H}(\sigma(T))$. Moreover, we investigate the close relationship among the B-Fredholm spectra of Drazin invertible operators.

Remark 4. *For a Drazin invertible operator $T \in L(X)$, we have:*

1. *The polaroid condition is transferred from T to their Drazin inverse S. Indeed, by Reference [6] (Theorem 4.22), if T is polaroid, then S is too and also S^* is polaroid (see [17] (Theorem 2.8)). Moreover, if T is polaroid and $f \in \mathcal{H}(\sigma(T))$, then $f(S)$ and $(f(S))^* = f(S^*)$ are polaroid (see [17] (Lemma 3.11)).*
2. *The SVEP is transferred from T to their Drazin inverse S. To see this, note that if T has SVEP at each $\lambda \neq 0$, then by Theorem 3, we conclude that S has SVEP at λ^{-1}. Since $p(S) < \infty$, we conclude that S has SVEP. Moreover, if T has SVEP and $f \in \mathcal{H}(\sigma(T))$, then $f(S)$ has SVEP (see [8] (Theorem 2.40)). Similarly, we deduce that S^* has SVEP whenever T^* has SVEP and, in this case, we have $(f(S))^* = f(S^*)$ has SVEP for each $f \in \mathcal{H}(\sigma(T))$.*

Some operators are polaroid and their dual operators have SVEP, but they are not Drazin invertible.

Example 1. *Let $H^2(\mathbf{T})$ denote the Hardy space of the unit circle $\mathbf{T}$ in the complex plane. Given $\phi \in L^\infty(\mathbf{T})$, the Toeplitz operator with symbol ϕ is the operator on $H^2(\mathbf{T})$ defined by*

$$T_\phi : f \longmapsto P(\phi f),$$

where $f \in H^2(\mathbf{T})$ and P is the orthogonal projection of $L^\infty(\mathbf{T})$ onto $H^2(\mathbf{T})$. We denote by $C(\mathbf{T})$ the algebra of all complex-valued continuous functions on $\mathbf{T}$. Consider $\phi \in C(\mathbf{T})$ and denote $\Gamma = \phi(\mathbf{T})$. By Reference [6] (Theorem 4.99), if ϕ is non-constant, then $\operatorname{iso}\sigma(T_\phi) = \emptyset$. Furthermore, it is shown in [6] (Theorem 4.100) that if the orientation of the curve Γ traced out by ϕ is clockwise, then T^*_ϕ has SVEP. Thus, if T_ϕ has continuous symbol ϕ non-constant and the orientation of the curve Γ traced out by ϕ is clockwise, then T_ϕ is a polaroid operator such that T^*_ϕ has SVEP, but T_ϕ is not a Drazin invertible operator.

In the following theorem, we use the notation (W) (resp. (aW), (gW), (gaW)) for the classic Weyl's (resp. a-Weyl's, generalized Weyl's, generalized a-Weyl's) theorem.

Theorem 12. *Let $T \in L(X)$ be a polaroid Drazin invertible operator with Drazin inverse S and let $f \in \mathcal{H}(\sigma(T))$ be not constant on each of the components of its domain. The following statements hold:*

1. *If T^* has SVEP, then properties (W), (aW), (R), (w), (gW), (gaW), (gR) and (gw) are equivalent for $f(S)$, and $f(S)$ satisfies each of these properties.*
2. *If T has SVEP, then properties (W), (aW), (R), (w), (gW), (gaW), (gR) and (gw) are equivalent for $f(S^*)$, and $f(S^*)$ satisfies each of these properties.*

Proof. (1). By hypothesis and Remark 4, we have that $f(S^*)$ is a polaroid operator and $f(S)$ has SVEP. Thus, by [17] (Theorem 3.12), it follows that $f(S)$ satisfies properties (W), (aW), (w), (gW), (gaW), (gw), and these properties are equivalent for $f(S)$. On the other hand, from Theorems 7 and 8, we obtain that properties (R), (gR) and (W) are equivalent for $f(S)$.

(2). Argue as in the proof of part (1). Just replace T^* with T, and S with S^*. □

Example 2. *If $T \in L(X)$ is a Drazin invertible $H(p)$-operator with Drazin inverse S, then $f(S^*)$ satisfies (ii) of Theorem 12, because in this case T is a polaroid operator having SVEP (see [17]). An example of Drazin invertible $H(p)$-operators is the class of algebraic operators, see [8] (Theorem 3.93 and Corollary 2.47). Nilpotent operators are special cases of algebraic operators. An extensive class of nilpotent operators is the class of the analytically quasi-$\mathcal{THN}$ operators, which are quasi-nilpotent over $L(H)$, where H is a Hilbert space (see [8] (Theorem 6.188)). In addition, idempotent operators are algebraic, similar to operators for which some power has a finite-dimensional range.*

Next, we establish some results that relate the B-Fredholm spectra of a Drazin invertible operator and those of its Drazin inverse. These are important because they will allow the transfer of spectral properties defined in terms of the B-Fredholm spectra from a Drazin invertible operator to its Drazin inverse.

Lemma 2. *If $T \in L(X)$ is a Drazin invertible operator with Drazin inverse S, then for each integer $n \geq 0$ and $\lambda \neq 0$ we have:*

1. $\alpha[(\lambda I - S)_{[n]}] = \alpha[(\lambda^{-1} I - T)_{[n]}]$.
2. $\beta[(\lambda I - S)_{[n]}] = \beta[(\lambda^{-1} I - T)_{[n]}]$.
3. $\operatorname{ind}((\lambda I - S)_{[n]}) = \operatorname{ind}((\lambda^{-1} I - T)_{[n]})$.

Proof. 1. For each integer $n \geq 0$ and $T \in L(X)$, we have $\ker(T_{[n]}) = \ker(T) \cap T^n(X)$. Then by Theorem 4 (case $k = n$), it follows that

$$\begin{aligned} \ker(\lambda I - S)_{[n]} &= \ker(\lambda I - S) \cap (\lambda I - S)^n(X) \\ &= \ker(\lambda^{-1} I - T) \cap (\lambda^{-1} I - T)^n(X) \\ &= \ker(\lambda^{-1} I - T)_{[n]}, \text{ for all } \lambda \neq 0. \end{aligned}$$

Therefore, $\alpha[(\lambda I - S)_{[n]}] = \alpha[(\lambda^{-1} I - T)_{[n]}]$.

2. For each integer $n \geq 0$ and $T \in L(X)$, we have $T_{[n]}(T^n(X)) = T^{n+1}(X)$. Then by Theorem 4 (case $k = n+1$), we have

$$\begin{aligned} (\lambda I - S)_{[n]}((\lambda I - S)^n(X)) &= (\lambda I - S)^{n+1}(X) = (\lambda^{-1} I - T)^{n+1}(X) \\ &= (\lambda^{-1} I - T)_{[n]}((\lambda^{-1} I - T)^n(X)). \end{aligned}$$

Therefore, $\beta[(\lambda I - S)_{[n]}] = \beta[(\lambda^{-1} I - T)_{[n]}]$.
3. Since ind $(T) = \alpha(T) - \beta(T)$, we have (3) follows from (1) and (2). □

In the following theorem, we show that for a Drazin invertible operator T, the relationship of reciprocity between the nonzero parts of the B-Fredholm spectra of T and the B-Fredholm spectra of its Drazin inverse, is true.

Theorem 13. *Let $T \in L(X)$ be a Drazin invertible operator with Drazin inverse S and $\lambda \neq 0$. The following statements hold:*

1. *$\lambda \notin \sigma_{usf}(T)$ if and only if $\lambda^{-1} \notin \sigma_{usf}(S)$.*
2. *$\lambda \notin \sigma_{ubw}(T)$ if and only if $\lambda^{-1} \notin \sigma_{ubw}(S)$.*
3. *$\lambda \notin \sigma_{ld}(T)$ if and only if $\lambda^{-1} \notin \sigma_{ld}(S)$.*
4. *$\lambda \notin \sigma_{ubf}(T)$ if and only if $\lambda^{-1} \notin \sigma_{ubf}(S)$.*

Proof. Without loss of generality, we prove only a sense of equivalences.

1. If $\lambda \notin \sigma_{usf}(T)$, then $\alpha(\lambda I - T) < \infty$ and $(\lambda I - T)(X)$ are closed. Hence, by Theorem 4, we get that $\alpha(\lambda^{-1} I - S) < \infty$ and $(\lambda^{-1} I - S)(X)$ are closed, which implies that $\lambda^{-1} \notin \sigma_{usf}(S)$.
2. If $\lambda \notin \sigma_{ubw}(T)$, then $\text{ind}(\lambda I - T)_{[n_0]} \leq 0$ and $(\lambda I - T)^{n_0}(X)$ are closed, for some integer $n_0 \geq 0$. Furthermore, by Lemma 2, ind $(\lambda^{-1} I - S)_{[n_0]} \leq 0$ and, by Theorem 4, we have $(\lambda^{-1} I - S)^{n_0}(X) = (\lambda I - T)^{n_0}(X)$ is closed. Hence, $\lambda^{-1} \notin \sigma_{ubw}(S)$.
3. If $\lambda \notin \sigma_{ld}(T)$, then $\lambda \notin \sigma_{ubw}(T)$ and by part (2), it follows that $\lambda^{-1} \notin \sigma_{ubw}(S)$. Also, by Remark 2, there exists an integer $n \geq 0$ such that $k := p((\lambda I - T)_{[n]}) < \infty$. Proceeding as in the proof of Lemma 2, we deduce that $\ker(\lambda^{-1} I - S)^k_{[n]} = \ker(\lambda I - T)^k_{[n]}$. Thus, we get that $p((\lambda^{-1} I - S)_{[n]}) < \infty$ and hence, $\lambda^{-1} \notin \sigma_{ld}(S)$.
4. The proof of "$\lambda^{-1} \notin \sigma_{ubf}(S)$ if $\lambda \notin \sigma_{ubf}(T)$" is similar to the proof of part (1). Just use Lemma 2.

□

Theorem 14. *Let $T \in L(X)$ be a Drazin invertible operator with Drazin inverse S and $\lambda \neq 0$. The following statements hold:*

1. *$\lambda \notin \sigma_{lsf}(T)$ if and only if $\lambda^{-1} \notin \sigma_{lf}(S)$.*
2. *$\lambda \notin \sigma_{lbw}(T)$ if and only if $\lambda^{-1} \notin \sigma_{lbw}(S)$.*
3. *$\lambda \notin \sigma_{rd}(T)$ if and only if $\lambda^{-1} \notin \sigma_{rd}(S)$.*
4. *$\lambda \notin \sigma_{lbf}(T)$ if and only if $\lambda^{-1} \notin \sigma_{lbf}(S)$.*

Proof. By Theorem 4 (case $k = 1$), we have $\beta(\lambda I - S) = \beta(\lambda^{-1} I - T)$, and by Theorem 5, $q(\lambda I - S) = q(\lambda^{-1} I - T)$. In addition, by Lemma 2, $\beta[(\lambda I - S)_{[n]}] = \beta[(\lambda^{-1} I - T)_{[n]}]$ and ind $(\lambda I - S)_{[n]} =$ ind $(\lambda^{-1} I - T)_{[n]}$ for each integer $n \geq 0$ and $\lambda \neq 0$. Thus, proceeding as in the proof of Theorem 13, the result is obtained. □

Combining Theorems 13 and 14, we obtain the following corollary.

Corollary 1. *Let $T \in L(X)$ be a Drazin invertible operator with Drazin inverse S and $\lambda \neq 0$. The following statements hold:*

1. *$\lambda \notin \sigma_e(T)$ if and only if $\lambda^{-1} \notin \sigma_e(S)$.*
2. *$\lambda \notin \sigma_{bf}(T)$ if and only if $\lambda^{-1} \notin \sigma_{bf}(S)$.*
3. *$\lambda \notin \sigma_{bw}(T)$ if and only if $\lambda^{-1} \notin \sigma_{bw}(S)$.*
4. *$\lambda \notin \sigma_d(T)$ if and only if $\lambda^{-1} \notin \sigma_d(S)$.*

Various spectral properties are defined through the spectral subsets $E_a(T)$ and $E(T)$, so it is also necessary to study the reciprocity relationship for these subsets.

Theorem 15. *Let $T \in L(X)$ be a Drazin invertible operator with Drazin inverse S and $\lambda \neq 0$. The following statements hold:*

1. *$\lambda \in E_a(S)$ if and only if $\lambda^{-1} \in E_a(T)$.*
2. *$\lambda \in E(S)$ if and only if $\lambda^{-1} \in E(T)$.*

Proof. 1. Let $\lambda \in E_a(S)$ then $\lambda \in \operatorname{iso} \sigma_a(S)$ and $0 < \alpha(\lambda I - S)$. By Theorem 4 (case $k = 1$), we get that $0 < \alpha(\lambda^{-1} I - T)$. Note that $\lambda^{-1} \in \operatorname{iso} \sigma_a(T)$, otherwise we have $\lambda \in \operatorname{acc} \sigma_a(S)$. Hence, $\lambda^{-1} \in E_a(T)$. The converse is clear.

2. It is similar to part (1). □

Remark 5. *Let $\mathcal{P}$ be the set of all spectral properties that appear in [5](Table 1). If T satisfies property (V_E), then by [5] (Theorem 2.27), all properties in $\mathcal{P}$ are equivalent and T satisfies each of these properties; in this case, T has SVEP at each $\lambda \notin \sigma_{ubw}(T)$ (resp. $\lambda \notin \sigma_{uw}(T)$), which implies by [11] (Theorem 2.7) that property (gR) (resp. property (R)) is part of the aforementioned equivalence.*

5. Some Applications

Throughout this section, let $T \in L(X)$ be a Drazin invertible operator with Drazin inverse S. According to [18], an operator $T \in L(X)$ is said to satisfy property (V_Π) if $\sigma_{uw}(T) = \sigma_d(T)$. It was shown in [16] (Theorem 4.3) that T satisfies property (V_Π) if and only if S satisfies property (V_Π). In this case, S is a Browder operator, because by [18] (Theorem 2.27), we have $\sigma_d(S) = \sigma_b(S)$. In this section, we transfer spectral properties defined in terms of B-Fredholm spectra from T to S. Specifically, we have the following results:

1. Recall that $T \in L(X)$ satisfies property (gaz) [15] if $\sigma(T) \setminus \sigma_{ubw}(T) = \sigma_a(T) \setminus \sigma_{ld}(T)$. It was shown in [15] (Theorem 3.4) that T satisfies property (gaz) if and only if $\sigma_{ubw}(T) = \sigma_{ld}(T)$ and $\sigma_a(T) = \sigma(T)$. In fact, this property is transferred from T to S. Indeed, if T satisfies property (gaz) and $\lambda \neq 0$, then combining Theorems 4, 5 and 13, we have

$$\begin{aligned} \lambda \in \sigma(S) \setminus \sigma_{\mathrm{ubw}}(S) &\Leftrightarrow \lambda^{-1} \in \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) \\ &\Leftrightarrow \lambda^{-1} \in \sigma_a(T) \setminus \sigma_{\mathrm{ld}}(T) \\ &\Leftrightarrow \lambda \in \sigma_a(S) \setminus \sigma_{ld}(S). \end{aligned}$$

   On the other hand, as S is a Drazin invertible operator, we have $0 \in \operatorname{iso} \sigma(S)$ and $0 \notin \sigma_{ubw}(S)$. Consequently, $0 \in \sigma(S) \setminus \sigma_{ubw}(S)$ and $0 \in \sigma_a(S) \setminus \sigma_{ld}(S)$. Therefore, $\sigma(S) \setminus \sigma_{ubw}(S) = \sigma_a(S) \setminus \sigma_{ld}(S)$ and S satisfies property (gaz). An example of this type of operators is the operator $T \in \ell^2(\mathbb{N})$ defined by $T(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, \ldots)$, for which $\sigma(T) = \sigma_a(T) = \{0, 1\}$, $\sigma_{ubw}(T) = \sigma_d(T) = \emptyset$. Hence, T is a Drazin invertible operator that satisfies property (gaz), it follows that S satisfies property (gaz), or equivalently, by [19] (Theorem 3.6), S^* has SVEP at each $\lambda \in \sigma_{ubw}(S)$.
2. Generalized a-Weyl's theorem is transferred from T to S. Indeed, if $\lambda \neq 0$, then combining Theorems 4, 13 and 15, we have

$$\lambda \in \sigma_a(S) \setminus \sigma_{\mathrm{ubw}}(S) \Leftrightarrow \lambda^{-1} \in \sigma_a(T) \setminus \sigma_{\mathrm{ubw}}(T) = E^a(T) \Leftrightarrow \lambda \in E_a(S).$$

   Note that by Lemma 1, for $\lambda = 0$, there is no problem because T and S are left Drazin invertible operators and hence, $T(X)$ and $S(X)$ are closed when $\alpha(T) = \alpha(S) = 0$. Therefore, $\sigma_a(S) \setminus \sigma_{\mathrm{ubw}}(S) = E^a(S)$, and S satisfies generalized a-Weyl's theorem. Similarly, we get that if T satisfies property (gR), then S satisfies property (gR). This is also true for generalized Weyl's theorem and property (gw).
3. Property (V_E) is transferred from T to S. If $\lambda \in \sigma(S) \setminus \sigma_{uw}(S)$ and $\lambda \neq 0$, then by Theorem 6, $\lambda^{-1} \in \sigma(T) \setminus \sigma_{uw}(T)$. Since T satisfies property (V_E), we have $\lambda^{-1} \in E(T)$, which implies, by Theorem 15, that $\lambda \in E(S)$. Therefore, $\sigma(S) \setminus \sigma_{uw}(S) \subseteq E(S)$. Similarly, if $\lambda \neq 0$ and $\lambda \in E(S)$, then $\lambda^{-1} \in E(T)$ and so, $\lambda^{-1} \in \sigma(T) \setminus \sigma_{uw}(T)$.

By Theorem 6, we have $\lambda \in \sigma(S) \setminus \sigma_{uw}(S)$. Hence, $E(S) \subseteq \sigma(S) \setminus \sigma_{uw}(S)$. Observe that by Lemma 1 and Theorem 2, for $\lambda = 0$, there is no difficulty. Thus, we conclude that $\sigma(S) \setminus \sigma_{uw}(S) = E(S)$ and S satisfies property (V_E). An example of this type of operator is a quasi-nilpotent operator $T \in L(\mathbb{C}^n)$, which is Drazin invertible and satisfies property (V_E).

4. If T satisfies properties (gz) and (Sb), then by Theorem 9, we have T satisfies property (V_E). By part (3), we obtain that S satisfies property (V_E), which implies, by Remark 5, that all properties in $\mathcal{P}$ are equivalent for S, and S satisfies each of these properties.
5. Let $T \in L(X)$ be a polaroid operator having SVEP at each $\lambda \notin \sigma_{lw}(T)$. If T satisfies property (V_E), then S^* satisfies property (V_E). Indeed, by Theorem 11, T^* satisfies property (V_E) and by part (3), S^* satisfies property (V_E).
6. An operator $T \in L(X)$ satisfies property (gbz) (resp. (bz)) [20] if $\sigma_{ubf}(T) = \sigma_{ld}(T)$ (resp. $\sigma_{usf}(T) = \sigma_{ub}(T)$). By Theorems 6 and 13, and Lemma 1, we get that if T satisfies property (gbz) (resp. (bz)), then S satisfies property (gbz) (resp. (bz)).

6. Conclusions

This paper studied the relationship between some spectra originating from the B-Fredholm theory of a Drazin operator invertible and its Drazin inverse; some applications concerning the polaroid property and SVEP of these operators were given. It should be noted that by the results established in this paper, it follows that each spectral property is defined in terms of Fredholm or B-Fredholm spectra; in particular, each property belonging to $\mathcal{P}$ is transferred from T to their Drazin inverse S.

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