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Comments on the Navier–Stokes Problem

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Abstract: The aim of this paper is to explain for broad audience the author's result concerning the Navier–Stokes problem (NSP) in $\mathbb{R}^3$ without boundaries. It is proved that the NSP is contradictory in the following sense: if one assumes that the initial data $v(x,0) \not\equiv 0$, $\nabla \cdot v(x,0) = 0$ and the solution to the NSP exists for all $t \geq 0$, then one proves that the solution $v(x,t)$ to the NSP has the property $v(x,0) = 0$. This paradox shows that the NSP is not a correct description of the fluid mechanics problem and the NSP does not have a solution. In the exceptional case, when the data are equal to zero, the solution $v(x,t)$ to the NSP exists for all $t \geq 0$ and is equal to zero, $v(x,t) \equiv 0$. Thus, one of the millennium problems is solved.

Keywords: hyper-singular integrals; Navier–Stokes problem

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1. Introduction

The results of this paper are proved in detail in the monograph [1]. In the author's papers, listed in the References (see [2–5]), some preliminary results are obtained. In paper [6] some of the results are summarized. These results are stated in the abstract. The aim of this paper is to explain for broad audience the author's result concerning the Navier–Stokes problem (NSP) in $\mathbb{R}^3$ without boundaries. The result, proven in detail in the book [1], can now be stated:

If the exterior force $f(x,t) = 0$, the initial velocity $v_0(x) := v(x,0) \not\equiv 0$, $\nabla \cdot v_0(x) = 0$ and the solution $v(x,t)$ of the NSP exists for all $t \geq 0$, then $v_0(x) = 0$.

This result, that we call **the NSP paradox** (or just paradox), shows that

The NSP is not a correct statement of the problem of motion of viscous incompressible fluid. The NSP is neither physically nor mathematically correct statement of the dynamics of incompressible viscous fluid.

Let us explain the steps of our proof. The NSP consists of solving the equations:

$$v' + (v,\nabla)v = -\nabla p + \nu\Delta vs. + f, \quad x \in \mathbb{R}^3,\ t \geq 0, \quad \nabla \cdot v = 0, \quad v(x,0) = v_0(x), \tag{1}$$

see, for example, books [1,7]. Here $v = v(x,t)$ is the velocity of incompressible viscous fluid, $p = p(x,t)$ is the pressure, $f = f(x,t)$ is the exterior force, $\nu = const > 0$ is the viscoucity coefficient, $v_0 = v_0(x)$ is the initial velocity, $\nabla \cdot v_0 = 0$. The data v_0 and f are given, the v and p are to be found. The fluid's density $\rho = 1$.

(a) First we reduce the NSP to an equivalent integral equation.

$$v(x,t) = F - \int_0^t ds \int_{\mathbb{R}^3} G(x-y,t-s)(v,\nabla)v dy, \tag{2}$$

where $F = F(x,t)$ depends only on the data f and v_0, see [1]. We assume that $f = 0$. This is done for simplicity only. Under this assumption one has (see [1]):

$$F(x,t) := \int_{\mathbb{R}^3} g(x-y,t)v_0(y)dy, \tag{3}$$

where

$$g(x,t) = \frac{e^{-\frac{|x|^2}{4\nu t}}}{(4\nu\pi t)^{3/2}}, \quad t>0; \quad g(x,t)=0, \quad t\le 0. \tag{4}$$

The tensor $G = G(x,t) = G_{jm}(x,t)$ is calculated explicitly in [1]:

$$G(x,t) = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{i\xi\cdot x}\Big(\delta_{pm} - \frac{\xi_p\xi_m}{\xi^2}\Big)e^{-\nu\xi^2 t}d\xi. \tag{5}$$

Let us define the Fourier transform:

$$\tilde{v} := \tilde{v}(\xi,t) := (2\pi)^{-3}\int_{\mathbb{R}^3} v(x,t)e^{-i\xi\cdot x}dx. \tag{6}$$

Take the Fourier transform of Equation (2) and get the integral equation:

$$\tilde{v}(\xi,t) = \tilde{F}(\xi,t) - \int_0^t ds\tilde{G}(\xi,t-s)\tilde{v}\bigstar(i\xi\tilde{v}), \tag{7}$$

where $\bigstar$ denotes the convolution in $\mathbb{R}^3$. The following inequality, that follows from the Cauchy inequality, is useful:

$$|\tilde{v}\bigstar(i\xi\tilde{v})| \le \|\tilde{v}\|\||\xi|\tilde{v}\|. \tag{8}$$

One proves a priori estimate (see [1]):

$$\sup_{t\ge 0}\|\tilde{v}\| < c. \tag{9}$$

By c here and throughout the paper various positive constants, independent of t, are denoted. We denote by $c_1 := |\Gamma(-\frac{1}{4})| > 0$ the special constant from Equation (27), see below.

Let us prove inequality (9).

We denote $v_{j,m} := \frac{\partial v_j}{\partial x_m}$, $\int := \int_{\mathbb{R}^3}$, write Equation (1) as

$$v_j' + v_m v_{j,m} = f_j - p_{,j} + \nu v_{,jj}, \quad v_{j,j} = 0, \tag{10}$$

where over the repeated indices summation is understood, $1 \le j \le 3$. We assume that $v = v(x,t)$ is real-valued and

$$\|v_0\| + \int_0^\infty \|f(x,t)\|dt < c. \tag{11}$$

Here $\|\cdot\|$ is $L^2(\mathbb{R}^3)$ norm.

Multiply Equation (10) by v_j, integrate over $\mathbb{R}^3$ and sum up over j to get

$$\frac{1}{2}(\|v\|^2)_{,t} \le |(f,v)| \le \|f\|\|v\|, \tag{12}$$

where $z_{,t} := \frac{\partial z}{\partial t}$. In deriving inequality (12) we have used integration by parts: $-\int p_{,j}v_j dx = \int p v_{j,j}dx = 0$, $\int \nu v_{,jj}v_j dx = -\nu\int v_{,j}v_{,j}dx \le 0$ and $\int v_m v_{j,m}v_j dx = -\frac{1}{2}\int v_{m,m}v_j v_j dx = 0$.

From inequality (12) it follows that $\|v\|_{,t} \le \|f\|$. Consequently,

$$\|v\| \le \|v_0\| + \int_0^\infty \|f\|dt.$$

This and our assumption (11) imply estimate $\sup_{t\ge 0}\|v\|$. By Parseval equality the desired estimate (9) follows. Estimate (9) is proved.

Inequalities (8) and (9) imply

$$|\tilde{v}\bigstar(i\xi\tilde{v})| \leq c\||\xi|\tilde{v}\|. \tag{13}$$

Therefore, Equation (7) implies inequality (22), see below.
From (5) it follows that

$$|\tilde{G}(\xi, t-s)| \leq ce^{-\nu(t-s)\xi^2}, \tag{14}$$

because $|\left(\delta_{pm} - \frac{\xi_p\xi_m}{\xi^2}\right)| < c$.

(b) Secondly, we prove that any solution to Equation (7) satisfies integral inequality (15), see below. The integral in this inequality is a convolution with the kernel that is hyper-singular; this integral diverges classically, that is, from the classical point of view. This inequality is derived in Section 2 (see also [1]):

$$b(t) \leq b_0(t) + c\int_0^t (t-s)^{-\frac{5}{4}} b(s)ds, \tag{15}$$

where

$$b_0(t) := \||\xi|\tilde{F}\|, \quad b(t) := \||\xi|\tilde{v}\| \geq 0. \tag{16}$$

The norm here and below is $L^2(\mathbb{R}^3)$ norm. Since the convolution integral in (15) diverges classically, we give a new definition to this integral in Section 3 and estimate the solution $b(t)$ to integral inequality (15) by the solution $q(t)$ to an integral equation with the same hyper-singular kernel:

$$q(t) = b_0(t) + c\int_0^t (t-s)^{-\frac{5}{4}} q(s)ds. \tag{17}$$

Namely, we prove the following inequality

$$b(t) \leq q(t). \tag{18}$$

The term $b_0(t)$ depends on the data only (on v_0 since we assume $f = 0$) and we may assume that $b_0(t)$ is smooth and rapidly decaying as $t \to \infty$.

(c) We prove a priori estimate

$$\sup_{t\geq 0}(\|\nabla v\| + \|v\|) < c, \tag{19}$$

part of which is inequality (9). One has

$$(2\pi)^{3/2}\|\tilde{v}\| = \|v\|, \quad (2\pi)^3\||\xi|\tilde{v}\|^2 = \|\nabla v\|^2, \tag{20}$$

by the Parseval equality. We prove that Equation (17) has a solution in the space $C(\mathbb{R}_+)$, $\sup_{t\geq 0} q(t) < c$, provided that the data $v_0(x)$ is smooth and rapidly decaying at infinity. Moreover, this solution is unique and

$$q(0) = 0. \tag{21}$$

(d) We prove that any solution $b(t) \geq 0$ of inequality (15) with $b_0(t)$ a smooth rapidly decaying function satisfies inequality (18). Since $q(0) = 0$ and $0 \leq b(t) \leq q(t)$ it follows that $b(0) = 0$. This yields the NSP paradox mentioned at the beginning of this section. Indeed, the initial data $v_0(x) \not\equiv 0$, so $b(0) > 0$ and we prove that $b(0) = 0$.

The NSP paradox impies the conclusions we have made:

The NSP is physically not a correct description of motion of incompressible viscous fluid in $\mathbb{R}^3$ and the NSP does not have a solution on the interval $[0, \infty)$ unless the data are equal to zero. In this case the solution to the NSP does exist on the whole inteval $[0, \infty)$ and is identically equal to zero.

The uniqueness of the solution to NSP is proved in Section 4, see Theorem 3.

2. Derivation of the Integral Inequality

Take the absolute value of both sides of Equation (7), then use inequalities (9) and (14) to get

$$u \le \mu + c\int_0^t e^{-\nu(t-s)\xi^2}\|u\|\||\xi|u\|ds \le \mu + c\int_0^t e^{-\nu(t-s)\xi^2}b(s)ds, \quad b(s) := \||\xi|u\|, \tag{22}$$

where the Parseval formula $(2\pi)^{3/2}\|\tilde{v}\| = \|v\| < c$ was used, and we denoted

$$|\tilde{v}(\xi,t)| := u, \quad |\tilde{F}| := \mu(\xi,t) := \mu. \tag{23}$$

In this paper, by c various constants, independent of t, are denoted. Multiply inequality (22) by $|\xi|$, take the norm $\|\cdot\|$ of both sides of the resulting inequality and get inequality (15). In this calculation one uses the formulas

$$\|e^{-\nu(t-s)\xi^2}\| = \frac{c}{(t-s)^{3/4}}, \quad \||\xi|e^{-\nu(t-s)\xi^2}\| = \frac{c}{(t-s)^{5/4}}, \quad 0 \le s < t, \tag{24}$$

which are easy to derive. The c are different in these formulas.

To study integral Equation (17) and integral inequality (15) we need to define the hyper-singular integral in this equation.

To do this, one needs some auxiliary material.

Let us define the function

$$\Phi_\lambda := \frac{t^{\lambda-1}}{\Gamma(\lambda)}, \tag{25}$$

where $\Gamma(\lambda)$ is the gamma function. Here and throughout $t = t_+$, that is, $t = 0$ for $t < 0$, $t := t$ for $t \ge 0$. It is known (see [8]) that $\Gamma(\lambda)$ is an analytic function of $\lambda \in \mathbb{C}$ except for the points $\lambda = 0, -1, -2, ...$, at which it has simple poles. The function $\frac{1}{\Gamma(\lambda)}$ is entire function of λ.

Consider the convolution operator

$$\Phi_\lambda \star b := \int_0^t \Phi_\lambda(t-s)b(s)ds. \tag{26}$$

One has

$$\int_0^t (t-s)^{-\frac{5}{4}}b(s)ds = \Gamma(-\frac{1}{4})\Phi_{-\frac{1}{4}} \star b = -c_1\Phi_{-\frac{1}{4}} \star b,$$

where $\star$ denotes the convolution on $\mathbb{R}_+$ and $c_1 := |\Gamma(-\frac{1}{4})|$. Inequality (15) can be written as

$$b(t) \le b_0(t) - cc_1\Phi_{-\frac{1}{4}} \star b. \tag{27}$$

Consider also the corresponding integral equation:

$$q(t) = b_0(t) - cc_1\Phi_{-\frac{1}{4}} \star q. \tag{28}$$

3. Investigation of Integral Equations and Inequalities with Hyper-Singular Kernel

In this section, we solve Equation (28) analytically and prove estimate (18). First, let us define the hyper-singular integral $\psi := \Phi_\lambda \star q$. We are especially interested in the value $\lambda = -\frac{1}{4}$ because it appears in Equation (28). For $\lambda > 0$ the convolution $\Phi_\lambda \star q$ is defined classically for $q \in L^2(\mathbb{R}_+)$ and one has $L(\psi) = L(q)p^{-\lambda}$, where L is the Laplace transform operator defined as

$$Q(p) := L(q) := \int_0^\infty e^{-pt}q(t)dt, \tag{29}$$

which is analytic in the region Re$p > 0$ if $q \in L^2(\mathbb{R}_+)$. If $q(t)e^{-at} \in L^2(\mathbb{R}_+)$ and $a = const > 0$, then $L(q)$ is an analytic function of p in the region Re$p > a$. The Laplace transform is injective on any domain of its definition. Therefore the inverse operator L^{-1} is well defined on the range of L. The inversion formula is known:

$$q(t) := L^{-1}L(q) := \frac{1}{2\pi i}\int_{C_\sigma} e^{pt}L(q)dp, \tag{30}$$

where C_σ is the straight line $\sigma = const > a$, $p = \sigma + i\omega$, ω changes from $-\infty$ to ∞ and $L(q)$ is a function of p. In Appendix 3 of [1] one finds information on the Laplace transform used in this paper. In particular, the following Lemmas 1–3 will be used. Their proofs can be found in Appendix 3 of [1].

Lemma 1. *If $Q(p)$ is analytic in the region $Rep > 0$ and*

$$|Q(p)| < c(1+|p|)^{-b}, \quad b > 1/2, \quad |p| \gg 1, \quad Re\, p > 0, \tag{31}$$

then $q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}Q(i\omega)d\omega$, $q(t) \in L^2(\mathbb{R}_+)$ and $L(q) = Q(p)$.

In Lemma 1 sufficient conditions are given for a function, analytic in the region $Rep > 0$ to be the Laplace transform of an $L^2(\mathbb{R}_+)$ function.

Lemma 2. *Let the assumptions of Lemma 1 hold with $b > 1$. Then $q(0) = 0$.*

Lemma 3. *One has*

$$L(\Phi_\lambda \star q) = L(q)p^{-\lambda}. \tag{32}$$

Here we have used the known result (see [1] or [9]):

$$L(\Phi_\lambda) = p^{-\lambda}, \tag{33}$$

and the known formula

$$L(\Phi_\lambda \star q) = L(\Phi_\lambda)L(q). \tag{34}$$

For $\lambda > 0$ and q smooth and decaying at infinity this formula can be understood classically. For $\lambda < 0$ it is defined by the analytic continuation with respect to $\lambda \in \mathbb{C}$ where $L(\Phi_\lambda)$ is given in formula (33). Formula (33) is valid for all $\lambda \in \mathbb{C}$ by the analytic continuation from the region Re$\lambda > 0$, where it is valid classically.

Let us define convolution $\psi := \Phi_\lambda \star q$ by the formula:

$$\psi(t) := L^{-1}(L(q)p^{-\lambda}). \tag{35}$$

The expression under the sign L^{-1} is an entire function of λ. For $\lambda > 0$ the $\psi(t)$ is well defined classically if $q \in C(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. The function $L(\psi)$ admits analytic continuation with respect to λ to the whole complex plane $\mathbb{C}$. Therefore, the convolution ψ is defined for all $\lambda \in \mathbb{C}$. We are especially interested in the value $\lambda = -\frac{1}{4}$ because it appears in Equation (27).

To illustrate the argument with analytic continuation, consider a simple example:

$$\int_0^\infty t^{z-1}e^{-pt}dt = \int_0^\infty s^{z-1}e^{-s}ds p^{-z} = \Gamma(z)p^{-z}, \tag{36}$$

where $s = pt$. Formula (36) is valid classically for Re$z > 0$, but remains valid for all $z \in \mathbb{C}$, $z \neq 0, -1, -2, \ldots\ldots$, by the analytic continuation with respect to z because $\Gamma(z)$ is analytic for $z \in \mathbb{C}$, $z \neq 0, -1, -2, \ldots\ldots$, and p^{-z} is an entire function of z. Formula (33) follows from (36) immediately: just divide both sides of (36) by $\Gamma(z)$. The integral (36) diverges classically for Re$z \leq 0$, but formula (36) is valid by analytic continuation for all $z \in \mathbb{C}$

except for $z \neq 0, -1, -2, \ldots \ldots$. In [10], a regularization method is described for defining divergent integrals. By this method one writes

$$\int_0^\infty s^{z-1} e^{-s} ds = \int_0^1 s^{z-1}(e^{-s} - 1 - s) ds + \int_1^\infty s^{z-1} e^{-s} ds + (\frac{s^z}{z} + 0.5\frac{s^{z+1}}{z+1})|_0^1, \quad (37)$$

and uses analytic continuation with respect to z. The third term of the right side of Equation (37) for Re$z > 0$ can be written as $\frac{1}{z} + 0.5\frac{1}{z+1}$. The first integral on the right side of (37) is analytic with respect to z in the region Re$z > -2$, the second integral is also analytic with respect to z in this region and the third term, $\frac{1}{z} + 0.5\frac{1}{z+1}$, admits analytic continuation with respect to z from the region Re$z > 0$ to the complex plane $\mathbb{C}$ except for the points $z = 0$ and $z = -1$. Thus, the right side of Equation (37) admits analytic continuation with respect to z to the region Re$z > -2$, except for the points $z = 0$ and $z = -1$ at which it has simple poles. So, this right side is well defined at $z = -\frac{1}{4}$.

However, the right side of (37) is much less convenient than $\Gamma(z)$, the expression we use. If one deals with the integral $\Phi_\lambda \star q$, then the advantage of our definition, based on the Laplace transform, is even greater because the three terms, analogous to the terms on the right side of Equation (37), will depend on p and on z and there is no separation of z-dependence similar to the one we have in Equation (36). Furthermore, these three terms are not all the Laplace transforms. Consequently, it is wrong to use the regularization procedure from [10] in our problem.

Lemma 4. *One has*

$$\Phi_\lambda \star \Phi_\mu = \Phi_{\lambda+\mu}. \quad (38)$$

for any $\lambda, \mu \in \mathbb{C}$. If $\lambda + \mu = 0$ then

$$\Phi_0(t) = \delta(t), \quad (39)$$

where $\delta(t)$ is the Dirac distribution.

Proof. By formulas (32) and (33) one gets

$$L(\Phi_\lambda \star \Phi_\mu) = \frac{1}{p^{\lambda+\mu}}. \quad (40)$$

By formula (33) one has

$$L^{-1}\Big(\frac{1}{p^{\lambda+\mu}}\Big) = \Phi_{\lambda+\mu}. \quad (41)$$

This proves formula (38).

If $\lambda + \mu = 0$ then

$$p^{-(\lambda+\mu)} = 1, \quad L^{-1}1 = \delta(t). \quad (42)$$

This proves formula (39).

Lemma 4 is proved. □

This proof is taken from [1].

Our plan is to prove that Equation (28) has a solution $q(t) \in C(\mathbb{R}_+)$ provided that $b_0(t)$ is smooth and rapidly decaying as $t \to \infty$. Moreover, this solution is unique in $C(\mathbb{R}_+)$ and $q(0) = 0$. Any solution $b(t) \geq 0$ to inequality (27) satisfies the relation $b(t) \leq q(t)$.

In particular, $b(0) = 0$. This is the NSP paradox because a priori $b(0) \neq 0$.

To realize this plan, let us investigate Equation (28). First, let us apply to (28) the operator $\Phi_{1/4}\star$ and use Lemma 4 to get

$$\Phi_{1/4} \star q = \Phi_{1/4} \star b_0 - cc_1 q. \quad (43)$$

This implies

$$q = c_3(\Phi_{1/4} \star b_0 - \Phi_{1/4} \star q), \quad c_3 := (cc_1)^{-1}, \quad c_3 > 0. \tag{44}$$

Take the Laplace transform of (44) to get

$$L(q) = c_3 L(b_0) p^{-\frac{1}{4}} - c_3 L(q) p^{-\frac{1}{4}}. \tag{45}$$

Therefore

$$L(q) = \frac{c_3 L(b_0)}{p^{1/4} + c_3}. \tag{46}$$

The function $p^{1/4} := |p|e^{i\phi}$ is analytic function of p in the region $-\pi/2 \le \phi \le \pi/2$, where ϕ is the argument of p. One can check that the function $\frac{1}{p^{1/4}+c_3}$, $c_3 > 0$, is an analytic function of p in the region Re$p > 0$ and is bounded in this region. To check this, denote $r := |p|$ and write

$$|re^{i\phi/4} + c_3|^2 = r^2 + 2rc_3 \cos(\phi/4) + c_3^2 \ge c_3^2(1 - \cos^2(\phi/4)) + (r\cos(\phi/4) + c_3)^2 > c > 0.$$

This inequality is valid for all $-\pi/2 \le \phi \le \pi/2$. The function $L(b_0)$ is also analytic in this region. Therefore, the function $L(q)$ in formula (46) is analytic in this region. We assumed that $b_0(t)$ is smooth and rapidly decaying as $t \to \infty$. Thus, $L(b_0)$ is analytic in the region Re$p > 0$ and

$$|L(b_0)| < c(1 + |p|)^{-1}, \quad Re\, p > 0, \quad |p| \gg 1. \tag{47}$$

Therefore, $L(q)$ is analytic in the region Re$p > 0$ and

$$|L(q)| < c(1 + |p|)^{-\frac{5}{4}}, \quad Re\, p > 0, \quad |p| \gg 1. \tag{48}$$

By Lemma 1, the function $L(q)$ is the Laplace transform of the function $q(t) \in C(\mathbb{R}_+)$ and $q(0) = 0$. We have proved the following result.

Theorem 1. *Assume that $v_0(x)$ is smooth and rapidly decaying as $|x| \to \infty$, $f(x,t) = 0$ and $x \in \mathbb{R}^3$. Then estimate (48) holds, Equation (28) is solvable in $C(\mathbb{R}_+)$, its solution $q(t)$ is unique in this space and $q(0) = 0$.*

Let us now prove that $b(t) \le q(t)$, where $b(t) \ge 0$ solves inequality (27).

Theorem 2. *Any solution $b(t) \ge 0$ of inequality (27) satisfies the inequality $b(t) \le q(t)$.*

Proof of Theorem 2 requires the following lemma.

Lemma 5. *The operator $Af := \int_0^t (t-s)^a f(s)ds$ in the space $X := C(0,T)$ for any fixed $T \in [0,\infty)$ and $a > -1$ has spectral radius $r(A)$ equal to zero. The equation $f = Af + g$ is uniquely solvable in X. Its solution can be obtained by iterations*

$$f_{n+1} = Af_n + g, \quad f_0 = g; \quad \lim_{n\to\infty} f_n = f, \quad f = \sum_{j=0}^{\infty} A^j g, \tag{49}$$

for any $g \in X$ and the convergence holds in X.

Proof. The spectral radius of a linear operator A is defined by the formula

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

By induction one proves that

$$|A^n f| \le t^{n(p+1)} \frac{\Gamma^n(p+1)}{\Gamma(n(p+1)+1)} \|f\|_X, \quad n \ge 1. \tag{50}$$

From this formula and the known asymptotic of the gamma function $\Gamma(z)$ for $z \to \infty$ (see [8]) the conclusion $r(A) = 0$ follows. If $r(A) = 0$ then the solution to equation $f = Af + g$ is unique and can be calculated by the iterative process (49).

This proof is taken from [1] where more details are provided.

Lemma 5 is proved. □

By Lemma 5 the solution to Equation (44) can be obtained as

$$q = \sum_{j=0}^{\infty} (-c_3 \Phi_{1/4} \star)^j c_3 \Phi_{1/4} \star b_0, \tag{51}$$

and any solution to inequality (27) satisfies the inequality

$$b \le \sum_{j=0}^{\infty} (-c_3 \Phi_{1/4} \star)^j c_3 \Phi_{1/4} \star b_0, \tag{52}$$

which is checked by iterations.

Proof of Theorem 2. From (51) and (52) the inequality $b(t) \le q(t)$ follows.

Theorem 2 is proved. □

It follows from Theorems 1 and 2 that $\sup_{t \ge 0} q(t) < c$, $b(t) \le q(t)$. This and the Parseval equality implies $\sup_{t \ge 0} \|\nabla \cdot v\| < c$. Together with the estimate (9) this proves the a priori estimate (19). So, solutions to Equation (7) belong to $W_2^1(\mathbb{R}^3) \times C(\mathbb{R}_+)$, where $W_2^1(\mathbb{R}^3)$ is the Sobolev space.

4. Uniqueness of the Solution to the NSP

Theorem 3. *There is no more than one solution to the NSP in the space* $W_2^1(\mathbb{R}^3) \times C(\mathbb{R}_+)$.

Proof. Let there be two solutions $\tilde{v}$ and $\tilde{w}$ to (7) and $z := \tilde{v} - \tilde{w}$. Then, subtracting from the first equation the second, one gets:

$$z = -\int_0^t ds \tilde{G}(\xi, t-s)\Big(z \bigstar (i\xi\tilde{v}) + \tilde{w} \bigstar (i\xi z)\Big). \tag{53}$$

Using estimate (13) and (19), one obtains from (53) the following inequality:

$$|z| \le c \int_0^t e^{-\nu(t-s)\xi^2} \eta(s) ds, \quad \eta := \|z\| + \||\xi| z\|. \tag{54}$$

From (54), taking the norm $\|\cdot\|$ and using (24), one obtains:

$$\|z\| \le c \int_0^t (t-s)^{-\frac{3}{4}} \eta(s) ds. \tag{55}$$

Multiply (54) by $|\xi|$ and take the norm $\|\cdot\|$. One gets:

$$\||\xi| z\| \le c \int_0^t (t-s)^{-\frac{5}{4}} \eta(s) ds. \tag{56}$$

Taking the Laplace transform of (55) and of (56) and summing the results yields:

$$L(\eta) \le c\Big(\Gamma(-\frac{1}{4}) p^{\frac{1}{4}} + p^{-\frac{1}{4}} \Gamma(1/4)\Big) L(\eta) = c\Big(-c_1 p^{\frac{1}{4}} + p^{-\frac{1}{4}} \Gamma(1/4)\Big) L(\eta), \quad c_1 > 0. \tag{57}$$

Since $L(\eta) \geq 0$, one concludes that $1 \leq c(-c_1 p^{\frac{1}{4}} + p^{-\frac{1}{4}}\Gamma(1/4))$. If $L(\eta) \not\equiv 0$, then one has a contradiction: take $p \to +\infty$, then the above inequality yields $1 \leq -\infty$. This contradiction proves that $L(\eta) = 0$, so $z = 0$. Theorem 3 is proved. □

Theorem 3 is not used in the derivation of our basic conclusions. This theorem is new. Earlier uniqueness theorems were proved under different assumptions on the spaces to which the solution to the NSP belongs, see [2,11].

5. Conclusions

From Theorems 1 and 2 the NSP paradox follows. From the NSP paradox we conclude that the NSP is physically and mathematically contradictive and is not a correct description of the dynamics of incompressible viscous fluid.

Thus, one of the millennium problems is solved.

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References

1. Ramm, A.G. *The Navier–Stokes Problem*; Morgan & Claypool Publishers: San Rafael, CA, USA, 2021.
2. Ramm, A.G. Concerning the Navier–Stokes problem. *Open J. Math. Anal. (OMA)* **2020**, *4*, 89–92. [CrossRef]
3. Ramm, A.G. Existence of the solutions to convolution equations with distributional kernels. *Glob. J. Math. Anal.* **2018**, *6*, 1–2
4. Ramm, A.G. On a hyper-singular equation. *Open J. Math. Anal.* **2020**, *4*, 8–10. [CrossRef]
5. Ramm, A.G. On hyper-singular integrals. *Open J. Math. Anal.* **2020**, *4*, 101–103. [CrossRef]
6. Ramm, A.G. Theory of Hyper-Singular Integrals and Its Application to the Navier–Stokes Problem. *Contrib. Math.* **2020**, *2*, 47–54.
7. Landau, L.; Lifshitz, E. *Fluid Mechanics*; Pergamon Press: New York, NY, USA, 1964.
8. Lebedev, N. *Special Functions and Their Applications*; Dover: New York, NY, USA, 1972.
9. Brychkov, Y.; Prudnikov, A. *Integral Tranforms of Generalized Functions*; Nauka: Moskow, Russia, 1977. (In Russian)
10. Gel'fand, I.; Shilov, G. *Generalized Functions*; GIFML: Moscow, Russia, 1959; Volume 1. (In Russian)
11. Ladyzhenskaya, O. *The Mathematical Theory of Viscous Incompressible Fluid*; Gordon and Breach: New York, NY, USA, 1969.