axioms

MDPI

Article

On Periodic Fractional (p, q)-Integral Boundary Value Problems for Sequential Fractional (p, q)-Integrodifference Equations

Jarunee Soontharanon [1] and Thanin Sitthiwirattham [2,*]

[1] Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; jarunee.s@sci.kmutnb.ac.th
[2] Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand
* Correspondence: thanin_sit@dusit.ac.th

Abstract: We study the existence results of a fractional (p, q)-integrodifference equation with periodic fractional (p, q)-integral boundary condition by using Banach and Schauder's fixed point theorems. Some properties of (p, q)-integral are also presented in this paper as a tool for our calculations.

Keywords: fractional (p, q)-integral; fractional (p, q)-difference; periodic boundary value problems; existence

Citation: Soontharanon, J.; Sitthiwirattham, T. On Periodic Fractional (p, q)-Integral Boundary Value Problems for Sequential Fractional (p, q)-Integrodifference Equations. *Axioms* **2021**, *10*, 264. https://doi.org/10.3390/axioms10040264

Academic Editor: Natália Martins

Received: 8 September 2021
Accepted: 18 October 2021
Published: 19 October 2021

1. Introduction

The studies of quantum calculus with integer order were presented in the last three decades, and many researchers extensively studied calculus without a limit that deals with a set of nondifferentiable functions, the so-called quantum calculus. Many types of quantum difference operators are employed in several applications of mathematical areas, such as the calculus of variations, particle physics, quantum mechanics, and theory of relativity. The q-calculus, one type of quantum initiated by Jackson [1–5], was employed in several fields of applied sciences and engineering such as physical problems, dynamical system, control theory, electrical networks, economics, and so on [6–14].

For fractional quantum calculus, Agarwal [15] and Al-Salam [16] proposed fractional q-calculus, and Díaz and Osler [17] proposed fractional difference calculus. In 2017, Brikshavana and Sitthiwirattham [18] introduced fractional Hahn difference calculus. In 2019, Patanarapeelert and Sitthiwirattham [19] studied fractional symmetric Hahn difference calculus.

Later, the motivation of quantum calculus based on two parameters (p, q)-integer was presented. The (p, q)-calculus (postquantum calculus) was introduced by Chakrabarti and Jagannathan [20]. This calculus was used in many fields such as special functions, approximation theory, physical sciences, Lie group, hypergeometric series, Bézier curves, and surfaces. For some recent papers about (p, q)-differenceequations, we refer to [21–33] and the references therein. For example, the fundamental theorems of (p, q)-calculus and some (p, q)-Taylor formulas were studied in [21]. In [32], the (p, q)-Melin transform and its applications were studied. The Picard and Gauss–Weierstrass singular integral in (p, q)-calculus were introduced in [33]. For the boundary value problem for (p, q)-difference equations were studied in [34–36]. For example, the nonlocal boundary value problems for first-order (p, q)-difference equations were studied in [34]. The second-order (p, q)-difference equations with separated boundary conditions were studied in [35]. In [36], the authors studied the first-order and second-order (p, q)-difference equations with impulse.

Recently, Soontharanon and Sitthiwirattham [37] introduced the fractional (p, q)-difference operators and its properties. Now, this calculus was used in the inequalities [38,39] and the boundary value problems [40–42]. However, the study of the boundary value problems for fractional (p, q)-difference equation in the beginning, there are a few literature on this knowledge. In [40], the existence results of a fractional (p, q)-integrodifference

equation with Robin boundary condition were studied in 2020. In 2021 [41], the authors investigated the boundary value problem of a class of fractional (p, q)-difference Schrödinger equations. In the same year, the existence results of solution and positive solution for the boundary value problem of a class of fractional (p, q)-difference equations involving the Riemann–Liouville fractional derivative [42] were studied.

Motivated by the above papers, we seek to enrich the contributions in this new research area. In this paper, we introduce and study the boundary value problem involving function F, which depends on fractional (p, q)-integral and fractional (p, q)-difference, and the boundary condition is nonlocal. Our problem is sequential fractional (p, q)-integrodifference equation with periodic fractional (p, q)-integral boundary conditions of the form

$$\begin{aligned} &D^{\alpha}_{p,q}D^{\beta}_{p,q}u(t) = F\Big[t, u(t), \Psi^{\gamma}_{p,q}u(t), D^{\nu}_{p,q}u(t)\Big], \quad t \in I^{T}_{p,q}, \\ &u(0) = u\left(\frac{T}{p}\right) \\ &\mathcal{I}^{\theta}_{p,q}g(\eta)u(\eta) = \varphi(u), \quad \eta \in I^{T}_{p,q} - \{0, T\}, \end{aligned} \tag{1}$$

where $I^{T}_{p,q} := \left\{\left(\frac{q}{p}\right)^{k}\frac{T}{p} : k \in \mathbb{N}_0\right\} \cup \{0\}$; $0 < q < p \leq 1$; $\alpha, \beta, \gamma, \nu, \theta \in (0, 1]$; $F \in C\big(I^{T}_{p,q} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$, $g \in \big(I^{T}_{p,q}, \mathbb{R}^{+}\big)$ are given functions; $\varphi : C\big(I^{T}_{p,q}, \mathbb{R}\big) \to \mathbb{R}$ is given functional; and for $\phi \in C\big(I^{T}_{p,q} \times I^{T}_{p,q}, [0, \infty)\big)$, we define an operator of the (p, q)-integral of the product of functions ϕ and u as

$$\Psi^{\gamma}_{p,q}u(t) := \big(\mathcal{I}^{\gamma}_{p,q}\phi\, u\big)(t) = \frac{1}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)}\int_0^t (t - qs)^{\underline{\gamma-1}}_{p,q}\phi\left(t, \frac{s}{p^{\gamma-1}}\right)u\left(\frac{s}{p^{\gamma-1}}\right)d_{p,q}s.$$

We aim to show the existence results to the problem (1). Firstly, we convert the given nonlinear problem (1) into a fixed point problem related to (1), by considering a linear variant of the problem at hand. Once the fixed point operator is available, we make use the classical Banach's and Schauder's fixed point theorems to establish existence results.

The paper is organized as follows: Section 2 contains some preliminary concepts related to our problem. We present the existence and uniqueness result in Section 2, and the existence of at least one solution in Section 4. To illustrate our results, we provide some examples in Section 5. Finally, Section 6 discusses our conclusions.

2. Preliminaries

In this section, we provide some basic definitions, notations, and lemmas as follows. For $0 < q < p \leq 1$, we define

$$\begin{aligned} [k]_q &:= \begin{cases} \dfrac{1 - q^k}{1 - q}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases} \\ [k]_{p,q} &:= \begin{cases} \dfrac{p^k - q^k}{p - q} = p^{k-1}[k]_{\frac{q}{p}}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases} \\ [k]_{p,q}! &:= \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q} = \displaystyle\prod_{i=1}^{k}\frac{p^i - q^i}{p - q}, & k \in \mathbb{N} \\ 1, & k = 0. \end{cases} \end{aligned}$$

The (p, q)-forward jump and the (p, q)-backward jump operators are defined as

$$\sigma^{k}_{p,q}(t) := \left(\frac{q}{p}\right)^{k}t \quad \text{and} \quad \rho^{k}_{p,q}(t) := \left(\frac{p}{q}\right)^{k}t, \quad \text{for } k \in \mathbb{N}, \text{ respectively.}$$

The q-analogue of the power function $(a - b)^{\underline{n}}_{q}$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is given by

$$(a-b)_q^{\underline{0}} := 1, \qquad (a-b)_q^{\underline{n}} := \prod_{i=0}^{n-1}(a-bq^i), \qquad a,b \in \mathbb{R}.$$

The (p,q)-analogue of the power function $(a-b)_{p,q}^{\underline{n}}$ with $n \in \mathbb{N}_0$ is given by

$$(a-b)_{p,q}^{\underline{0}} := 1, \qquad (a-b)_{p,q}^{\underline{n}} := \prod_{k=0}^{n-1}(ap^k - bq^k), \qquad a,b \in \mathbb{R}.$$

Generally, for $\alpha \in \mathbb{R}$, we define

$$(a-b)_q^{\underline{\alpha}} = a^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^i}{1-\left(\frac{b}{a}\right)q^{\alpha+i}}, \ a \neq 0.$$

$$(a-b)_{p,q}^{\underline{\alpha}} = p^{\binom{\alpha}{2}}(a-b)_{\frac{q}{p}}^{\underline{\alpha}} \ = \ a^\alpha \prod_{i=0}^{\infty} \frac{1}{p^\alpha}\left[\frac{1-\frac{b}{a}\left(\frac{q}{p}\right)^i}{1-\frac{b}{a}\left(\frac{q}{p}\right)^{i+\alpha}}\right], \ a \neq 0.$$

In particular, $a_q^{\underline{\alpha}} = a^\alpha$, $a_{p,q}^{\underline{\alpha}} = \left(\frac{a}{p}\right)^\alpha$ and $(0)_q^{\underline{\alpha}} = (0)_{p,q}^{\underline{\alpha}} = 0$ for $\alpha > 0$.

The (p,q)-gamma and (p,q)-beta functions are defined by

$$\begin{aligned}
\Gamma_{p,q}(x) &:= \begin{cases} \frac{(p-q)_{p,q}^{\underline{x-1}}}{(p-q)^{x-1}} = \frac{\left(1-\frac{q}{p}\right)_{p,q}^{\underline{x-1}}}{\left(1-\frac{q}{p}\right)^{x-1}}, & x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\} \\ [x-1]_{p,q}!, & x \in \mathbb{N}, \end{cases} \\
B_{p,q}(x,y) &:= \int_0^1 t^{x-1}(1-qt)_{p,q}^{\underline{y-1}} d_{p,q}t = p^{\frac{1}{2}(y-1)(2x+y-2)} \frac{\Gamma_{p,q}(x)\Gamma_q p,q(y)}{\Gamma_{p,q}(x+y)},
\end{aligned}$$

respectively.

Definition 1. *For $0 < q < p \leq 1$ and $f : [0,T] \to \mathbb{R}$, we define the (p,q)-difference of f as*

$$D_{p,q}f(t) := \begin{cases} \frac{f(pt)-f(qt)}{(p-q)(t)}, & \text{for } t \neq 0 \\ f'(0), & \text{for } t = 0 \end{cases}$$

provided that f is differentiable at 0 and f is called (p,q)-differentiable on $I_{p,q}^T$ if $D_{p,q}f(t)$ exists for all $t \in I_{p,q}^T$.

Observe that the function $g(t) = D_{p,q}f(t)$ is defined on $[0, T/p]$.

Definition 2. *Let I be any closed interval of $\mathbb{R}$ containing a, b and 0. Assuming that $f : I \to \mathbb{R}$ is a given function, we define (p,q)-integral of f from a to b by*

$$\int_a^b f(t)d_{p,q}t := \int_0^b f(t)d_{p,q}t - \int_0^a f(t)d_{p,q}t,$$

where

$$\mathcal{I}_{p,q}f(x) = \int_0^x f(t)d_{p,q}t = (p-q)x\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$ and f is called (p,q)-integrable on $[a,b]$ if it is (p,q)-integrable on $[a,b]$ for all $a, b \in I$.

An operator $\mathcal{I}_{p,q}^{N}$ is defined as

$$\mathcal{I}_{p,q}^{0} f(x) = f(x) \text{ and } \mathcal{I}_{p,q}^{N} f(x) = \mathcal{I}_{p,q} \mathcal{I}_{p,q}^{N-1} f(x), N \in \mathbb{N}.$$

The relations between (p, q)-difference and (p, q)-integral operators are given by

$$D_{p,q} \mathcal{I}_{p,q} f(x) = f(x) \text{ and } \mathcal{I}_{p,q} D_{p,q} f(x) = f(x) - f(0).$$

Fractional (p, q)-integral and fractional (p, q)-difference of Riemann–Liouville type are defined as follows.

Definition 3. *For $\alpha > 0$, $0 < q < p \leq 1$ and f defined on $I_{p,q}^{T}$, the fractional (p,q)-integral is defined by*

$$\begin{aligned} \mathcal{I}_{p,q}^{\alpha} f(t) &:= \frac{1}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \int_0^t (t - qs)^{\underline{\alpha-1}}_{p,q} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q} s \\ &= \frac{(p-q)t}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(t - \left(\frac{q}{p}\right)^{k+1} t\right)^{\underline{\alpha-1}}_{p,q} f\left(\frac{q^k}{p^{k+\alpha}} t\right), \end{aligned}$$

and $(\mathcal{I}_{p,q}^{0} f)(t) = f(t)$.

Definition 4. *For $\alpha > 0$, $0 < q < p \leq 1$ and f defined on $I_{p,q}^{T}$, the fractional (p,q)-difference operator of Riemann–Liouville type of order α is defined by*

$$\begin{aligned} D_{p,q}^{\alpha} f(t) &:= D_{p,q}^{N} \mathcal{I}_{p,q}^{N-\alpha} f(t) \\ &= \frac{1}{p^{\binom{-\alpha}{2}} \Gamma_{p,q}(-\alpha)} \int_0^t (t - qs)^{\underline{-\alpha-1}}_{p,q} f\left(\frac{s}{p^{-\alpha-1}}\right) d_{p,q} s, \end{aligned}$$

and $D_{p,q}^{0} f(t) = f(t)$, where $N - 1 < \alpha < N$, $N \in \mathbb{N}$.

Lemma 1 ([37]). *Let $\alpha \in (N-1, N)$, $N \in \mathbb{N}$, $0 < q < p \leq 1$ and $f : I_{p,q}^{T} \to \mathbb{R}$. Then,*

$$\mathcal{I}_{p,q}^{\alpha} D_{p,q}^{\alpha} f(t) = f(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$.

Lemma 2 ([37]). *Let $0 < q < p \leq 1$ and $f : I_{p,q}^{T} \to \mathbb{R}$ be continuous at 0. Then,*

$$\int_0^x \int_0^s f(\tau)\, d_{p,q}\tau\, d_{p,q} s = \int_0^{\frac{x}{p}} \int_{pq\tau}^x f(\tau)\, d_{p,q} s\, d_{p,q}\tau.$$

Lemma 3 ([37]). *Let $\alpha, \beta > 0$, $0 < q < p \leq 1$. Then,*

$$\begin{aligned} (a) \quad & \int_0^t (t - qs)^{\underline{\alpha-1}}_{p,q} s^{\beta}\, d_{p,q} s = t^{\alpha+\beta} B_{p,q}(\beta + 1, \alpha), \\ (b) \quad & \int_0^t \int_0^x (t - qx)^{\underline{\alpha-1}}_{p,q} (x - qs)^{\underline{\beta-1}}_{p,q} d_{p,q} s\, d_{p,q} x = \frac{B_{p,q}(\beta+1, \alpha)}{[\beta]_{p,q}} t^{\alpha+\beta}. \end{aligned}$$

Lemma 4 ([40]). *Let $\alpha, \beta > 0$, $0 < q < p \leq 1$ and $n \in \mathbb{Z}$. Then,*

$$(a)\quad \int_0^t (t-qs)^{\overline{\frac{\alpha-1}{p,q}}} d_{p,q}s \;=\; p^{\binom{\alpha}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha+1)}\, t^{\alpha},$$

$$(b)\quad \int_0^t \int_0^{\frac{x}{p^{-\beta-1}}} (t-qx)^{\overline{\frac{-\beta-1}{p,q}}} \left(\frac{x}{p^{-\beta-1}} - qs\right)_{p,q}^{\overline{\alpha-1}} d_{p,q}s\, d_{p,q}x \;=\; p^{\binom{\alpha}{2}+\binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha+1)}\, t^{\alpha+\beta},$$

$$(c)\quad \int_0^t (t-qs)^{\overline{\frac{-\beta-1}{p,q}}} \left(\frac{s}{p^{-\beta-1}}\right)^{\overline{\alpha-n}} d_{p,q}s \;=\; p^{\binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha-n+1)\Gamma_{p,q}(-\beta)}{\Gamma_{p,q}(\alpha-\beta-n+1)}\, t^{\alpha-\beta-n}.$$

Lemma 5. *Let* $\alpha, \beta, \theta > 0,\ 0 < q < p \le 1$ *and* $n \in \mathbb{Z}$. *Then,*

$$(a)\quad \int_0^t \int_0^{\frac{x}{p^{\theta-1}}} (t-qx)^{\overline{\frac{\theta-1}{p,q}}} \left(\frac{x}{p^{\theta-1}} - qs\right)_{p,q}^{\overline{\beta-1}} \left(\frac{s}{p^{\alpha-1}}\right)^{\overline{\alpha-n}}_{p,q} d_{p,q}s\, d_{p,q}x$$
$$= p^{\binom{\beta}{2}+\binom{\theta}{2}} \frac{\Gamma_{p,q}(\alpha-n+1)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}{\Gamma_{p,q}(\alpha+\beta+\theta-n+1)}\, t^{\alpha+\beta+\theta-n},$$

$$(b)\quad \int_0^t \int_0^{\frac{y}{p^{\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (t-qy)^{\overline{\frac{\theta-1}{p,q}}} \left(\frac{y}{p^{\theta-1}} - qx\right)_{p,q}^{\overline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\overline{\alpha-1}} d_{p,q}s\, d_{p,q}x\, d_{p,q}y$$
$$= p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{\theta}{2}} \frac{\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}{\Gamma_{p,q}(\alpha+\beta+\theta+1)}\, t^{\alpha+\beta+\theta}.$$

Proof. By Lemmas 2, 3 and 4 and definition of the (p,q)-beta function, we have

$$(a)\quad \int_0^t \int_0^{\frac{x}{p^{\theta-1}}} (t-qx)^{\overline{\frac{\theta-1}{p,q}}} \left(\frac{x}{p^{\theta-1}} - qs\right)_{p,q}^{\overline{\beta-1}} \left(\frac{s}{p^{\alpha-1}}\right)^{\overline{\alpha-n}}_{p,q} d_{p,q}s\, d_{p,q}x$$
$$= \int_0^t (t-qx)^{\overline{\frac{\theta-1}{p,q}}} \left[\int_0^{\frac{x}{p^{\theta-1}}} \left(\frac{x}{p^{\theta-1}} - qs\right)_{p,q}^{\overline{\beta-1}} \left(\frac{s}{p^{\alpha-1}}\right)^{\overline{\alpha-n}}_{p,q} d_{p,q}s\right] d_{p,q}x$$
$$= \frac{p^{\binom{\beta}{2}}}{p^{(\theta-1)(\alpha+\beta-n)}} \cdot \frac{\Gamma_{p,q}(\alpha-n+1)\Gamma_{p,q}(\beta)}{\Gamma_{p,q}(\alpha+\beta-n+1)} \int_0^t (t-qx)^{\overline{\frac{\theta-1}{p,q}}} x^{\alpha+\beta-n}\, d_{p,q}x$$
$$= p^{\binom{\beta}{2}+\binom{\theta}{2}} \frac{\Gamma_{p,q}(\alpha-n+1)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}{\Gamma_{p,q}(\alpha+\beta+\theta-n+1)}\, t^{\alpha+\beta+\theta-n},$$

$$(b)\quad \int_0^t \int_0^{\frac{y}{p^{\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (t-qy)^{\overline{\frac{\theta-1}{p,q}}} \left(\frac{y}{p^{\theta-1}} - qx\right)_{p,q}^{\overline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\overline{\alpha-1}} d_{p,q}s\, d_{p,q}x\, d_{p,q}y$$
$$= \int_0^t \int_0^{\frac{y}{p^{\theta-1}}} (t-qy)^{\overline{\frac{\theta-1}{p,q}}} \left(\frac{y}{p^{\theta-1}} - qx\right)_{p,q}^{\overline{\beta-1}} \left[\int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\overline{\alpha-1}} d_{p,q}s\right] d_{p,q}x\, d_{p,q}y$$
$$= p^{\binom{\alpha}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha+1)} \int_0^t \int_0^{\frac{y}{p^{\theta-1}}} (t-qy)^{\overline{\frac{\theta-1}{p,q}}} \left(\frac{y}{p^{\theta-1}} - qx\right)_{p,q}^{\overline{\beta-1}} \left(\frac{x^{\alpha}}{p^{\beta-1}}\right) d_{p,q}x\, d_{p,q}y$$
$$= p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{\theta}{2}} \frac{\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}{\Gamma_{p,q}(\alpha+\beta+\theta+1)}\, t^{\alpha+\beta+\theta}.$$

The proof is complete. □

The following lemma, dealing with a linear variant of problem (1), plays an important role in the forthcoming analysis.

Lemma 6. *Let* $\Omega \neq 0,\ \alpha, \beta, \theta \in (0,1],\ 0 < q < p \le 1,\ h \in C\left(I_{p,q}^T, \mathbb{R}\right)$ *and* $g \in C\left(I_{p,q}^T, \mathbb{R}^+\right)$ *be given functions,* $\varphi : C\left(I_{p,q}^T, \mathbb{R}\right) \to \mathbb{R}$ *be given functional. Then, the problem*

$$D^{\alpha}_{p,q}D^{\beta}_{p,q}u(t) = h(t), \quad t \in I^{T}_{p,q}, \tag{2}$$

$$u(0) = u\left(\frac{T}{p}\right) \tag{3}$$

$$\mathcal{I}^{\theta}_{p,q}g(\eta)u(\eta) = \varphi(u), \quad \eta \in I^{T}_{p,q} - \left\{0, \frac{T}{p}\right\} \tag{4}$$

has the unique solution:

$$\begin{aligned} u(t) = {} & \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q(\beta)}}\int_0^t \int_0^{\frac{x}{p^{\beta-1}}} (t-qx)^{\frac{\beta-1}{p,q}}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\frac{\alpha-1}{p,q}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x \\ & - \frac{t^{\beta-1}}{\Omega}\left\{\boldsymbol{B}_{\eta}\mathbb{P}[h] + \boldsymbol{A}_{T}\left(\varphi(u) - \mathbb{Q}[h]\right)\right\} \\ & + \frac{t^{\alpha+\beta-1}}{\Omega\Gamma_{p,q}(\alpha+\beta)}\left\{\boldsymbol{A}_{\eta}\mathbb{P}[h] + \left(\frac{T}{p}\right)^{\beta-1}\left(\varphi(u) - \mathbb{Q}[h]\right)\right\} \end{aligned} \tag{5}$$

where the functionals $\mathbb{P}[h]$ *and* $\mathbb{Q}[h]$ *are defined by*

$$\begin{aligned} \mathbb{P}[h] := {} & \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{T}{p}-qx\right)^{\frac{\beta-1}{p,q}}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\frac{\alpha-1}{p,q}} \times \\ & h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x \end{aligned} \tag{6}$$

$$\begin{aligned} \mathbb{Q}[h] := {} & \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}\int_0^{\eta} \int_0^{\frac{y}{p^{\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (\eta-qy)^{\frac{\theta-1}{p,q}} \times \\ & \left(\frac{y}{p^{\theta-1}}-qx\right)^{\frac{\beta-1}{p,q}}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\frac{\alpha-1}{p,q}} g\left(\frac{y}{p^{\theta-1}}\right) h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x\, d_{p,q}y \end{aligned} \tag{7}$$

and the constants $\boldsymbol{A}_{T}, \boldsymbol{A}_{\eta}, \boldsymbol{B}_{\eta}$ *and* Ω *are defined by*

$$\boldsymbol{A}_{T} := \frac{1}{p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)}\int_0^{\frac{T}{p}} \left(\frac{T}{p}-qs\right)^{\frac{\beta-1}{p,q}}\left(\frac{s}{p^{\theta-1}}\right)^{\frac{\alpha-1}{}} d_{p,q}s = \frac{\left(\frac{T}{p}\right)^{\alpha+\beta-1}}{\Gamma_{p,q}(\alpha+\beta)} \tag{8}$$

$$\boldsymbol{A}_{\eta} := \frac{1}{p^{\binom{\theta}{2}}\Gamma_{p,q}(\theta)}\int_0^{\eta} (\eta-qs)^{\frac{\theta-1}{p,q}} g\left(\frac{s}{p^{\theta-1}}\right)\left(\frac{s}{p^{\theta-1}}\right)^{\frac{\beta-1}{}} d_{p,q}s \tag{9}$$

$$\begin{aligned} \boldsymbol{B}_{\eta} := {} & \frac{1}{p^{\binom{\beta}{2}+\binom{\theta}{2}}\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}\int_0^{\eta} \int_0^{\frac{x}{p^{\theta-1}}} (\eta-qx)^{\frac{\theta-1}{p,q}}\left(\frac{x}{p^{\theta-1}}-qs\right)^{\frac{\beta-1}{p,q}} g\left(\frac{x}{p^{\theta-1}}\right) \times \\ & \left(\frac{s}{p^{\alpha-1}}\right)^{\frac{\alpha-1}{}} d_{p,q}s\, d_{p,q}x \end{aligned} \tag{10}$$

$$\Omega := \left(\frac{T}{p}\right)^{\beta-1}\boldsymbol{B}_{\eta} - \boldsymbol{A}_{T}\boldsymbol{A}_{\eta}. \tag{11}$$

Proof. Taking fractional (p, q)-integral of order α for (2) and using Lemma 1, we then have

$$\begin{aligned} D^{\beta}_{p,q}u(t) = {} & C_1 t^{\alpha-1} + \mathcal{I}^{\alpha}_{p,q}h(t) \\ = {} & C_1 t^{\alpha-1} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)^{\frac{\alpha-1}{p,q}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s. \end{aligned} \tag{12}$$

Next, taking fractional (p, q)-difference of order β for (12), we have

$$u(t) = C_0 t^{\beta-1} + C_1 \frac{t^{\alpha+\beta-1}}{\Gamma_{p,q}(\alpha+\beta)} + \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \times \int_0^t \int_0^{\frac{x}{p^{\beta-1}}} (t-qx)^{\frac{\beta-1}{p,q}} \left(\frac{x}{p^{\beta-1}} - qs\right)^{\frac{\alpha-1}{p,q}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x. \tag{13}$$

Substituting $t = 0, \frac{T}{p}$ into (13) and employing the condition (3), we have

$$C_0 \left(\frac{T}{p}\right)^{\beta-1} + C_1 \mathbf{A}_T = -\mathbb{P}[h]. \tag{14}$$

By taking fractional (p, q)-integral of order θ for (13), we have

$$\begin{aligned} \mathcal{I}^{\theta}_{p,q} u(t) = {} & C_0 \frac{t^{\beta+\theta-1}}{\Gamma_{p,q}(\beta+\theta)} + \frac{C_1}{p^{\binom{\beta}{2}+\binom{\theta}{2}}\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)} \times \\ & \int_0^t \int_0^{\frac{x}{p^{\theta-1}}} (t-qx)^{\frac{\theta-1}{p,q}} \left(\frac{x}{p^{\theta-1}} - qs\right)^{\frac{\beta-1}{p,q}} \left(\frac{s}{p^{\alpha-1}}\right)^{\frac{\alpha-1}{p,q}} d_{p,q}s\, d_{p,q}x \\ & + \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)} \int_0^t \int_0^{\frac{y}{p^{\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (t-qy)^{\frac{\theta-1}{p,q}} \times \\ & \left(\frac{y}{p^{\theta-1}} - qx\right)^{\frac{\beta-1}{p,q}} \left(\frac{x}{p^{\beta-1}} - qs\right)^{\frac{\alpha-1}{p,q}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x\, d_{p,q}y. \end{aligned} \tag{15}$$

From the condition (4) we have

$$C_0 \mathbf{A}_\eta + C_1 \mathbf{B}_\eta = \varphi(u) - \mathbb{Q}[h] \tag{16}$$

Solving the system of linear Equations (14) and (16),we obtain

$$C_0 = \frac{-\mathbf{B}_\eta \mathbb{P}[h] - \mathbf{A}_T(\varphi(u) - \mathbb{Q}[h])}{\Omega} \quad \text{and} \quad C_1 = \frac{\left(\frac{T}{p}\right)^{\beta-1}(\varphi(u) - \mathbb{Q}[h]) + \mathbf{A}_\eta \mathbb{P}[h]}{\Omega},$$

where $\mathbb{P}[h], \mathbb{Q}[h], \mathbf{A}_T, \mathbf{A}_\eta, \mathbf{B}_\eta$ and Ω are defined by (6)–(11), respectively.

After substituting C_0, C_1 into (13), we obtain (5). We can prove the converse by direct computation. The proof is complete. □

3. Existence and Uniqueness Result

In this section, we prove the existence and uniqueness result for problem (1) by using Banach fixed point theorem as follows.

Lemma 7 ([43] Banach fixed point theorem)**.** *Let a nonempty closed subset C of a Banach space X, then there is a unique fixed point for any contraction mapping P of C into itself.*

Let $\mathcal{C} = C\left(I^T_{p,q}, \mathbb{R}\right)$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \max\left\{\|u\|, \left\|D^{\nu}_{p,q}u\right\|\right\},$$

where $\|u\| = \max_{t\in I^T_{p,q}}\left\{|u(t)|\right\}$ and $\|D^{\nu}_{p,q}u\|_{\mathcal{C}} = \max_{t\in I^T_{p,q}}\left\{\left|D^{\nu}_{p,q}u(t)\right|\right\}$.

By Lemma 6, replacing $h(t)$ by $F\left[t, u(t), \Psi^{\gamma}_{p,q}u(t), D^{\nu}_{p,q}u(t)\right]$, we define an operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$
\begin{aligned}
(\mathcal{A}u)(t) :=\ & \frac{\varphi(u)}{\Omega}\left[\frac{(\frac{T}{p})^{\beta-1}}{\Gamma_{p,q}(\alpha+\beta)}t^{\alpha+\beta-1}-\mathbf{A}_T t^{\beta-1}\right]\\
&+\frac{\mathbb{Q}^*[F_u]}{\Omega}\left[\mathbf{A}_T t^{\beta-1}-\frac{(\frac{T}{p})^{\beta-1}}{\Gamma_{p,q}(\alpha+\beta)}t^{\alpha+\beta-1}\right]\\
&+\frac{\mathbb{P}^*[F_u]}{\Omega}\left[\mathbf{A}_\eta\frac{t^{\alpha+\beta-1}}{\Gamma_{p,q}(\alpha+\beta)}-\mathbf{B}_\eta t^{\beta-1}\right]\\
&+\frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^t\int_0^{\frac{x}{p^{\beta-1}}}(t-qx)_{p,q}^{\frac{\beta-1}{}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\alpha-1}\times\\
&F\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\Psi_{p,q}^{\gamma}u\left(\frac{s}{p^{\alpha-1}}\right),D_{p,q}^{\nu}u\left(\frac{s}{p^{\alpha-1}}\right)\right]d_{p,q}s\,d_{p,q}x \qquad (17)
\end{aligned}
$$

where the functionals $\mathbb{P}^*[F_u]$ and $\mathbb{Q}^*[F_u]$ are defined by

$$
\begin{aligned}
\mathbb{P}^*[F_u] :=\ & \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^{\frac{T}{p}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qx\right)_{p,q}^{\frac{\beta-1}{}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\alpha-1}\times\\
&F\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\Psi_{p,q}^{\gamma}u\left(\frac{s}{p^{\alpha-1}}\right),D_{p,q}^{\nu}u\left(\frac{s}{p^{\alpha-1}}\right)\right]d_{p,q}s\,d_{p,q}x \qquad (18)\\
\mathbb{Q}^*[F_u] :=\ & \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}\times\\
&\int_0^{\eta}\int_0^{\frac{y}{p^{\theta-1}}}\int_0^{\frac{x}{p^{\beta-1}}}(\eta-qy)_{p,q}^{\frac{\theta-1}{}}\left(\frac{y}{p^{\theta-1}}-qx\right)_{p,q}^{\frac{\beta-1}{}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\alpha-1}g\left(\frac{y}{p^{\theta-1}}\right)\times\\
&F\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\Psi_{p,q}^{\gamma}u\left(\frac{s}{p^{\alpha-1}}\right),D_{p,q}^{\nu}u\left(\frac{s}{p^{\alpha-1}}\right)\right]d_{p,q}s\,d_{p,q}x\,d_{p,q}y \qquad (19)
\end{aligned}
$$

and the constants $\mathbf{A}_T, \mathbf{A}_\eta, \mathbf{B}_\eta$ and Ω are defined by (8)–(11), respectively.

We see that the problem (1) has solution if and only if the operator $\mathcal{A}$ has fixed point.

Theorem 1. *Assume that* $F : I_{p,q}^T \times \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *is continuous,* $\phi : I_{p,q}^T \times I_{p,q}^T \to [0,\infty)$ *is continuous with* $\phi_0 = \max\left\{\phi(t,s) : (t,s) \in I_{p,q}^T \times I_{p,q}^T\right\}$, *and* $\varphi : C\big(I_{p,q}^T,\mathbb{R}\big) \to \mathbb{R}$ *is given functional. Suppose that the following conditions hold:*

(H_1) *There exist positive constants* L_1, L_2, L_3 *such that for each* $t \in I_{p,q}^T$ *and* $u_i, v_i \in \mathbb{R}$, $i = 1,2,3$,

$$\Big|F[t,u_1,u_2,u_3] - F[t,v_1,v_2,v_3]\Big| \le L_1|u_1-v_1| + L_2|u_2-v_2| + L_3|u_3-v_3|.$$

(H_2) *There exists a positive constant* ω *such that for each* $u, v \in \mathcal{C}$,

$$|\varphi(u)-\varphi(v)| \le \omega\|u-v\|_{\mathcal{C}}\,.$$

(H_3) *For each* $t \in I_{p,q}^T$, $0 < g < g(t) < G$.

(H_4) $\mathcal{X} := \omega\boldsymbol{O}_T + (\mathcal{L}+L_3)\Theta \le 1$,

where

$$\mathcal{L} := L_1 + L_2 \frac{\phi_0 (\frac{T}{p})^{\gamma}}{\Gamma_{p,q}(\gamma+1)}, \tag{20}$$

$$\Theta := \frac{\boldsymbol{O}_T G \eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta-1)} + \frac{(\frac{T}{p})^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}(\boldsymbol{O}_\eta + 1), \tag{21}$$

$$\boldsymbol{O}_T := \left[\frac{\left(\frac{T}{p}\right)^{\alpha+\beta-1}}{\Gamma_{p,q}(\alpha+\beta)} + \boldsymbol{A}_T \right] \frac{\left(\frac{T}{p}\right)^{\beta-1}}{\min|\Omega|}, \tag{22}$$

$$\boldsymbol{O}_\eta := \left[\frac{\left(\frac{T}{p}\right)^{\alpha}}{\Gamma_{p,q}(\alpha+\beta)} \max \boldsymbol{A}_\eta + \max \boldsymbol{B}_\eta \right] \frac{\left(\frac{T}{p}\right)^{\beta-1}}{\min|\Omega|}. \tag{23}$$

Then, problem (1) *has a unique solution in* $I^T_{p,q}$.

Proof. For each $t \in I^T_{p,q}$ and $u, v \in \mathcal{C}$,

$$\begin{aligned}
\left|\Psi^{\gamma}_{p,q}u(t) - \Psi^{\gamma}_{p,q}v(t)\right| &\leq \frac{\phi_0}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)} \int_0^t (t-qs)^{\underline{\gamma-1}}_{p,q} \left| u\left(\frac{s}{p^{\gamma-1}}\right) - v\left(\frac{s}{p^{\gamma-1}}\right)\right| d_{p,q}s \\
&\leq \frac{\phi_0}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)} \|u-v\| \int_0^{\frac{T}{p}} \left(\frac{T}{p} - qs\right)^{\underline{\gamma-1}}_{p,q} d_{p,q}s \\
&= \frac{\phi_0 \left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)} \|u-v\|.
\end{aligned}$$

Denote that

$$\mathcal{F}|u-v|(t) := \left| F\left[t, u(t), \Psi^{\gamma}_{p,q}u(t), D^{\nu}_{p,q}u(t)\right] - F\left[t, v(t), \Psi^{\gamma}_{p,q}v(t), D^{\nu}_{p,q}v(t)\right]\right|.$$

By using Lemma 5(a), we obtain

$$\begin{aligned}
&\left|\mathbb{P}^*[F_u] - \mathbb{P}^*[F_v]\right| \\
&\leq \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{T}{p} - qx\right)^{\underline{\beta-1}}_{p,q} \left(\frac{x}{p^{\beta-1}} - qs\right)^{\underline{\alpha-1}}_{p,q} \times \\
&\quad \mathcal{F}|u-v|\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x \\
&\leq \frac{\left[L_1|u-v| + L_2\left|\Psi^{\gamma}_{p,q}u - \Psi^{\gamma}_{p,q}v\right| + L_3\left|D^{\nu}_{p,q}u - D^{\nu}_{p,q}v\right|\right]}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{T}{p} - qx\right)^{\underline{\beta-1}}_{p,q} \times \\
&\quad \left(\frac{x}{p^{\beta-1}} - qs\right)^{\underline{\alpha-1}}_{p,q} \left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x \\
&\leq \left(\left[L_1 + L_2\phi_0 \frac{\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)} \right] |u-v| + L_3\left|D^{\nu}_{p,q}u - D^{\nu}_{p,q}v\right| \right) \frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)} \\
&\leq \frac{(\mathcal{L}+L_3)\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)} \|u-v\|_{\mathcal{C}},
\end{aligned} \tag{24}$$

and by using Lemma 5(b), we have

$$
\begin{aligned}
&\Big|\mathbb{Q}^*[F_u]-\mathbb{Q}^*[F_v]\Big| \\
&\leq \frac{G}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}\int_0^{\eta}\int_0^{\frac{y}{p^{\theta-1}}}\int_0^{\frac{x}{p^{\beta-1}}}(\eta-qy)^{\underline{\theta-1}}_{p,q}\left(\frac{y}{p^{\theta-1}}-qx\right)^{\underline{\beta-1}}_{p,q}\times \\
&\quad \left(\frac{x}{p^{\beta-1}}-qs\right)^{\underline{\alpha-1}}_{p,q}\mathcal{F}|u-v|\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}x\,d_{p,q}y \\
&\leq \frac{G\Big[L_1|u-v|+L_2\Big|\Psi^{\gamma}_{p,q}u-\Psi^{\gamma}_{p,q}v\Big|+L_3\big|D^{\nu}_{p,q}u-D^{\nu}_{p,q}v\big|\Big]}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)}\times \\
&\quad \int_0^{\eta}\int_0^{\frac{y}{p^{\theta-1}}}\int_0^{\frac{x}{p^{\beta-1}}}(\eta-qy)^{\underline{\theta-1}}_{p,q}\left(\frac{y}{p^{\theta-1}}-qx\right)^{\underline{\beta-1}}_{p,q}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\underline{\alpha-1}}_{p,q}d_{p,q}s\,d_{p,q}x\,d_{p,q}y \\
&\leq \left[\left(L_1+L_2\frac{\phi_0\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}\right)|u-v|+L_3\big|D^{\nu}_{p,q}u-D^{\nu}_{p,q}v\big|\right]\frac{G\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta+1)} \\
&\leq \frac{G(\mathcal{L}+L_3)\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta+1)}\,\|u-v\|_{\mathcal{C}}. \qquad (25)
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\big|(\mathcal{A}u)(t)-(\mathcal{A}v)(t)\big| \\
&\quad\leq \frac{\omega\|u-v\|_{\mathcal{C}}\left(\frac{T}{p}\right)^{\beta-1}}{|\Omega|}\left[\frac{\left(\frac{T}{p}\right)^{\alpha+\beta-1}}{\Gamma_{p,q}(\alpha+\beta)}+\mathbf{A}_T\right] \\
&\qquad+(\mathcal{L}+L_3)\|u-v\|_{\mathcal{C}}\,\frac{G\,\eta^{\alpha+\beta+\theta}}{|\Omega|\Gamma_{p,q}(\alpha+\beta+\theta-1)}\cdot\left(\frac{T}{p}\right)^{\beta-1}\left[\frac{\left(\frac{T}{p}\right)^{\alpha+\beta-1}}{\Gamma_{p,q}(\alpha+\beta)}+\mathbf{A}_T\right] \\
&\qquad+(\mathcal{L}+L_3)\|u-v\|_{\mathcal{C}}\,\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{|\Omega|\Gamma_{p,q}(\alpha+\beta+1)}\cdot\left(\frac{T}{p}\right)^{\beta-1}\left[\frac{\left(\frac{T}{p}\right)^{\alpha}}{\Gamma_{p,q}(\alpha+\beta)}\mathbf{A}_{\eta}+\mathbf{B}_{\eta}\right] \\
&\qquad+\frac{(\mathcal{L}+L_3)\|u-v\|_{\mathcal{C}}}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^{\frac{T}{p}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qx\right)^{\underline{\beta-1}}_{p,q}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\underline{\alpha-1}}_{p,q}d_{p,q}s\,d_{p,q}x \\
&\quad\leq \left\{\mathbf{O}_T\left[\omega+(\mathcal{L}+L_3)\,\frac{G\,\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta-1)}\right]\right. \\
&\qquad\left.+\mathbf{O}_{\eta}\left[(\mathcal{L}+L_3)\,\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}\right]+(\mathcal{L}+L_3)\,\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}\right\}\|u-v\|_{\mathcal{C}} \\
&\quad=\mathcal{X}\|u-v\|_{\mathcal{C}}. \qquad (26)
\end{aligned}
$$

Taking fractional (p,q)-difference of order ν for (17), we get

$$
\begin{aligned}
&(D^{\nu}_{p,q}\mathcal{A}u)(t)\\
&= \frac{\varphi(u)}{\Omega}\left[\frac{(\frac{T}{p})^{\beta-1}\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta)\Gamma_{p,q}(\alpha+\beta-\nu)}t^{\alpha+\beta-\nu-1} - \mathbf{A}_T\frac{\Gamma_{p,q}(\beta)}{\Gamma_{p,q}(\beta-\nu)}t^{\beta-\nu-1}\right]\\
&\quad + \frac{\mathbb{Q}^*[F_u]}{\Omega}\left[\mathbf{A}_T\frac{\Gamma_{p,q}(\beta)}{\Gamma_{p,q}(\beta-\nu)}t^{\beta-\nu-1} - \frac{(\frac{T}{p})^{\beta-1}\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta)\Gamma_{p,q}(\alpha+\beta-\nu)}t^{\alpha+\beta-\nu-1}\right]\\
&\quad + \frac{\mathbb{P}^*[F_u]}{\Omega}\left[\mathbf{A}_\eta\frac{\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)\Gamma_{p,q}(\alpha+\beta)}t^{\alpha+\beta-\nu-1} - \mathbf{B}_\eta\frac{\Gamma_{p,q}(\beta)}{\Gamma_{p,q}(\beta-\nu)}t^{\beta-\nu-1}\right]\\
&\quad + \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\nu}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\nu)}\times\\
&\quad \int_0^t\int_0^{\frac{y}{p^{-\nu-1}}}\int_0^{\frac{x}{p^{\beta-1}}}(t-qy)^{\underline{-\nu-1}}_{p,q}\left(\frac{y}{p^{-\nu-1}}-qx\right)^{\underline{\beta-1}}_{p,q}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\underline{\alpha-1}}_{p,q}\times\\
&\quad F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \Psi^{\gamma}_{p,q}u\left(\frac{s}{p^{\alpha-1}}\right), D^{\nu}_{p,q}u\left(\frac{s}{p^{\alpha-1}}\right)\right]d_{p,q}s\,d_{p,q}x\,d_{p,q}y. \qquad (27)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|(D^{\nu}_{p,q}\mathcal{A}u)(t) - (D^{\nu}_{p,q}\mathcal{A}v)(t)\right|\\
&\qquad \le \omega\|u-v\|_{\mathcal{C}}\left(\frac{T}{p}\right)^{-\nu}\frac{\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)}\mathbf{O}_T\\
&\qquad \le (\mathcal{L}+L_3)\|u-v\|_{\mathcal{C}}\frac{G\,\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta-1)}.\left(\frac{T}{p}\right)^{-\nu}\frac{\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)}\mathbf{O}_T\\
&\qquad\quad + (\mathcal{L}+L_3)\|u-v\|_{\mathcal{C}}\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}.\left(\frac{T}{p}\right)^{-\nu}\frac{\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)}\mathbf{O}_\eta\\
&\qquad\quad + (\mathcal{L}+L_3)\|u-v\|_{\mathcal{C}}\frac{(\frac{T}{p})^{\alpha+\beta-\nu}}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\\
&\qquad \le \left\{\mathbf{O}_T\left[\omega+(\mathcal{L}+L_3)\frac{G\,\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta-1)}\right]\frac{\left(\frac{T}{p}\right)^{-\nu}\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)}\right.\\
&\qquad\quad + \mathbf{O}_\eta\left[(\mathcal{L}+L_3)\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}\right]\frac{\left(\frac{T}{p}\right)^{-\nu}\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)}\\
&\qquad\quad \left. + (\mathcal{L}+L_3)\frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right\}\|u-v\|_{\mathcal{C}}\\
&\qquad < \mathcal{X}\|u-v\|_{\mathcal{C}}. \qquad (28)
\end{aligned}
$$

From (26) and (28), we have

$$\|\mathcal{A}u-\mathcal{A}v\|_{\mathcal{C}} \le \mathcal{X}\|u-v\|_{\mathcal{C}}.$$

By (H_4), we can conclude that $\mathcal{A}$ is a contraction. Thus, by using Banach fixed point theorem in lemma 7, $\mathcal{A}$ has a fixed point, which is a unique solution of problem (1) on $I^T_{p,q}$. □

4. Existence of at Least One Solution

In this section, we prove the existence of at least one solution to (1). The following lemmas reviewing the Schauder's fixed point theorem are also provided.

Lemma 8 ([43] Arzelá-Ascoli theorem). *A collection of functions in $C[a,b]$ with the sup norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 9 ([43]). *If a set is closed and relatively compact, then it is compact.*

Lemma 10 ([44] Schauder's fixed point theorem). *Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T: D \to D$ be the map such that the set $Tu: u \in U$ is relatively compact in D. Then, the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

Theorem 2. *Assume that $F: I_{p,q}^T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and $\varphi: C\left(I_{p,q}^T, \mathbb{R}\right) \to \mathbb{R}$ is given functional. Suppose that the following conditions hold:*

(H_5) There exists a positive constant M such that for each $t \in I_{p,q}^T$ and $u_i \in \mathbb{R}$, $i = 1,2,3$,

$$\left|F[t,u_1,u_2,u_3]\right| \leq M.$$

(H_6) There exists a positive constant N such that for each $u \in \mathcal{C}$,

$$|\varphi(u)| \leq N.$$

Then, problem (1) has at least one solution on $I_{p,q}^T$.

Proof. To prove this theorem, we proceed as follows.

Step I. Verify $\mathcal{A}$ maps bounded sets into bounded sets in $B_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \leq R\}$. Let us prove that for any $R > 0$, there exists a positive constant L such that for each $x \in B_R$, we have $\|\mathcal{A}u\|_{\mathcal{C}} \leq L$. By using Lemma 5, for each $t \in I_{p,q}^T$ and $u \in B_R$, we have

$$\begin{aligned}
&\left|\mathbb{P}^*[F_u]\right| \\
&\leq \frac{M}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{T}{p} - qx\right)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s\, d_{p,q}x \\
&\leq \frac{M\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}, \qquad (29) \\
&\left|\mathbb{Q}^*[F_u]\right| \\
&\leq \frac{GM}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(\theta)} \int_0^{\eta} \int_0^{\frac{y}{p^{\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (\eta - qy)_{p,q}^{\underline{\theta-1}} \left(\frac{y}{p^{\theta-1}} - qx\right)_{p,q}^{\underline{\beta-1}} \times \\
&\quad \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s\, d_{p,q}x\, d_{p,q}y \\
&\leq \frac{GM\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta+1)}. \qquad (30)
\end{aligned}$$

From (29) and (30), we have

$$\begin{aligned}
\left|(\mathcal{A}u)(t)\right| \leq\ & N\mathbf{O}_T + \frac{GM\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta-1)}\mathbf{O}_T \\
& + \frac{M\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}\mathbf{O}_\eta \\
& + \frac{M}{p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^{\frac{T}{p}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qx\right)_{p,q}^{\underline{\beta-1}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s\, d_{p,q}x \\
\leq\ & N\mathbf{O}_T + M\left[\frac{\mathbf{O}_T\, G\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta-1)} + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}(\mathbf{O}_\eta+1)\right] \\
\leq\ & N\mathbf{O}_T + M\Theta := L. \qquad (31)
\end{aligned}$$

We find that

$$\begin{aligned}
\left|\left(D_{p,q}^{\nu}\mathcal{A}u\right)(t)\right| \leq\ & N\mathbf{O}_T\left(\frac{T}{p}\right)^{-\nu}\frac{\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)} \\
& + M\left[\frac{\mathbf{O}_T\, G\eta^{\alpha+\beta+\theta}}{\Gamma_{p,q}(\alpha+\beta+\theta-1)}\left(\frac{T}{p}\right)^{-\nu}\frac{\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)}\right. \\
& \left. + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}(\mathbf{O}_\eta+1)\left(\frac{T}{p}\right)^{-\nu}\frac{\Gamma_{p,q}(\alpha+\beta)}{\Gamma_{p,q}(\alpha+\beta-\nu)}\right] \\
<\ & L. \qquad (32)
\end{aligned}$$

Thus, $\|(\mathcal{A}u)\|_{\mathcal{C}} \leq L$, which implies that $\mathcal{A}$ is uniformly bounded.

Step II. Since F is continuous, we can conclude that the operator $\mathcal{A}$ is continuous on B_R.

Step III. For any $t_1, t_2 \in I_{p,q}^T$ with $t_1 < t_2$, we find that

$$\begin{aligned}
\left|(\mathcal{A}u)(t_1)-(\mathcal{A}u)(t_2)\right| \leq\ & \frac{\left|t_2^{\beta-1}-t_1^{\beta-1}\right|}{|\Omega|}\left[\mathbf{A}_T(N+\mathbb{Q}^*[F_u])+\mathbf{B}_\eta\mathbb{P}^*[F_u]\right] \\
& + \frac{\left|t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}\right|}{|\Omega|\Gamma_{p,q}(\alpha+\beta)}\left[\left(\frac{T}{p}\right)^{\beta-1}(N+\mathbb{Q}^*[F_u])+\mathbf{A}_\eta\mathbb{P}^*[F_u]\right] \\
& + \frac{M}{\Gamma_{p,q}(\alpha+\beta)}\left|t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\right|, \qquad (33)
\end{aligned}$$

and

$$\begin{aligned}
& \left|(D_{p,q}{}^{\nu}\mathcal{A}u)(t_2)-(D_{p,q}^{\nu}\mathcal{A}u)(t_1)\right| \\
& \leq \frac{\left|t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\right|}{|\Omega|\Gamma_{p,q}(\beta-\nu)}\left[\mathbf{A}_T(N+\mathbb{Q}^*[F_u])+\mathbf{B}_\eta\mathbb{P}^*[F_u]\right] + \frac{\left|t_2^{\alpha+\beta-\nu-1}-t_1^{\alpha+\beta-\nu-1}\right|}{|\Omega|\Gamma_{p,q}(\alpha+\beta-\nu)}\times \\
& \left[\left(\frac{T}{p}\right)^{\beta-1}(N+\mathbb{Q}^*[F_u])+\mathbf{A}_\eta\mathbb{P}^*[F_u]\right] + \frac{M}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\left|t_2^{\alpha+\beta-\nu}-t_1^{\alpha+\beta-\nu}\right|. \qquad (34)
\end{aligned}$$

We see that the right-hand side of (33) and (34) tends to be zero when $|t_2 - t_1| \to 0$. Thus, $\mathcal{A}$ is relatively compact on B_R. This implies that $\mathcal{A}(B_R)$ is an equicontinuous set. By Arzelá-Ascoli theorem in Lemma 8, Lemma 9, and the above steps, we see that $\mathcal{A} : \mathcal{C} \to \mathcal{C}$

is completely continuous. Hence, we can conclude from Schauder fixed point theorem in Lemma 10 that problem (1) has at least one solution. □

5. Examples

In this section, to illustrate our results, we consider some examples.

Example 1. *Consider the following fractional (p,q)-integrodifference equation as*

$$\begin{aligned} D^{\frac{3}{4}}_{\frac{2}{3},\frac{1}{2}} D^{\frac{1}{2}}_{\frac{2}{3},\frac{1}{2}} u(t) &= \frac{1}{(100e^2+t^3)(1+|u(t)|)}\left[e^{-3t}\left(u^2+2|u|\right)+e^{-(\pi+\sin^2\pi t)}\left|\Psi^{\frac{1}{3}}_{\frac{2}{3},\frac{1}{2}} u(t)\right|\right. \\ &\quad \left.+e^{-(2\pi+\cos^2\pi t)}\left|D^{\frac{1}{4}}_{\frac{2}{3},\frac{1}{2}} u(t)\right|\right],\ t\in I^{10}_{\frac{2}{3},\frac{1}{2}} = \left\{\frac{10\left(\frac{1}{2}\right)^k}{\left(\frac{2}{3}\right)^{k+1}} : k\in\mathbb{N}_0\right\}\cup\{0\} \end{aligned} \tag{35}$$

with periodic fractional (p,q)-integral boundary condition

$$\begin{aligned} u(0) &= u(15) \\ \mathcal{I}^{\frac{2}{3}}_{\frac{2}{3},\frac{1}{2}}\left(2e+\sin\left(\frac{1215}{256}\right)\right)^2 u\left(\frac{1215}{256}\right) &= \sum_{i=0}^{\infty}\frac{C_i|u(t_i)|}{1+|u(t_i)|},\ t_i=\sigma^i_{\frac{2}{3},\frac{1}{2}}(10), \end{aligned} \tag{36}$$

where C_i is given constants with $\frac{1}{1000}\le\sum_{i=0}^{\infty}C_i\le\frac{e}{1000}$ and $\phi(t,s)=\frac{e^{-|t-s|}}{(t+e)^3}$.

Letting $\alpha=\frac{3}{4}$, $\beta=\frac{1}{2}$, $\gamma=\frac{1}{3}$, $\nu=\frac{1}{4}$, $\theta=\frac{2}{3}$, $p=\frac{2}{3}$, $q=\frac{1}{2}$, $T=10$, $\eta=\sigma^4_{\frac{2}{3},\frac{1}{2}}(10)=\frac{1215}{256}$, $g(t)=(20e+\sin t)^2$ and $F\left[t,u(t),\Psi^{\gamma}_{p,q}u(t),D^{\nu}_{p,q}u(t)\right]=\frac{1}{(100e^2+t^3)(1+|u(t)|)}\times\left[e^{-3t}\left(u^2+2|u|\right)+e^{-(\pi+\sin^2\pi t)}\left|\Psi^{\frac{1}{3}}_{\frac{2}{3},\frac{1}{2}}u(t)\right|+e^{-(2\pi+\cos^2\pi t)}\left|D^{\frac{1}{4}}_{\frac{2}{3},\frac{1}{2}}u(t)\right|\right]$.

Using above values, we find that

$$\phi_0=0.0498,\quad |\mathbf{A}_T|=2.06344,\quad |\mathbf{A}_\eta|\le 264.588,\quad |\mathbf{B}_\eta|\le 196.777\ \text{ and }\ |\Omega|\ge 283.525.$$

For all $t\in I^{10}_{\frac{2}{3},\frac{1}{2}}$ and $u,v\in\mathbb{R}$, we find that

$$\begin{aligned} &\left|F\left[t,u,\Psi^{\gamma}_{p,q}u,D^{\nu}_{p,q}u\right]-F\left[t,v,\Psi^{\gamma}_{p,q}v,D^{\nu}_{p,q}v\right]\right| \\ &\le \frac{1}{100e^2}|u-v|+\frac{1}{1000e^{2+\pi}}|\Psi^{\gamma}_{p,q}u-\Psi^{\gamma}_{p,q}v|+\frac{1}{100e^{2+2\pi}}\left|D^{\nu}_{p,q}u-D^{\nu}_{p,q}v\right|. \end{aligned}$$

Thus, (H_1) holds with $L_1=0.001353$, $L_2=5.848\times10^{-5}$ and $L_3=2.5273\times10^{-6}$. So $\mathcal{L}=0.00136$.

For all $u,v\in\mathcal{C}$,

$$|\varphi(u)-\varphi(v)|\le\frac{e}{1000}\|u-v\|_{\mathcal{C}}.$$

Thus, (H_2) holds with $\omega=0.002718$.

In addition, (H_3) holds with $g=19.6831$, $G=41.42935$.

Since

$$\mathbf{O}_T=0.001885\quad \mathbf{O}_\eta=2.10484\quad\text{and}\quad \Theta=89.5277,$$

therefore, (H_4) holds with

$$\mathcal{X}=0.121989<1.$$

Hence, by Theorem 1 this problem has a unique solution.

Example 2. *Consider the following fractional (p,q)-integrodifference equation as*

$$D^{\frac{3}{4}}_{\frac{2}{3},\frac{1}{2}} D^{\frac{1}{2}}_{\frac{2}{3},\frac{1}{2}} u(t) = \frac{1}{10}\left(t+\frac{1}{3}\right) e^{-(t+\frac{1}{10})\left[u(t)+\left|\Psi^{\frac{1}{3}}_{\frac{2}{3},\frac{1}{2}} u(t)\right|+\left|D^{\frac{1}{4}}_{\frac{2}{3},\frac{1}{2}} u(t)\right|\right]}, \ t \in I^{10}_{\frac{2}{3},\frac{1}{2}} \tag{37}$$

with periodic fractional (p,q)-integral boundary condition

$$\begin{aligned} u(0) &= u(15) \\ \mathcal{I}^{\frac{2}{3}}_{\frac{2}{3},\frac{1}{2}}\left(2e+\sin\left(\frac{1215}{256}\right)\right)^2 u\left(\frac{1215}{256}\right) &= \sum_{i=0}^{\infty} C_i e^{-|u(t_i)|}, \ t_i = \sigma^i_{\frac{2}{3},\frac{1}{2}}(10), \end{aligned} \tag{38}$$

where D_i is given constants with $\frac{1}{500} \leq \sum_{i=0}^{\infty} D_i \leq \frac{e}{500}$.

Letting $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{3}$, $\nu = \frac{1}{4}$, $\theta = \frac{2}{3}$, $p = \frac{2}{3}$, $q = \frac{1}{2}$, $T = 10$, $\eta = \frac{1215}{256}$. It is clear that $\left|F\left[t,u,\Psi^{\gamma}_{p,q}u, D^{\nu}_{p,q}u\right]\right| \leq \frac{23}{15} = M$ for $t \in I^{10}_{\frac{2}{3},\frac{1}{2}}$, and $|\varphi(u)| \leq \frac{e}{500} = N$ for $u \in \mathcal{C}$. Thus, we can conclude from Theorem 2 that our problem has at least one solution.

6. Conclusions

A fractional (p, q)-integrodifference equation with periodic fractional (p, q)-integral boundary condition (1) is studied. Our problem contains three fractional (p, q)-difference operators, and two fractional (p, q)-integral operators. We establish the conditions for the existence and uniqueness of solution for problem (1) by using the Banach fixed point theorem, and this result is shown in Theorem 1. We also established the conditions of at least one solution by using the Schauder's fixed point theorem, and this result is shown in Theorem 2. The choice to use of Theorems 1 or 2 depends on the conditions of the assumptions. The main results are illustrated by a numerical example. Some properties of fractional (p, q)-integral needed in our study are also discussed. The results of the paper are new and enrich the subject of boundary value problems for fractional (p, q)-difference equations. In the future work, we may extend this work by considering new boundary value problems.

Author Contributions: Conceptualization, J.S. and T.S.; methodology, J.S. and T.S.; formal analysis, J.S. and T.S.; funding acquisition, J.S and T.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok. Contract no.KMUTNB-62-KNOW-22.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their gratitude to anonymous referees for very helpful suggestions and comments which led to improvements of our original manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jackson, F.H. On q-difference equations. *Am. J. Math.* **1910**, *32*, 305–314. [CrossRef]
2. Jackson, F.H. On q-difference integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
3. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Springer: Basel, Switzerland, 2012.
4. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
5. Ernst, T. *A New Notation for q-Calculus and a New q-Taylor Formula*; U.U.D.M. Report 1999:25; Department of Mathematics, Uppsala University: Uppsala, Sweden, 1999; ISSN 1101-3591.
6. Finkelstein, R.J. Symmetry group of the hydrogen atom. *J. Math. Phys.* **1967**, *8*, 443–449. [CrossRef]
7. Finkelstein, R.J. q-gauge theory. *Int. J. Mod. Phys. A* **1996**, *11*, 733–746. [CrossRef]

8. Ilinski, K.N.; Kalinin, G.V.; Stepanenko, A.S. q-functional field theory for particles with exotic statistics. *Phys. Lett. A* **1997**, *232*, 399–408. [CrossRef]
9. Finkelstein, R.J. q-field theory. *Lett. Math. Phys.* **1995**, *34*, 169–176. [CrossRef]
10. Finkelstein, R.J. The q-Coulomb problem. *J. Math. Phys.* **1996**, *37*, 2628–2636. [CrossRef]
11. Cheng, H. Canonical Quantization of Yang-Mills theories. *Perspectives in Mathematical Physics*; International Press: Boston, MA, USA, 1996.
12. Finkelstein, R.J. q-gravity. *Lett. Math. Phys.* **1996**, *38*, 53–62. [CrossRef]
13. Negadi, T.; Kibler, M. A q-deformed Aufbau Prinzip. *J. Phys. A Math. Gen.* **1992**, *25*, 157–160. [CrossRef]
14. Siegel, W. *Introduction to String Field Theory, Volume 8 of Advanced Series in Mathematical Physics*; World Scientific Publishing: Teaneck, NJ, USA, 1988.
15. Agarwal, R.P. Certain fractional q-integrals and q-derivatives. *Proc. Camb. Phil. Soc.* **1969**, *66*, 365–370. [CrossRef]
16. Al-Salam, W.A. Some fractional q-integrals and q-derivatives. *Proc. Edin. Math. Soc.* **1966**, *15*, 135–140. [CrossRef]
17. Díaz, J.B.; Osler, T.J. Differences of fractional order. *Math. Comp.* **1974**, *28*, 185–202. [CrossRef]
18. Brikshavana, T.; Sitthiwirattham, T. On Fractional Hahn Calculus. *Adv. Differ. Equ.* **2017**, *354*, 1–15. [CrossRef]
19. Patanarapeelert, N.; Sitthiwirattham, T. On Fractional Symmetric Hahn Calculus. *Mathematics* **2019**, *7*, 873. [CrossRef]
20. Chakrabarti, R.; Jagannathan, R. A (p, q)-oscillator realization of two-parameter quantum algebras. *J. Math. Phys. Math. Gen.* **1991**, *24*, 5683–5701. [CrossRef]
21. Sadjang, P.N. On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas. *Results Math.* **2018**, *73*, 1–21.
22. Mursaleen, M.; Ansari, K.J.; Khan, A. On (p, q)-analogue of Bernstein operators. *Appl. Math. Comput.* **2015**, *266*, 874–882. [CrossRef]
23. Mursaleen, M.; Ansari, K.J.; Khan, A. Some approximation results by (p, q)-analogue of Bernstein-Stancu operators. *Appl. Math. Comput.* **2015**, *264*, 392–402; Corrigendum in *Appl. Math. Comput.* **2015**, *269*, 744–746. [CrossRef]
24. Mursaleen, M.; Khan, F.; Khan, A. Approximation by (p, q)-Lorentz polynomials on a compact disk. *Complex Anal. Oper. Theory* **2016**, *10*, 1725–1740. [CrossRef]
25. Rahman, S.; Mursaleen, M.; Alkhaldi, A.H. Convergence of iterates of q-Bernstein and (p, q)-Bernstein operators and the Kelisky-Rivlin type theorem. *Filomat* **2018**, *32*, 4351–4364. [CrossRef]
26. Jebreen, H.B.; Mursaleen, M.; Ahasan, M. On the convergence of Lupas (p, q)-Bernstein operators via contraction principle. *J. Ineq. Appl.* **2019**, *2019*. [CrossRef]
27. Khan, A.; Sharma, V. Statistical approximation by (p, q)-analogue of Bernstein-Stancu operators. *Azerbaijan J. Math.* **2018**, *8*, 100–121.
28. Khan, K.; Lobiyal, D.K. Bezier curves based on Lupas (p, q)-analogue of Bernstein functions in CAGD. *J. Comput. Appl. Math.* **2017**, *317*, 458–477. [CrossRef]
29. Milovanovic, G.V.; Gupta, V.; Malik, N. (p, q)-Beta functions and applications in approximation. *Bol. Soc. Mat. Mex.* **2018**, *24*, 219–237. [CrossRef]
30. Cheng, W.T.; Zhang, W.H.; Cai, Q.B. (p, q)-gamma operators which preserve x^2. *J. Inequal. Appl.* **2019**, *2019*, 1–14. [CrossRef]
31. Nasiruzzaman, M.; Mukheimer, A.; Mursaleen, M. Some Opial-type integral inequalities via (p, q)-calculus. *J. Ineq. Appl.* **2019**, *2019*, 1–15. [CrossRef]
32. Jain, P.; Basu, C.; Panwar, V. On the (p, q)-Melin Transform and Its Applications. *Acta Math. Sci. Ser. B* **2021**, *41*, 1719–1732. [CrossRef]
33. Aral, A.; Deniz, E.; Erbay, H. The Picard and Gauss-Weierstrass singular integral in (p, q)-calculus. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 1569–1583. [CrossRef]
34. Kamsrisuk, N.; Promsakon, C.; Ntouyas, S.K.; Tariboon, J. Nonlocal boundary value problems for (p, q)-difference equations. *Differ. Equ. Appl.* **2018**, *10*, 183–195. [CrossRef]
35. Promsakon, C.; Kamsrisuk, N.; Ntouyas, S.K.; Tariboon, J. On the second-order quantum (p, q)-difference equations with separated boundary conditions. *Adv. Math. Phys.* **2018**, *2018*, 9089865. [CrossRef]
36. Nuntigrangjana, T.; Putjuso, S.N.; Ntouyas, S.K.; Tariboon, J. Impulsive quantum (p, q)-difference equations. *Adv. Differ. Equ.* **2020**, *2020*, 1–20. [CrossRef]
37. Soontharanon, J.; Sitthiwirattham, T. On Fractional (p, q)-Calculus. *Adv. Differ. Equ.* **2020**, *2020*, 1–18. [CrossRef]
38. Pheak, N.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. Fractional (p, q)-Calculus on Finite Intervals and Some Integral Inequalities. *Symmetry* **2021**, *13*, 504.
39. Pheak, N.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K. Praveen Agarwal, Some trapezoid and midpoint type inequalities via fractional (p, q)-calculus. *Adv. Differ. Equ.*, **2021**, *2021*, 1–22.
40. Soontharanon, J.; Sitthiwirattham, T. Existence Results of Nonlocal Robin Boundary Value Problems for Fractional (p, q)-Integrodifference Equations. *Adv. Differ. Equ.* **2020**, *2020*, 1–17. [CrossRef]
41. Qin, Z.; Sun, S. On a nonlinear fractional (p, q)-difference Schrödinger equation. *J. Appl. Math. Comput.* **2021**. [CrossRef]
42. Qin, Z.; Sun, S. Positive solutions for fractional (p, q)-difference boundary value problems. *J. Appl. Math. Comput.* **2021**. [CrossRef]
43. Griffel, D.H. *Applied Functional Analysis*; Ellis Horwood Publishers: Chichester, UK, 1981.
44. Guo, D.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cone*; Academic Press: Orlando, FL, USA, 1988.