On a Nonlinear Mixed Problem for a Parabolic Equation with a Nonlocal Condition

Abdelkader Djerad [1], Ameur Memou [2,*] and Ali Hameida [3]

1   Laboratoire du Mathematique Pure et Appliquée, Department of Mathematics, University of Msila, Msila 28000, Algeria; abdelkader.djerad@univ-msila.dz
2   Department of Mathematics, University of Msila, Msila 28000, Algeria
3   Department of Mathematics, University of Constantine 1, Constantine 25000, Algeria; a_hameida@yahoo.fr
*   Correspondence: ameur.memou@univ-msila.dz

Abstract: The aim of this work is to prove the well-posedness of some linear and nonlinear mixed problems with integral conditions defined only on two parts of the considered boundary. First, we establish for the associated linear problem a priori estimate and prove that the range of the operator generated by the considered problem is dense using a functional analysis method. Then by applying an iterative process based on the obtained results for the linear problem, we establish the existence, uniqueness and continuous dependence of the weak solution of the nonlinear problem.

Keywords: energy inequality; integral boundary conditions; strong solution; weak solution; second order parabolic equation

MSC: 35B45; 35K61

1. Introduction and Statement of the Problem

Some problems related to physical and technical issues can be described in terms of nonlocal problems with integral conditions in partial differential equations. Nonlocal conditions arise mainly when the values of the studied function on the boundary cannot be measured directly, while their average are known. Parabolic equation, describe several physical phenomena belongs to this of problem i.e., nonlocal problems. Therefore, the problem of parabolic equation with integral condition is stated as follows:

We consider in the rectangular domain $\Omega = [0,1] \times [0,T]$, where the problem is to find a solution $\sigma(x,t)$ of the following non-classical boundary value problem such that

$$\pounds\sigma = \frac{\partial \sigma}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial \sigma}{\partial x}\right) = g\left(x,t,\sigma,\frac{\partial \sigma}{\partial x}\right), \qquad \text{for } (x,t) \in [0,1] \times [0,T], \tag{1}$$

with the initial condition

$$l\sigma = \sigma(x,0) = \varphi(x), \qquad \text{for } x \in [0,1], \tag{2}$$

and Dirichlet boundary condition

$$\sigma(0,t) = \sigma(1,t), \qquad \text{for } t \in [0,T], \tag{3}$$

and the nonlocal condition

$$\int_0^\alpha K_1(x)\sigma(x,t)dx + \int_\alpha^1 K_2(x)\sigma(x,t)dx = 0, \qquad 0 < \alpha < 1 \qquad \forall t \in [0,T]. \tag{4}$$

where the functions $g\left(x,t,\sigma,\dfrac{\partial\sigma}{\partial x}\right)$, $\varphi(x)$ are known, and we assume that the following matching conditions are satisfied

$$\begin{cases} \varphi(0) = 0, \\ \int_0^\alpha K_1(x)\varphi(x)dx + \int_\alpha^1 K_2(x)\varphi(x)dx = 0. \end{cases}$$

We also assume that there exists a positive constant d such that

$$\left| g\left(x,t,\sigma_1,\dfrac{\partial\sigma_1}{\partial x}\right) - g\left(x,t,\sigma_2,\dfrac{\partial\sigma_2}{\partial x}\right) \right| \leq d\left(|\sigma_1 - \sigma_2| + \left| \dfrac{\partial\sigma_1}{\partial x} - \dfrac{\partial\sigma_2}{\partial x} \right| \right),$$

for all $(x,t) \in \Omega$.

In addition, we assume that the functions $a(x,t)$, $K_1(x)$, $K_2(x)$ are differentiable on Ω, respectively on $[0,1]$ and their derivatives satisfy the conditions

$$\begin{cases} 0 < a_0 \leq a(x,t) \leq a_1 & \forall(x,t) \in \Omega, \\[2mm] c_2 \leq \dfrac{\partial a}{\partial t}(x,t) \leq c_1, & \forall(x,t) \in \Omega, \\[2mm] \left| \dfrac{\partial a}{\partial x}(x,t) \right| \leq b, & \forall(x,t) \in \Omega. \\[2mm] 0 < m_0 \leq K_1(x) \leq m_1, 0 < m_2 \leq K_2(x) \leq m_3, & \forall x \in [0,1], \\[2mm] \left| \dfrac{dK_1(x)}{dx} \right| \leq b_1, \left| \dfrac{dK_2(x)}{dx} \right| \leq b_2. & \forall x \in [0,1]. \end{cases} \quad (5)$$

This type of problem can be found in various problems arising from physics, such as heat conduction [1–4], plasma physics [5], thermoelasticity [6], electrochemistry [7], chemical diffusion [8] and underground water flow [9–11]. Several results published in the literature such as in [1–4,7,12–24] have solved the parabolic equation by combining the integral condition with Dirichlet condition or Newmann condition, or with purely integral conditions, using various methods. For hyperbolic equations, the unicity and existence of the solution have been studies in [13,25–32] and the mixed type equations in [33–37] . The elliptic equations were considered in [38–40].

The linear problem associated with the problem stated in Equations (1)–(4), for $\alpha = 0$ and $K_2(x) = 1$, has been studied in [20] and for $\alpha = 1$ and $K_1(x) = 1$ in [17] with Dirichlet condition.

The main purpose of the present paper is to study and found a solution to the posed problem without imposing any conditions on the functions $K_1(x)$, $K_2(x)$ and on the constant α in the interval $]0,1[$. In addition, the nonlinear problem of the parabolic equation with weighted integral condition defined on two parts of the boundary is solved.

The following methodology to solve the posed problem is summarized below
First, an a priori estimate is established for the associated linear problem and the density of the operator range generated by the considered problem is proved using the functional analysis method. Subsequently, by applying an iterative process based on the obtained results for the linear problem, the existence and uniqueness of the weak solution of the nonlinear problems is established.

The rest of the paper is organized as follows: In Section 2, the associated linear problem is stated. Section 3 deals with the proof of the uniqueness of the solution using an a priori estimate, while Section 4 gives the solvability of the considered linear problem. Finally, in Section 5, based on the obtained results in Sections 3 and 4, and on the use of an iterative process, we prove the existence and uniqueness of the solution of the nonlinear problem .

2. Statement of the Associated Linear Problem

Let us in this section give the position of the linear problem and introduce the different function spaces needed to investigate the mixed nonlocal problem given by Equation (6), bellow

$$\pounds u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) = f(x,t), \tag{6}$$

and supplemented by the conditions Equations (2)–(4).

The given problem Equations (6), (2)–(4) can be considered to be a solving of the operator equation $Lu = (\pounds u, lu) = \mathcal{F}$, where the operator L has a domain of definition $D(L)$ consisting of functions $u \in L^2(\Omega)$ such that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial t}(x,t) \in L^2(\Omega)$ and satisfying the conditions Equations (3) and (4).

The operator L is an operator defined on E into $\mathbb{F}$, where E is the Banach space of functions $u \in L^2(\Omega)$, with the finite norm

$$\|u\|_E^2 = \int_\Omega \Phi(x)\left[\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial^2 u}{\partial x^2}\right|^2\right]dxdt + \sup_t \int_0^1 \left[\Phi(x)\left|\frac{\partial u}{\partial x}\right|^2 + |u|^2\right]dx. \tag{7}$$

$\mathbb{F}$ is the Hilbert space of functions $F = (f, \varphi)$, $f \in L^2(\Omega)$, $\varphi \in H^1(0,1)$ with the finite norm

$$\|\mathbb{F}\|_\mathbb{F}^2 = \int_\Omega \Phi(x)|f(x,t)|^2dxdt + \int_0^1 \left[\Phi(x)\left|\frac{d\varphi}{dx}\right|^2 + |\varphi|^2\right]dx. \tag{8}$$

where

$$\Phi(x) = \begin{cases} \dfrac{x^2}{\alpha^2}, & 0 \leq x \leq \alpha, \\ \dfrac{(1-x)^2}{(1-\alpha)^2}, & \alpha \leq x \leq 1. \end{cases}$$

Then, we show that the operator L has a closure $\overline{L}$ and establish an energy inequality:

$$\|u\|_E \leq k\|Lu\|_\mathbb{F} \qquad \forall u \in D(L). \tag{9}$$

Definition 1. *A solution of the operator equation $\overline{L}u = F = (f, \varphi)$ is called a strong solution of problem Equations (6), (2)–(4).*

Since the points of the graph of the operator $\overline{L}$ are limits of sequences of points of the graph of L, we can extend the a priori estimate Equation (9) to be applied to strong solutions by taking limits, i.e., we have the inequality

$$\|u\|_E \leq k\|\overline{L}u\|_\mathbb{F}, \qquad \forall u \in D(\overline{L}). \tag{10}$$

From this inequality, we deduce the uniqueness of a strong solution, if it exists, and that the range of the operator $\overline{L}$ coincides with the closure of the range of L.

Proposition 1. *The operator $L : E \longrightarrow F$ admits a closure $\overline{L}$.*

The following a priori estimate gives the uniqueness of the solution if it exists of the posed linear problem.

3. An Energy Inequality and Its Application

Theorem 1. *There exists a positive constant k, such that for each function $u \in D(L)$ we have*

$$\|u\|_E \le k\|Lu\|_{\mathbb{F}}. \tag{11}$$

Proof of Theorem 1. Let

$$Mu = \begin{cases} \dfrac{x^2 K_1}{a(0,t)} e^{\lambda x} \dfrac{\partial u}{\partial t} + \dfrac{2x}{a(0,t)} e^{\lambda x} \int_x^\alpha K_1(\mu) \dfrac{\partial u}{\partial t} d\mu, & 0 \le x \le \alpha, \\[3mm] \dfrac{(1-x)^2 K_2}{a(1,t)} e^{\beta(1-x)} \dfrac{\partial u}{\partial t} + \dfrac{2(1-x)}{a(1,t)} e^{\beta(1-x)} \int_\alpha^x K_2(\mu) \dfrac{\partial u}{\partial t} d\mu, & \alpha \le x \le 1. \end{cases}$$

where

$$\begin{cases} \dfrac{\alpha^2}{a(0,t)} e^{\lambda \alpha} K_1(\alpha) = \dfrac{(1-\alpha)^2}{a(1,t)} e^{\beta(1-\alpha)} K_2(\alpha), \\[3mm] \lambda > \max\left(0, \sup_{0 \le x \le \alpha} \dfrac{k_1'(x)}{k_1(x)} \right), \\[3mm] \beta > \max(0, \sup_{\alpha \le x \le 1} \dfrac{k_2'(x)}{k_2(x)}. \end{cases} \tag{12}$$

We consider the quadratic form obtained by multiplying Equation (6) by $e^{-ct}\overline{Mu}$, with $0 \le s \le T$, $c > 0$, integrating over $\Omega^s = [0,1] \times [0,s]$ and taking the real part, we obtain

$$\Phi(u,u) = \text{Re} \int_{\Omega^s} e^{-ct} \frac{\partial u}{\partial t} \overline{Mu} dx dt - \text{Re} \int_{\Omega^s} e^{-ct} \frac{\partial}{\partial x} \left(a(x,t) \frac{\partial u}{\partial x} \right) \overline{Mu} dx dt. \tag{13}$$

Substituting Mu by its expression in the first term in the right-hand side of Equation (13), integrating with respect to x, using the boundary condition Equation (3), the integral condition Equations (4) and (12), we obtain

$$\text{Re} \int_{\Omega^s} e^{-ct} \frac{\partial u}{\partial t} \overline{Mu} dx dt = \int_0^s \int_0^\alpha \frac{x^2 K_1}{a(0,t)} e^{-ct+\lambda x} \left| \frac{\partial u}{\partial t} \right|^2 dx dt$$

$$+ \int_0^s \int_\alpha^1 \frac{(1-x)^2 K_2}{a(1,t)} e^{-ct+\beta(1-x)} \left| \frac{\partial u}{\partial t} \right|^2 dx dt + \int_0^s \int_0^\alpha \frac{K_1 + (\lambda K_1 - K_1')x}{a(0,t)K_1^2} e^{-ct+\lambda x} \left| \int_x^\alpha K_1 \frac{\partial u}{\partial t} \right|^2 dx dt \tag{14}$$

$$+ \int_0^s \int_\alpha^1 \frac{K_2 + (\beta K_2 + K_2')(1-x)}{a(1,t)K_2^2} e^{-ct+\beta(1-x)} \left| \int_\alpha^x K_2 \frac{\partial u}{\partial t} \right|^2 dx dt,$$

$$- \text{Re} \int_{\Omega^s} e^{-ct} \frac{\partial}{\partial x} \left(a(x,t) \frac{\partial u}{\partial x} \right) \overline{Mu} \, dx dt =$$

$$\text{Re} \int_0^s \int_0^\alpha \frac{x^2 K_1}{a(0,t)} a e^{-ct+\lambda x} \frac{\partial u}{\partial x} \frac{\overline{\partial^2 u}}{\partial x \partial t} \, dx dt + \text{Re} \int_0^s \int_\alpha^1 \frac{(1-x)^2 K_2}{a(1,t)} a e^{-ct+\beta(1-x)} \frac{\partial u}{\partial x} \frac{\overline{\partial^2 u}}{\partial x \partial t} \, dx dt$$

$$+2 \text{Re} \int_0^s \int_0^\alpha \frac{(1+\lambda x) K_1}{a(0,t)} a e^{-ct+\lambda x} u \frac{\overline{\partial u}}{\partial t} \, dx dt + 2 \text{Re} \int_0^s \int_\alpha^1 \frac{(1+\beta(1-x)) K_2}{a(1,t)} a e^{-ct+\beta(1-x)} u \frac{\overline{\partial u}}{\partial t} \, dx dt \qquad (15)$$

$$+ \text{Re} \int_0^s \int_0^\alpha \frac{\left(K_1' + \lambda K_1\right) x^2}{a(0,t)} a e^{-ct+\lambda x} \frac{\partial u}{\partial x} \frac{\overline{\partial u}}{\partial t} \, dx dt + \text{Re} \int_0^s \int_\alpha^1 \frac{\left(K_2' - \beta K_2\right)(1-x)^2}{a(1,t)} a e^{-ct+\beta(1-x)} \frac{\partial u}{\partial x} \frac{\overline{\partial u}}{\partial t} \, dx dt$$

$$-2 \text{Re} \int_0^s \int_0^\alpha \frac{\lambda a + (1+\lambda x)\left(\lambda a + \frac{\partial a}{\partial x}\right)}{a(0,t)} e^{-ct+\lambda x} u \int_x^\alpha K_1 \frac{\overline{\partial u}}{\partial t} \, d\zeta \, dx dt$$

$$-2 \text{Re} \int_0^s \int_\alpha^1 \frac{\beta a + (1+\beta(1-x))\left(\beta a - \frac{\partial a}{\partial x}\right)}{a(1,t)} e^{-ct+\beta(1-x)} u \int_\alpha^x K_2 \frac{\overline{\partial u}}{\partial t} \, d\zeta \, dx dt.$$

Integrating the first four terms with respect to t in Equation (15) using the condition Equation (2) we have

$$\text{Re} \int_0^s \int_0^\alpha \frac{x^2 K_1}{a(0,t)} a e^{-ct+\lambda x} \frac{\partial u}{\partial x} \frac{\overline{\partial^2 u}}{\partial x \partial t} \, dx dt = \frac{1}{2} \int_0^s \int_0^\alpha \frac{caa(0,t) - \left(\frac{\partial a}{\partial t} a(0,t) - a\frac{\partial a(0,t)}{\partial t}\right)}{a^2(0,t)} x^2 K_1 e^{-ct+\lambda x} \left|\frac{\partial u}{\partial x}\right|^2 \, dx dt$$

$$+ \frac{1}{2} \int_0^\alpha \frac{x^2 a K_1}{a(0,t)} e^{-ct+\lambda x} \left|\frac{\partial u}{\partial x}\right|^2 \bigg|_{t=s} \, dx - \frac{1}{2} \int_0^\alpha \frac{x^2 a(x,0) K_1}{a(0,0)} e^{\lambda x} \left|\frac{d\varphi}{dx}\right|^2 \, dx,$$

$$\text{Re} \int_0^s \int_\alpha^1 \frac{(1-x)^2 K_2}{a(1,t)} a e^{-ct+\beta(1-x)} \frac{\partial u}{\partial x} \frac{\overline{\partial^2 u}}{\partial x \partial t} \, dx dt =$$

$$\frac{1}{2} \int_0^s \int_\alpha^1 \frac{caa(1,t) - \left(\frac{\partial a}{\partial t} a(1,t) - a\frac{\partial a(1,t)}{\partial t}\right)}{a^2(1,t)} (1-x)^2 K_2 e^{-ct+\beta(1-x)} \left|\frac{\partial u}{\partial x}\right|^2 \, dx dt$$

$$+ \frac{1}{2} \int_\alpha^1 \frac{(1-x)^2 a K_2}{a(1,t)} e^{-ct+\beta(1-x)} \left|\frac{\partial u}{\partial x}\right|^2 \bigg|_{t=s} \, dx - \frac{1}{2} \int_\alpha^1 \frac{(1-x)^2 a(x,0) K_2}{a(1,0)} e^{+\beta(1-x)} \left|\frac{d\varphi}{dx}\right|^2 \, dx,$$

$$2 \text{Re} \int_0^s \int_0^\alpha \frac{(1+\lambda x) a K_1}{a(0,t)} e^{-ct+\lambda x} u \frac{\overline{\partial u}}{\partial t} \, dx dt =$$

$$\frac{1}{2} \int_0^s \int_0^\alpha \frac{caa(0,t) - \left(\frac{\partial a}{\partial t} a(0,t) - a\frac{\partial a(0,t)}{\partial t}\right)}{a^2(0,t)} (1+\lambda x) K_1 e^{-ct+\lambda x} |u|^2 \, dx dt$$

$$+ \frac{1}{2} \int_0^\alpha \frac{(1+\lambda x) a K_1}{a(0,t)} e^{-ct+\lambda x} |u|^2 \bigg|_{t=s} \, dx - \frac{1}{2} \int_0^\alpha \frac{(1+\lambda x) a(x,0) K_1}{a(0,t)} e^{\lambda x} |\varphi|^2 \, dx,$$

and

$$2 \text{Re} \int_0^s \int_\alpha^1 \frac{(1+\beta(1-x)) K_2}{a(1,t)} a e^{-ct+\beta(1-x)} u \frac{\overline{\partial u}}{\partial t} \, dx dt =$$

$$\int_0^s \int_\alpha^1 \frac{caa(1,t) - \left(\frac{\partial a}{\partial t} a(1,t) - a\frac{\partial a(1,t)}{\partial t}\right)}{a^2(1,t)} (1+\beta(1-x)) K_2 e^{-ct+\beta(1-x)} |u|^2 \, dx dt$$

$$+ \frac{1}{2} \int_\alpha^1 \frac{(1+\beta(1-x)) a K_2}{a(1,t)} e^{-ct+\beta(1-x)} |u|^2 \bigg|_{t=s} \, dx - \frac{1}{2} \int_\alpha^1 \frac{(1+\beta(1-x)) a(x,0) K_2}{a(1,0)} e^{+\beta(1-x)} |\varphi|^2 \, dx.$$

Combining the previous equalities with Equations (14) and (15), Equation (13) becomes

$$
\int_0^s \int_0^\alpha \frac{x^2 K_1}{a(0,t)} e^{-ct+\lambda x} \left|\frac{\partial u}{\partial t}\right|^2 dxdt + \int_0^s \int_\alpha^1 \frac{(1-x)^2 K_2}{a(1,t)} e^{-ct+\beta(1-x)} \left|\frac{\partial u}{\partial t}\right|^2 dxdt
$$

$$
+ \frac{1}{2} \int_0^s \int_0^\alpha \frac{caa(0,t) - \left(\frac{\partial a}{\partial t} a(0,t) - a\frac{\partial a(0,t)}{\partial t}\right)}{a^2(0,t)} x^2 K_1 e^{-ct+\lambda x} \left|\frac{\partial u}{\partial x}\right|^2 dxdt
$$

$$
+ \frac{1}{2} \int_0^s \int_\alpha^1 \frac{caa(1,t) - \left(\frac{\partial a}{\partial t} a(1,t) - a\frac{\partial a(1,t)}{\partial t}\right)}{a^2(1,t)} (1-x)^2 K_2 e^{-ct+\beta(1-x)} \left|\frac{\partial u}{\partial x}\right|^2 dxdt
$$

$$
+ \int_0^s \int_0^\alpha \frac{K_1 + (\lambda K_1 - K_1')x}{a(0,t)K_1^2} e^{-ct+\lambda x} \left|\int_x^\alpha K_1 \frac{\partial u}{\partial t}\right|^2 dxdt + \int_0^s \int_\alpha^1 \frac{K_2 + (\beta K_2 + K_2')(1-x)}{a(1,t)K_2^2} e^{-ct+\beta(1-x)} \left|\int_\alpha^x K_2 \frac{\partial u}{\partial t}\right|^2 dxdt
$$

$$
+ \int_0^s \int_0^\alpha \frac{caa(0,t) - \left(\frac{\partial a}{\partial t} a(0,t) - a\frac{\partial a(0,t)}{\partial t}\right)}{a^2(0,t)} (1+\lambda x) K_1 e^{-ct+\lambda x} |u|^2 dxdt
$$

$$
+ \int_0^s \int_\alpha^1 \frac{caa(1,t) - \left(\frac{\partial a}{\partial t} a(1,t) - a\frac{\partial a(1,t)}{\partial t}\right)}{a^2(1,t)} (1+\beta(1-x)) K_2 e^{-ct+\beta(1-x)} |u|^2 dxdt
$$

$$
+ \frac{1}{2} \int_0^\alpha \frac{x^2 a K_1}{a(0,t)} e^{-ct+\lambda x} \left|\frac{\partial u}{\partial x}\right|^2 \bigg|_{t=s} dx + \frac{1}{2} \int_\alpha^1 \frac{(1-x)^2 a K_2}{a(1,t)} e^{-ct+\beta(1-x)} \left|\frac{\partial u}{\partial x}\right|^2 \bigg|_{t=s} dx \tag{16}
$$

$$
+ \int_0^\alpha \frac{(1+\lambda x) a K_1}{a(0,t)} e^{-ct+\lambda x} |u|^2 \bigg|_{t=s} dx + \frac{1}{2} \int_\alpha^1 \frac{(1+\beta(1-x)) a K_2}{a(1,t)} e^{-ct+\beta(1-x)} |u|^2 \bigg|_{t=s} dx
$$

$$
+ \mathrm{Re} \int_0^s \int_0^\alpha \frac{(K_1' + \lambda K_1)x^2}{a(0,t)} a e^{-ct+\lambda x} \frac{\partial u}{\partial x} \overline{\frac{\partial u}{\partial t}} dxdt + \mathrm{Re} \int_0^s \int_\alpha^1 \frac{(K_2' - \beta K_2)(1-x)^2}{a(1,t)} a e^{-ct+\beta(1-x)} \frac{\partial u}{\partial x} \overline{\frac{\partial u}{\partial t}} dxdt
$$

$$
- 2 \, \mathrm{Re} \int_0^s \int_0^\alpha \frac{\lambda a + (1+\lambda x)\left(\lambda a + \frac{\partial a}{\partial x}\right)}{a(0,t)} e^{-ct+\lambda x} u \int_x^\alpha K_1 \overline{\frac{\partial u}{\partial t}} d\zeta dxdt
$$

$$
- 2 \, \mathrm{Re} \int_0^s \int_\alpha^1 \frac{\beta a + (1+\beta(1-x))\left(\beta a - \frac{\partial a}{\partial x}\right)}{a(1,t)} e^{-ct+\beta(1-x)} u \int_\alpha^x K_2 \overline{\frac{\partial u}{\partial t}} d\zeta dxdt =
$$

$$
\mathrm{Re} \int_{\Omega^s} e^{-ct} f \overline{Mu} dxdt + \frac{1}{2} \int_0^\alpha \frac{x^2 a(x,0) K_1}{a(0,0)} e^{\lambda x} \left|\frac{d\varphi}{dx}\right|^2 dx + \frac{1}{2} \int_\alpha^1 \frac{(1-x)^2 a(x,0) K_2}{a(1,0)} e^{\beta(1-x)} \left|\frac{d\varphi}{dx}\right|^2 dx
$$

$$
+ \int_0^\alpha \frac{(1+\lambda x) a(x,0) K_1}{a(0,t)} e^{\lambda x} |\varphi|^2 dx + \int_\alpha^1 \frac{(1+\beta(1-x)) a(x,0) K_2}{a(1,0)} e^{\beta(1-x)} |\varphi|^2 dx.
$$

Using Young inequalities and using the fact that

$$
\int_0^s \int_0^\alpha e^{-ct+\lambda x} \left|\int_x^\alpha K_1 \frac{\partial u}{\partial t} d\zeta\right|^2 dxdt \leq 4 m_1 a_1 \int_0^s \int_0^\alpha \frac{K_1}{a(0,t)} x^2 e^{-ct+\lambda x} \left|\frac{\partial u}{\partial t}\right|^2 dxdt,
$$

$$
\int_0^s \int_\alpha^1 e^{-ct+\lambda x} \left|\int_x^\alpha K_2 \frac{\partial u}{\partial t} d\zeta\right|^2 dxdt \leq 4 m_2 a_1 \int_0^s \int_\alpha^1 \frac{K_2}{a(1,t)} (1-x)^2 e^{-ct+\lambda x} \left|\frac{\partial u}{\partial t}\right|^2 dxdt,
$$

we obtain

$$
\mathrm{Re} \int_0^s \int_0^\alpha \frac{(K_1' + \lambda K_1)x^2}{a(0,t)} a e^{-ct+\lambda x} \frac{\partial u}{\partial x} \overline{\frac{\partial u}{\partial t}} dxdt \leq
$$

$$
\frac{1}{4} \int_0^s \int_0^\alpha \frac{x^2 K_1}{a(0,t)} e^{-ct+\lambda x} \left|\frac{\partial u}{\partial t}\right|^2 dxdt + \frac{a_1^2 (b_1 + \lambda m_1)^2}{a_0 m_0^2} \int_0^s \int_0^\alpha x^2 K_1 e^{-ct+\lambda x} \left|\frac{\partial u}{\partial x}\right|^2 dxdt,
$$

$$
\mathrm{Re} \int_0^s \int_\alpha^1 \frac{(K_2' - \beta K_2)(1-x)^2}{a(1,t)} a e^{-ct+\beta(1-x)} \frac{\partial u}{\partial x} \overline{\frac{\partial u}{\partial t}} dxdt \leq
$$

$$\frac{1}{4}\int_0^s\int_\alpha^1\frac{(1-x)^2 K_2}{a(1,t)}e^{-ct+\lambda x}\left|\frac{\partial u}{\partial t}\right|^2 dxdt+\frac{a_1^2(b_2+\beta m_3)^2}{a_0 m_2^2}\int_0^s\int_\alpha^1(1-x)^2 K_2 e^{-ct+\lambda x}\left|\frac{\partial u}{\partial x}\right|^2 dxdt,$$

$$2\,\mathrm{Re}\int_0^s\int_0^\alpha\frac{\lambda a+(1+\lambda x)\left(\lambda a+\frac{\partial a}{\partial x}\right)}{a(0,t)}e^{-ct+\lambda x}u\int_x^\alpha K_1\frac{\overline{\partial u}}{\partial t}d\zeta dxdt\le$$

$$16\int_0^s\int_0^\alpha\frac{(\lambda a_1+(1+\lambda\alpha)(\lambda a_1+b))^2}{a(0,t)}\frac{m_0}{m_1 a_1}K_1 e^{-ct+\lambda x}|u|^2 dxdt+\frac{1}{4}\int_0^s\int_0^\alpha\frac{K_1}{a(0,t)}e^{-ct+\lambda x}\left|\frac{\partial u}{\partial t}\right|^2 dxdt,$$

$$2\,\mathrm{Re}\int_0^s\int_\alpha^1\frac{\beta a+(1+\beta(1-x))\left(\beta a-\frac{\partial a}{\partial x}\right)}{a(1,t)}e^{-ct+\beta(1-x)}u\int_\alpha^x K_2\frac{\overline{\partial u}}{\partial t}d\zeta dxdt\le$$

$$16\int_0^s\int_\alpha^1\frac{(\beta a_1+(1+\beta(1-\alpha))(\beta a_1+b))^2}{a(1,t)}\frac{m_2}{m_3 a_1}K_2 e^{-ct+\beta(1-x)}|u|^2 dxdt+\frac{1}{4}\int_0^s\int_\alpha^1\frac{K_2}{a(1,t)}e^{-ct+\beta(1-x)}\left|\frac{\partial u}{\partial t}\right|^2 dxdt,$$

$$\mathrm{Re}\int_0^s\int_0^\alpha e^{-ct}f\overline{Mu}dxdt\le$$

$$\left(\frac{8m_1}{a_0}+\frac{32}{m_1 a_0^2 a_1}\right)\int_0^s\int_0^\alpha x^2 e^{-ct+\lambda x}|f|^2 dxdt+\frac{1}{4}\int_0^s\int_0^\alpha\frac{K_1}{a(0,t)}x^2 e^{-ct+\lambda x}\left|\frac{\partial u}{\partial t}\right|^2 dxdt,$$

$$\mathrm{Re}\int_0^s\int_\alpha^1 e^{-ct}f\overline{Mu}dxdt\le$$

$$\left(\frac{8m_3}{a_0}+\frac{32}{m_3 a_0^2 a_1}\right)\int_0^s\int_\alpha^1(1-x)^2 e^{-ct+\beta(1-x)}|f|^2 dxdt+\frac{1}{4}\int_0^s\int_\alpha^1\frac{(1-x)^2 K_2}{a(1,t)}e^{-ct+\beta(1-x)}\left|\frac{\partial u}{\partial t}\right|^2 dxdt.$$

We choose the constant c such that

$$c>\frac{a_1}{a_0}\max\left\{\begin{array}{c}\left(\frac{M_1}{a_0^2}+16\frac{(\lambda a_1+(\lambda a_1+b))^2}{a_0}\right)(1+\lambda\alpha),\ \left(\frac{M_2}{a_0^2}+16\frac{\beta a_1+(\beta a_1+b)^2}{a_0}\right)(1+\beta(1-\alpha)),\\ M_1+\frac{a_1^2(b_1+\lambda m_1)^2}{a_0 m_0},M_2+\frac{a_1^2(b_2+\beta m_3)^2}{a_0 m_2}.\end{array}\right\}\tag{17}$$

where

$$M_1=\sup_{(x,t)\in\Omega}\left(a(0,t)\frac{\partial a}{\partial t}-a\frac{\partial a(0,t)}{\partial t}\right),\ M_2=\sup_{(x,t)\in\Omega}\left(\frac{\partial a}{\partial t}a(1,t)-a\frac{\partial a(1,t)}{\partial t}\right).$$

Then by combining the previous inequalities with Equation (16), we obtain

$$\int_{\Omega^s}\Phi(x)\left[\left|\frac{\partial u}{\partial t}\right|^2+\left|\frac{\partial u}{\partial x}\right|^2\right]dxdt+\int_0^1\left(\Phi(x)\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right)dx\bigg|_{t=s}\le$$
$$K^2\left[\int_\Omega\Phi(x)|f|^2 dxdt+\int_0^1\left(\Phi(x)\left|\frac{\partial\varphi}{\partial x}\right|^2+|\varphi|^2\right)dx\right],\tag{18}$$

where

$$K^2=\frac{\max\left\{\left(\frac{8m_1}{a_0}+\frac{32}{m_1 a_0^2 a_1}\right)\alpha^2 e^{\lambda\alpha},\left(\frac{8m_3}{a_0}+\frac{32}{m_3 a_0^2 a_1}\right)(1-\alpha)^2 e^{\beta(1-\alpha)},\frac{(1+\lambda\alpha)a_1 m_1}{a_0}e^{\lambda\alpha},\frac{(1+\beta(1-\alpha))a_1 m_3}{a_0}e^{\beta(1-\alpha)}\right\}}{\min\left\{\frac{\alpha^2 m_0}{2a_1},\alpha^2 m_0\frac{ca_0^2-M_1}{2a_1^2},\frac{(1-\alpha)^2 m_2}{2a_1},(1-\alpha)^2 m_2\frac{ca_0^2-M_2}{2a_1^2},\frac{a_0 m_2(1-\alpha)^2}{2a_1},\frac{\alpha^2 a_0 m_0}{2a_1}\right\}}e^{cT}.$$

From Equations (6) and (18), we deduce

$$\int_{\Omega^s} \Phi(x)\left[\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial^2 u}{\partial x^2}\right|^2\right]dxdt + \int_{\Omega^s} \Phi(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt + \int_0^1 \left(\Phi(x)\left|\frac{\partial u}{\partial x}\right|^2 + |u|^2\right)dx\bigg|_{t=s} \leq$$

$$k^2\left[\int_Q \Phi(x)|f|^2 dxdt + \int_0^1\left(\Phi(x)\left|\frac{\partial \varphi}{\partial x}\right|^2 + |\varphi|^2\right)dx\right],$$

If we drop the second term in the last inequality and by taking the least upper bound of the left side with respect to s from 0 to T, we obtain the desired estimate Equation (11) with $k^2 = K^2 + \frac{4+2K^2+4b^2K^2}{a_0^2}$.

Then, we deduce the uniqueness and continuous dependence of the solution on the input data of the problem Equations (6), (2)–(4). □

Corollary 1. *If a strong solution of Equations (6), (2)–(4) exists, it is unique and depends continuously on $F = (f, \varphi)$.*

Corollary 2. *The range $R(\overline{L})$ of $\overline{L}$ is closed in $\mathbb{F}$ and $\overline{R(L)} = R(\overline{L})$.*

Corollary (2) shows that to prove that problem Equations (6), (2)–(4) has a strong solution for arbitrary F, it suffices to prove that the set $R(L)$ is dense in $\mathbb{F}$.

4. Solvability of Problem Equations (6), (2)–(4)

To prove the solvability of problem Equations (6), (2)–(4) it is sufficient to show that $R(L)$ is dense in $\mathbb{F}$. The proof is based on the following lemma

Lemma 1. *Suppose that the function a and its derivatives are bounded.*
Let $u \in D_0(L) = \{u \in D(L), u(x,0) = 0\}$. If, for $u \in D_0(L)$ and some functions $w \in L^2(\Omega)$, we have

$$\int_\Omega \Phi(x)f\overline{w}dxdt = 0, \tag{19}$$

then w vanish almost everywhere in Ω.

Proof of Lemma 1. Equation (19), can be written as follows

$$\int_\Omega \frac{\partial u}{\partial t}\overline{\rho}dxdt = \int_\Omega A(t)u\overline{\rho}dxdt, \tag{20}$$

where

$$\rho = \Phi(x)w,$$

and

$$A(t)u = \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right).$$

We introduce the smoothing operators [17] $J_\varepsilon^{-1} = \left(I - \varepsilon\frac{\partial}{\partial t}\right)^{-1}$ and $\left(J_\varepsilon^{-1}\right)^* = \left(I + \varepsilon\frac{\partial}{\partial t}\right)^{-1}$ Substituting the function u in Equation (20) by the smoothing function u_ε and using the relation

$$A(t)u_\varepsilon = J_\varepsilon^{-1}A(t)u - \varepsilon J_\varepsilon^{-1}B_\epsilon(t)u_\varepsilon,$$

where

$$B_\epsilon(t)u_\epsilon = \frac{\partial A(t)}{\partial t}u_\epsilon = \frac{\partial}{\partial x}\left(\frac{\partial a}{\partial t}\frac{\partial u_\epsilon}{\partial x}\right),$$

we obtain

$$-\int_\Omega u\overline{\frac{\partial \rho_\epsilon^*}{\partial t}}dxdt = \int_\Omega (A(t)u - \epsilon B_\epsilon(t)u_\epsilon)\overline{\rho_\epsilon^*}dxdt. \tag{21}$$

Since the operator $A(t)$ has a continuous inverse in $L^2(0,1)$ defined by

$$A^{-1}(t)g = \int_0^x \frac{d\zeta}{a}\int_0^\zeta g(\eta)d\eta + C_1(t)\int_0^x \frac{d\zeta}{a} + C_2(t),$$

the functions $C_1(t)$ and $C_2(t)$ satisfies

$$\begin{cases} C_1(t) = -\dfrac{\int_0^1 K(x)dx \int_0^x \frac{\int_0^\zeta g(\eta)d\eta}{a}d\zeta + \int_0^1 K(x)dx\int_0^1 \frac{\int_0^x g(\eta)d\eta}{a}dx}{\int_0^1 K(x)dx\int_0^x \frac{d\zeta}{a} + \int_0^1 K(x)dx}, \\[4mm] C_2(t) = -\dfrac{\int_0^1 K(x)dx \int_0^x \frac{\int_0^\zeta g(\eta)d\eta}{a}d\zeta + \int_0^1 K(x)dx\int_0^1 \frac{\int_0^x g(\eta)d\eta}{a}dx}{\int_0^1 K(x)dx\int_0^x \frac{d\zeta}{a} + \int_0^1 K(x)dx}\int_0^1 \frac{1}{a}dx + \int_0^1 \frac{\int_0^\zeta g(\eta)d\eta}{a}d\zeta, \end{cases}$$

where

$$K(x) = \begin{cases} K_1(x) & 0 \le x \le \alpha, \\ K_2(x) & \alpha \le x \le 1. \end{cases}$$

Then, we have $\int_0^\alpha K_1(x)A^{-1}(t)udx + \int_\alpha^1 K_2(x)A^{-1}(t)udx = 0$, hence, the function $J_\epsilon^{-1}u = u_\epsilon$ can be represented in the form

$$u_\epsilon = J_\epsilon^{-1}A^{-1}(t)A(t)u,$$

then

$$B_\epsilon(t)g = \frac{\partial^2 a}{\partial t\partial x}J_\epsilon^{-1}\frac{C_1(t) + \int_0^\zeta g(\eta)d\eta}{a} + \frac{\partial a}{\partial t}J_\epsilon^{-1}\frac{g}{a} - \frac{\partial a}{\partial t}J_\epsilon^{-1}\frac{\frac{\partial a}{\partial x}}{a}\frac{C_1(t) + \int_0^\zeta g(\eta)d\eta}{a}.$$

Consequently, Equation (21), becomes

$$-\int_\Omega u\overline{\frac{\partial \rho_\epsilon^*}{\partial t}}dxdt = \int_\Omega A(t)u\overline{h_\epsilon}dxdt, \tag{22}$$

where

$$h_\epsilon = \rho_\epsilon^* - \epsilon B_\epsilon^*(t)\rho_\epsilon^*,$$

and $B_\epsilon^*(t)$ is the adjoint operator of $B_\epsilon(t)$.

The left-hand side of Equation (22) is a continuous linear functional of u, hence the function h_ϵ has the derivatives $\frac{\partial h_\epsilon}{\partial x}, \frac{\partial^2 h_\epsilon}{\partial x^2} \in L^2(\Omega)$ and the following condition are satisfied

$$h_\epsilon(0,t) = h_\epsilon(1,t) = 0, \ a(0,t)\frac{\partial h_\epsilon}{\partial x}(0,t) = a(1,t)\frac{\partial h_\epsilon}{\partial x}(1,t).$$

For a sufficiently small ϵ and the operator $\dfrac{(J_\epsilon^{-1})^*\dfrac{\partial a}{\partial t}}{a}$ is bounded in $L^2(\Omega)$, we have

$\left\| \epsilon \dfrac{(J_\epsilon^{-1})^*\dfrac{\partial a}{\partial t}}{a} \right\|_{L^2(\Omega)} < 1$, hence, the operator $I - \epsilon \dfrac{(J_\epsilon^{-1})^*\dfrac{\partial a}{\partial t}}{a}$ has a bounded inverse in

$L^2(\Omega)$, we deduce that $\dfrac{\partial \rho_\epsilon^*}{\partial x}, \dfrac{\partial^2 \rho_\epsilon^*}{\partial x^2} \in L^2(\Omega)$ and the following condition is satisfied

$$\rho_\epsilon^*(0,t) = \rho_\epsilon^*(1,t) = 0, \quad a(0,t)\frac{\partial \rho_\epsilon^*}{\partial x}(0,t) = a(1,t)\frac{\partial \rho_\epsilon^*}{\partial x}(1,t). \tag{23}$$

We introduce the function v such that

$$\begin{cases} K_1 v = \dfrac{x}{\alpha}w + \dfrac{1}{\alpha}\int_\alpha^x w d\zeta, & x \in (0,\alpha), \\[3mm] K_2 v = \dfrac{1-x}{1-\alpha}w - \dfrac{1}{1-\alpha}\int_\alpha^x w d\zeta, & x \in (\alpha,1), \end{cases}$$

then, the function $\rho(x)$ can be expressed as follows

$$\rho(x) = \begin{cases} \dfrac{x^2}{\alpha^2}w = \dfrac{x}{\alpha}K_1 v - \dfrac{1}{\alpha}\int_\alpha^x K_1 v dx, & x \in (0,\alpha), \\[3mm] \dfrac{(1-x)^2}{(1-\alpha)^2}w = \dfrac{1-x}{1-\alpha}K_2 v + \dfrac{1}{1-\alpha}\int_\alpha^x K_2 v dx, & x \in (\alpha,1), \end{cases}$$

and we deduce that

$$\begin{cases} v(0,t) = v(1,t) = 0, \quad \int_0^\alpha K_1 v dx = \int_\alpha^1 K_2 v dx = 0, \\[3mm] a(0,t)K_1(0)\dfrac{\partial v}{\partial x}(0,t) = a(1,t)K_2(1)\dfrac{\partial v}{\partial x}(1,t). \end{cases}$$

and

$$\frac{\partial \rho}{\partial x} = H(x)\frac{\partial[K(x)v]}{\partial x}, \text{ where } H(x) = \begin{cases} \dfrac{x}{\alpha}, & x \in (0,\alpha), \\[3mm] \dfrac{1-x}{1-\alpha}, & x \in (\alpha,1). \end{cases}$$

Putting

$$u = \int_0^t \exp(c\tau)v d\tau,$$

in Equation (20) and integrating with respect to x and t, using Equation (23) we obtain

$$\mathrm{Re}\int_\Omega A(t)u\bar{\rho}dxdt = -\int_\Omega \frac{K(x)H(x)}{2}\left(ca - \frac{\partial a}{\partial t}\right)e^{-ct}\left|\frac{\partial u}{\partial x}\right|^2 dxdt$$

$$-\int_0^1 \frac{K(x)H(x)}{2}ae^{-ct}\left|\frac{\partial u}{\partial x}\right|^2 dx\bigg|_{t=T} - \mathrm{Re}\int_\Omega K(x)H(x)ae^{-ct}\frac{\partial u}{\partial x}\bar{v}dxdt,$$

and

$$\mathrm{Re}\int_0^s \int_0^1 \frac{\partial u}{\partial t}\bar{\rho}dxdt = \int_0^s \int_0^1 e^{ct}H(x)|v|^2 dxdt,$$

we choose

$$c > \frac{a_3}{a_0} + \frac{1}{2},$$

then, we obtain

$$\int_Q \exp(ct)K(x)H(x)|v|^2 dxdt = 0,$$

then $v = 0$ a.e., which implies $w = 0$. $\square$

Theorem 2. *The range $R(L)$ of the operator L is dense in $\mathbb{F}$.*

Proof of Theorem 2. Since F is a Hilbert space, we have $\overline{R(L)} = \mathbb{F}$ if and only if the relation

$$\int_Q \Phi(x) f\overline{g} dxdt + \int_0^1 \Phi(x) \frac{dlu}{dx} \frac{\overline{d\varphi}}{dx} dx + \int_0^1 lu\overline{\varphi} dx, \tag{24}$$

for arbitrary $u \in D(L)$ and $(g, \varphi) \in \mathbb{F}$, implies that $g = 0$ and $\varphi = 0$.

Putting $u \in D_0(L)$ in Equation (24), we conclude from the Lemma 1 that $g = w = 0$, a.e. then $g = 0$.

Taking $u \in D(L)$ in Equation (24) yields

$$\int_0^1 \Phi(x) \frac{dlu}{dx} \frac{\overline{d\varphi}}{dx} dx + \int_0^1 lu\overline{\varphi} dx = 0, \tag{25}$$

Since the two terms in the previous equality vanish independently and since the range of the trace operator l is everywhere dense in Hilbert space with the norm

$$\int_0^1 \Phi(x) \left| \frac{d\varphi}{dx} \right|^2 dx + \int_0^1 |\varphi|^2 dx,$$

hence, $\varphi = 0$. Thus, $\overline{R(A)} = \mathbb{F}$. Then the problem Equations (6), (2)–(4) has a strong solution for an arbitrary F. $\square$

5. Study of the Nonlinear Problem

This section is devoted to the proof of the existence, uniqueness and continuous dependence of the solution on the data of the problem Equations (1)–(4).

If the solution of problem Equations (1)–(4) exists, it can be expressed in the form $u = w + U$, where

U is a solution of the homogeneous problem

$$\pounds U = \frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial U}{\partial x}\right) = 0, \qquad \forall(x, t) \in \Omega, \tag{26}$$

$$U_0 = U(x, 0) = \varphi(x), \qquad \forall x \in [0, 1], \tag{27}$$

$$U(0, t) = U(1, t), \qquad \forall t \in [0, T], \tag{28}$$

$$\int_0^\alpha K_1(x)U(x, t)dx + \int_\alpha^1 K_2(x)U(x, t)dx = 0, \qquad \forall t \in [0, T]. \tag{29}$$

and w is a solution of the problem

$$\pounds w = \frac{\partial w}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right) = F\left(x, t, w, \frac{\partial w}{\partial x}\right), \qquad \forall (x, t) \in \Omega, \tag{30}$$

$$w(x, 0) = 0, \qquad \forall x \in [0, 1], \tag{31}$$

$$w(0, t) = w(1, t), \qquad \forall t \in [0, T], \tag{32}$$

$$\int_0^\alpha K_1(x)w(x, t)dx + \int_\alpha^1 K_2(x)w(x, t)dx = 0, \qquad \forall t \in [0, T]. \tag{33}$$

where $F\left(x, t, w, \frac{\partial w}{\partial x}\right) = f\left(x, t, w + U, \frac{\partial(w + U)}{\partial x}\right)$ and it satisfies the condition

$$|F(x, t, u_1, v_1) - F(x, t, u_2, v_2)| \le d(|u_1 - u_2| + |v_1 - v_2|) \quad \text{for all } x, t \in \Omega. \tag{34}$$

According to Theorem 1 and Lemma 1, the problem Equations (26)–(29) has a unique solution that depends continuously on $U_0 \in V^{1,0}(0, 1)$ where $V^{1,0}(0, 1)$ is a Hilbert space with the scalar product

$$(u, v)_{V^{1,0}(0, 1)} = \int_0^1 \Phi(x)\frac{\partial u}{\partial x}\frac{\overline{\partial v}}{\partial x}dx + \int_0^1 u\bar{v}dx.$$

and with associated norm

$$\|u\|_{V^{1,0}(0, 1)} = \int_0^1 \Phi(x)\left|\frac{\partial u}{\partial x}\right|^2 dx + \int_0^1 |u|^2 dx.$$

We shall prove that the problem Equations (30)–(33) has a weak solution using an approximation process and passage to the limit.

Assume that v and $w \in C^1(\Omega)$, and the following conditions are satisfied

$$\begin{cases} v(x, T) = 0, \ \int_0^\alpha K_1(x)v(x, t)dx + \int_\alpha^1 K_2(x)v(x, t)dx = 0, \\ w(x, 0) = w(1, t), w(0, t) = 0. \end{cases} \tag{35}$$

Taking the scalar product in $L^2(\Omega)$ of Equation (30) and the integrodifferential operator

$$\Lambda v = \begin{cases} \dfrac{x}{a(0, t)}\int_x^\alpha K_1(\mu)v d\mu, & 0 \le x \le \alpha, \\ \dfrac{(1 - x)}{a(1, t)}\int_\alpha^x K_2(\mu)v d\mu & \alpha \le x \le 1. \end{cases},$$

by taking the real part, we obtain

$$H(w, v) = \text{Re} \int_\Omega F\left(x, t, w, \frac{\partial w}{\partial x}\right)\overline{\Lambda v}dxdt$$

$$= \text{Re} \int_\Omega \frac{\partial w}{\partial t}\overline{\Lambda v}dxdt - \text{Re} \int_\Omega \frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right)\overline{\Lambda v}dxdt. \tag{36}$$

Substituting the expression of Λv in the first integral of the right hind-side of Equation (36), integrating with respect to t, using the condition Equation (35), we obtain

$$\text{Re} \int_\Omega \frac{\partial w}{\partial t}\overline{\Lambda v}dxdt = - \text{Re} \int_0^T \int_0^\alpha w\left(\frac{x}{a(0, t)}\int_x^\alpha K_1 \frac{\overline{\partial v}}{\partial t}d\zeta - \frac{xa_t(0, t)}{a^2(0, t)}\int_x^\alpha K_1\bar{v}d\zeta\right)dxdt$$

$$- \text{Re} \int_0^T \int_\alpha^1 w\left(\frac{(1 - x)}{a(1, t)}\int_\alpha^x K_2\frac{\overline{\partial v}}{\partial t}d\zeta - \frac{(1 - x)a_t(1, t)}{a^2(1, t)}\int_\alpha^x K_2(\mu)v d\mu\right)dxdt. \tag{37}$$

Substituting the expression of Mv in the second integral of the right hind-side of Equation (36), integrating with respect to x, using the condition Equation (35), we obtain

$$-\text{Re}\int_{\Omega}\frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right)\overline{\Lambda v}dxdt = \text{Re}\int_0^T\int_0^\alpha\frac{K_1a}{a(0,t)}\left[-x\frac{\partial w}{\partial x}+w\right]\overline{v}dxdt - \int_0^T\int_0^\alpha\frac{1}{a(0,t)}\frac{\partial a}{\partial x}w\int_x^\alpha K_1\frac{\overline{\partial v}}{\partial t}d\zeta dxdt$$

$$+ \text{Re}\int_0^T\int_\alpha^1\frac{K_2a}{a(0,t)}\left[(1-x)\frac{\partial w}{\partial x}+w\right]\overline{v}dxdt + \int_0^T\int_\alpha^1\frac{1}{a(1,t)}\frac{\partial a}{\partial x}w\int_\alpha^x K_2\frac{\overline{\partial v}}{\partial t}d\zeta dxdt. \tag{38}$$

Insertion of Equation (37), Equation (38) into Equation (36) yields

$$H(w,v) = -\text{Re}\int_0^T\int_0^\alpha w\left(\frac{x}{a(0,t)}\int_x^\alpha K_1\frac{\overline{\partial v}}{\partial t}d\zeta - \frac{xa_t(0,t)}{a^2(0,t)}\int_x^\alpha K_1\overline{v}d\zeta\right)dxdt$$

$$-\text{Re}\int_0^T\int_\alpha^1 w\left(\frac{(1-x)}{a(1,t)}\int_\alpha^x K_2\frac{\overline{\partial v}}{\partial t}d\zeta - \frac{(1-x)a_t(1,t)}{a^2(1,t)}\int_\alpha^x K_2(\mu)\overline{v}d\mu\right)dxdt$$

$$+\text{Re}\int_0^T\int_0^\alpha\frac{K_1a}{a(0,t)}\left[-x\frac{\partial w}{\partial x}+w\right]\overline{v}dxdt - \int_0^T\int_0^\alpha\frac{1}{a(0,t)}\frac{\partial a}{\partial x}w\int_x^\alpha K_1\frac{\overline{\partial v}}{\partial t}d\zeta dxdt$$

$$+\text{Re}\int_0^T\int_\alpha^1\frac{K_2a}{a(0,t)}\left[(1-x)\frac{\partial w}{\partial x}+w\right]\overline{v}dxdt + \int_0^T\int_\alpha^1\frac{1}{a(1,t)}\frac{\partial a}{\partial x}w\int_\alpha^x K_2\frac{\overline{\partial v}}{\partial t}d\zeta dxdt ,$$

where

$$H(w,v) =$$
$$\text{Re}\int_0^T\int_0^\alpha\frac{K_1}{a(0,t)}v\int_0^x\zeta\overline{F}\left(\zeta,t,w,\frac{\partial w}{\partial \zeta}\right)d\zeta dxdt + \text{Re}\int_0^T\int_\alpha^1\frac{K_2}{a(1,t)}v\int_x^1(1-\zeta)\overline{F}\left(\zeta,t,w,\frac{\partial w}{\partial \zeta}\right)d\zeta dxdt, \tag{39}$$

obtained by intrgating the right-hand side of Equation (36) with respect to x.

Definition 2. *By a weak solution of problem Equations (30)–(33) we mean a function $w \in L^2(0, T : V^{1,0}(0,1))$ satisfying the identity Equation (39) and the integral condition Equation (33).*

We will construct an iteration sequence in the following way.
Starting with $w_0 = 0$, the sequence $(w_n)_{n\in\mathbb{N}}$ is defined as follows: given w_{n-1}, then for $n \geq 1$, we solve the problem

$$£w_n = \frac{\partial w_n}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial w_n}{\partial x}\right) = F\left(x,t,w_{n-1},\frac{\partial w_{n-1}}{\partial x}\right), \qquad \forall(x,t)\in\Omega, \tag{40}$$

$$w_n(x,0) = 0, \qquad \forall x\in[0,1], \tag{41}$$

$$w_n(0,t) = w_n(1,t), \qquad \forall t\in[0,T], \tag{42}$$

$$\int_0^\alpha K_1(x)w_n(x,t)dx + \int_\alpha^1 K_2(x)w_n(x,t)dx = 0, \qquad \forall t\in[0,T]. \tag{43}$$

From Theorem 1 and Lemma 1, we deduce that for fixed n, each problem Equations (40)–(43) has a unique solution $w_n(x,t)$. If we set $V_n(x,t) = w_{n+1}(x,t) - w_n(x,t)$, we obtain the new problem

$$£V_n = \frac{\partial V_n}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial V_n}{\partial x}\right) = \sigma_{n-1}, \qquad \forall(x,t)\in\Omega, \tag{44}$$

$$V_n(x,0) = 0, \qquad \forall x\in[0,1], \tag{45}$$

$$V_n(0,t) = V_n(1,t), \qquad \forall t\in[0,T], \tag{46}$$

$$\int_0^\alpha K_1(x)V_n(x,t)dx + \int_\alpha^1 K_2(x)V_n(x,t)dx = 0, \qquad \forall t\in[0,T]. \tag{47}$$

where

$$\sigma_{n-1} = F\left(x, t, w_n, \frac{\partial w_n}{\partial x}\right) - F\left(x, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial x}\right). \tag{48}$$

Theorem 3. *Assume that the condition Equation* (34) *holds, for the linearized problem Equations* (44)–(47), *there exists a positive constant k, such that*

$$\|V_n\|_{L^2\left(0, T: V^{1,0}(0, 1)\right)} \leq k\|V_{n-1}\|_{L^2\left(0, T: V^{1,0}(0, 1)\right)}. \tag{49}$$

Proof of Theorem 3. We denote by

$$QV_n = \begin{cases} \dfrac{x^2 K_1}{a(0,t)}e^{\lambda x}\dfrac{\partial V_n}{\partial t} + \dfrac{2x}{a(0,t)}e^{\lambda x}\int_x^\alpha K_1(\mu)\dfrac{\partial V_n}{\partial t}d\mu, & 0 \leq x \leq \alpha, \\[3mm] \dfrac{(1-x)^2 K_2}{a(1,t)}e^{\beta(1-x)}\dfrac{\partial V_n}{\partial t} + \dfrac{2(1-x)}{a(1,t)}e^{\beta(1-x)}\int_\alpha^x K_2(\mu)\dfrac{\partial V_n}{\partial t}d\mu & \alpha \leq x \leq 1. \end{cases}$$

where

$$\begin{cases} \dfrac{\alpha^2}{a(0,t)}e^{\lambda\alpha}K_1(\alpha) = \dfrac{(1-\alpha)^2}{a(1,t)}e^{\beta(1-\alpha)}K_2(\alpha), \\[3mm] \lambda > \max\left(0, \sup_{0 \leq x \leq \alpha}\dfrac{k_1'(x)}{k_1(x)}\right), \\[3mm] \beta > \max(0, \sup_{\alpha \leq x \leq 1}\dfrac{k_2'(x)}{k_2(x)}. \end{cases} \qquad ,$$

We consider the quadratic form obtained by multiplying Equation (44) by $e^{-ct}\overline{QV_n}$, with the constant c satisfying Equation (17), integrating over $\Omega_s = [0, 1] \times [0, s]$, with $0 \leq s \leq T$, taking the real part, we obtain

$$\Phi(V_n, V_n) = \text{Re}\int_{\Omega_s} e^{-ct}\sigma_{n-1}\overline{QV_n}dxdt$$

$$= \text{Re}\int_{\Omega_s} e^{-ct}x^2(1-x)^2\frac{\partial V_n}{\partial t}\overline{QV_n}dxdt - \text{Re}\int_{\Omega_s} e^{-ct}\frac{\partial}{\partial x}\left(a\frac{\partial V_n}{\partial x}\right)\overline{QV_n}dxdt. \tag{50}$$

Following the same procedure done in establishing the proof of Theorem 1, using Equation (34), we obtain

$$\|V_n\|^2_{L^2\left(0, T: V^{1,0}(0, 1)\right)} \leq k^2\|V_{n-1}\|^2_{L^2\left(0, T: V^{1,0}(0, 1)\right)}, \tag{51}$$

where

$$K^2 = \frac{\max\left\{\left(\dfrac{8m_1}{a_0} + \dfrac{32}{m_1 a_0^2 a_1}\right)\alpha^2 e^{\lambda\alpha}, \left(\dfrac{8m_3}{a_0} + \dfrac{32}{m_3 a_0^2 a_1}\right)(1-\alpha)^2 e^{\beta(1-\alpha)}\right\}}{\min\left\{\alpha^2 m_0 \dfrac{ca_0^2 - M_1}{2a_1^2}, (1-\alpha)^2 m_2 \dfrac{ca_0^2 - M_2}{2a_1^2}, \dfrac{a_0 m_2(1-\alpha)^2}{2a_1}, \dfrac{a_0 m_0\alpha^2}{2a_1}\right\}}d^2 e^{cT},$$

Since $V_n(x, t) = w_{n+1}(x, t) - w_n(x, t)$, then the sequence $w_n(x, t)$ can be written as follows

$$w_n(x, t) = \sum_{k=1}^{k=n-1} V_k + w_0(x, t),$$

the sequence $w_n(x, t)$ converge to an element $w \in L^2\left(0, T: V^{1,0}(0, 1)\right)$ if

$$d^2 < \frac{\min\left\{\alpha^2 m_0 \frac{ca_0^2 - M_1}{2a_1^2}, (1-\alpha)^2 m_2 \frac{ca_0^2 - M_2}{2a_1^2}, \frac{a_0 m_2 (1-\alpha)^2}{2a_1}, \frac{a_0 m_0 \alpha^2}{2a_1}\right\}}{\max\left\{\left(\frac{8m_1}{a_0} + \frac{32}{m_1 a_0^2 a_1}\right)\alpha^2 e^{\lambda\alpha}, \left(\frac{8m_3}{a_0} + \frac{32}{m_3 a_0^2 a_1}\right)(1-\alpha)^2 e^{\beta(1-\alpha)}\right\}} e^{-cT}.$$

Now to prove that this limit function w is a solution of the problem under consideration Equations (44)–(47), we should show that w satisfies Equations (33) and (39).

For problem Equations (40)–(43), we have

$$H(w_n - w, v) + H(w, v) =$$

$$\text{Re } \int_0^T \int_0^\alpha \frac{K_1}{a(0,t)} v \int_0^x \zeta\left(\overline{F}\left(\eta, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial \eta}\right) - \overline{F}\left(\eta, t, w, \frac{\partial w}{\partial \eta}\right) d\eta\right) dx dt$$

$$\text{Re } \int_0^T \int_\alpha^1 \frac{K_2}{a(1,t)} v \int_x^1 (1-\zeta)\left(\overline{F}\left(\eta, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial \eta}\right) - \overline{F}\left(\eta, t, w, \frac{\partial w}{\partial \eta}\right)\right) d\eta dx dt \tag{52}$$

$$\text{Re}\int_0^T\int_0^\alpha \frac{K_1}{a(0,t)} v \int_0^x \zeta\overline{F}\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) d\zeta dx dt + \text{Re}\int_0^T\int_\alpha^1 \frac{K_2}{a(1,t)} v \int_x^1 (1-\zeta)\overline{F}\left(\zeta, t, w, \frac{\partial w}{\partial \zeta}\right) d\zeta dx dt.$$

From Equation (40), we have

$$H(w_n - w, v) = \text{Re } \int_\Omega \frac{\partial(w_n - w)}{\partial t}\overline{M}v dx dt - \text{Re } \int_\Omega \frac{\partial}{\partial x}\left(a\frac{\partial(w_n - w)}{\partial x}\right)\overline{M}v dx dt.$$

Integrating with respect to t and x using the conditions Equation (35), we obtain

$$H(w_n - w, v) = -\text{Re } \int_0^T \int_0^\alpha (w_n - w)\left(\frac{x}{a(0,t)}\int_x^\alpha K_1 \overline{\frac{\partial v}{\partial t}} d\zeta - \frac{x a_t(0,t)}{a^2(0,t)}\int_x^\alpha K_1 \overline{v} d\zeta\right) dx dt$$

$$- \text{Re } \int_0^T \int_\alpha^1 (w_n - w)\left(\frac{(1-x)}{a(1,t)}\int_\alpha^x K_2 \overline{\frac{\partial v}{\partial t}} d\zeta - \frac{(1-x)a_t(1,t)}{a^2(1,t)}\int_\alpha^x K_2(\mu)\overline{v} d\mu\right) dx dt \tag{53}$$

$$+\text{Re}\int_0^T\int_0^\alpha \frac{K_1 a}{a(0,t)}\left[-x\frac{\partial(w_n - w)}{\partial x} + (w_n - w)\right]\overline{v} dx dt - \int_0^T\int_0^\alpha \frac{1}{a(0,t)}\frac{\partial a}{\partial x}(w_n - w)\int_x^\alpha K_1 \overline{\frac{\partial v}{\partial t}} d\zeta dx dt$$

$$+\text{Re}\int_0^T\int_\alpha^1 \frac{K_2 a}{a(0,t)}\left[(1-x)\frac{\partial(w_n - w)}{\partial x} + w\right]\overline{v} dx dt + \int_0^T\int_\alpha^1 \frac{1}{a(1,t)}\frac{\partial a}{\partial x}(w_n - w)\int_\alpha^x K_2 \overline{\frac{\partial v}{\partial t}} d\zeta dx dt,$$

Each term of the left-hand side of Equation (53) is controlled by

$$-\text{Re } \int_0^T \int_0^\alpha (w_n - w)\left(\frac{x}{a(0,t)}\int_x^\alpha K_1 \overline{\frac{\partial v}{\partial t}} d\zeta - \frac{x a_t(0,t)}{a^2(0,t)}\int_x^\alpha K_1 \overline{v} d\zeta\right) dx dt \leq$$

$$\max\left(\frac{\alpha^2 m_1 \max(|c_1|, |c_2|)}{a_0^2}, \frac{\alpha^2 m_1}{a_0}\right)\left(\int_\Omega |w_n - w|^2 dx dt\right)^{\frac{1}{2}}\left(\int_\Omega \left|\frac{\partial v}{\partial t}\right|^2 + |v|^2 dx dt\right)^{\frac{1}{2}},$$

$$-\text{Re } \int_0^T \int_\alpha^1 (w_n - w)\left(\frac{(1-x)}{a(1,t)}\int_\alpha^x K_2 \overline{\frac{\partial v}{\partial t}} d\zeta - \frac{(1-x)a_t(1,t)}{a^2(1,t)}\int_\alpha^x K_2(\mu)v d\mu\right) dx dt \leq$$

$$\max\left(\frac{(1-\alpha)^2 m_3 \max(|c_1|, |c_2|)}{a_0^2}, \frac{(1-\alpha)^2 m_3}{a_0}\right)\left(\int_\Omega |w_n - w|^2 dx dt\right)^{\frac{1}{2}}\left(\int_\Omega \left|\frac{\partial v}{\partial t}\right|^2 + |v|^2 dx dt\right)^{\frac{1}{2}},$$

$$\text{Re}\int_0^T \int_0^\alpha \frac{K_1 a}{a(0,t)}\left[-x\frac{\partial(w_n - w)}{\partial x} + (w_n - w)\right]\overline{v} dx dt - \int_0^T\int_0^\alpha \frac{1}{a(0,t)}\frac{\partial a}{\partial x}(w_n - w)\int_x^\alpha K_1 \overline{\frac{\partial v}{\partial t}} d\zeta dx dt \leq$$

$$\max\left(\frac{\alpha m_1 a_1 \sqrt{2}}{a_0}, \frac{\alpha b m_1}{a_0}\right)\left(\int_0^T\int_0^\alpha \left(x^2\left|\frac{\partial(w_n - w)}{\partial x}\right|^2 + |w_n - w|^2\right) dx dt\right)^{\frac{1}{2}}\left(\int_0^T\int_0^\alpha \left(\left|\frac{\partial v}{\partial t}\right|^2 + |v|^2\right) dx dt\right)^{\frac{1}{2}}$$

$$\text{Re} \int_0^T \int_\alpha^1 \frac{K_2 a}{a(0,t)} \left[(1-x) \frac{\partial(w_n - w)}{\partial x} + w \right] \overline{v} dx dt + \int_0^T \int_\alpha^1 \frac{1}{a(1,t)} \frac{\partial a}{\partial x} (w_n - w) \int_\alpha^x K_2 \overline{\frac{\partial v}{\partial t}} d\zeta dx dt \le$$

$$\max \left(\frac{(1-\alpha)m_3 a_1 \sqrt{2}}{a_0}, \frac{\alpha b m_3}{a_0} \right) \left(\int_0^T \int_\alpha^1 \left((1-x)^2 \left| \frac{\partial(w_n-w)}{\partial x} \right|^2 + |w_n - w|^2 \right) dx dt \right)^{\frac{1}{2}} \left(\int_0^T \int_\alpha^1 \left(\left| \frac{\partial v}{\partial t} \right|^2 + |v|^2 \right) dx dt \right)^{\frac{1}{2}}$$

from the previous inequalities, we deduce that

$$|H(w_n - w, v)| \le$$

$$C \left(\|w_n - w\|_{L^2\left(0,T:V^{1,0}(0,1)\right)} \right) \left(\int_\Omega \left(\left| \frac{\partial v}{\partial t} \right|^2 + |v|^2 \right) dx dt \right)^{\frac{1}{2}}, \tag{54}$$

Using the condition Equation (34) and Cauchy-Schwartz inequality in the first two terms in the left-hand side in Equation (52), we obtain

$$\text{Re} \int_0^T \int_0^\alpha \frac{K_1}{a(0,t)} v \int_0^x \zeta \left(F \left(\eta, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial \eta} \right) - F \left(\eta, t, w, \frac{\partial w}{\partial \eta} \right) d\eta \right) dx dt$$

$$\text{Re} \int_0^T \int_\alpha^1 \frac{K_2}{a(1,t)} v \int_x^1 (1-\zeta) \left(F \left(\eta, t, w_{n-1}, \frac{\partial w_{n-1}}{\partial \eta} \right) - F \left(\eta, t, w, \frac{\partial w}{\partial \eta} \right) \right) d\eta dx dt \le \tag{55}$$

$$\max \left(\frac{\alpha d m_1}{a_0}, \frac{(1-\alpha) d m_3}{a_0} \right) \|w_n - w\|_{L^2\left(0,T:V_0^1(0,1)\right)} \left(\int_\Omega |v|^2 dx dt \right)^{\frac{1}{2}},$$

From Equations (54) and (55) and passing to the limit in Equation (52) as $n \to +\infty$, we deduce that

$$H(w,v) = \text{Re} \int_0^T \int_0^\alpha \frac{K_1}{a(0,t)} v \int_0^x \zeta \overline{F} \left(\zeta, t, w, \frac{\partial w}{\partial \zeta} \right) d\zeta dx dt$$

$$+ \text{Re} \int_0^T \int_\alpha^1 \frac{K_2}{a(1,t)} v \int_x^1 (1-\zeta) \overline{F} \left(\zeta, t, w, \frac{\partial w}{\partial \zeta} \right) d\zeta dx dt.$$

Now we show that Equation (33) holds. Since $\lim_{n \to +\infty} \|w_n - w\|_{L^2\left(0,T:V^{1,0}(0,1)\right)} = 0$, then

$$\lim_{n \to +\infty} \left| \int_0^\alpha K_1(w_n - w) dx + \int_\alpha^1 K_2(w_n - w) dx \right|^2 \le \left(m_1 \sqrt{\alpha} + m_3 \sqrt{1-\alpha} \right) \lim_{n \to +\infty} \int_0^1 |w_n - w|^2 dx \to 0. \tag{56}$$

from Equation (56) we conclude that $\int_0^\alpha K_1 w dx + \int_\alpha^1 K_2 w dx = 0$. Then the problem Equations (30)–(33) has a weak solution for arbitrary F. □

Thus, we have proved the following

Theorem 4. *If condition Equation (34) is satisfied, then the solution of problem Equations (30)–(33) is unique.*

Proof of Theorem 4. Suppose that $w_1, w_2 \in L^2\left(0, T : V^{1,0}(0,1)\right)$ are two solution of Equations (30)–(33), the function $v = w_1 - w_2$ is in $L^2\left(0, T : V^{1,0}(0,1)\right)$ and satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x} \left(a \frac{\partial v}{\partial x} \right) = G(x,t), \qquad (x,t) \in \Omega, \tag{57}$$

$$v(x,0) = 0, \qquad x \in [0,1], \tag{58}$$

$$v(0,t) = 0, \qquad t \in [0,T], \tag{59}$$

$$\int_0^\alpha K_1 v dx + \int_\alpha^1 K_2 v dx = 0, \qquad t \in [0,T], \tag{60}$$

where $G(x,t) = F \left(x, t, w_1, \frac{\partial w_1}{\partial x} \right) - F \left(x, t, w_2, \frac{\partial w_2}{\partial x} \right)$.

Taking the inner product in $L^2(\Omega)$ of Equation (57) and the integro-differential operator

$$
Mu = \begin{cases}
\dfrac{x^2 K_1}{a(0,t)} e^{\lambda x} \dfrac{\partial v}{\partial t} + \dfrac{2x}{a(0,t)} e^{\lambda x} \int_x^\alpha K_1(\mu) \dfrac{\partial v}{\partial t} d\mu, & 0 \leq x \leq \alpha, \\[3mm]
\dfrac{(1-x)^2 K_2}{a(1,t)} e^{\beta(1-x)} \dfrac{\partial v}{\partial t} + \dfrac{2(1-x)}{a(1,t)} e^{\beta(1-x)} \int_\alpha^x K_2(\mu) \dfrac{\partial v}{\partial t} d\mu & \alpha \leq x \leq 1.
\end{cases}
$$

where

$$
\begin{cases}
\dfrac{\alpha^2}{a(0,t)} e^{\lambda \alpha} K_1(\alpha) = \dfrac{(1-\alpha)^2}{a(1,t)} e^{\beta(1-\alpha)} K_2(\alpha), \\[3mm]
\lambda > \max\left(0, \sup_{0 \leq x \leq \alpha} \dfrac{k_1'(x)}{k_1(x)} \right), \\[3mm]
\beta > \max(0, \sup_{\alpha \leq x \leq 1} \dfrac{k_2'(x)}{k_2(x)}.
\end{cases}
$$

Following the same procedure done in establishing the proof of Theorem 1, we obtain

$$
\|v\|^2_{L^2\left(0,\,T:V^{1,0}(0,1)\right)} \leq k^2 \|v\|^2_{L^2\left(0,\,T:V^{1,0}(0,1)\right)}.
$$

where

$$
K^2 = \frac{\max\left\{ \left(\dfrac{8m_1}{a_0} + \dfrac{32}{m_1 a_0^2 a_1} \right) \alpha^2 e^{\lambda \alpha}, \left(\dfrac{8m_3}{a_0} + \dfrac{32}{m_3 a_0^2 a_1} \right) (1-\alpha)^2 e^{\beta(1-\alpha)} \right\}}{\min\left\{ \alpha^2 m_0 \dfrac{ca_0^2 - M_1}{2a_1^2}, (1-\alpha)^2 m_2 \dfrac{ca_0^2 - M_2}{2a_1^2}, \dfrac{a_0 m_2 (1-\alpha)^2}{2a_1}, \dfrac{a_0 m_0 \alpha^2}{2a_1} \right\}} d^2 e^{cT},
$$

Since $k^2 < 1$, then $v = 0$, which implies that $w_1 = w_2 \in L^2\left(0, T : V^{1,0}(0,1)\right)$. Then the uniqueness of the weak solution holds. $\square$

6. Conclusions

In this work we studied the existence, uniqueness and continuous dependence of a weak solution for some classes of mixed nonlinear problems with nonlocal conditions (boundary integral conditions). The used method is one of the most efficient functional analysis methods for solving linear partial differential equations with boundary integral conditions, the so-called energy-integral method or a priori estimates method. We constructed for each problem suitable multiplicators, which provide the a priori estimate, from which it was possible to establish the solvability of the problem. For the nonlocal mixed problems for nonlinear equations, we first established the a priori estimate for an associated linear problem and hence the solvability of this associated linear problem. Then, by applying an iterative process based on the obtained results for the linear problem, we proved the existence, uniqueness and continuous dependence of a weak solution of the considered nonlinear problem.

References

1. Cahlon, B.; Kulkarni, D.M.; Shi, P. Stepwise stability for the heat equation with nonlocal constraint. *SIAM J. Numer. Anal.* **1995**, *32*, 571–593. [CrossRef]
2. Cannon, J.R. The solution of the heat equation subject to the specification of energy. *Quart. Appl. Math.* **1963**, *21*, 155–160. [CrossRef]
3. Ionkin, N.I. The solution of a certain boundary value problem of the theory of heat conduction with a nonclasical boundary condition. *Differ. Uravn.* **1977**, *13*, 294–304. (In Russian)
4. Kamynin, N.I. A boundary value problem in the theory of the heat conduction with non classical boundary conditions. *Th. Vychisl. Mat. Fiz.* **1964**, *43*, 1006–1024.
5. Samarski, A.A. Some problems in the modern theory of differential equations. *Differ. Uravn.* **1980**, *16*, 1221–1228. (In Russian)
6. Shi, P.; Shilor, M. *Design of Contact Patterns in one Dimensional Thermoelasticity in Theoretical Aspects of Industrial Design*; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 1992.
7. Choi, Y.S.; Chan K.Y. A parabolic equation with nonlocal boundary conditions arising from electrochemistry. *Nonlinear Anal.* **1992**, *18*, 317–331. [CrossRef]
8. Cushman, J.H.; Xu, H.; Deng, F. Nonlocal reactive transport with physical and chemical heterogeneity: Localization error. *Water Resour. Res.* **1995**, *31*, 2219–2237. [CrossRef]
9. Ewing, R.E.; Lin, T. A class of parameter estimation techniques for fluid flow in porous media. *Adv. Water Resour.* **1997**, *14*, 89–97. [CrossRef]
10. Nakhushev, A.M. On a certain approximate method for boundary-value problems for differential equations and their applications in ground waters dynamics. *Differ. Uravn.* **1982**, *18*, 72–81.
11. Vodakhova, V.A. A boundary-value problem with Nakhuchev nonlocal condition for a certain pseudoparabolic water transfer equation. *Differ. Uravn.* **1982**, *18*, 280–285.
12. Batten, G.W., Jr. Second-order correct boundary conditions for the numerical solution of mixed boundary problem for parabolic equations. *Math. Comp.* **1963**, *17*, 405–413. [CrossRef]
13. Beilin, A.S. Existence of solutions for one-dimensional wave equation with a nonlocal condition. *Electron. J. Differ. Equ.* **2001**, *2001*, 1–8.
14. Beilin, A.S. On a mixed nonlocal problem for a wave equation. *Electron. J. Differ. Equ.* **2006**, *2006*, 1–10.
15. Belmouloud, I.; Memou, A. On the solvability of a class of nonlinear singular parabolic equation with integral boundary condition. *Appl. Math. Comput.* **2020**, *373*, 124999. [CrossRef]
16. Cannon, J.R. *The One-Dimensional Heat Equation*; Encyclopedia of Mathematics and Its Applications; Addison-Wesley Publishing: Boston, MA, USA, 1984; Volume 23.
17. Kartynnik, A.V. A three-point boundary-value problem with an integral space-variable condition for a second-order parabolic equations. *Differ. Equ.* **1990**, *26*, 1160–1166.
18. Korzyuk, V.I.; Kozlovskaya, I.S.; Naumavets, S.N. Classical solution of a problem with integral congitions of the second kind for the one-dimensional wave equation. *Differ. Equ.* **2019**, *3*, 353–362. [CrossRef]
19. Shi, P. Weak solution to evolution problem with a nonlocal constraint. *SIAM J. Math. Anal.* **1993**, *24*, 46–58. [CrossRef]
20. Yurchuk, N.I. Mixed problem with an integral condition for certain parabolic equations. *Differ. Equ.* **1986**, *22*, 1457–1463.
21. Denche, M.; Memou, A. Boundary value problem with integral conditions for a linear third order equation. *J. Appl. Math.* **2003**, *11*, 553–567. [CrossRef]
22. Makarov, V.L.; Kulyev, D.T. Solution of a boundary value problem for a quasi-parabolic equation with a nonclassical boundary condition. *Differ. Equ.* **1985**, *21*, 296–305.
23. Marhoune, A.L.; Lakhal, F. A boundary value problem with multivariables integral type condition for parabolic equations. *J. Appl. Math. Stoch. Anal.* **2009**, [CrossRef]
24. Marhoune, A.L.; Memou, A. Nonlocal singular problem with integral condition for a second-order parabolic equation. *Electron. J. Differ. Equ.* **2015**, *2015*, 1–9.
25. Ashyralyev, A.; Aggez, N. On the Solution of Multipoint NBVP for Hyperbolic Equation with Integral Condition. *Malays. J. Math. Sci.* **2012**, *6*, 111–121.
26. Assanova, A.T. Nonlocal problem with integral conditions for the system of hyperbolic equations in the characteristic rectangle. *Russ. Math.* **2017**, *5*, 7–20. [CrossRef]
27. Avalishvili, G.; Avalishvili, M.; Gordeziani, D. On Integral Nonlocal Boundary Value Problems for some Partial Di erential Equations. *Bull. Georgian Natl. Acad. Sci.* **2011**, *5*, 31–37.
28. Pulkina, L.S. A non-local problem with integral conditions for hyperbolic equations. *Electron. J. Differ. Equ.* **1999**, *1999*, 1–6.
29. Pulkina, L.S. A non-local problem with integral conditions for the quasilinear hyperbolic equation. *Math. Notes* **2001**, *70*, 88–95
30. Pulkina, L.S. A Mixed Problem With Integral Condition for the Hyperbolic Equation. *Math. Notes* **2003**, *74*, 411–421 [CrossRef]
31. Pulkina, L.S. Nonlocal Problems for Hyperbolic Equation from the viewpoint of strongly regular boundary conditions. *Electron. J. Differ. Equ.* **2020**, *2020*, 1–20
32. Volkodavov, V.F.; Zhukov, V.E. Two problems for the string vibration equation with integral conditions and special matching conditions on the characteristic. *Differ. Equ.* **1998**, *34*, 501–505.

33. Denche, M.; Marhoune, A.L. A three-point boundary value problem with an integral condition for parabolic equations with the Bessel operator. *Appl. Math. Lett.* **2000**, *13*, 85–89. [CrossRef]
34. Denche, M.; Marhoune A.L. High order mixed-type differential equations with weighted integral boundary conditions. *Electron. J. Differ. Equ.* **2000**, *2000*, 1–10.
35. Denche, M.; Marhoune, A.L. Mixed problem with nonlocal boundary conditions for a third order partial differential equation of mixed type. *Int. J. Math. Math. Sci.* **2001**, *26*, 417–426. [CrossRef]
36. Denche, M.; Marhoune, A.L. Mixed problem with integral boundary condition for a high order mixed type partial differential equation. *J. Appl. Math. Stoch. Anal.* **2003**, *16*, 69–79. [CrossRef]
37. Latrous, C.; Memou, A. A Three- point Boundary Value Problem with an Integral condition for a third order partial differential equation. *Abstr. Appl. Anal.* **2005**, *1*, 33–43. [CrossRef]
38. Skubachevski, A.L.; Steblov, G.M. On spectrum of differential operators with domain non-dense in L_2. *Doklad. Akad. Nauk USSR* **1991**, *321*, 1158–1163.
39. Gushin, A.K.; Mikhailov, V.P. On solvability of nonlocal problem for second-order elliptic equation. *Matem. Sbornik.* **1994**, *185*, 121–160.
40. Paneiah, B.P. On certain nonlocal boundary problem's for linear differential operators. *Mat. Zametki* **1984**, *35*, 425–434.