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Degenerate Canonical Forms of Ordinary Second-Order Linear Homogeneous Differential Equations

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Abstract: For each fundamental and widely used ordinary second-order linear homogeneous differential equation of mathematical physics, we derive a family of associated differential equations that share the same "degenerate" canonical form. These equations can be solved easily if the original equation is known to possess analytic solutions, otherwise their properties and the properties of their solutions are de facto known as they are comparable to those already deduced for the fundamental equation. We analyze several particular cases of new families related to some of the famous differential equations applied to physical problems, and the degenerate eigenstates of the radial Schrödinger equation for the hydrogen atom in N dimensions.

Keywords: ordinary differential equations; analytical methods; mathematical models; Riccati equation; radial Schrödinger equation; transformations

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1. Introduction

The ordinary second-order linear homogeneous (OSLH) differential equations of mathematical physics have the general form [1–4]

$$y_0'' + b_0(x)y_0' + c_0(x)y_0 = 0\,, \tag{1}$$

where primes denote derivatives with respect to the independent variable x and $b_0(x)$ and $c_0(x)$ are functions of x. Equation (1) can be transformed to the canonical form [5–8]

$$u_0'' + q_0(x)u_0 = 0\,, \tag{2}$$

where

$$q_0 \equiv c_0 - \frac{1}{4}\left(2b_0' + b_0^2\right), \tag{3}$$

and then the solutions $y_0(x)$ are given by

$$y_0(x) = u_0(x)\exp\left(-\frac{1}{2}\int b_0(x)dx\right). \tag{4}$$

Equation (2) is degenerate in the sense that it can also be obtained from another equation of the form

$$y'' + b(x)y' + c(x)y = 0\,, \tag{5}$$

in which the functions $b \neq b_0$ and c obey the condition that

$$q \equiv c - \frac{1}{4}\left(2b' + b^2\right) = q_0\,, \tag{6}$$

and then the solutions $y(x)$ of Equation (5) are given by

$$y(x) = u_0(x)\exp\left(-\frac{1}{2}\int b(x)dx\right). \tag{7}$$

Therefore, the original transformation $(b_0, c_0) \to q_0$ is not uniquely invertible as there exist an infinite number of function pairs $(b,\, c)$ that result in the same q_0 coefficient in Equation (2). The solutions $y_0(x)$ and $y(x)$ of the two differential equations still differ in their exponential factors, but the $u_0(x)$ function is the same in Equations (4) and (7) and generally ascribes similar qualitative properties to the solutions.

The degeneracy of the canonical form (2) effectively provides a new method of solution or at least of investigation of an enormous number of potentially useful OSLH differential equations. In what follows, we determine some of these families of associated equations that may prove to be of current or future interest in applied mathematics and in physics applications. In Section 2, we describe the general theory and some notable special cases derived from degenerate canonical forms. In Sections 3 and 4, we analyze specific examples of such families with closely related properties and solutions. In particular, we revisit 15 fundamental OSLH equations of mathematical physics listed in [3] and the degeneracies of the radial Schrödinger equation across $N \geq 1$ spatial dimensions. In Section 5, we summarize and discuss our results.

2. Exploiting the Degeneracy of the Canonical Form

We consider Equations (1) and (5) with $b \neq b_0$ and/or $c \neq c_0$ leading to the same canonical form (2) with coefficient $q_0(x)$. Ibragimov [5] calls $q_0(x)$ the invariant function and the associated equations equivalent by function (his Theorem 3.3.2, page 112) in the Lie symmetry group of second-order linear equations [6], but he does not pursue the classification further, as we do. We assume that the solutions (or at least their properties) are known for Equation (1) and we determine all other OSLH equations of the form (5) that are closely related due to the appearance of the same $u_0(x)$ function in their solutions (7). Combining Equations (3) and (6), we find that

$$\left(2b' + b^2\right) - \left(2b_0' + b_0^2\right) = 4(c - c_0)\,. \tag{8}$$

The coefficients $b_0(x)$ and $c_0(x)$ are known functions of x, whereas $b(x)$ and $c(x)$ are generally unknown functions to be determined. If $b = b_0$, then $c = c_0$ also, in which case there is no family of associated equations. If $c = c_0$, then $b = b_0$ is only a particular solution of Equation (8). We examine this case in Section 2.1, two special cases with $c \neq c_0$ in Section 2.2, and the general case for arbitrary $b(x)$ and $c(x)$ in Section 2.3 below.

Written as a Riccati equation for $b(x)$, Equation (8) takes the form

$$b' = 2(c - q_0) - \frac{1}{2}b^2\,, \tag{9}$$

where q_0 is known by virtue of Equation (3). A given $c(x)$ and the general solution $b(x)$ of the Riccati equation determine together a family of coefficients for the associated Equation (5); some examples of important differential equations from mathematical physics with $c = c_0$ are analyzed in Section 3 below. Furthermore, two chosen functions $b(x)$ and $c(x)$ such that they satisfy Equation (9) identically (i.e., $q \equiv q_0$) produce additional (and generally more complicated) members of the same family; a physically interesting problem from multidimensional quantum mechanics is analyzed in Section 4 below.

2.1. The Case for $b(x)$ When $c = c_0$

When the Riccati Equation (9) is solved to obtain $b(x)$, a particular solution $b_P(x)$ is needed [7,8]. In the case with $c = c_0$, we already know that $b_P = b_0$. In this case:

Theorem 1. *The general solution of Equation (9) is given by*

$$b = b_0 + \frac{1}{z}\,, \tag{10}$$

where $z(x)$ is the general solution of the linear differential equation

$$z' - b_0(x)z = \frac{1}{2}\,. \tag{11}$$

Proof. See Procedure 2 in page 392 of [8]. □

This result appears to be important for physics applications using equations of the form (1) with predetermined coefficients $b_0(x)$ and $c_0(x)$. It shows that when the new term $1/z(x)$ is added to the coefficient $b_0(x)$ of the first derivative (Equation (10)), the complexity of the mathematical problem does not increase at all; and the new problem remains just as mathematically tractable as the original problem since the two equations share the exact same canonical form (Equation (2)).

2.2. Additional Riccati Cases with Particular Solutions $b_P = b_0$

(a) For $c = Kb$ and $c_0 = Kb_0$, where K is a constant, the Riccati Equation (9) takes the form

$$b' = -2q_0(x) + 2Kb - \frac{1}{2}b^2\,, \tag{12}$$

for which $b_P = b_0$ is a particular solution. Then Equation (10) is the general solution, where $z(x)$ is the general solution of the linear equation

$$z' + [2K - b_0(x)]z = \frac{1}{2}\,. \tag{13}$$

Example 1. *In the special case with $b_0 = c_0 = 0$, the method generates a family of damped harmonic oscillators (associated with the basic equation $y_0'' = 0$ [5]) whose simplest member has constant coefficients $b = 4K$ and $c = 4K^2$ in Equation (5).*

(b) For $c = Kb^2$ and $c_0 = Kb_0^2$, where K is a constant, the Riccati Equation (9) takes the form

$$b' = -2q_0(x) + \frac{1}{2}(4K - 1)b^2\,, \tag{14}$$

for which $b_P = b_0$ is a particular solution. Then Equation (10) is again the general solution, where $z(x)$ is the general solution of the linear equation

$$z' + (4K - 1)b_0(x)z = \frac{1}{2}(1 - 4K)\,. \tag{15}$$

Example 2. *In the special case with $K = 1/4$, then $z = 1/C =$ constant, and the known function $b_0(x)$ is shifted vertically in order to produce the family of associated coefficients, i.e., $b = b_0 + C$ and $c = \frac{1}{4}(b_0 + C)^2$, in Equation (5).*

Example 3. *On the other hand, for $K \neq 1/4$ and for $b_0 = c_0 = 0$, the method generates a family of Cauchy–Euler equations (associated with $y_0'' = 0$ [5]) whose simplest member has coefficients $b = B_0/x$ and $c = K(B_0/x)^2$ in Equation (5), where $B_0 = 2/(1 - 4K) =$ constant.*

By comparing the associated families in Examples 1 and 3 above, we see how complexity is being built up into the coefficients of the general OSLH form (5), starting merely from the simplest possible OSLH equation $y_0'' = 0$; but without causing any serious difficulties to the investigations of properties or solutions of the associated equations (see also related examples in [5], pages 112 and 114).

2.3. The General Case for $b(x)$ and $c(x)$

2.3.1. Solving a Riccati Equation

For arbitrary coefficients $c(x)$ and $c_0(x)$ (not related to b and b_0, respectively), Equation (8) or (9) can be written as a Riccati equation without a linear b-term, viz.

$$b' = p(x) - \frac{1}{2}b^2\,, \tag{16}$$

where

$$p \equiv 2(c - q_0) = 2(c - c_0) + b_0' + \frac{1}{2}b_0^2\,, \tag{17}$$

is a function of x with no particular dependencies among the functions involved or any special symmetries. This function does not appear explicitly in the calculations that follow, but it does affect the determination of the sought-after particular solution. The general solution of Equation (16) from Theorem 1 is

$$b = b_P + \frac{1}{z}\,, \tag{18}$$

where $b_P(x)$ is a particular solution and $z(x)$ is the general solution of the linear equation

$$z' - b_P(x)z = \frac{1}{2}\,. \tag{19}$$

The particular solution $b_P(x)$ cannot be specified in general terms. Its form will depend on the details of the given fundamental differential Equation (1) and on the coefficient $c(x)$ that will be chosen for the family of the associated Equation (5).

2.3.2. Solving a Canonical Equation

If a particular solution $b_P(x)$ cannot be found, then there is one more transformation that one can try ([8], Section 86, page 392):

Theorem 2. *Equation (16) can be recast as an OSLH equation in canonical form (since there is no linear b-term, and the coefficient of b^2 is a constant), viz.*

$$v'' - \frac{1}{2}p(x)v = 0\,, \tag{20}$$

where p is given by Equation (17), and $b(x)$ will then be determined from the general solution $v(x)$, viz.

$$b = \frac{2v'}{v}. \tag{21}$$

Proof. See Procedure 1 in page 392 of [8]. □

It is important that this $b(x)$ coefficient will finally contain only one arbitrary constant, just as the solution (18). The two integration constants in the solution of Equation (20) will always combine into one constant in Equation (21), thus the solutions (18) and (21) are equivalent, as shown following Example 4.

Example 4. *An example of such a reduction to one arbitrary constant is provided by the simplest case with* $p = 0$*. In this case,* $v(x)$ *is a linear function of* x*, i.e.,* $v = C_1 x + C_2$*, where* C_1 *and* C_2 *are the integration constants, and then Equation (21) gives*

$$b = \frac{2C_1}{C_1 x + C_2} = \frac{2}{x + C}\,, \tag{22}$$

where $C \equiv C_2/C_1$*. Thus, Equation (21) produces a function* $b(x)$ *that depends on only one arbitrary constant* C*.*

In the general case, $v = C_1 v_1 + C_2 v_2$, where $v_1(x)$ and $v_2(x)$ are two nontrivial linearly-indepenent particular solutions of Equation (20). Then Equation (21) gives

$$b = \frac{2\left(C_1 v_1' + C_2 v_2'\right)}{C_1 v_1 + C_2 v_2} = \frac{2\left(v_1' + C v_2'\right)}{v_1 + C v_2}, \tag{23}$$

where, again, $C \equiv C_2/C_1$. In this case as well, the determined $b(x)$ coefficient depends on only one arbitrary constant C.

Example 5. *A simple choice that results in complicated associated equations is* $b_0 = 0$ *and* $c - c_0 = x \geq 0$*. Then,* $p = 2x$ *from Equation (17), and Equation (16) gives* $b' = 2x - b^2/2$*, a Riccati equation for which a particular solution* b_P *cannot be readily found. Thus, we turn to Equation (20) which takes the form of Airy's differential equation* $v'' - xv = 0$ *with particular solutions* $v_1 = \mathcal{A}i(x)$ *and* $v_2 = \mathcal{B}i(x)$*, where* $\mathcal{A}i$ *and* $\mathcal{B}i$ *are the Airy functions [3]; and the general solution of* $b(x)$ *is then given by Equation (23), where* C *is an arbitrary constant. For* $C = 0$*, the principal solution is* $b = 2(\ln \mathcal{A}i)'$*, which is much more involved as compared to the initial choice of* $b_0 = 0$*.*

3. Families of Associated Differential Equations with $c = c_0$

We analyze several examples of families of associated OSLH differential equations of the form (5) that are closely related to well-known and widely used equations of mathematical physics that take the form of Equation (1). In this section, we limit ourselves to families with

$$c = c_0\,, \tag{24}$$

hence the methodology of Section 2.1 is applicable. The new differential equations have significantly more complicated coefficients $b(x)$ due to the addition of nontrivial terms $1/z$ (see Equation (10)) for which $z(x)$ is determined by solving the first-order linear differential Equation (11).

In physics applications of the standard form (1), the term $b_0 y_0'$ usually represents damping due to friction or other resisting forces [7,9], unless it was created by the specific choice of a curvilinear coordinate system [4], as for example the inertial term y_0'/x in the cylindrical Bessel differential equation [4,10]. The new coefficient $b = b_0 + 1/z$ then generally represents a significantly more sophisticated model of resistance to motion that surprisingly has a similar effect on the dynamics of the physical system as the original simpler damping coefficient b_0 (see Table 1 for a summary). The similarity is not precise however because the solutions (4) and (7) also contain differing exponential factors. The differences in the exponential factors, $\exp(-\int [b(x) - b_0(x)]dx/2)$, are also summarized in Table 1.

Table 1. Exponential factors and $1/z$ terms that appear in the solutions (4) and (7) of the OSLH differential Equations (1) and (5) with $c = c_0$, due to transformations to the canonical form (2).

Differential Equation(s) (1)	**Section (2)**	$\boldsymbol{b_0(x)}$ $\boldsymbol{1/z(x) = b(x) - b_0(x)}$ **(3)**	$\boldsymbol{\exp(-\int b_0(x)dx/2)}$ $\boldsymbol{\exp(-\int [b(x) - b_0(x)]dx/2)}$ **(4)**
Canonical Equations ($b_0 = 0$)	3.1	0 $2/(x + C)$	1 $1/\lvert x + C\rvert$
Damped Harmonic Oscillator	3.2	$2k$ $4k/[C\exp(2kx) - 1]$	$\exp(-kx)$ $1/\lvert \exp(-2kx) - C\rvert$
Cauchy–Euler ($B_0 \neq 1$)	3.3	B_0/x $2(1 - B_0)/\left(x + Cx^{B_0}\right)$	$\lvert x\rvert^{-B_0/2}$ $1/\lvert C + x^{1-B_0}\rvert$
Cauchy–Euler ($B_0 = 1$) and (Modified) Bessel	3.3 3.4	$1/x$ $2/(x\ln\lvert Cx\rvert)$	$1/\sqrt{\lvert x\rvert}$ $1/\lvert\ln\lvert Cx\rvert\rvert$
Legendre and Associated Legendre	3.5	$-2x/(1 - x^2)$ $4/\left[(1 - x^2)(C + \ln[(1 + x)/(1 - x)])\right]$	$(1 - x^2)^{-1/2}$ $1/\lvert C + \ln[(1 + x)/(1 - x)]\rvert$
Chebyshev	3.6	$-x/(1 - x^2)$ $2/[\sqrt{1 - x^2}\left(C + \sin^{-1} x\right)]$	$(1 - x^2)^{-1/4}$ $1/\lvert C + \sin^{-1} x\rvert$
Hermite (Physics)	3.7	$-2x$ $2/[C\exp(-x^2) + \mathcal{D}i(x)]$	$\exp(x^2/2)$ $1/\lvert C + \exp(x^2)\mathcal{D}i(x)\rvert$
Hermite (Probability)	3.7	$-x$ $\sqrt{2}/[C\exp(-x^2/2) + \mathcal{D}i(x/\sqrt{2})]$	$\exp(x^2/4)$ $1/\lvert C + \exp(x^2/2)\mathcal{D}i(x/\sqrt{2})\rvert$
Laguerre	3.8	$(1 - x)/x$ $2\exp(x)/[x(C + \mathcal{E}i(x))]$	$x^{-1/2}\exp(x/2)$ $1/\lvert C + \mathcal{E}i(x)\rvert$
Associated Laguerre	3.8	$(\nu + 1 - x)/x$ $2\exp(x)/[x^{\nu+1}(C + (-1)^{\nu+1}\Gamma(-\nu, -x))]$	$x^{-(\nu+1)/2}\exp(x/2)$ $1/\lvert C + (-1)^{\nu+1}\Gamma(-\nu, -x)\rvert$
3-D Radial Schrödinger (Hydrogen Atom)	3.9	$2/x$ $2/[x(Cx - 1)]$	$1/x$ $x/\lvert Cx - 1\rvert$
3-D Radial Schrödinger (Kummer's Form)	3.9	$2(\ell + 1)/x - 1$ $2\exp(x)/[x^{2(\ell+1)}(C + \Gamma(-2\ell - 1, -x))]$	$x^{-(\ell+1)}\exp(x/2)$ $1/\lvert C + \Gamma(-2\ell - 1, -x)\rvert$
3-D Radial Schrödinger (Whittaker's Form)	3.9	1 $2/[C\exp(x) - 1]$	$\exp(-x/2)$ $1/\lvert\exp(-x) - C\rvert$

Notes: (a) To obtain the coefficient $b(x)$, add the two functions in column (3) in each case. (b) To obtain the factor $\exp(-\int b(x)dx/2)$, multiply the two functions in column (4) in each case. (c) In Sections 3.5 and 3.6, $|x| < 1$. In Sections 3.8 and 3.9, $x > 0$ and $-\nu, \ell \geq 0$ are integers. (d) Whittaker's form with $b_0 = 1 =$ constant is a form of damped harmonic oscillator with $k = \frac{1}{2}$. Definitions (Ref. [3]): (1) Dawson's Integral: $\mathcal{D}i(x) = \int_0^x dt\, \exp(t^2 - x^2)$ (2) Exponential Integral: $\mathcal{E}i(x) = -\int_{-x}^{\infty} dt\, \exp(-t)/t,\ x > 0$ (3) Upper Incomplete Gamma Function: $\Gamma(a, x) = \int_x^{\infty} dt\, t^{a-1}\exp(-t)$.

3.1. Canonical Equations of Physics with $b_0 = 0$

There are quite a few OSLH equations of mathematical physics that lack a first derivative term ($b_0 = 0$ in Equation (1)) [3,4] and their properties and solutions depend only on the single remaining coefficient $c_0(x)$. For such equations, we find that the associated Equation (5) admit nonzero terms of the form $b(x)y'$ that complicate their appearances but not their studies. For $b_0 = 0$ and $c = c_0$, Equation (11) reduces to $z' = 1/2$ and Equation (10) provides a nonzero coefficient $b(x)$ of the form (22) (since $p = 0$ from Equation (17)), viz.

$$b = 2/(x + C)\,, \tag{25}$$

where C is an arbitrary constant. This is not a trivial result. The principal ($C = 0$) particular solution $b = 2/x$ is ubiquitous in physical models [1,4,9] and the degeneracy of the canonical form was first discovered in this case: transformations of equations with $b_0 = 2/x$ to their canonical forms would eliminate the b_0-terms from q_0, thus leading to $q_0 \equiv c_0$ in such models (Equation (3) with $b_0 = 2/x$; see also Section 6.2 in [4]).

3.2. Damped Harmonic Oscillator

The damped harmonic oscillator [7] is described by Equation (1) with $b_0 = 2k =$ constant and $c_0 = \omega_0^2 =$ constant. A family of associated differential equations is obtained from Equations (10) and (11). We find that the family members with $c = c_0$ have coefficients $b(x)$ of the form

$$b = 2k\left[\frac{C\exp(2kx) + 1}{C\exp(2kx) - 1}\right], \tag{26}$$

where C is an arbitrary constant. The result can also be written in terms of hyperbolic functions (Appendix 2 in [11]). It may be surprising that such a complicated damping coefficient can be introduced to the harmonic oscillator, yet the problem remains analytically solvable. We have seen analogous "harmless" complications in the past (hyperbolic tangents in $b(x)$; Equations (56) and (59) in [4]) when we solved analytically the CDOS differential equation [11,12].

3.3. Cauchy–Euler Equation

The Cauchy–Euler equation [2,7] is described by Equation (1) with $b_0 = B_0/x$ and $c_0 = C_0/x^2$, where B_0 and C_0 are constants. We find that its family members with $c = c_0$ have

$$b = \begin{cases} 1/x + 2/(x\ln|Cx|), & \text{for } B_0 = 1 \\ B_0/x + 2(1 - B_0)/\left(x + Cx^{B_0}\right), & \text{for } B_0 \neq 1 \end{cases}, \tag{27}$$

where C is an arbitrary constant. As with the Bessel differential equation [10], the $1/x$ term in the $B_0 = 1$ case does not represent damping if x is a cylindrical radial coordinate [4]. This must be the case for the new term as well, because b and b_0 lead to the same canonical form with $q_0 = (C_0 + 1/4)/x^2$, which implies that $q_0 > 1/(4x^2)$ for $C_0 > 0$; thus, the solutions are oscillatory in $x > 0$ for any positive value of the constant C_0 (see [4] for details).

Example 6. *The cases with $B_0 = 0$ and $B_0 = 2$ are also notable and consistent with the results obtained in Section 3.1 above and in Section 3.9 below, respectively:*

(a) *For $B_0 = 0$ (i.e., $b_0 = 0$), then $b = 2/(x + C)$, a renowned coefficient [1,4,9].*
(b) *For $B_0 = 2$ (i.e., $b_0 = 2/x$), then $b = 2C/(Cx + 1)$, a coefficient that includes the special forms $b = 0$ (for $C = 0$) and $b = 2/(x + C)$ (for $C \to 1/C$).*

It is important to note here that both Cauchy–Euler special cases with $B_0 = 0$ and $B_0 = 2$ include the ubiquitous result that $b = 2/(x + C)$.

3.4. Bessel Equations

The Bessel equation of order n [10,13] is described by Equation (1) with $b_0 = 1/x$ and $c_0 = 1 - n^2/x^2$, where n is a constant. We find that its family members with $c = c_0$ have

$$b = 1/x + 2/(x\ln|Cx|)\,, \tag{28}$$

where C is an arbitrary constant. In this case too, the new coefficient $b(x)$ does not represent damping in a cylindrical coordinate frame (see also equation (77) in [4]).

The modified Bessel equation of order n [10,13] also has $b_0 = 1/x$, but it differs in the form of $c_0 = -(1 + n^2/x^2)$. Members of this family are described by the same coefficient $b(x)$ as that in Equation (28) and they are distinguished from the corresponding Bessel family members only because of their "modified" coefficient $c(x) = -(1 + n^2/x^2)$.

3.5. Legendre Equations

The Legendre ($m = 0$) and associated Legendre ($m \neq 0$) equations [13] are described by Equation (1) with $b_0 = -2x/(1 - x^2)$ and $c_0 = \ell(\ell + 1)/(1 - x^2) - m^2/(1 - x^2)^2$, where $|x| < 1$ and ℓ, m are constants. We find that their family members with $c = c_0$ have

$$b = -2x/(1 - x^2) + 4/\Big[(1 - x^2)(C + \ln[(1 + x)/(1 - x)])\Big]\,, \tag{29}$$

where C is an arbitrary constant other than zero. The condition $C \neq 0$ eliminates a singularity at $x = 0$ where $b(0) = 4/C$.

3.6. Chebyshev Equation

The Chebyshev equation [13] is described by Equation (1) with $b_0 = -x/(1 - x^2)$ and $c_0 = n^2/(1 - x^2)$, where $|x| < 1$ and n is a constant. We find that its family members with $c = c_0$ have

$$b = -x/(1 - x^2) + 2/[\sqrt{1 - x^2}\Big(C + \sin^{-1} x\Big)]\,, \tag{30}$$

where C is an arbitrary constant other than zero. The condition $C \neq 0$ eliminates a singularity at $x = 0$ where $b(0) = 2/C$. The Chebyshev equation and the associated differential equations can all be solved analytically by a transformation to their degenerate canonical form [4,11].

3.7. Hermite Equations

The Hermite differential equation [13] for the so-called $H_\lambda(x)$ polynomials in physics applications is described by Equation (1) with $b_0 = -2x$ and $c_0 = 2\lambda$, where $\lambda \geq 0$ is an integer. We find that its family members with $c = c_0$ have

$$b = -2x + 2/[C \exp(-x^2) + \mathcal{D}i(x)]\,, \tag{31}$$

where $C \neq 0$ is an arbitrary constant and $\mathcal{D}i(x)$ is Dawson's integral [3,14].

In probability applications, the Hermite differential equation for the so-called $He_\lambda(x)$ polynomials is written with $b_0 = -x$ and an integer $c_0 = \lambda \geq 0$ [3]. In this case, we find that family members with $c = c_0$ have

$$b = -x + \sqrt{2}/[C \exp(-x^2/2) + \mathcal{D}i(x/\sqrt{2})]\,, \tag{32}$$

where, again, $C \neq 0$ is an arbitrary constant and $\mathcal{D}i(x/\sqrt{2})$ is Dawson's integral [3,14]. In both of the above $b(x)$ coefficients, the condition that $C \neq 0$ eliminates the singularity at $x = 0$ introduced by $\mathcal{D}i(0) = 0$.

3.8. Laguerre Equations

The Laguerre equation [13] is described by Equation (1) with $b_0 = (1 - x)/x$ and $c_0 = \lambda/x$, where $x > 0$ and $\lambda \geq 0$ is a constant. We find that its family members with $c = c_0$ have

$$b = (1 - x)/x + 2\exp(x)/[x(C + \mathcal{E}i(x))]\,, \tag{33}$$

where C is an arbitrary constant and $\mathcal{E}i(x)$ is the exponential integral [3,11].

The associated Laguerre equation [13] is described by Equation (1) with $b_0 = (\nu + 1 - x)/x$ and $c_0 = \lambda/x$, where $x > 0$ and $\lambda \geq 0,\ \nu$ are real constants. Here we take ν to be a negative integer so that the coefficients $b(x)$ will be real (on the other hand, $\nu = 0$ leads back to Equation (33)). We find that family members with $c = c_0$ have

$$b = (\nu + 1 - x)/x + 2\exp(x)/[x^{\nu+1}(C + (-1)^{\nu+1}\Gamma(-\nu, -x))]\,, \tag{34}$$

where C is an arbitrary constant and $\Gamma(-\nu, -x)$ is the upper incomplete Gamma function [3]. We note that the coefficient $b(x)$ in Equation (34) is not a real function of $x > 0$ if ν is taken to be a real number other than a negative integer or zero.

3.9. Radial Schrödinger Equation in Three Dimensions

The radial Schrödinger equation for the hydrogen atom [1–3,15–17] is described by Equation (1) with $b_0 = 2/x$ and $c_0 = n/x - \ell(\ell+1)/x^2 - 1/4$, where $x > 0$ is a spherical radial coordinate and the integers $n \geq 1$ and $0 \leq \ell \leq n-1$ are the principal and secondary quantum numbers, respectively. It is often written in alternative forms such as in Kummer's form of the confluent hypergeometric equation (Section 67 in [15]) with $b_0 = 2(\ell+1)/x - 1$ and $c_0 = (n - \ell - 1)/x$; and as Whittaker's differential equation (Section 16.1 in [1]) with $b_0 = 1$, $c_0 = (-m^2 + 1/4)/x^2$, and $m = \ell + 1/2$. All three equations share the same canonical form (2) with $q_0 = n/x - \ell(\ell+1)/x^2 - 1/4$ [16].

For $c = c_0$, the above forms produce three distinct families of associated differential equations having $b(x)$ coefficients (Equation (10))

$$b = \frac{2}{x + C}\,, \tag{35}$$

$$b = 2(\ell+1)/x - 1 + 2\exp(x)/[x^{2(\ell+1)}(C + \Gamma(-2\ell-1, -x))]\,, \tag{36}$$

and

$$b = \frac{C\exp(x) + 1}{C\exp(x) - 1}\,, \tag{37}$$

respectively, where C is an arbitrary constant and $\Gamma(-2\ell-1, -x)$ is the upper incomplete Gamma function [3]. The coefficient (35) with $C = 0$ is ubiquitous in mathematical physics [1,4,9]. On the other hand, we find that, as in Equation (34) above with integer $-\nu < 0$, the coefficient (36) here is not a real function of $x > 0$ since $-2\ell - 1 < 0$ in the Gamma function for all quantum numbers $\ell \geq 0$. Finally, $b(x)$ in Equation (37) (derived from the original $b_0 = 1$) corresponds to the associated coefficient (26) of a damped harmonic oscillator derived from an original constant damping of $b_0 = 2k = 1$.

4. Radial Schrödinger Equations in N Dimensions

Here we consider the eigenvalue problem posed by the radial Schrödinger equation in N dimensions with quantum numbers $n \geq 1$ and $0 \leq \ell \leq n-1$ and radial scale $x > 0$. The fundamental N-dimensional equation [17] takes the form (1) with $b_0 = (N-1)/x$ and $c_0 = E^N_{n_r\ell} - V(x) - \ell(\ell + N - 2)/x^2$, where V is the potential and $E^N_{n_r\ell}$ is the discrete spectrum of the eigenvalues with radial quantum numbers $n_r = n - \ell - 1$ such that $0 \leq n_r \leq n-1$.

The corresponding eigenfunctions $\psi(x) \in L^2(\mathbb{R}^+)$, $\psi(0) = 0$, $\psi(x) \to 0$ as $x \to \infty$, and they have n_r radial nodes, not counting the boundary node at $x = 0$. For $N = 3$ and $V = -k/x$, where $k > 0$ is a constant, and with the proper normalization of variables, the main differential equation in [17] reduces to the spherical form discussed at the top of Section 3.9 above and in [16] for the hydrogen atom. In this transformation, the eigenvalues (usually denoted by E_n) are absorbed by the scaling (Section 67 in [15]) and they can be obtained from $E_n = -1/(2n^2)$ in atomic units (Section 3.9.1 in [18]) or, more commonly, from $E_n \simeq -13.6/n^2$ in electron-volts, where $n \geq 1$. (We note that, in the metric system of units [19], 13.6 eV = 2.18×10^{-18} J.)

Our interest in this differential equation stems from the comparison theorems of Hall and Katatbeh [17] who showed that the eigenvalues and the corresponding eigenstates with the same number of radial nodes n_r are related across different dimensions because the associated differential equations share effectively the same canonical form. Using our formulation, we recover and extend their Theorem 2 that quantifies the degeneracies between eigenvalues across dimensions N and $M \neq N$ (and within the 1-dimensional case itself) for the same potential function $V(x)$ and with quantum numbers n_r, ℓ and n_r, ℓ', respectively. We note that, although n_r is taken to be the same in degenerate eigenstates, their principal quantum numbers may still differ since n depends also on ℓ [15], viz.

$$n \equiv n_r + \ell + 1 \quad (0 \leq n_r, \ell \leq n-1)\,. \tag{38}$$

On the other hand, it is the number of radial nodes that determines the number of oscillations in the corresponding eigenfunctions, causing thus the appearance of similar qualitative characteristics in the degenerate eigenstates [16].

4.1. The Case with $\ell' = 0$

For the given b_0 and c_0 functions, the coefficient of the N-dimensional canonical form (2) is

$$q_0 = E^N_{n_r\ell} - V(x) - \frac{1}{4x^2}[(N-1)(N-3) + 4\ell(\ell+N-2)]\,. \tag{39}$$

Degeneracy occurs between these eigenstates with discrete eigenvalues $E^N_{n_r\ell}$ and the families of the corresponding eigenstates in M dimensions with eigenvalues $E^M_{n_r0}$ and canonical coefficients

$$q = E^M_{n_r0} - V(x) - \frac{1}{4x^2}(M-1)(M-3)\,, \tag{40}$$

in which the secondary quantum number is $\ell' = 0$ [17]. The condition $q = q_0$ then results in two intersecting sets of degenerate solutions with eigenvalues $E^M_{n_r0} = E^N_{n_r\ell}$: (i) $M = N + 2\ell$ and (ii) $M = 4 - (N + 2\ell)$. Set (ii) is finite (since $M \geq 1$ requires that $N + 2\ell \leq 3$ and $M + N \geq 2$ requires that $\ell \leq 1$) and its elements are also contained in set (i), except for one particular solution: $M = 1$ for $N = 3$ and $\ell = 0$. This solution indicates that in three-dimensional and in one-dimensional spaces, the corresponding coefficients $b_0 = 2/x$ and $b_0 = 0$ result in the same canonical form for $\ell = \ell' = 0$. This occurs because the s-orbitals effectively respond to the same radial potential $V(x)$ in one and three dimensions. The same property does not extend to the s-orbitals in two dimensions because the electron sees a different effective potential, $V(x) - 1/(4x^2)$, when we restrict its motion to be on a plane (Equation (39) with $N = 2$ and $\ell = 0$).

We note that Equations (39) and (40) allow for more sets of solutions with $\ell = 2 - N$ and/or $\ell' = 2 - M$ for $1 \leq M, N \leq 2$. These sets are finite and their solutions are included in the fundamental set (i). We conclude that in the N-dimensional radial Schrödinger equation, given an eigenstate with eigenvalue $E^N_{n_r\ell}$ for potential $V(x)$, the degenerate eigenstates with $\ell' = 0$ are described by the conditions

$$E^{N+2\ell}_{n_r0} = E^N_{n_r\ell} \quad \text{and} \quad E^1_{n_r0} = E^3_{n_r0}\,, \tag{41}$$

where the integers $N \geq 1$, $0 \leq n_r \leq n-1$, and $1 \leq \ell \leq n-1$ (because using $\ell = 0$ for s-orbitals in the first condition leads to a tautology).

4.2. The Case with $c = c_0$

For $c = E^M_{n_r\ell'} - V(x) - \ell'(\ell' + M - 2)/x^2 = c_0$, the inferred equation

$$\ell'(\ell' + M - 2) = \ell(\ell + N - 2)\,, \tag{42}$$

can be rewritten in the convenient form

$$\big[(M+N-4) + 2(\ell'+\ell)\big]\big[(M-N) + 2(\ell'-\ell)\big] = (M+N-4)(M-N)\,, \tag{43}$$

which has three nontrivial solution sets with degenerate eigenvalues $E^M_{n_r\ell'} = E^N_{n_r\ell}$ in the case $M + N = 4$: (iii) $M = 1$ for $N = 3$, $\ell' = \ell + 1$ (where $\ell' \geq 1$); (iv) $M = 3$ for $N = 1$, $\ell' = \ell - 1$ (where $\ell \geq 1$); and (v) $\ell' + \ell = 0$ and $M \neq N$ (which is identical to the second condition in Equation (41) obtained in Section 4.1). Sets (iii) and (iv) are equivalent, thus the degenerate eigenstates with $c = c_0$, $M + N = 4$, and $\ell' + \ell > 0$ are described by the condition

$$E^1_{n_r\ell+1} = E^3_{n_r\ell} \quad (0 \leq n_r, \ell \leq n - 1)\,, \tag{44}$$

that associates the $(\ell + 1)$-orbitals in 1 dimension with the corresponding ℓ-orbitals in three dimensions and the same number of radial nodes.

Equation (42) also has a solution set (vi) for $M = N = 1$: In one dimension, we find that $\ell' + \ell = 1$, which gives the degeneracy condition

$$E^1_{n_r1} = E^1_{n_r0} \quad (0 \leq n_r \leq n - 1)\,. \tag{45}$$

This condition shows that the s- and p-orbitals are degenerate in one dimension, if they have the same number of radial nodes n_r (i.e., if their principal quantum numbers differ by 1). Finally, combining Equations (44) and (45) and with $\ell = 0$, we infer the second condition in Equation (41) which also results from set (v) above.

4.3. The General Case for Any $c(x)$ Function

For the same potential function $V(x)$, the degeneracy condition $q = q_0$ in the general case takes the form

$$(M-1)(M-3) + 4\ell'(\ell' + M - 2) = (N-1)(N-3) + 4\ell(\ell + N - 2)\,, \tag{46}$$

which can be recast as a quadratic equation for $M + 2\ell'$ in terms of $N + 2\ell$, viz.

$$\left(M + 2\ell' - 2\right)^2 = (N + 2\ell - 2)^2\,, \tag{47}$$

that has two sets of solutions: (I) $M + 2\ell' = N + 2\ell$ and (II) $M + 2\ell' = 4 - (N + 2\ell)$.

The two sets are intesecting and the combinations $(M \pm N)$ are even integers in all solutions, just as in the subsets of solutions with $\ell' = 0$ studied in Section 4.1. Similarly here, set (II) is finite and small in size since its solutions are valid only for $2 \leq M + N \leq 4$ and $0 \leq \ell + \ell' \leq 1$. Sets (I) and (II) include all special cases found in Sections 4.1 and 4.2 above:

(a) From set (II) and for $M + N = 2$, we recover the solution set (45);
(b) whereas for $M + N = 4$ in set (II), we recover the second condition (41).
(c) Finally, condition (44) is recovered here from set (I) for $N - M = 2$;
(d) and the first condition (41) is recovered also from set (I) for $\ell' = 0$.

5. Summary and Discussion

For OSLH differential equations of the form (1), we have determined entire families of associated differential Equation (5) of the same form, but with generally different coefficients $b(x)$ and/or $c(x)$, that exhibit comparable qualitative properties in their solutions. All such equations belonging to the same family share the same canonical form (see Equations (2) and (6)) and their general solutions $y(x)$ differ only by the introduction of exponential factors in Equations (4) and (7), such as those listed in the $\exp(-\int[b(x) - b_0(x)]dx/2)$ entries of the summarizing Table 1. Given an original well-studied and widely used differential equation, the methods for determining associated equations with comparable qualitative properties were described in Section 2, and several examples known from physics applications were analyzed in Section 3 ($c = c_0$; see also Table 1) and Section 4 (generally $b \neq b_0$ and $c \neq c_0$ in $L^2(\mathbb{R}^+)$ Hilbert spaces with different spatial dimensions).

Although one may generally create arbitrarily complicated differential equations (as in Sections 2.2 and 2.3), we focused here on the "tip of the iceberg," that is, on the multidimensional radial Schrödinger equations of quantum mechanics (Section 4), as well as on other physically-important OSLH differential equations (Section 3) in which the y-coefficients of Equations (1) and (5) remain the same (Equation (24)) within each family of associated equations. In the latter case, the transformations of coefficients $b_0(x) \to b(x)$ that we carried out are not iterative: If the derived function $b(x)$ is used in place of the original $b_0(x)$, then the new derived function is equivalent to the input $b(x)$, that is, repeated transformations produce the sequence $b_0 \to b \to b$ and only one general solution $b(x)$.

Example 7. *For instance, in the canonical case with $b_0 = 0$ (Section 3.1):*

$$b_0 = 0 \;\to\; b = 2/(x + C) \;\to\; b = 2/(x + \overline{C});$$

and similarly in the Bessel case with $b_0 = 1/x$ (Section 3.4):

$$b_0 = 1/x \;\to\; b = 1/x + 2/(x \ln |Cx|) \;\to\; b = 1/x + 2/(x \ln |\overline{C}x|),$$

where C and $\overline{C}$ are arbitrary constants.

This property arises from the method of solution of the Riccati Equation (9). For any choice of the arbitrary constant $C = C_1$, $b(x)$ becomes a particular solution and if it is used in place of $b_0(x)$ in Equation (11), then this equation will produce the same general solution (10) for $b(x)$ that will contain yet another arbitrary constant $\overline{C}$ which absorbs both C_1 and the new integration constant C_2. In particular, in the two cases of Example 7, we have $\overline{C} = C_1 + C_2$ and $\overline{C} = C_1 C_2$, respectively.

The results listed in Table 1 indicate that $b \to b_0$ as $C \to \pm\infty$, and then the listed $\exp(-\int [b(x) - b_0(x)]dx/2)$ entries are not applicable; as $b \to b_0$, these exponential factors tend to 1. On the other hand, for $C = 0$, the principal solutions $b(x)$ are described mostly by elementary functions and by three notable special functions (Dawson's integral $\mathcal{D}i(x)$, the exponential integral $\mathcal{E}i(x)$, and the upper incomplete Gamma function $\Gamma(a, x)$; their standard definitions are given in [3] and in the notes to Table 1). Because of their appearance in the corresponding families of associated differential equations, these special functions have just grown somewhat in importance to mathematical physics. Of the three special functions appearing in Table 1, $\mathcal{E}i(x)$ and $\Gamma(-\nu, -x)$ (for $x > 0$ and $-\nu > 0$ an even integer) contain singular points other than the familiar $x = 0$ in the coefficients $b(x)$ of the (associated) Laguerre equation for $C = 0$ (Equations (33) and (34), respectively, in Section 3.8). In particular, the only root of $\mathcal{E}i(x) = 0$ is $x \approx 0.372507$ and it lies in the domain $x > 0$ of the Laguerre equation; and the root of $\Gamma(2, -x) = 0$ is $x = 1$ and it lies in the domain $x > 0$ of the associated Laguerre equation; similarly, the real roots of $\Gamma(-\nu, -x) = 0$ for $-\nu = 4, 6, 8, 10$ are $x \approx 1.596072, 2.180607, 2.759003, 3.333551$, respectively.

The coefficients $b(x)$ derived from Equation (10) for $c = c_0$ and listed in Table 1 (one has to add up the two b-entries in each case) generally describe damping of motion due to friction ([20], Section 3.4, page 172), or air resistance ([20], Section 2.3, page 93), or other dissipative processes (e.g., [21], Section 17.9, page 603) in physics applications (unless the $b(x)y'$ term is inertial created by the curvature of the coordinate system; see [4]). At present, there is no general theory of friction or such resisting forces [22]. Then, these new functions $b(x)$ would potentially represent more complicated and more sophisticated models of resisting forces acting on the corresponding dynamical systems. Despite their intimidating look at first sight (owing to the overly complicated $b(x)y'$ terms), the associated differential equations of the various families are quite easily mathematically tractable, provided that the original models involving simpler damping terms of the form $b_0(x)y_0'$ in Equation (1) are already well-studied and their qualitative properties are fully understood.

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Abbreviations

The following abbreviations are used in this manuscript:

OSLH	Ordinary Second-order Linear Homogeneous
CDOS	Chuaqui, Duren, Osgood, Stowe (Ref. [12])
$\mathcal{A}i$, $\mathcal{B}i$	Airy Functions (Ref. [3])
$\mathcal{D}i$	Dawson's Integral (Refs. [3,14])
$\mathcal{E}i$	Exponential Integral (Refs. [3,11])
Γ	Upper Incomplete Gamma Function (Ref. [3])

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