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*Article*

# Explicit Formulas for Some Infinite ${}_3F_2(1)$-Series

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**Abstract:** We establish two recurrence relations for some Clausen's hypergeometric functions with unit argument. We solve them to give the explicit formulas. Additionally, we use the moments of Ramanujan's generalized elliptic integrals to obtain these recurrence relations.

**Keywords:** Clausen's hypergeometric functions; elliptic integrals; moments

**MSC:** 33C20; 33C75



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## 1. Introduction

The (generalized) hypergeometric function [1] is defined to be the complex analytic function

$$ {}_{p+1}F_p\left[\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, b_2, \ldots, b_p \end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{p+1})_n}{(b_1)_n (b_2)_n \cdots (b_p)_n} \frac{z^n}{n!}, $$

where

$$ (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1) $$

denotes the The Pochhammer symbol, $\Gamma(s)$ is the Euler's $\Gamma$-function, $p$ is a non-negative integer, the complex numbers $a_i$, $b_j$ are called, respectively, the numerator and denominator parameters, and $z$ is called the variable. The denominator parameters are not allowed to be zero or negative integers ($b_j \notin \mathbb{Z}_{\le 0}$). If the numerator parameters $a_i \in \mathbb{Z}_{\le 0}$, then the series ${}_{p+1}F_p$ reduces to a finite sum. The series ${}_{p+1}F_p$ converges when $|z| < 1$ for all choices of $a_i$, $b_j$. If $z = 1$, the series converges for

$$ \Re(b_1 + b_2 + \cdots + b_p - a_1 - a_2 - \cdots - a_{p+1}) > 0. $$

The case ${}_{p+1}F_p$ when $p = 1$ is called the Gauss hypergeometric function. The following well-known and celebrated summation formula for ${}_2F_1(1)$ is due to Gauss:

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0, c \notin \mathbb{Z}_{\le 0}. \tag{1} $$

Another interesting formula for ${}_3F_2(1)$ is due to Ramanujan:

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, n \\ 1, n+1 \end{matrix} \,\middle|\, 1\right] = \frac{16^n}{\pi n \binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{16^k}. \tag{2} $$

where $n$ is a positive integer and which is obtained by replacing $n$ by $n-1$ in Entry 29(b) in ([2], p. 39). This formula was stated without proof by Ramanujan in his first letter to Hardy. There are numerous hypergeometric series identities in mathematical literature (see [3,4]). The evaluation of the hypergeometric sum ${}_3F_2(1)$ (the Clausenian hypergeometric function with unit argument) is of ongoing interest, since it appears ubiquitously in many physics and statistics problems [5–7]. The q-extension of the 3F2(1)-series is also very interesting and has been studied by many researchers; for example, see [8] and the references therein.

Recently, Asakura et al. [9] proved that

$$B(a,b)\,{}_3F_2\left[\begin{matrix} a,b,q \\ a+b,q+1 \end{matrix}\middle| 1\right] \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\log\overline{\mathbb{Q}}^{\times}, \tag{3}$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and the right hand side denotes the $\overline{\mathbb{Q}}$-linear subspace of $\mathbb{C}$ generated by $1, 2\pi i$, and $\log(\alpha)$'s, $\alpha \in \overline{\mathbb{Q}}^{\times}$ under some conditions on $a,b,q \in \mathbb{Q}\backslash\mathbb{Z}$.

To obtain an explicit description of Equation (3) has not been completed except some cases. Asakura, Yabu [10] evaluated the cases with $a = 1/6$, $b = 5/6$ for the examples of their works. For example,

$${}_3F_2\left[\begin{matrix} \frac{1}{6},\frac{5}{6},\frac{1}{2} \\ 1,\frac{3}{2} \end{matrix}\middle| 1\right] = \frac{3\sqrt{3}}{2\pi}\log(2+\sqrt{3}), \tag{4}$$

and

$${}_3F_2\left[\begin{matrix} \frac{1}{6},\frac{5}{6},\frac{1}{3} \\ 1,\frac{4}{3} \end{matrix}\middle| 1\right] = \frac{\sqrt{3}\sqrt[3]{2}}{2\pi}A - \frac{\sqrt[3]{2}}{\pi}B, \quad \text{and} \quad {}_3F_2\left[\begin{matrix} \frac{1}{6},\frac{5}{6},\frac{2}{3} \\ 1,\frac{5}{3} \end{matrix}\middle| 1\right] = \frac{\sqrt{3}\sqrt[3]{4}}{3\pi}A + \frac{2\sqrt[3]{4}}{3\pi}B, \tag{5}$$

where

$$\begin{aligned} A &= \log\Big((1-2^{-\frac{2}{3}})^2 + (1+2^{-\frac{2}{3}}\sqrt{3})^2\Big) - \log\Big((1-2^{-\frac{2}{3}})^2 + (1-2^{-\frac{2}{3}}\sqrt{3})^2\Big), \\ B &= \arctan\left(\frac{3}{3+\sqrt[3]{2}+3\sqrt[3]{4}}\right). \end{aligned} \tag{6}$$

They list all the explicit values of the cases $q = \frac{1}{2}, \frac{i}{3}, \frac{j}{4}, \frac{k}{5}$, where $i \in \{1,2\}$, $j \in \{1,2,3\}$, and $k \in \{1,2,3,4\}$ by applying their method to the elliptic fibration $y^2 = 2x^3 - 3x^2 + t^{\ell}$ where $\ell = 2,3,4,5$, respectively. Motivated by their works, it is interesting to give an explicit formula for the corresponding general form. In this paper, we aim to give an explicit formula for

$${}_3F_2\left[\begin{matrix} \frac{1}{6},\frac{5}{6},q+n \\ 1,q+n+1 \end{matrix}\middle| 1\right] = \sum_{k=0}^{\infty} \frac{\binom{6k}{3k}\binom{3k}{2k}}{16^k 27^k}\frac{q+n}{k+q+n},$$

where $n$ is an arbitrary integer and $q = \frac{1}{2}, \frac{i}{3}, \frac{j}{4}$, where $i \in \{1,2\}$, and $j \in \{1,2,3\}$. For the sake of brevity and our convenience, we sometimes will denote ${}_3F_2\left[\begin{matrix} \frac{1}{6},\frac{5}{6},x \\ 1,x+1 \end{matrix}\middle| 1\right]$ as $F(x)$. For example, for any non-negative integer $n$, we have the following explicit formulas:

$${}_3F_2\left[\begin{matrix} \frac{1}{6},\frac{5}{6},\frac{1}{2}+n \\ 1,\frac{3}{2}+n \end{matrix}\middle| 1\right] = \frac{2n+1}{2\pi(3n+1)}\frac{27^n}{16^n}\frac{\binom{2n}{n}}{\binom{3n}{2n}} \times \left\{ \begin{matrix} 3\sqrt{3}\log(2+\sqrt{3}) \\ +2\displaystyle\sum_{k=0}^{n-1}\frac{3k+1}{(2k+1)^2}\frac{16^k}{27^k}\frac{\binom{3k}{2k}}{\binom{2k}{k}} \end{matrix} \right\},$$

$${}_3F_2\left[\begin{matrix} \frac{1}{6},\frac{5}{6},-\frac{1}{2}-n \\ 1,\frac{1}{2}-n \end{matrix}\middle| 1\right] = \frac{(3n+1)}{3\pi(2n+1)}\frac{16^n}{27^n}\frac{\binom{3n}{2n}}{\binom{2n}{n}} \times \left\{ \begin{matrix} 2\sqrt{3}\log(2+\sqrt{3}) \\ +3\displaystyle\sum_{k=0}^{n}\frac{27^k\binom{2k}{k}}{16^k\binom{3k}{2k}(3k+1)} \end{matrix} \right\},$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{3}+n\\ 1,\frac{4}{3}+n\end{matrix}\,\middle|\,1\right]=\frac{3n+1}{2\pi}\frac{\left(\frac{1}{3}\right)_n^2}{\left(\frac{1}{2}\right)_n\left(\frac{7}{6}\right)_n}\left\{\sqrt{3}\sqrt[3]{2}A-2\sqrt[3]{2}B+3\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)_k\left(\frac{7}{6}\right)_k}{(3k+1)^2\left(\frac{1}{3}\right)_k^2}\right\}, $$

where $A, B$ is stated in Equation (6).

For $0 \le s < \frac{1}{2}$ and $0 \le k \le 1$, let

$$ K^s(k)=\frac{\pi}{2}{}_2F_1\left[\begin{matrix}\frac{1}{2}-s,\frac{1}{2}+s\\ 1\end{matrix}\,\middle|\,k^2\right] $$

be Ramanujan's generalized elliptic integral of the first kind of order $s$. The moment $K_{n,s}$ is given by

$$ K_{n,s}=\int_0^1 k^n K^s(k)\,dk, $$

where $n$ is a real number. Borwein et al. ([11], Theorem 2) proved that for $0 \le s < \frac{1}{2}$,

$$ K_{n,s}=\frac{\pi}{2(n+1)}{}_3F_2\left[\begin{matrix}\frac{1}{2}-s,\frac{1}{2}+s,\frac{n+1}{2}\\ 1,\frac{n+3}{2}\end{matrix}\,\middle|\,1\right]. $$

Thus, our hypergeometric series $F(x)$ can be got by setting $s = 1/3$, and we have

$$ F(x)={}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x\\ 1,x+1\end{matrix}\,\middle|\,1\right]=\frac{4x}{\pi}K_{2x-1,\frac{1}{3}}. $$

In the last section, we will use the moments of Ramanujan's generalized elliptic integral to give another method of obtaining the explicit evaluations.

The organization of this paper is as follows. In Section 2, we give some preliminaries. We provide two recurrence relations for the hypergeometric series $F(x)$. Then, we solve these recurrence relations to obtain explicit evaluations of the hypergeometric series $F(n \pm q)$ for $n \in \mathbb{Z}$ and $q = 1, 1/2, 1/3, 2/3, 1/4, 3/4$ in Section 3. In Section 4, we list the explicit forms of $F(n \pm q)$ for $n = \pm 1, \pm 2, \pm 3$. In the final section, we use the moments of Ramanujan's generalized elliptic integral to give another method of obtaining the same evaluations.

## 2. Preliminaries

We list an explicit formula in ([12], Equation 3.13-(41)) which we need to use later.

$$ {}_3F_2\left[\begin{matrix}a,b,1\\ c,2\end{matrix}\,\middle|\,1\right]=\frac{(c-1)}{(a-1)(b-1)}\left[\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}-1\right], $$

where $a \ne 1$, $b \ne 1$, and $\Re(c-a-b+1) > 0$. Thus, let $a = \frac{1}{6}, b = \frac{5}{6}, c = 2$ we have

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},1\\ 2,2\end{matrix}\,\middle|\,1\right]=\frac{36(18-5\pi)}{25\pi}. \tag{7} $$

We first prove a useful lemma.

**Lemma 1.** *Let $x$ be a complex number with $x \notin \mathbb{Z}_{\le 0}$. Then*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x\\ 1,x+1\end{matrix}\,\middle|\,1\right]=\frac{(6x+5)(6x+1)}{36x(x+1)}{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x+1\\ 1,x+2\end{matrix}\,\middle|\,1\right]-\frac{1}{2\pi x}, \tag{8} $$

**Proof.** We rewrite this hypergeometric series $F(x)$ as

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x\\1,x+1\end{matrix}\middle|\,1\right] &= 1+\sum_{n=0}^{\infty}\frac{(\frac{1}{6})_{n+1}(\frac{5}{6})_{n+1}x}{(1)_{n+1}^2(n+1+x)}\\
&= 1+\sum_{n=0}^{\infty}\frac{(\frac{1}{6})_n(\frac{5}{6})_n(n+\frac{1}{6})(n+\frac{5}{6})x}{(1)_n^2(n+1)^2(n+1+x)}.
\end{aligned}
$$

We use the partial fraction decomposition of

$$
\frac{(n+\frac{1}{6})(n+\frac{5}{6})x}{(n+1)^2(n+1+x)} = \frac{5}{36(n+1)^2} - \frac{36x+5}{36x(n+1)} + \frac{(6x+1)(6x+5)}{36x(n+1+x)}
$$

to the above identity, we have

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x\\1,x+1\end{matrix}\middle|\,1\right] &= 1+\frac{5}{36}\,{}_4F_3\left[\begin{matrix}\frac{1}{6},\frac{5}{6},1,1\\1,2,2\end{matrix}\middle|\,1\right] - \frac{36x+5}{36x}\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},1\\1,2\end{matrix}\middle|\,1\right]\\
&\quad + \frac{(6x+1)(6x+5)}{36x(x+1)}\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x+1\\1,x+2\end{matrix}\middle|\,1\right].
\end{aligned}
$$

The first two hypergeometric series in the right-hand side of the above equation can be evaluated by Equation (7) and the Gauss formula Equation (1):

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}\frac{1}{6},\frac{5}{6},1,1\\1,2,2\end{matrix}\middle|\,1\right] &= {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},1\\2,2\end{matrix}\middle|\,1\right] = \frac{36(18-5\pi)}{25\pi}, \quad \text{and}\\
{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},1\\1,2\end{matrix}\middle|\,1\right] &= {}_2F_1\left[\begin{matrix}\frac{1}{6},\frac{5}{6}\\2\end{matrix}\middle|\,1\right] = \frac{18}{5\pi}.
\end{aligned}
$$

Substituting these values into the last equation of our ${}_3F_2(1)$, we can get the required formula. □

We reverse Equation (8) and get another recurrence relation for our ${}_3F_2(1)$.

**Lemma 2.** *Let $x$ be a complex number with $x \notin \mathbb{Z}_{\le 0}$ and $x \neq \frac{1}{6},\frac{5}{6}$. Then*

$$
{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x\\1,x+1\end{matrix}\middle|\,1\right] = \frac{36x(x-1)}{(6x-1)(6x-5)}\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x-1\\1,x\end{matrix}\middle|\,1\right] + \frac{18x}{\pi(6x-1)(6x-5)}. \tag{9}
$$

We have listed the explicit formulas of

$$
{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{2}\\1,\frac{3}{2}\end{matrix}\middle|\,1\right], {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{3}\\1,\frac{4}{3}\end{matrix}\middle|\,1\right], \text{ and } {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{2}{3}\\1,\frac{5}{3}\end{matrix}\middle|\,1\right]
$$

which Asakura, Yabu have obtained in [10], in Equations (4) and (5). Here we list the remaining explicit formulas for $q = 1/4, 3/4, 1/5, 2/5, 3/5, 4/5$, which is given in [10].

$$
{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{4}\\1,\frac{5}{4}\end{matrix}\middle|\,1\right] = \frac{12^{3/4}(C-D)}{2\pi}, \quad \text{and} \quad {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{3}{4}\\1,\frac{7}{4}\end{matrix}\middle|\,1\right] = \frac{9\cdot 12^{3/4}(C+D)}{14\sqrt{3}\pi},
$$

where

$$C = \frac{1}{2}\log\left(\frac{3^{5/4} - 3^{3/4} + \sqrt{2}}{3^{5/4} - 3^{3/4} - \sqrt{2}}\right), \quad \text{and} \quad D = \arccos\left(\frac{3^{5/4} + 3^{3/4}}{2\sqrt{5+3\sqrt{3}}}\right). \tag{10}$$

Let $\zeta = e^{2\pi i/5}$, $\zeta_{20} = e^{2\pi i/20}$, $\alpha = 1/\sqrt[10]{24} > 0$,

$$e_j = \frac{\sqrt{2}\alpha^3\zeta_{20}^3\zeta^j + \frac{\sqrt{2}}{4}\alpha^{-3}\zeta_{20}^{-3}\zeta^j - \sqrt{3}(\alpha^2\zeta_{20}^2\zeta^j - 1)}{\sqrt{2}\alpha^3\zeta_{20}^3\zeta^j + \frac{\sqrt{2}}{4}\alpha^{-3}\zeta_{20}^{-3}\zeta^j + \sqrt{3}(\alpha^2\zeta_{20}^2\zeta^j - 1)},$$

$$f_j = \frac{\zeta^{2j} - 1}{5}\begin{pmatrix} (\zeta^{2j} - 1)\log(e_0) + (\zeta^{2j} - \zeta^{3j})\log(e_1) \\ +(\zeta^{2j} - \zeta^j)\log(e_2) + (\zeta^{2j} - \zeta^{4j})\log(e_3) + 4\pi i\zeta^{2j} \end{pmatrix}$$

and

$$A_j = \frac{\Gamma(\frac{j}{5} + \frac{1}{6})\Gamma(\frac{j}{5} + \frac{5}{6})}{\Gamma(\frac{j}{5})^2},$$

where $\log(x)$ takes the principal values,

$$\log(x) = \log|x| + \arg(x)i \quad (-\pi < \arg(x) \le \pi).$$

Then Asakura, Yabu [10] gave

$${}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{k}{5} \\ 1, \frac{k}{5} + 1 \end{matrix} \middle| 1\right] = \frac{kf_k}{2\pi A_k}. \tag{11}$$

This formula for $q = k/5$ is complicated. It can be seen that, although the results in Theorems 1 and 2 can be used to obtain their general formulas, the formulas will be more cumbersome and complicated, so we will not deal with the formula and its general form for $q = k/5$ in this paper.

We give examples applying Equation (8) and note that

$$\frac{(\frac{1}{6})_n(\frac{5}{6})_n}{(1)_n^2} = \frac{\binom{6n}{3n}\binom{3n}{2n}}{16^n 27^n}.$$

Thus,

$${}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{-1}{2} \\ 1, \frac{1}{2} \end{matrix} \middle| 1\right] = \sum_{n=0}^{\infty} \frac{\binom{6n}{3n}\binom{3n}{2n}}{16^n 27^n}\frac{1}{1-2n} = \frac{3 + 2\sqrt{3}\log(2+\sqrt{3})}{3\pi}, \tag{12}$$

$${}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{-1}{3} \\ 1, \frac{2}{3} \end{matrix} \middle| 1\right] = \sum_{n=0}^{\infty} \frac{\binom{6n}{3n}\binom{3n}{2n}}{16^n 27^n}\frac{1}{1-3n} = \frac{\sqrt[3]{4}\sqrt{3}A + 2\sqrt[3]{4}B + 12}{8\pi}, \tag{13}$$

$${}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{-2}{3} \\ 1, \frac{1}{3} \end{matrix} \middle| 1\right] = \sum_{n=0}^{\infty} \frac{\binom{6n}{3n}\binom{3n}{2n}}{16^n 27^n}\frac{2}{2-3n} = \frac{3\sqrt[3]{2}\sqrt{3}A - 6\sqrt[3]{2}B + 12}{16\pi}, \tag{14}$$

$${}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{-1}{4} \\ 1, \frac{3}{4} \end{matrix} \middle| 1\right] = \sum_{n=0}^{\infty} \frac{\binom{6n}{3n}\binom{3n}{2n}}{16^n 27^n}\frac{1}{1-4n} = \frac{\sqrt{2}\sqrt[4]{3}(C+D) + 6}{3\pi}, \tag{15}$$

$${}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{-3}{4} \\ 1, \frac{1}{4} \end{matrix} \middle| 1\right] = \sum_{n=0}^{\infty} \frac{\binom{6n}{3n}\binom{3n}{2n}}{16^n 27^n}\frac{3}{3-4n} = \frac{7\sqrt{2}\sqrt[4]{27}(C-D) + 18}{27\pi}, \tag{16}$$

where $A, B$ are defined in Equation (6) and $C, D$ are defined in Equation (10).

## 3. Explicit Formulas

We will solve the recurrence relations in Lemmas 1 and 2 as explicit formulas in this section.

**Theorem 1.** *Let $m$ be a non-negative integer, $p$ be a non-zero complex number such that $m+p+1$ is not a non-positive integer. Then*

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p+m\\1,p+1+m\end{matrix}\,\middle|\,1\right]\\
&=\frac{(p+m)\cdot(p)_m^2}{\left(\frac{1}{6}+p\right)_m\left(\frac{5}{6}+p\right)_m}\left\{\frac{1}{p}{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p\\1,p+1\end{matrix}\,\middle|\,1\right]+\frac{1}{2\pi}\sum_{k=0}^{m-1}\frac{\left(\frac{1}{6}+p\right)_k\left(\frac{5}{6}+p\right)_k}{(p+k)^2(p)_k^2}\right\}.
\end{aligned}
$$

**Proof.** Consider the hypergeometric series $F(x)$ with $x>1$. So we can decomposite $x=m+p$, where $m\in\mathbb{N}$. Applying the recurrence relation in Lemma 2 a positive integer $\ell$ times, we get

$$
{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x\\1,x+1\end{matrix}\,\middle|\,1\right]=T(\ell)\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},x-\ell\\1,x+1-\ell\end{matrix}\,\middle|\,1\right]+\frac{1}{2\pi}\sum_{k=0}^{\ell-1}\frac{T(k+1)}{x-k-1},
$$

where

$$
T(\ell)=\prod_{j=1}^{\ell}\frac{(x+1-j)(x-j)}{(x-j+\frac{5}{6})(x-j+\frac{1}{6})}.
$$

Using the mathematical induction on the integer $\ell$, it is easy to prove that the above formula is correct. We use the Pochhammer symbols to rewrite the function $T$ and note that $x=m+p$, we have

$$
T(m-k)=\frac{(p+1)_m(p)_m}{(p+\frac{5}{6})_m(p+\frac{1}{6})_m}\frac{(p+\frac{5}{6})_k(p+\frac{1}{6})_k}{(p+1)_k(p)_k}.
$$

Therefore,

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},m+p\\1,m+p+1\end{matrix}\,\middle|\,1\right]&=T(m)\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p\\1,p+1\end{matrix}\,\middle|\,1\right]+\frac{1}{2\pi}\sum_{k=0}^{m-1}\frac{T(k+1)}{x-k-1}\\
&=T(m)\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p\\1,p+1\end{matrix}\,\middle|\,1\right]+\frac{1}{2\pi}\sum_{\ell=0}^{m-1}\frac{T(m-\ell)}{p+\ell}\\
&=\frac{(p+1)_m(p)_m}{(p+\frac{5}{6})_m(p+\frac{1}{6})_m}\times\left\{\begin{matrix}{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p\\1,p+1\end{matrix}\,\middle|\,1\right]\\+\frac{1}{2\pi}\sum_{\ell=0}^{m-1}\frac{(p+\frac{5}{6})_\ell(p+\frac{1}{6})_\ell}{(p+\ell)(p+1)_\ell(p)_\ell}\end{matrix}\right\}.
\end{aligned}
$$

Our result is followed by the fact $p(p+1)_m=(p)_m(p+m)$. □

Followed by using the similar method to the recurrence relation in Lemma 1, we have the explicit formula for $F(p-m)$:

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p-m\\1,p+1-m\end{matrix}\,\middle|\,1\right]\\
&=\frac{(p-m+\frac{1}{6})_m(p-m+\frac{5}{6})_m}{(p-m)_m(p-m+1)_m}\times\left\{\begin{matrix}{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p\\1,p+1\end{matrix}\,\middle|\,1\right]\\-\frac{1}{2\pi}\sum_{k=1}^{m}\frac{(p-k)_k(p-k+1)_k}{(p-k+\frac{1}{6})_k(p-k+\frac{5}{6})_k(p-k)}\end{matrix}\right\}.
\end{aligned}
$$

Furthermore, we use the Pochhammer symbols at the negative integer index $(a)_{-k}$, which is defined by

$$(a)_{-k}=\frac{\Gamma(a-k)}{\Gamma(a)}=\frac{1}{(a-k)(a-k+1)\cdots(a-1)}=\frac{1}{(a-k)_k}.$$

Then, the explicit formula $F(p-m)$ is symmetry to the formula $F(p+m)$.

**Theorem 2.** *Let $m$ be a non-negative integer, $p$ be a non-zero complex number such that $p+1-m$ is not a non-positive integer. Then*

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p-m\\1,p+1-m\end{matrix}\,\middle|\,1\right]\\&=\frac{(p-m)\cdot(p)^2_{-m}}{\left(\frac{1}{6}+p\right)_{-m}\left(\frac{5}{6}+p\right)_{-m}}\left\{\frac{1}{p}{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},p\\1,p+1\end{matrix}\,\middle|\,1\right]-\frac{1}{2\pi}\sum_{k=1}^{m}\frac{\left(\frac{1}{6}+p\right)_{-k}\left(\frac{5}{6}+p\right)_{-k}}{(p-k)^2(p)^2_{-k}}\right\},\end{aligned}$$

*where $(a)_{-k}$ is defined by*

$$(a)_{-k}=\frac{\Gamma(a-k)}{\Gamma(a)}=\frac{1}{(a-k)(a-k+1)\cdots(a-1)}=\frac{1}{(a-k)_k}.$$

In the end of this section we give the explicit formulas $F(q+n)$ with $q=1/2,1/3,2/3,1/4,3/4$, and $n\in\mathbb{Z}$.

**Proposition 1.** *For any non-negative integer $n$, we have*

$${}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{2}+n\\1,\frac{3}{2}+n\end{matrix}\,\middle|\,1\right]=\frac{(2n+1)}{2\pi(3n+1)}\frac{27^n}{16^n}\frac{\binom{2n}{n}}{\binom{3n}{2n}}\times\left\{\begin{matrix}3\sqrt{3}\log(2+\sqrt{3})\\+2\sum_{k=0}^{n-1}\frac{(3k+1)}{(2k+1)^2}\frac{16^k}{27^k}\frac{\binom{3k}{2k}}{\binom{2k}{k}}\end{matrix}\right\},\tag{17}$$

$${}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},-\frac{1}{2}-n\\1,\frac{1}{2}-n\end{matrix}\,\middle|\,1\right]=\frac{(3n+1)}{3\pi(2n+1)}\frac{16^n}{27^n}\frac{\binom{3n}{2n}}{\binom{2n}{n}}\times\left\{\begin{matrix}2\sqrt{3}\log(2+\sqrt{3})\\+3\sum_{k=0}^{n}\frac{27^k\binom{2k}{k}}{16^k\binom{3k}{2k}(3k+1)}\end{matrix}\right\},\tag{18}$$

$${}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{3}+n\\1,\frac{4}{3}+n\end{matrix}\,\middle|\,1\right]=\frac{(3n+1)(\frac{1}{3})_n^2}{2\pi(\frac{1}{2})_n(\frac{7}{6})_n}\times\left\{\begin{matrix}\sqrt{3}\sqrt[3]{2}A-2\sqrt[3]{2}B\\+3\sum_{k=0}^{n-1}\frac{(\frac{1}{2})_k(\frac{7}{6})_k}{(3k+1)^2(\frac{1}{3})_k^2}\end{matrix}\right\},\tag{19}$$

$${}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},-\frac{1}{3}-n\\1,\frac{2}{3}-n\end{matrix}\,\middle|\,1\right]=\frac{(3n+1)(\frac{-1}{3})_{-n}^2}{8\pi(\frac{-1}{6})_{-n}(\frac{1}{2})_{-n}}\times\left\{\begin{matrix}\sqrt[3]{4}\sqrt{3}A+2\sqrt[3]{4}B\\+12\sum_{k=0}^{n}\frac{(\frac{-1}{6})_{-k}(\frac{1}{2})_{-k}}{(3k+1)^2(\frac{-1}{3})_{-k}^2}\end{matrix}\right\},\tag{20}$$

$${}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{2}{3}+n\\1,\frac{5}{3}+n\end{matrix}\,\middle|\,1\right]=\frac{(3n+2)\left(\frac{2}{3}\right)_n^2}{6\pi\left(\frac{5}{6}\right)_n\left(\frac{3}{2}\right)_n}\times\left\{\begin{matrix}\sqrt{3}\sqrt[3]{4}A+2\sqrt[3]{4}B\\+9\sum_{k=0}^{n-1}\frac{\left(\frac{5}{6}\right)_k\left(\frac{3}{2}\right)_k}{(3k+2)^2\left(\frac{2}{3}\right)_k^2}\end{matrix}\right\},\tag{21}$$

$${}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},-\frac{2}{3}-n\\1,\frac{1}{3}-n\end{matrix}\,\middle|\,1\right]=\frac{(3n+2)(\frac{-2}{3})_{-n}^2}{32\pi(\frac{-1}{2})_{-n}(\frac{1}{6})_{-n}}\times\left\{\begin{matrix}3\sqrt[3]{2}\sqrt{3}A-6\sqrt[3]{2}B\\+48\sum_{k=0}^{n}\frac{(\frac{-1}{2})_{-k}(\frac{1}{6})_{-k}}{(3k+2)^2(\frac{-2}{3})_{-k}^2}\end{matrix}\right\},\tag{22}$$

$${}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{1}{4}+n\\1,\frac{5}{4}+n\end{matrix}\,\middle|\,1\right]=\frac{(4n+1)(\frac{1}{4})_n^2}{\pi(\frac{5}{12})_n(\frac{13}{12})_n}\times\left\{\begin{matrix}\sqrt{2}\sqrt[4]{27}(C-D)\\+2\sum_{k=0}^{n-1}\frac{(\frac{5}{12})_k(\frac{13}{12})_k}{(4k+1)^2(\frac{1}{4})_k^2}\end{matrix}\right\},\tag{23}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},-\frac{1}{4}-n\\ 1,\frac{3}{4}-n\end{matrix}\middle|\,1\right]=\frac{(4n+1)(\frac{-1}{4})^2_{-n}}{3\pi(\frac{-1}{12})_{-n}(\frac{7}{12})_{-n}}\times\left\{\begin{matrix}\sqrt{2}\sqrt[4]{3}(C+D)\\ +6\sum_{k=0}^{n}\frac{(\frac{-1}{12})_{-k}(\frac{7}{12})_{-k}}{(4k+1)^2(\frac{-1}{4})^2_{-k}}\end{matrix}\right\}, \tag{24}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{3}{4}+n\\ 1,\frac{7}{4}+n\end{matrix}\middle|\,1\right]=\frac{(4n+3)(\frac{3}{4})^2_{n}}{7\pi(\frac{11}{12})_{n}(\frac{19}{12})_{n}}\times\left\{\begin{matrix}3\sqrt{2}\sqrt[4]{3}(C+D)\\ +14\sum_{k=0}^{n-1}\frac{(\frac{11}{12})_{k}(\frac{19}{12})_{k}}{(4k+3)^2(\frac{3}{4})^2_{k}}\end{matrix}\right\}, \tag{25}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},-\frac{3}{4}-n\\ 1,\frac{1}{4}-n\end{matrix}\middle|\,1\right]=\frac{(4n+3)(\frac{-3}{4})^2_{-n}}{81\pi(\frac{-7}{12})_{-n}(\frac{1}{12})_{-n}}\times\left\{\begin{matrix}7\sqrt{2}\sqrt[4]{27}(C-D)\\ +162\sum_{k=0}^{n}\frac{(\frac{-7}{12})_{-k}(\frac{1}{12})_{-k}}{(4k+3)^2(\frac{-3}{4})^2_{-k}}\end{matrix}\right\}, \tag{26}$$

*where $A, B$ are defined in Equation (6) and $C, D$ are defined in Equation (10).*

## 4. Examples

In this section, we list some concrete examples of our results by using Equations (17)–(26) with $n = 1, 2$. First we indicate that our formula also cover the most well-known formula, for $n \in \mathbb{N}$,

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},n\\ 1,n+1\end{matrix}\middle|\,1\right]=\frac{27^n16^n}{2\pi n\binom{6n}{3n}\binom{3n}{2n}}\sum_{k=0}^{n-1}\frac{\binom{6k}{3k}\binom{3k}{2k}}{27^k16^k} \tag{27}$$

with the parameters $p = 1$ and $m = n-1 \in \mathbb{N}_0$ in Theorem 1. This identity can be derived by using the more general formula about ${}_3F_2\left[\begin{matrix}a,b,n\\ c,n+p\end{matrix}\middle|\,1\right]$ in ([13], Equation (1.7)), or ([3], Equation (16)), where $n, p$ are positive integers. It is note that we use the notations $A, B$ defined in Equation (6) and $C, D$ defined in Equation (10) in the following examples.

**Example 1.** *The Cases $p = 1/2, -1/2$*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{3}{2}\\ 1,\frac{5}{2}\end{matrix}\middle|\,1\right]=\frac{54+81\sqrt{3}\log\left(\sqrt{3}+2\right)}{64\pi}, \tag{28}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},-\frac{3}{2}\\ 1,-\frac{1}{2}\end{matrix}\middle|\,1\right]=\frac{123+64\sqrt{3}\log\left(\sqrt{3}+2\right)}{81\pi}, \tag{29}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{5}{2}\\ 1,\frac{7}{2}\end{matrix}\middle|\,1\right]=\frac{2034+2187\sqrt{3}\log\left(\sqrt{3}+2\right)}{1792\pi}, \tag{30}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},-\frac{5}{2}\\ 1,-\frac{3}{2}\end{matrix}\middle|\,1\right]=\frac{19407+8960\sqrt{3}\log\left(\sqrt{3}+2\right)}{10935\pi}. \tag{31}$$

**Example 2.** *The Case $p = 1/3, 2/3$*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{4}{3}\\ 1,\frac{7}{3}\end{matrix}\middle|\,1\right]=\frac{8\sqrt[3]{2}\sqrt{3}A-16\sqrt[3]{2}B+24}{21\pi}, \tag{32}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{5}{3}\\ 1,\frac{8}{3}\end{matrix}\middle|\,1\right]=\frac{8\sqrt[3]{4}\sqrt{3}A+16\sqrt[3]{4}B+18}{27\pi}, \tag{33}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{5}{6},\frac{7}{3}\\ 1,\frac{10}{3}\end{matrix}\middle|\,1\right]=\frac{128\sqrt[3]{2}\sqrt{3}A-256\sqrt[3]{2}B+510}{351\pi}, \tag{34}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{8}{3} \\ 1, \frac{11}{3}\end{matrix} \middle| 1\right] = \frac{1280\sqrt[3]{4}\sqrt{3}A + 2560\sqrt[3]{4}B + 4176}{4455\pi}. \tag{35} $$

**Example 3.** *The Case $p = -1/3, -2/3$*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, -\frac{4}{3} \\ 1, -\frac{1}{3}\end{matrix} \middle| 1\right] = \frac{21\ \sqrt[3]{4}\sqrt{3}\ A + 42\ \sqrt[3]{4}\ B + 300}{128\pi}, \tag{36} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, -\frac{5}{3} \\ 1, -\frac{2}{3}\end{matrix} \middle| 1\right] = \frac{732 + 135\sqrt[3]{2}\sqrt{3}\ A - 270\sqrt[3]{2}\ B}{640\pi}, \tag{37} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, -\frac{7}{3} \\ 1, -\frac{4}{3}\end{matrix} \middle| 1\right] = \frac{38172 + 2457\sqrt[3]{4}\sqrt{3}\ A + 4914\sqrt[3]{4}\ B}{14336\pi}, \tag{38} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, -\frac{8}{3} \\ 1, -\frac{5}{3}\end{matrix} \middle| 1\right] = \frac{27996 + 4455\sqrt[3]{2}\sqrt{3}\ A - 8910\sqrt[3]{2}\ B}{20480\pi}. \tag{39} $$

**Example 4.** *The Cases $p = 1/4$ and $p = 3/4$*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{5}{4} \\ 1, \frac{9}{4}\end{matrix} \middle| 1\right] = \frac{9\sqrt[4]{27}\sqrt{2}(C - D) + 18}{13\pi}, \tag{40} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{7}{4} \\ 1, \frac{11}{4}\end{matrix} \middle| 1\right] = \frac{243\sqrt[4]{3}\sqrt{2}(C + D) + 126}{209\pi}, \tag{41} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{9}{4} \\ 1, \frac{13}{4}\end{matrix} \middle| 1\right] = \frac{3645\sqrt[4]{27}\sqrt{2}(C - D) + 9396}{5525\pi}, \tag{42} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, \frac{11}{4} \\ 1, \frac{15}{4}\end{matrix} \middle| 1\right] = \frac{15309\sqrt[4]{3}\sqrt{2}(C + D) + 11700}{13547\pi}. \tag{43} $$

**Example 5.** *The Cases $p = -1/4$ and $p = -3/4$*

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, -\frac{5}{4} \\ 1, -\frac{1}{4}\end{matrix} \middle| 1\right] = \frac{444 + 65\sqrt{2}\sqrt[4]{3}(C + D)}{135\pi}, \tag{44} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, -\frac{7}{4} \\ 1, -\frac{3}{4}\end{matrix} \middle| 1\right] = \frac{5220 + 1463\sqrt{2}\sqrt[4]{27}(C - D)}{5103\pi}, \tag{45} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, -\frac{9}{4} \\ 1, -\frac{5}{4}\end{matrix} \middle| 1\right] = \frac{8034 + 1105\sqrt{2}\sqrt[4]{3}(C + D)}{2187\pi}, \tag{46} $$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{6}, \frac{5}{6}, -\frac{11}{4} \\ 1, -\frac{7}{4}\end{matrix} \middle| 1\right] = \frac{4364838 + 1043119\sqrt{2}\sqrt[4]{27}(C - D)}{3536379\pi}. \tag{47} $$

## 5. Moments of Ramanujan's Generalized Elliptic Integrals

For $0 \le s < \frac{1}{2}$ and $0 \le k \le 1$, let

$$ K^s(k) = \frac{\pi}{2}\,{}_2F_1\left[\begin{matrix}\frac{1}{2} - s, \frac{1}{2} + s \\ 1\end{matrix} \middle| k^2\right] $$

be Ramanujan's generalized elliptic integral of the first kind of order $s$. The moment $K_{n,s}$ is given by

$$ K_{n,s} = \int_0^1 k^n K^s(k)\, dk, $$

where $n$ is a real number. Borwein et al. ([11], Theorem 2) proved that for $0 \le s < \frac{1}{2}$,

$$K_{n,s} = \frac{\pi}{2(n+1)} {}_3F_2\left[\begin{matrix} \frac{1}{2}-s, \frac{1}{2}+s, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{matrix} \middle| 1\right].$$

Thus, our hypergeometric series $F(x)$ can be got by setting $s = 1/3$, and we have

$$F(x) = {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, x \\ 1, x+1 \end{matrix} \middle| 1\right] = \frac{4x}{\pi} K_{2x-1,\frac{1}{3}}. \tag{48}$$

The following formula is in ([11], Equation (29)).

$${}_3F_2\left[\begin{matrix} a, 1-a, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1\right] = \frac{2\sin(\pi a)}{\pi(1-2a)}\gamma(a) - \frac{1}{1-2a},$$

where

$$\gamma(a) = \frac{1}{2}\left[\Psi(\frac{a+1}{2}) - \Psi(\frac{a}{2})\right],$$

and $\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ is the digamma function. We substitute $a = 1/6$ in the above identity and use the fact

$$\gamma(\frac{1}{6}) = \pi + \sqrt{3}\log(2+\sqrt{3}),$$

we have

$$\begin{aligned} {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1\right] &= \frac{6\sin(\frac{\pi}{6})}{2\pi}\gamma(\frac{1}{6}) - \frac{3}{2} \\ &= \frac{3\sqrt{3}}{2\pi}\log(2+\sqrt{3}). \end{aligned}$$

Therefore, we give another evaluation of Equation (4), which recently was obtained in ([10], Equation (4.1)). In fact, this number is related to the generalized Catalan constant $G_{1/3}$ which was defined in [11].

$${}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \middle| 1\right] = \frac{4}{\pi} G_{1/3}.$$

Borwen et al. ([11], Equation (76)) found a result that followed by Carlson's Theorem:

$$((2r+1)^2 - 4s^2)K_{2r+1,s} - (2r)^2 K_{2r-1,s} = \cos \pi s. \tag{49}$$

Using Equation (48) we transform the above identity into the following

$$F(r) = \frac{(6r+1)(6r+5)}{36r(r+1)}F(r+1) - \frac{1}{2r\pi}.$$

This is exactly the same recurrence relation Equation (8) in Lemma 1. This provides a new approach to our results. Moreover, if we use the formula for odd moments of $K^s$ in ([11], Theorem 3), we could get Equation (27):

$$\begin{aligned} {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{5}{6}, n \\ 1, n+1 \end{matrix} \middle| 1\right] &= \frac{4n}{\pi} K_{2n-1,\frac{1}{3}} \\ &= \frac{4n}{\pi}\left[\frac{1}{8}\frac{(n-1)!^2}{\Gamma(\frac{1}{6}+n)\Gamma(\frac{5}{6}+n)}\sum_{k=0}^{n-1}\frac{\Gamma(\frac{1}{6}+k)\Gamma(\frac{5}{6}+k)}{k!^2}\right] \end{aligned}$$

$$= \frac{27^n 16^n}{2\pi n \binom{6n}{3n}\binom{3n}{2n}} \sum_{k=0}^{n-1} \frac{\binom{6k}{3k}\binom{3k}{2k}}{27^k 16^k}.$$

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