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Article

Revisiting a Classic Identity That Implies the Rogers–Ramanujan Identities II

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Abstract: We give a new proof of an identity due to Ramanujan. From this identity, he deduced the famous Rogers–Ramanujan identities. We prove this identity by establishing a simple recursion $J_k = q^k J_{k-1}$, where $|q| < 1$. This is a sequel to our recent work.

Keywords: q-series; Rogers–Ramanujan identities

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1. Introduction

The Rogers–Ramanujan identities, given by (with $|q| < 1$)

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})}, \tag{1}$$

$$1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}, \tag{2}$$

are perhaps one of the most important and celebrated results in the theory of partition. See [1] (in particular, Entries 38 (i) and (ii)) for an account of the fascinating history behind these identities. See [2] for an account of the state-of-the-art research on these identities. See also the introduction in [3].

Note that we use the notation of Hardy and Wright [4] for (1) and (2). See Appendix A for a brief discussion on notation.

In our recent work [3], we revisited the following identity: for a being a variable and $|q| < 1$, we have

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k} a^k = \sum_{r=0}^{\infty} (-1)^r a^{2r} q^{r(5r-1)/2} \left(1 - a\, q^{2r}\right) \frac{1}{(q)_r (a\, q^r; q)_\infty}, \tag{3}$$

where the q-Pochhammer symbol is defined by

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \qquad \text{if } n \geq 1 \tag{4}$$

and $(a;q)_0 := 1$. Note also the frequently used notation

$$(q)_n := (q;q)_n.$$

Identity (3) is a classic and spectacular result: it was proven by Ramanujan in [5]. From (3), he proved (1) and (2).

Note that it was Hardy who arranged the publication of [5], in which Rogers and Ramanujan independently contributed proofs of (1) and (2). Interestingly, both proofs are essentially the same.

For an account of Ramanujan's proof of (3), see [6] (in particular, Section 15.2). See also [2,4,7–14].

The left-hand side of (3) is already a power series in a. It is natural to ask: what is the coefficient of a^k on the right-hand side?

By comparing the coefficients of a^k on both sides of (3) (details can be found in Appendix A in [3]) we have the following polynomial identity: for $k \geq 1$, we have

$$\begin{aligned} q^{k^2} &= 1 + \sum_{r=1}^{\lfloor k/2 \rfloor} \begin{bmatrix} k-r \\ r \end{bmatrix} (-1)^r q^{r(r-1)/2+kr} \left(q^{k+1-r}; q\right)_r \\ &\quad + \sum_{r=1}^{\lfloor (k+1)/2 \rfloor} \begin{bmatrix} k-r \\ r-1 \end{bmatrix} (-1)^r q^{r(r-1)/2+k(r-1)} \left(q^{k+1-r}; q\right)_r. \end{aligned} \tag{5}$$

(Note that we have multiplied $(q)_k$ to both sides.) Here, the q-binomial numbers, also known as the Gaussian polynomials, are defined by

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} \dfrac{(1-q^A)(1-q^{A-1})\cdots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\cdots(1-q)} = \dfrac{(q)_A}{(q)_B\,(q)_{A-B}} & \text{if } 0 \leq B \leq A \\ 0 & \text{otherwise.} \end{cases}$$

Identity (5) is far from obvious. After all, it is equivalent to (3), and the latter implies the Rogers–Ramanujan identities. All the proofs of the Rogers–Ramanujan identities are known to be far from obvious.

In [3], we give a new and elementary proof of (5). The motivation of that proof is as follows. The right-hand side of (5) is far more complicated than the left-hand side. Many terms, when we multiply out the right-hand side, should cancel each other. However, it is not clear how the single term on the left emerges at the end. See Appendix B for an illustration of this process. The proof in [3] is aimed at understanding the cancellation process.

While the overall strategy of the proof in [3] occupies less than a page (see p. 300 of [3] after Theorem 1.3), the most challenging parts rely on how terms are grouped and paired up for cancellations (hence the title "Chasing after cancellations"). The following is one of the key decomposition results (see Theorem 1.1 in [3]): For $k \geq 2$ and $1 \leq r \leq \lfloor k/2 \rfloor$, we have

$$\alpha(k)_r \;=\; \alpha^+(k)_r + \alpha^-(k)_r - q^{(2r-1)k}, \tag{6}$$

where

$$\alpha(k)_r = (-1)^r q^{r(r-1)/2+kr} \begin{bmatrix} k-r \\ r \end{bmatrix} \left(q^{k+1-r}; q\right)_{r-1} \qquad 1 \leq r \leq \lfloor k/2 \rfloor, \tag{7}$$

and

$$\alpha^\pm(k)_r := \sum_{s=0}^{r-1} \sum_{j=1}^{s+1} \alpha^\pm(k)_{rsj} \qquad 1 \leq r \leq \lfloor k/2 \rfloor, \tag{8}$$

with

$$\alpha^+(k)_{rsj} := \gamma(k)_{rs} \begin{bmatrix} k-r-j \\ r-j+1 \end{bmatrix} \begin{bmatrix} r-j \\ s-j+1 \end{bmatrix} q^{r-j+1} \qquad \begin{matrix} 1 \leq r \leq \lfloor (k-1)/2 \rfloor, \\ 0 \leq s \leq r-1, \\ 1 \leq j \leq s+1, \end{matrix} \tag{9}$$

$$\alpha^-(k)_{rsj} := \gamma(k)_{rs} \begin{bmatrix} k-r-j \\ r-j \end{bmatrix} \begin{bmatrix} r-j-1 \\ s-j+1 \end{bmatrix} q^{s-j+1} \qquad \begin{matrix} 1 \leq r \leq \lfloor k/2 \rfloor, \\ 0 \leq s \leq r-2, \\ 1 \leq j \leq s+1, \end{matrix} \tag{10}$$

where $\gamma(k)_{rs} := (-1)^{r+s} q^{r(r-1)/2 + kr + s(s-1)/2 + (k+1-r)s}$ (see (2.2)–(2.7) in [3]). For each of these functions, if one or more of its arguments lies outside the range specified on the right-hand side, the function is defined to be zero. It turns out $\alpha^{\pm}(k)_r$ are canceled by their counterparts (see Theorem 1.2 and Lemma 3.2 in [3]).

In this paper, we give yet another proof of (5), which is motivated by the simplicity of the left-hand side (which has only one term, q^{k^2}). Since $q^{k^2} = q^{2k-1} q^{(k-1)^2}$, we expect the right-hand side of (5) would behave the same way–see (12) in the next section for details.

One may wonder if one of these proofs is better than or superior to the other. Each proof focuses and relies on different aspects of (5). The proof in [3] focuses on cancellations: it is based on how the terms on the right-hand side of (5), when multiplied out, cancel each other. The present proof relies on how the right-hand side of (5) depends on k (in the sense of (12)). We believe that each proof has its own merit. Given the importance of (5), as it implies the Rogers–Ramanujan identities, both of these proofs should be of interest.

We may add that tracking down cancellations (as in the proof in [3]) is a rather tedious process (and we need to deal with many expressions that involve bi-Gaussian polynomials, such as (9) and (10) above, and sort out which terms cancel each other). In the present proof, this is avoided and the main calculations involved are those of proving (13) and (14), which are relatively simpler when compared with the proof in [3].

2. Proof of the Main Result

For $k \geq 1$, let

$$J_k := \text{the right-hand side of (5)}, \tag{11}$$

and $J_0 := 1$. We are to prove that, for $k \geq 1$,

$$J_k = q^{2k-1} J_{k-1}. \tag{12}$$

Observe that this equation implies (5): applying (12) recursively gives

$$\begin{aligned} J_k &= q^{2k-1} J_{k-1} \\ &= q^{(2k-1)+(2k-3)} J_{k-2} \\ &\;\;\vdots \\ &= q^{(2k-1)+\cdots+3+1} J_0. \end{aligned}$$

Thanks to the simple fact that

$$1 + 3 + 5 + \cdots + (2k-1) = k^2,$$

the above implies $J_k = q^{k^2}$, the left-hand side of (5).

Let us turn to the proof of (12). We start with some definitions. These symbols are to mask out the complexity and help make the main derivation more transparent. For each of the functions defined below, its *admissible domain* for its arguments is stated on the right. Whenever its argument lies outside the stated admissible domain, the function is defined to be zero.

$$a(k)_r := \frac{(q)_k}{(q)_r (q)_{k-2r}} (-1)^r q^{r(r-1)/2+kr-r} \qquad 0 \leq r \leq \lfloor k/2 \rfloor,$$

$$b(k)_r := \frac{(q)_k}{(q)_{r-1} (q)_{k-2r}} (-1)^r q^{r(r-1)/2+kr-k} \qquad 1 \leq r \leq \lfloor k/2 \rfloor,$$

$$c(k)_r := \frac{(q)_k}{(q)_r (q)_{k-2r}} (-1)^r q^{r(r-1)/2+kr} \qquad 0 \leq r \leq \lfloor k/2 \rfloor,$$

$$d(k)_r := \frac{(q)_k}{(q)_{r-1} (q)_{k-2r+1}} (-1)^r q^{r(r-1)/2+kr-k} \qquad 1 \leq r \leq \lfloor (k+1)/2 \rfloor.$$

Here is the proof of (12):

$$\begin{aligned}
J_k &\overset{(e1)}{=} \sum_r c(k)_r + d(k)_r \\
&\overset{(e2)}{=} \sum_r c(k)_r + d(k)_{r+1} \\
&\overset{(e3)}{=} q^k \sum_r (a(k)_r + b(k)_r) \\
&= \frac{q^k}{1-q^k} \sum_r (1-q^k)\,(a(k)_r + b(k)_r) \\
&= \frac{q^k}{1-q^k} \sum_r \Big\{ (1-q^k)\, a(k)_r - q^k b(k)_r + b(k)_r \Big\} \\
&\overset{(e4)}{=} \frac{q^k}{1-q^k} \sum_r \Big\{ (1-q^k)\, a(k)_r - q^k b(k)_r + b(k)_{r+1} \Big\} \\
&\overset{(e5)}{=} \frac{q^k}{1-q^k} \sum_r (1-q^k)\, q^{k-1} (c(k-1)_r + d(k-1)_r) \\
&= q^{2k-1} \sum_r (c(k-1)_r + d(k-1)_r) \\
&= q^{2k-1} J_{k-1}.
\end{aligned}$$

In the above derivation, $\sum_r$ denotes a generic sum with $r \geq 0$. The actual non-zero terms are determined by the admissible domains involved. For example, $\sum_r d(k)_r$ covers all r from 1 to $\lfloor (k+1)/2 \rfloor$.

The details for (e1) to (e5) are as follows:

(e1) The fact that $J_k = \sum_r c(k)_r + d(k)_r$ follows from the definition of $c(k)_r$ and $d(k)_r$. For example, the last sum of J_k (see (5)) gives $\sum_r d(k)_r$, as

$$\begin{aligned}
&\begin{bmatrix} k-r \\ r-1 \end{bmatrix} (-1)^r q^{r(r-1)/2+k(r-1)} \left(q^{k+1-r};q\right)_r \\
&= \frac{(q)_{k-r}}{(q)_{r-1}(q)_{k-2r-1}} (-1)^r q^{r(r-1)/2+k(r-1)} \left(q^{k+1-r};q\right)_r \\
&= \frac{(q)_k}{(q)_{r-1}(q)_{k-2r-1}} (-1)^r q^{r(r-1)/2+k(r-1)},
\end{aligned}$$

which is $d(k)_r$. To obtain the last equality, we have used the fact that

$$(q)_{k-r}\left(q^{k+1-r};q\right)_r = (q)_k.$$

Note that the constant term 1 on the right-hand side of (5) is $c(k)_0$.

(e2) The summand $c(k)_r$ is paired up with $d(k)_{r+1}$.

(e3) We have used the following identity. Let $l := \lfloor (k+1)/2 \rfloor$, then, for $r = 0, 1, \cdots, l$,

$$c(k)_r + d(k)_{r+1} = q^k (a(k)_r + b(k)_r). \tag{13}$$

Note that, when taking the admissible domains into account, this equation is given by:

$$\begin{aligned}
c(k)_0 + d(k)_1 &= q^k a(k)_0, \\
c(k)_r + d(k)_{r+1} &= q^k (a(k)_r + b(k)_r) \qquad 1 \leq r \leq l-1, \\
c(k)_l &= q^k (a(k)_l + b(k)_l).
\end{aligned}$$

To prove these identities, one simply applies the definition of these functions ($a(k)_r$, etc.) and checks that these identities are satisfied. For example, for $1 \leq r \leq l-1$, we have

$$
\begin{aligned}
&c(k)_r + d(k)_{k+1} \\
&= \frac{(q)_k}{(q)_r (q)_{k-2r}} (-1)^r q^{r(r-1)/2+kr} \\
&\qquad\qquad + \frac{(q)_k}{(q)_r (q)_{k-2r-1}} (-1)^{r+1} q^{r(r+1)/2+kr} \\
&= \frac{(q)_k}{(q)_r (q)_{k-2r}} (-1)^r q^{r(r-1)/2+kr} \left(1 - (1-q^{k-2r}) q^r\right) \\
&= q^k \frac{(q)_k}{(q)_r (q)_{k-2r}} (-1)^r q^{r(r-1)/2+kr} \left((1-q^r) q^{-k} + q^{-r}\right) \\
&= q^k (b(k)_r + a(k)_r).
\end{aligned}
$$

The cases for $r = 0$ and l can be verified directly by using the definition of the functions involved.

(e4) The summand $(1-q^k)\, a(k)_r - q^k b(k)_r$ is paired up with $b(k)_{r+1}$ (instead of $b(k)_r$).

(e5) We have used the following: for $r \geq 0$,

$$
\begin{aligned}
&(1-q^k)\, a(k)_r - q^k b(k)_r + b(k)_{r+1} \\
&\qquad = (1-q^k)\, q^{k-1} (c(k-1)_r + d(k-1)_r).
\end{aligned} \tag{14}
$$

The proof is similar to that of (13) and will be omitted. As in (e3), the boundary cases, $r = 0$ and $r = \lfloor k/2 \rfloor$, need to be checked separately because some of the terms are zero (due to the admissibility conditions).

This completes the proof of (5). □

3. Conclusions

In this paper we give a new proof of (5). This, in turn, gives a new proof of (3) (by reversing the process outlined in Appendix A in [3]), from which the Rogers–Ramanujan identities are derived.

An Open Question: Find a combinatorial proof of (12). The simplicity of (12) cries out for a combinatorial explanation. One interesting approach would be the method of combinatorial telescoping; see [15]. Such a proof should shed light on the combinatorial nature of (5). We believe that this should also lead to a new combinatorial proof of (1) and (2), the Rogers–Ramanujan identities.

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Appendix A

In this appendix, we briefly discuss the notation for (1) and (2). As mentioned in the Introduction, our notation for (1) and (2) follows that of Hardy and Wright [4]. Other ways of writing these identities include (as in Gasper and Rahman [16])

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty}, \tag{A1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}. \tag{A2}$$

Here, the q-Pochhammer symbol, (4), is used. In addition, the notation on the right-hand side means

$$(a_1,a_2,\cdots,a_k;q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty. \tag{A3}$$

With this understood, we note that

$$\begin{aligned}
&\frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty} \quad \text{(The RHS of (A1))} \\
&= \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty}{(q;q^5)_\infty (q^2;q^5)_\infty (q^3;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty} \\
&= \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \\
&= \prod_{k=1}^{\infty} \frac{1}{(1-q^{5k-4})(1-q^{5k-1})},
\end{aligned}$$

giving the right-hand side of (1). A similar calculation shows the equivalence of (2) and (A2).

Appendix B

We illustrate (5) with an explicit calculation. The first few terms of (5) are given by

$$\begin{aligned}
q^{k^2} \quad = \quad & 1 - \begin{bmatrix} k-1 \\ 1 \end{bmatrix} q^k \left(q^k;q\right)_1 + \begin{bmatrix} k-2 \\ 2 \end{bmatrix} q^{2k+1} \left(q^{k-1};q\right)_2 + \cdots \\
& - \begin{bmatrix} k-1 \\ 0 \end{bmatrix} \left(q^k;q\right)_1 + \begin{bmatrix} k-2 \\ 1 \end{bmatrix} q^{k+1} \left(q^{k-1};q\right)_2 + \cdots
\end{aligned} \tag{A4}$$

Showing all the terms on the right-hand side of (A5) reduce to the left is the most challenging part of our previous paper [3]. A glimpse of the cancellation process can be seen in the following example.

$$\begin{aligned}
&\text{The RHS of (A5) with } k = 3 \\
&= 1 - \begin{bmatrix} 2 \\ 1 \end{bmatrix} q^3 \left(q^3;q\right)_1 \\
&\qquad - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \left(q^3;q\right)_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} q^4 \left(q^2;q\right)_2 \\
&= 1 - (1+q)q^3(1-q^3) \\
&\qquad - (1-q^3) + q^4(1-q^2)(1-q^3) \\
&= 1 - (1-q^3)\Big((1+q)q^3 + 1 - q^4(1-q^2) \Big) \\
&= 1 - (1-q^3)\Big(q^3 + q^4 + 1 - q^4 + q^6 \Big) \\
&\overset{(\spadesuit)}{=} 1 - (1-q^3)\Big(1 + q^3 + q^6 \Big) \\
&= 1 - (1-q^9) = q^9.
\end{aligned}$$

Note that, for the general case of k, the line (♠) becomes

$$1 - (1 - q^k)(1 + q^k + q^{2k} + \cdots + q^{(k-1)k}).$$

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