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Article

Some New Post-Quantum Integral Inequalities Involving Twice (p,q)-Differentiable ψ-Preinvex Functions and Applications

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Abstract: The main motivation of this article is derive a new post-quantum integral identity using twice (p,q)-differentiable functions. Using the identity as an auxiliary result, we will obtain some new variants of Hermite–Hadamard's inequality essentially via the class of ψ-preinvex functions. To support our results, we offer some applications to a special means of positive real numbers and twice (p,q)-differentiable functions that are in absolute value bounded as well.

Keywords: Hermite–Hadamard inequality; Hölder's inequality; power mean inequality; ψ-preinvex functions; post-quantum calculus; special means; bounded functions

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1. Introduction and Preliminaries

Inequalities play a pivotal role in almost all branches of mathematics. For instance, the inequalities arising from the convexity property of related functions have numerous applications in the study of qualitative theory of differential equations and partial differential equations (see, for example, the papers of [1,2] for more details). In modern analysis, a significant amount of inequalities can be obtained by using the convexity property of the functions. Hermite–Hadamard's inequality is one of the most studied inequalities pertaining to convexity. This result reads as

$$\Upsilon\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\Upsilon(x)\mathrm{d}x \leq \frac{\Upsilon(\mu_1)+\Upsilon(\mu_2)}{2}$$

if $\Upsilon : [\mu_1,\mu_2] \mapsto \mathbb{R}$ is a convex function on closed interval $[\mu_1,\mu_2]$.

In recent years, the improvements, generalizations, and variants of Hermite–Hadamard's inequality have been the subject of much research. In this regard, a variety of novel and innovative approaches have been utilized in obtaining new refinements of Hermite–Hadamard's inequality. For the first time, Tariboon and Ntouyas [3] obtained a q-analogue of Hermite–Hadamard's inequality using the concepts of quantum calculus, which is also known as calculus without limits. In quantum calculus, we establish the q-analogues of classical mathematical objects that can be recaptured by taking $q \to 1^-$. Alp et al. [4] obtained a corrected q-analogue of Hermite–Hadamard's inequality. Noor et al. [5] and Sudsutad et al. [6] derived some more q-analogues of Hermite–Hadamard-like inequalities involving first order q-differentiable convex functions, and Liu and Zhuang [7] established these analogues via second order q-differentiable convex functions. Zhang et al. [8] obtained a new generalized q-integral identity and obtained several new q-analogues of a first order q-differentiable convex function.

Chakarabarti and Jagannathan [9] studied post-quantum calculus, which is another significant generalization of quantum calculus is the post-quantum calculus. In quantum calculus, we deal with a q-number with one base q, but post-quantum calculus includes p and q-numbers with two independent variables p and q. Tunç and Gov [10] introduced the concepts of (p,q)-derivatives ${}_{\mu_1}\mathcal{D}_{\mathrm{p,q}}\Upsilon(x)$ and (p,q)-integrals on finite intervals $\int_{\mu_1}^{x}\Upsilon(\tau)\,{}_{\mu_1}\mathrm{d}_{\mathrm{p,q}}\tau$ for all $x \neq \mu_1$, where $x \in \mathcal{K} \subset \mathbb{R}$, as follows.

Definition 1 ([10]). *Let $\Upsilon : \mathcal{K} \subset \mathbb{R} \mapsto \mathbb{R}$ be a continuous function and let $x \in \mathcal{K}$ and $0 < \mathrm{q} < \mathrm{p} \leq 1$. The (p,q)-derivative on $\mathcal{K}$ of function Υ at x is then defined as*

$$ {}_{\mu_1}\mathcal{D}_{\mathrm{p,q}}\Upsilon(x) = \frac{\Upsilon(\mathrm{p}x + (1-\mathrm{p})\mu_1) - \Upsilon(\mathrm{q}x + (1-\mathrm{q})\mu_1)}{(\mathrm{p}-\mathrm{q})(x-\mu_1)}, \quad x \neq \mu_1. \tag{1} $$

Definition 2 ([10]). *Let $\Upsilon : \mathcal{K} \subset \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. The (p,q)-integral on $\mathcal{K}$ is then defined as*

$$ \int_{\mu_1}^{x}\Upsilon(\tau)\,{}_{\mu_1}\mathrm{d}_{\mathrm{p,q}}\tau = (\mathrm{p}-\mathrm{q})(x-\mu_1)\sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\Upsilon\left(\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}x + \left(1-\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\right)\mu_1\right), $$

for $x \in \mathcal{K}$ and $x \neq \mu_1$.

Since then, several new variants of classical integral inequalities have been obtained using the concepts of post-quantum calculus. For example, Awan et al. [11] obtained a generalized (p,q)-integral identity and obtained several new (p,q)-analogues of trapezium-like inequalities. Kunt et al. [12] obtained some (p,q)-analogues of Hermite–Hadamard and mid-point type inequalities. Yu et al. [13] derived several new (p,q)-analogues of some classical integral inequalities and discussed applications as well.

Definition 3. *A set $\mathcal{K} \subset \mathbb{R}$ is said to be an invex set with respect to the mapping $\zeta : \mathcal{K} \times \mathcal{K} \times [0,1] \mapsto \mathbb{R}$ if $\psi\mu_1 + \tau\zeta(\mu_2, \mu_1, \psi) \in \mathcal{K}$ for every μ_1, $\mu_2 \in \mathcal{K}$ and $\psi, \tau \in [0,1]$. The invex set $\mathcal{K}$ is also called an ζ-connected set.*

Before we proceed further, let us recall the definition of ψ-preinvex functions.

Definition 4 ([14]). *A function Υ on the invex set $\mathcal{K}$ is said to be ψ-preinvex with respect to $\zeta(\mu_2, \mu_1, \psi)$ if*

$$ \Upsilon(\psi\mu_1 + \tau\zeta(\mu_2, \mu_1, \psi)) \leq \psi\,(1-\tau)\Upsilon(\mu_1) + \tau\Upsilon(\mu_2), \quad \forall\, \mu_1, \mu_2 \in \mathcal{K}, \quad \psi, \tau \in [0,1]. $$

Remark 1. *Note the following:*

I. *If we take $\zeta(\mu_2, \mu_1, \psi) = \mu_2 - \psi\mu_1$ in Definition 4, then we have the definition of a ψ-convex function, see [15].*

II. *If we choose $\psi = 1$ in Definition 4, then we obtain the class of classical preinvex functions, see [16].*

The main motivation of this article is to derive a new post-quantum integral identity using twice (p,q)-differentiable functions. Using the identity as an auxiliary result, we will obtain some new variants of Hermite–Hadamard's inequality essentially via the class of ψ-preinvex functions. To support our results, we also present some applications to a special means of positive real numbers and twice (p,q)-differentiable functions that are in absolute value bounded. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

2. Main Results

In this section, we derive new post-quantum integral identity. This result will be helpful in obtaining main results of this paper.

Lemma 1. *Let $\Upsilon : \mathcal{K} \mapsto \mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$ (the interior of set $\mathcal{K}$), and let ${}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon$ be continuous and (p,q)-integrable on $\mathcal{K}$, where $0<\mathrm{q}<\mathrm{p}\le 1$. Thus,*

$$\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi))}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p},\mathrm{q}}x$$
$$=\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int_0^1\tau(1-\mathrm{q}\tau)\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))\,{}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\tau.$$

Proof. Applying Definition 1, we have

$$\begin{aligned}
&{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))\\
&={}_{\psi\mu_1}\mathcal{D}_{\mathrm{p},\mathrm{q}}\big({}_{\psi\mu_1}\mathcal{D}_{\mathrm{p},\mathrm{q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))\big)\\
&=\frac{{}_{\psi\mu_1}\mathcal{D}_{\mathrm{p},\mathrm{q}}\Upsilon(\psi\mu_1+\mathrm{p}\tau\zeta(\mu_2,\mu_1,\psi))-{}_{\psi\mu_1}\mathcal{D}_{\mathrm{p},\mathrm{q}}\Upsilon(\psi\mu_1+\mathrm{q}\tau\zeta(\mu_2,\mu_1,\psi))}{\tau(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)}\\
&=\frac{1}{\tau(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)}\bigg[\frac{\Upsilon(\psi\mu_1+\mathrm{p}^2\tau\zeta(\mu_2,\mu_1,\psi))-\Upsilon(\psi\mu_1+\mathrm{p}\mathrm{q}\tau\zeta(\mu_2,\mu_1,\psi))}{\tau p(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)}\\
&-\frac{\Upsilon(\psi\mu_1+\mathrm{p}\mathrm{q}\tau\zeta(\mu_2,\mu_1,\psi))-\Upsilon(\psi\mu_1+\mathrm{q}^2\tau\zeta(\mu_2,\mu_1,\psi))}{\tau q(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)}\bigg]\\
&=\frac{\mathrm{q}\Upsilon(\psi\mu_1+\mathrm{p}^2\tau\zeta(\mu_2,\mu_1,\psi))-(\mathrm{p}+\mathrm{q})\Upsilon(\psi\mu_1+\mathrm{p}\mathrm{q}\tau\zeta(\mu_2,\mu_1,\psi))+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{q}^2\tau\zeta(\mu_2,\mu_1,\psi))}{\mathrm{p}\mathrm{q}\tau^2(\mathrm{p}-\mathrm{q})^2\zeta^2(\mu_2,\mu_1,\psi)}.
\end{aligned}$$

Now by using Definition 2, we obtain

$$\begin{aligned}
&\int_0^1\tau(1-\mathrm{q}\tau)\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))\,{}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\tau\\
&=\int_0^1\tau(1-\mathrm{q}\tau)\\
&\times\frac{\mathrm{q}\Upsilon(\psi\mu_1+\mathrm{p}^2\tau\zeta(\mu_2,\mu_1,\psi))-(\mathrm{p}+\mathrm{q})\Upsilon(\psi\mu_1+\mathrm{q}\tau\zeta(\mu_2,\mu_1,\psi))+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{q}^2\tau\zeta(\mu_2,\mu_1,\psi))}{\tau^2\mathrm{p}\mathrm{q}(\mathrm{p}-\mathrm{q})^2\zeta^2(\mu_2,\mu_1,\psi)}\,{}_0\mathrm{d}_{\mathrm{p},\mathrm{q}}\tau\\
&=\frac{1}{\mathrm{p}\mathrm{q}(\mathrm{p}-\mathrm{q})\zeta^2(\mu_2,\mu_1,\psi)}\left[\mathrm{q}\sum_{n=0}^{\infty}\Upsilon\left(\psi\mu_1+\mathrm{p}^2\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\right)-(\mathrm{p}+\mathrm{q})\sum_{n=0}^{\infty}\Upsilon\left(\psi\mu_1+\mathrm{p}\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\right)\right.\\
&\left.+\mathrm{p}\sum_{n=0}^{\infty}\Upsilon\left(\psi\mu_1+\frac{\mathrm{q}^{n+2}}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\right)\right]\\
&-\mathrm{q}\left\{\frac{\mathrm{q}(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)\sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\Upsilon\left(\psi\mu_1+\mathrm{p}^2\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\right)}{\mathrm{p}\mathrm{q}(\mathrm{p}-\mathrm{q})^2\zeta^3(\mu_2,\mu_1,\psi)}\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(\mathrm{p}+\mathrm{q})(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)\sum_{n=0}^{\infty}\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\Upsilon\Big(\psi\mu_1+\mathrm{p}\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\Big)}{\mathrm{p}\mathrm{q}^2(\mathrm{p}-\mathrm{q})^2\zeta^3(\mu_2,\mu_1,\psi)} \\
&\left.+\frac{\mathrm{p}(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)\sum_{n=0}^{\infty}\frac{\mathrm{q}^{n+2}}{\mathrm{p}^{n+1}}\Upsilon\Big(\psi\mu_1+\frac{\mathrm{q}^{n+2}}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\Big)}{\mathrm{p}\mathrm{q}^3(\mathrm{p}-\mathrm{q})^2\zeta^3(\mu_2,\mu_1,\psi)}\right\} \\
&=\frac{\mathrm{q}\Big[\sum_{n=0}^{\infty}\Upsilon\Big(\psi\mu_1+\mathrm{p}^2\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\Big)-\sum_{n=0}^{\infty}\Upsilon\Big(\psi\mu_1+\mathrm{p}\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\Big)\Big]}{\mathrm{p}\mathrm{q}(\mathrm{p}-\mathrm{q})\zeta^2(\mu_2,\mu_1,\psi)} \\
&-\frac{\mathrm{p}\Big[\sum_{n=0}^{\infty}\Upsilon\Big(\psi\mu_1+\mathrm{p}\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\Big)-\sum_{n=0}^{\infty}\Upsilon\Big(\psi\mu_1+\frac{\mathrm{q}^{n+2}}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\Big)\Big]}{\mathrm{p}\mathrm{q}(\mathrm{p}-\mathrm{q})\zeta^2(\mu_2,\mu_1,\psi)} \\
&-\mathrm{q}\left\{\frac{\mathrm{q}(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)\sum_{n=0}^{\infty}\frac{\mathrm{q}^{n}}{\mathrm{p}^{n+1}}\Upsilon\Big(\psi\mu_1+\mathrm{p}^2\frac{\mathrm{q}^{n}}{\mathrm{p}^{n+1}}\zeta(\mu_2,\mu_1,\psi)\Big)}{\mathrm{p}\mathrm{q}(\mathrm{p}-\mathrm{q})^2\zeta^3(\mu_2,\mu_1,\psi)}\right. \\
&-\frac{\mathrm{p}^2(\mathrm{p}+\mathrm{q})(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)\sum_{n=0}^{\infty}\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+2}}\Upsilon\Big(\psi\mu_1+\mathrm{p}^2\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+2}}\zeta(\mu_2,\mu_1,\psi)\Big)}{\mathrm{p}\mathrm{q}^2(\mathrm{p}-\mathrm{q})^2\zeta^3(\mu_2,\mu_1,\psi)} \\
&\left.+\frac{\mathrm{p}^3(\mathrm{p}-\mathrm{q})\zeta(\mu_2,\mu_1,\psi)\sum_{n=0}^{\infty}\frac{\mathrm{q}^{n+2}}{\mathrm{p}^{n+3}}\Upsilon\Big(\psi\mu_1+\mathrm{p}^2\frac{\mathrm{q}^{n+2}}{\mathrm{p}^{n+3}}\zeta(\mu_2,\mu_1,\psi)\Big)}{\mathrm{p}\mathrm{q}^3(\mathrm{p}-\mathrm{q})^2\zeta^3(\mu_2,\mu_1,\psi)}\right\} \\
&=\frac{\mathrm{q}[\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)-\Upsilon(\psi\mu_1)]-\mathrm{p}[\Upsilon(\psi\mu_1+\mathrm{q}\zeta(\mu_2,\mu_1,\psi))-\Upsilon(\psi\mu_1)]}{\mathrm{p}\mathrm{q}(\mathrm{p}-\mathrm{q})\zeta^2(\mu_2,\mu_1,\psi)} \\
&-\frac{\mathrm{p}+\mathrm{q}}{\mathrm{p}^3\mathrm{q}^2\zeta^3(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_0d_{\mathrm{p},\mathrm{q}}\tau-\frac{\mathrm{q}^2+\mathrm{p}\mathrm{q}-\mathrm{p}}{\mathrm{p}\mathrm{q}^2(\mathrm{p}-\mathrm{q})\zeta^2(\mu_2,\mu_1,\psi)}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)) \\
&+\frac{\Upsilon(\psi\mu_1+\mathrm{q}\zeta(\mu_2,\mu_1,\psi))}{\mathrm{q}(\mathrm{p}-\mathrm{q})\zeta^2(\mu_2,\mu_1,\psi)} \\
&=\frac{\Upsilon(\psi\mu_1)}{\mathrm{p}\mathrm{q}\zeta^2(\mu_2,\mu_1,\psi)}+\frac{\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi))}{\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}-\frac{\mathrm{p}+\mathrm{q}}{\mathrm{p}^3\mathrm{q}^2\zeta^3(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{\mathrm{p},\mathrm{q}}x.
\end{aligned}
$$

Multiplying both sides of the above equality by $\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}$, we obtain the required result. □

Using Lemma 1, we can obtain the following new results.

Theorem 1. *Let $\Upsilon:\mathcal{K}\mapsto\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$, and let ${}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon$ be continuous and (p,q)-integrable on $\mathcal{K}$, where $0<\mathrm{q}<\mathrm{p}\le 1$. Assume that $|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon|$ is a ψ-preinvex function. Thus,*

$$
\begin{aligned}
&\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{\mathrm{p},\mathrm{q}}x\right| \\
&\le\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)(\psi(\mathrm{p}^4-\mathrm{p}^3+\mathrm{p}^2\mathrm{q}^2)|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_1)|+\mathrm{p}^3|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_2)|)}{(\mathrm{p}+\mathrm{q})^2(\mathrm{p}^2+\mathrm{q}^2)(\mathrm{q}^2+\mathrm{p}\mathrm{q}+\mathrm{p}^2)}. \qquad (2)
\end{aligned}
$$

Proof. Using Lemma 1, the ψ-preinvexity of $|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|$, and the properties of the modulus, we have

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int\limits_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|$$
$$\leq\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int\limits_0^1\tau(1-\mathrm{q}\tau)|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|\,_0\mathrm{d}_{\mathrm{p,q}}\tau$$
$$\leq\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\psi|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|\int\limits_0^1\tau(1-\tau)(1-\mathrm{q}\tau)\,_0\mathrm{d}_{\mathrm{p,q}}\tau+|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|\int\limits_0^1\tau^2(1-\mathrm{q}\tau)\,_0\mathrm{d}_{\mathrm{p,q}}\tau\right)$$
$$=\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)(\psi(\mathrm{p}^4-\mathrm{p}^3+\mathrm{p}^2\mathrm{q}^2)|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|+\mathrm{p}^3|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|)}{(\mathrm{p}+\mathrm{q})^2(\mathrm{p}^2+\mathrm{q}^2)(\mathrm{q}^2+\mathrm{pq}+\mathrm{p}^2)}.$$

This completes the proof. □

Theorem 2. *Let $\Upsilon:\mathcal{K}\mapsto\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$, and let $_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon$ be continuous and integrable on $\mathcal{K}$, where $0<\mathrm{q}<\mathrm{p}\leq 1$. Suppose that $|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$ is a ψ-preinvex function for $r\geq 1$. Thus,*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int\limits_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|$$
$$\leq\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}}\left(\psi d_1|\,_{\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1)|^r+d_2|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r\right)^{\frac{1}{r}},\tag{3}$$

where

$$d_1:=(\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\left(\frac{\mathrm{q}^{2n}}{\mathrm{p}^{2n+2}}-\frac{\mathrm{q}^{3n}}{\mathrm{p}^{3n+3}}\right)\left(1-\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\right)^r$$

and

$$d_2:=(\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\frac{\mathrm{q}^{3n}}{\mathrm{p}^{3n+3}}\left(1-\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\right)^r.$$

Proof. Using Lemma 1, the power mean inequality, the ψ-preinvexity of $|\,_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$, and the properties of the modulus, we have

$$
\begin{aligned}
&\left|\frac{q\Upsilon(\psi\mu_1)+p\Upsilon(\psi\mu_1+p\zeta(\mu_2,\mu_1,\psi)}{p+q}-\frac{1}{p^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+p^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{p,q}x\right| \\
&\le \frac{pq^2\zeta^2(\mu_2,\mu_1,\psi)}{p+q}\int_0^1 \tau(1-q\tau)|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|\,{}_0d_{p,q}\tau \\
&\le \frac{pq^2\zeta^2(\mu_2,\mu_1,\psi)}{p+q}\left(\int_0^1 \tau\,{}_0d_{p,q}\tau\right)^{1-\frac{1}{r}}\left(\int_0^1 \tau(1-q\tau)^r|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|^r\,{}_0d_{p,q}\tau\right)^{\frac{1}{r}} \\
&\le \frac{pq^2\zeta^2(\mu_2,\mu_1,\psi)}{p+q}\left(\frac{1}{p+q}\right)^{1-\frac{1}{r}}\left(\psi|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_1)|^r\int_0^1 \tau(1-\tau)(1-q\tau)^r\,{}_0d_{p,q}\tau\right. \\
&\quad\left.+|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_2)|^r\int_0^1 \tau^2(1-q\tau)^r\,{}_0d_{p,q}\tau\right)^{\frac{1}{r}} \\
&= \frac{pq^2\zeta^2(\mu_2,\mu_1,\psi)}{(p+q)^{2-\frac{1}{r}}}\left(\psi d_1|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_1)|^r+d_2|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_2)|^r\right)^{\frac{1}{r}}.
\end{aligned}
$$

This completes the proof. □

Theorem 3. *Let $\Upsilon:\mathcal{K}\mapsto\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$, and let ${}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon$ be continuous and integrable on $\mathcal{K}$, where $0<q<p\le 1$. Assume that $|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon|^r$ is a ψ-preinvex function for $r>1$ and $\frac{1}{s}+\frac{1}{r}=1$. Thus,*

$$
\begin{aligned}
&\left|\frac{q\Upsilon(\psi\mu_1)+p\Upsilon(\psi\mu_1+p\zeta(\mu_2,\mu_1,\psi)}{p+q}-\frac{1}{p^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+p^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{p,q}x\right| \\
&\le \frac{pq^2\zeta^2(\mu_2,\mu_1,\psi)}{(p+q)}h^{\frac{1}{s}}\left(\frac{\psi(q^2+p^2+pq-p-q)|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_1)|^r+(p+q)|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_2)|^r}{(p+q)(q^2+pq+p^2)}\right)^{\frac{1}{r}}, \qquad (4)
\end{aligned}
$$

where

$$
h:=(p-q)\sum_{n=0}^{\infty}\frac{q^{2n}}{p^{2n+2}}\left(1-\frac{q^n}{p^{n+1}}\right)^s.
$$

Proof. Using Lemma 1, Hölder's inequality, the ψ-preinvexity of $|\,{}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon|^r$, and the properties of the modulus, we have

$$\begin{aligned}
&\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|\\
&\le\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int_0^1\tau(1-\mathrm{q}\tau)|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\\
&\le\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1\tau(1-\mathrm{q}\tau)^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}\left(\int_0^1\tau|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|^r\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}\\
&\le\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1\tau(1-\mathrm{q}\tau)^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}\\
&\quad\times\left(\psi|\,{}_{\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1)|^r\int_0^1\tau(1-\tau)\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau+|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r\int_0^1\tau^2\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}\\
&=\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}h^{\frac{1}{s}}\left(\frac{\psi(\mathrm{q}^2+\mathrm{p}^2+\mathrm{pq}-\mathrm{p}-\mathrm{q})|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r+(\mathrm{p}+\mathrm{q})|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r}{(\mathrm{p}+\mathrm{q})(\mathrm{q}^2+\mathrm{pq}+\mathrm{p}^2)}\right)^{\frac{1}{r}}.
\end{aligned}$$

This completes the proof. □

Theorem 4. *Let $\Upsilon:\mathcal{K}\mapsto\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$, and let ${}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon$ be continuous and integrable on $\mathcal{K}$, where $0<\mathrm{q}<\mathrm{p}\le 1$. Suppose that $|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$ is a ψ-preinvex function for $r\ge 1$. Thus,*

$$\begin{aligned}
&\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|\\
&\le\frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\left(\psi k_1|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r+k_2|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r\right)^{\frac{1}{r}},
\end{aligned}\tag{5}$$

where

$$k_1:=(\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\left(\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\right)^{r+1}\left(1-\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\right)\left(1-\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\right)^r$$

and

$$k_2:=(\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\left(\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\right)^{r+3}\left(1-\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\right)^r.$$

Proof. Using Lemma 1, the power mean inequality, the ψ-preinvexity of $|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$, and the properties of the modulus, we have

$$\left|\frac{q\Upsilon(\psi\mu_1) + p\Upsilon(\psi\mu_1 + p\zeta(\mu_2, \mu_1, \psi)}{p+q} - \frac{1}{p^2\zeta(\mu_2, \mu_1, \psi)} \int_{\psi\mu_1}^{\psi\mu_1 + p^2\zeta(\mu_2,\mu_1,\psi)} \Upsilon(x)\ {}_{\psi\mu_1}d_{p,q}x\right|$$
$$\leq \frac{pq^2\zeta^2(\mu_2, \mu_1, \psi)}{p+q} \int_0^1 \tau(1-q\tau)|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\psi\mu_1 + \tau\zeta(\mu_2, \mu_1, \psi))|\ {}_0d_{p,q}\tau$$
$$\leq \frac{pq^2\zeta^2(\mu_2, \mu_1, \psi)}{p+q} \left(\int_0^1 1\ {}_0d_{p,q}\tau\right)^{1-\frac{1}{r}} \left(\int_0^1 \tau^r(1-q\tau)^r|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\psi\mu_1 + \tau\zeta(\mu_2, \mu_1, \psi))|^r\ {}_0d_{p,q}\tau\right)^{\frac{1}{r}}$$
$$\leq \frac{pq^2\zeta^2(\mu_2, \mu_1, \psi)}{p+q} \left(\psi|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_1)|^r \int_0^1 \tau^r(1-\tau)(1-q\tau)^r\ {}_0d_{p,q}\tau\right.$$
$$\left. +|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_2)|^r \int_0^1 \tau^{r+2}(1-q\tau)^r\ {}_0d_{p,q}\tau\right)^{\frac{1}{r}}$$
$$= \frac{pq^2\zeta^2(\mu_2, \mu_1, \psi)}{(p+q)} \left(\psi k_1|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_1)|^r + k_2|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_2)|^r\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 5. *Let $\Upsilon : \mathcal{K} \mapsto \mathbb{R}$ be a twice (p, q)-differentiable function on $\mathcal{K}^\circ$, and let ${}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon$ be continuous and integrable on $\mathcal{K}$, where $0 < q < p \leq 1$. Assume that $|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon|^r$ is a ψ-preinvex function for $r > 1$ and $\frac{1}{s} + \frac{1}{r} = 1$. Thus,*

$$\left|\frac{q\Upsilon(\psi\mu_1) + p\Upsilon(\psi\mu_1 + p\zeta(\mu_2, \mu_1, \psi)}{p+q} - \frac{1}{p^2\zeta(\mu_2, \mu_1, \psi)} \int_{\psi\mu_1}^{\psi\mu_1 + p^2\zeta(\mu_2,\mu_1,\psi)} \Upsilon(x)\ {}_{\psi\mu_1}d_{p,q}x\right|$$
$$\leq \frac{pq^2\zeta^2(\mu_2, \mu_1, \psi)}{(p+q)}\omega^{\frac{1}{s}} \left(\frac{\psi(p+q-1)|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_1)|^r + |\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon(\mu_2)|^r}{(p+q)}\right)^{\frac{1}{r}}, \tag{6}$$

where

$$\omega := (p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^{s+1}\left(1 - \frac{q^{n+1}}{p^{n+1}}\right)^s.$$

Proof. Using Lemma 1, Hölder's inequality, the ψ-preinvexity of $|\ {}_{\psi\mu_1}\mathcal{D}^2_{p,q}\Upsilon|^r$, and the properties of the modulus, we have

$$
\begin{aligned}
&\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|\\
&\leq \frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int_0^1 \tau(1-\mathrm{q}\tau)|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\\
&\leq \frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1 \tau^s(1-\mathrm{q}\tau)^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}\left(\int_0^1 |\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|^r\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}\\
&\leq \frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\\
&\times\left(\int_0^1 \tau^s(1-\mathrm{q}\tau)^s\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}\left(\psi|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r\int_0^1(1-\tau)\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau+|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r\int_0^1\tau\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}\\
&=\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\omega^{\frac{1}{s}}\left(\frac{\psi(\mathrm{p}+\mathrm{q}-1)|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r+|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r}{(\mathrm{p}+\mathrm{q})}\right)^{\frac{1}{r}}.
\end{aligned}
$$

This completes the proof. □

Theorem 6. *Let $\Upsilon : \mathcal{K} \mapsto \mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$, and let ${}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon$ be continuous and integrable on $\mathcal{K}$, where $0<\mathrm{q}<\mathrm{p}\leq 1$. Suppose that $|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$ is a ψ-preinvex function for $r>1$ and $\frac{1}{s}+\frac{1}{r}=1$. Thus,*

$$
\begin{aligned}
&\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|\\
&\leq \frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\left(\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{s+1}-\mathrm{q}^{s+1}}\right)^{\frac{1}{s}}\left(\psi\Delta_1|\,{}_{\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r+\Delta_2|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r\right)^{\frac{1}{r}}, \qquad (7)
\end{aligned}
$$

where

$$\Delta_1 := (\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\left(\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}-\frac{\mathrm{q}^{2n}}{\mathrm{p}^{2n+2}}\right)\left(1-\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\right)^r$$

and

$$\Delta_2 := (\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\frac{\mathrm{q}^{2n}}{\mathrm{p}^{2n+2}}\left(1-\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\right)^r.$$

Proof. Using Lemma 1, Hölder's inequality, the ψ-preinvexity of $|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$, and the properties of the modulus, we have

$$
\begin{aligned}
&\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|\\
&\leq \frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int_0^1 \tau(1-\mathrm{q}\tau)|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau
\end{aligned}
$$

$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1 \tau^s\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}\left(\int_0^1 (1-\mathrm{q}\tau)^r\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))\right|^r\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}$$

$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{s+1}-\mathrm{q}^{s+1}}\right)^{\frac{1}{s}}\left(\psi\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)\right|^r\int_0^1 (1-\tau)(1-\mathrm{q}\tau)^r\,{}_0d_{\mathrm{p,q}}\tau\right.$$

$$\left.+\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)\right|^r\int_0^1 \tau(1-\mathrm{q}\tau)^r\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}$$

$$= \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\left(\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{s+1}-\mathrm{q}^{s+1}}\right)^{\frac{1}{s}}\left(\psi\Delta_1\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)\right|^r+\Delta_2\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)\right|^r\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 7. *Let $\Upsilon : \mathcal{K} \mapsto \mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$, and let ${}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon$ be continuous and integrable on $\mathcal{K}$, where $0<\mathrm{q}<\mathrm{p}\leq 1$. Assume that $|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$ is a ψ-preinvex function for $r>1$ and $\frac{1}{s}+\frac{1}{r}=1$. Thus,*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{\mathrm{p,q}}x\right|$$

$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\gamma^{\frac{1}{s}}\left(\psi\left(\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+1}-\mathrm{q}^{r+1}}-\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}}\right)\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)\right|^r+\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}}\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)\right|^r\right)^{\frac{1}{r}}, \tag{8}$$

where

$$\gamma := (\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\left(1-\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\right)^s.$$

Proof. Using Lemma 1, Hölder's inequality, the ψ-preinvexity of $|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$, and the properties of the modulus, we have

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{\mathrm{p,q}}x\right|$$

$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int_0^1 (1-\mathrm{q}\tau)\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))\right|\,{}_0d_{\mathrm{p,q}}\tau$$

$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1 (1-\mathrm{q}\tau)^s\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}\left(\int_0^1 \tau^r\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))\right|^r\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}$$

$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1 (1-\mathrm{q}\tau)^s\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}\left(\psi\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)\right|^r\int_0^1 \tau^r(1-\tau)\,{}_0d_{\mathrm{p,q}}\tau\right.$$

$$\left.+\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)\right|^r\int_0^1 \tau^{r+1}\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}$$

$$= \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\gamma^{\frac{1}{s}}\left(\psi\left(\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+1}-\mathrm{q}^{r+1}}-\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}}\right)\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)\right|^r+\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}}\left|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)\right|^r\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 8. *Let* $\Upsilon : \mathcal{K} \mapsto \mathbb{R}$ *be a twice* (p,q)*-differentiable function on* $\mathcal{K}^{\circ}$*, and let* ${}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon$ *be continuous and integrable on* $\mathcal{K}$*, where* $0 < \mathrm{q} < \mathrm{p} \leq 1$*. Suppose that* $|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon|^r$ *is a* ψ*-preinvex function for* $r \geq 1$*. Thus,*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq \frac{\mathrm{p}^{2-\frac{1}{r}}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}}\left(\frac{\mathrm{p}^2}{\mathrm{q}^2+\mathrm{pq}+\mathrm{p}^2}\right)^{1-\frac{1}{r}}\left(\frac{\psi(\mathrm{p}^4-\mathrm{p}^3+\mathrm{p}^2\mathrm{q}^2)|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_1)|^r+\mathrm{p}^3|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_2)|^r}{(\mathrm{p}+\mathrm{q})(\mathrm{p}^2+\mathrm{q}^2)(\mathrm{q}^2+\mathrm{pq}+\mathrm{p}^2)}\right)^{\frac{1}{r}}. \quad (9)$$

Proof. Using Lemma 1, the power mean inequality, the ψ-preinvexity of $|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon|^r$, and the properties of the modulus, we have

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int_0^1\tau(1-\mathrm{q}\tau)|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|\,{}_0d_{\mathrm{p},\mathrm{q}}\tau$$
$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1\tau(1-\mathrm{q}\tau)\,{}_0d_{\mathrm{p},\mathrm{q}}\tau\right)^{1-\frac{1}{r}}\left(\int_0^1\tau(1-\mathrm{q}\tau)|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|^r\,{}_0d_{\mathrm{p},\mathrm{q}}\tau\right)^{\frac{1}{r}}$$
$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1\tau(1-\mathrm{q}\tau)\,{}_0d_{\mathrm{p},\mathrm{q}}\tau\right)^{1-\frac{1}{r}}$$
$$\times\left(\psi|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_1)|^r\int_0^1\tau(1-\tau)(1-\mathrm{q}\tau)\,{}_0d_{\mathrm{p},\mathrm{q}}\tau+|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_2)|^r\int_0^1\tau^2(1-\mathrm{q}\tau)\,{}_0d_{\mathrm{p},\mathrm{q}}\tau\right)^{\frac{1}{r}}$$
$$= \frac{\mathrm{p}^{2-\frac{1}{r}}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}}\left(\frac{\mathrm{p}^2}{\mathrm{q}^2+\mathrm{pq}+\mathrm{p}^2}\right)^{1-\frac{1}{r}}\left(\frac{\psi(\mathrm{p}^4-\mathrm{p}^3+\mathrm{p}^2\mathrm{q}^2)|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_1)|^r+\mathrm{p}^3|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_2)|^r}{(\mathrm{p}+\mathrm{q})(\mathrm{p}^2+\mathrm{q}^2)(\mathrm{q}^2+\mathrm{pq}+\mathrm{p}^2)}\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 9. *Let* $\Upsilon : \mathcal{K} \mapsto \mathbb{R}$ *be a twice* (p,q)*-differentiable function on* $\mathcal{K}^{\circ}$*, and let* ${}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon$ *be continuous and integrable on* $\mathcal{K}$*, where* $0 < \mathrm{q} < \mathrm{p} \leq 1$*. Assume that* $|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon|^r$ *is a* ψ*-preinvex function for* $r \geq 1$*. Thus,*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}d_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq \frac{\mathrm{p}^{2-\frac{1}{r}}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}}\left(\psi\left[\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+1}-\mathrm{q}^{r+1}}-\frac{(\mathrm{p}-\mathrm{q})(1+\mathrm{q})}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}}+\frac{\mathrm{q}(\mathrm{p}-\mathrm{q})}{\mathrm{p}^{r+3}-\mathrm{q}^{r+3}}\right]|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_1)|^r\right.$$
$$\left.+\left[\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}}-\frac{\mathrm{q}(\mathrm{p}-\mathrm{q})}{\mathrm{p}^{r+3}-\mathrm{q}^{r+3}}\right]|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon(\mu_2)|^r\right)^{\frac{1}{r}}. \quad (10)$$

Proof. Using Lemma 1, the power mean inequality, the ψ-preinvexity of $|{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon|^r$, and the properties of the modulus, we have

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|$$
$$\le\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int_0^1\tau(1-\mathrm{q}\tau)|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau$$
$$\le\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1(1-\mathrm{q}\tau)\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{1-\frac{1}{r}}\left(\int_0^1\tau^r(1-\mathrm{q}\tau)|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|^r\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}$$
$$\le\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1(1-\mathrm{q}\tau)\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{1-\frac{1}{r}}$$
$$\times\left(\psi|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r\int_0^1\tau^r(1-\tau)(1-\mathrm{q}\tau)\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau+|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r\int_0^1\tau^{r+1}(1-\mathrm{q}\tau)\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}$$
$$=\frac{\mathrm{p}^{2-\frac{1}{r}}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}}\left(\psi\left[\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+1}-\mathrm{q}^{r+1}}-\frac{(\mathrm{p}-\mathrm{q})(1+\mathrm{q})}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}}+\frac{\mathrm{q}(\mathrm{p}-\mathrm{q})}{\mathrm{p}^{r+3}-\mathrm{q}^{r+3}}\right]|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r\right.$$
$$\left.+\left[\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}}-\frac{\mathrm{q}(\mathrm{p}-\mathrm{q})}{\mathrm{p}^{r+3}-\mathrm{q}^{r+3}}\right]|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r\right)^{\frac{1}{r}}.$$

This completes the proof. □

Theorem 10. *Let $\Upsilon:\mathcal{K}\mapsto\mathbb{R}$ be a twice (p,q)-differentiable function on $\mathcal{K}^\circ$, and let ${}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon$ be continuous and integrable on $\mathcal{K}$, where $0<\mathrm{q}<\mathrm{p}\le 1$. Suppose that $|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$ is a ψ-preinvex function for $r>1$ and $\frac{1}{s}+\frac{1}{r}=1$. Thus,*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|$$
$$\le\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\lambda^{\frac{1}{s}}\left(\frac{\psi(\mathrm{p}^3-\mathrm{p}^2+\mathrm{p}\mathrm{q}^2+\mathrm{p}^2\mathrm{q})|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r+\mathrm{p}^2|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r}{(\mathrm{p}+\mathrm{q})(\mathrm{q}^2+\mathrm{p}\mathrm{q}+\mathrm{p}^2)}\right)^{\frac{1}{r}},\tag{11}$$

where

$$\lambda:=(\mathrm{p}-\mathrm{q})\sum_{n=0}^{\infty}\left(\frac{\mathrm{q}^n}{\mathrm{p}^{n+1}}\right)^{s+1}\left(1-\frac{\mathrm{q}^{n+1}}{\mathrm{p}^{n+1}}\right)^s.$$

Proof. Using Lemma 1, Hölder's inequality, the ψ-preinvexity of $|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon|^r$, and the properties of the modulus, we have

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\,{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p,q}}x\right|$$
$$\le\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\int_0^1\tau(1-\mathrm{q}\tau)|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|\,{}_0\mathrm{d}_{\mathrm{p,q}}\tau$$

$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1 \tau^s(1-\mathrm{q}\tau)\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}\left(\int_0^1(1-\mathrm{q}\tau)|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\psi\mu_1+\tau\zeta(\mu_2,\mu_1,\psi))|^r\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}$$

$$\leq \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}\left(\int_0^1 \tau^s(1-\mathrm{q}\tau)^s\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{s}}$$

$$\times\left(\psi|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r\int_0^1(1-\tau)(1-\mathrm{q}\tau)\,{}_0d_{\mathrm{p,q}}\tau+|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r\int_0^1\tau(1-\mathrm{q}\tau)\,{}_0d_{\mathrm{p,q}}\tau\right)^{\frac{1}{r}}$$

$$= \frac{\mathrm{pq}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\lambda^{\frac{1}{s}}\left(\frac{\psi(\mathrm{p}^3-\mathrm{p}^2+\mathrm{pq}^2+\mathrm{p}^2\mathrm{q})|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_1)|^r+\mathrm{p}^2|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p,q}}\Upsilon(\mu_2)|^r}{(\mathrm{p}+\mathrm{q})(\mathrm{q}^2+\mathrm{pq}+\mathrm{p}^2)}\right)^{\frac{1}{r}}.$$

This completes the proof. □

3. Applications

In this section, we will discuss some applications regarding our main results given in Section 2 for special means and bounded functions as well.

3.1. Application to Special Means

First, let us denote

$$\mathcal{M}^\diamond := \frac{\mathrm{q}((1-\mathrm{p}^2)\mu_1+\mathrm{p}^2\mu_2)^n-(\mathrm{p}+\mathrm{q})((1-\mathrm{pq})\mu_1+\mathrm{pq}\mu_2)^n+\mathrm{p}((1-\mathrm{q}^2)\mu_1+\mathrm{q}^2\mu_2)^n}{\mathrm{pq}(\mathrm{p}-\mathrm{q})^2(\mu_2-\mu_1)^2},$$

where $0<\mu_1<\mu_2$ are real numbers and $0<\mathrm{q}<\mathrm{p}\leq 1$.

- The arithmetic mean is defined as

$$\mathcal{A}(\mu_1,\mu_2) := \frac{\mu_1+\mu_2}{2}.$$

- The generalized logarithmic mean is given by

$$\mathcal{L}_n(\mu_1,\mu_2) := \left[\frac{\mu_2^{n+1}-\mu_1^{n+1}}{(n+1)(\mu_2-\mu_1)}\right]^{\frac{1}{n}}, \quad n\in\mathbb{R}\backslash\{-1,0\}.$$

Using the above special means, we can establish the following inequalities.

Proposition 1. *Let* $0<\mu_1<\mu_2$, $n>1$ *and* $0<\mathrm{q}<\mathrm{p}\leq 1$. *Thus,*

$$\left|\frac{2}{\mathrm{p}+\mathrm{q}}\mathcal{A}(\mu_1^n,\mathrm{p}((1-\mathrm{p})\mu_1+\mathrm{p}\mu_2)^n)-\frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})}\mathcal{L}_n^n(\mu_1,(1-\mathrm{p}^2)\mu_1+\mathrm{p}^2\mu_2)\right|$$
$$\leq \frac{\mathrm{pq}^2(\mu_2-\mu_1)^2((\mathrm{p}^4-\mathrm{p}^3+\mathrm{p}^2\mathrm{q}^2)n(n-1)\mu_1^{n-2}+\mathrm{p}^3|M^\diamond|)}{(\mathrm{p}+\mathrm{q})^2(\mathrm{p}^2+\mathrm{q}^2)(\mathrm{q}^2+\mathrm{pq}+\mathrm{p}^2)}. \quad (12)$$

Proof. The proof directly follows from Theorem 1, applying $\Upsilon(x)=x^n,\psi=1$ and $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$. □

Proposition 2. *Let* $0<\mu_1<\mu_2$, $n>1, r\geq 1$ *and* $0<\mathrm{q}<\mathrm{p}\leq 1$. *Thus,*

$$\left|\frac{2}{\mathrm{p}+\mathrm{q}}\mathcal{A}(\mu_1^n,\mathrm{p}((1-\mathrm{p})\mu_1+\mathrm{p}\mu_2)^n)-\frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})}\mathcal{L}_n^n(\mu_1,(1-\mathrm{p}^2)\mu_1+\mathrm{p}^2\mu_2)\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2(\mu_2-\mu_1)}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}}\left(d_1n(n-1)\mu_1^{r(n-2)}+d_2|\mathcal{M}^\diamond|^r\right)^{\frac{1}{r}}, \tag{13}$$

Proof. The proof directly follows from Theorem 2, applying $\Upsilon(x)=x^n,\psi=1$ and $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$. $\square$

Proposition 3. *Let* $0<\mu_1<\mu_2$, $n>1$, $r>1$, $\frac{1}{s}+\frac{1}{r}=1$ *and* $0<\mathrm{q}<\mathrm{p}\leq 1$. *Thus,*

$$\left|\frac{2}{\mathrm{p}+\mathrm{q}}\mathcal{A}(\mu_1^n,\mathrm{p}((1-\mathrm{p})\mu_1+\mathrm{p}\mu_2)^n)-\frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})}\mathcal{L}_n^n(\mu_1,(1-\mathrm{p}^2)\mu_1+\mathrm{p}^2\mu_2)\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2(\mu_2-\mu_1)2}{(\mathrm{p}+\mathrm{q})}h^{\frac{1}{s}}\left(\frac{(\mathrm{q}^2+\mathrm{p}^2+\mathrm{p}\mathrm{q}-\mathrm{p}-\mathrm{q})n(n-1)\mu_1^{r(n-2)}+(\mathrm{p}+\mathrm{q})|\mathcal{M}^\diamond|^r}{(\mathrm{p}+\mathrm{q})(\mathrm{q}^2+\mathrm{p}\mathrm{q}+\mathrm{p}^2)}\right)^{\frac{1}{r}}. \tag{14}$$

Proof. The proof directly follows from Theorem 3, applying $\Upsilon(x)=x^n,\psi=1$ and $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$. $\square$

Proposition 4. *Let* $0<\mu_1<\mu_2$, $n>1$, $r\geq 1$ *and* $0<\mathrm{q}<\mathrm{p}\leq 1$. *Thus,*

$$\left|\frac{2}{\mathrm{p}+\mathrm{q}}\mathcal{A}(\mu_1^n,\mathrm{p}((1-\mathrm{p})\mu_1+\mathrm{p}\mu_2)^n)-\frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})}\mathcal{L}_n^n(\mu_1,(1-\mathrm{p}^2)\mu_1+\mathrm{p}^2\mu_2)\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2(\mu_2-\mu_1)^2}{(\mathrm{p}+\mathrm{q})}\left(n(n-1)k_1\mu_1^{r(n-2)}+k_2|\mathcal{M}^\diamond|^r\right)^{\frac{1}{r}}. \tag{15}$$

Proof. The proof directly follows from Theorem 4, applying $\Upsilon(x)=x^n,\psi=1$ and $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$. $\square$

Proposition 5. *Let* $0<\mu_1<\mu_2$, $n>1$, $r>1$, $\frac{1}{s}+\frac{1}{r}=1$ *and* $0<\mathrm{q}<\mathrm{p}\leq 1$. *Thus,*

$$\left|\frac{2}{\mathrm{p}+\mathrm{q}}\mathcal{A}(\mu_1^n,\mathrm{p}((1-\mathrm{p})\mu_1+\mathrm{p}\mu_2)^n)-\frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})}\mathcal{L}_n^n(\mu_1,(1-\mathrm{p}^2)\mu_1+\mathrm{p}^2\mu_2)\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2(\mu_2-\mu_1)^2}{(\mathrm{p}+\mathrm{q})}\omega^{\frac{1}{s}}\left(\frac{n(n-1)(\mathrm{p}+\mathrm{q}-1)\mu_1^{r(n-2)}+|\mathcal{M}^\diamond|^r}{(\mathrm{p}+\mathrm{q})}\right)^{\frac{1}{r}}. \tag{16}$$

Proof. The proof directly follows from Theorem 5, applying $\Upsilon(x)=x^n,\psi=1$ and $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$. $\square$

Proposition 6. *Let* $0<\mu_1<\mu_2$, $n>1$, $r>1$, $\frac{1}{s}+\frac{1}{r}=1$ *and* $0<\mathrm{q}<\mathrm{p}\leq 1$. *Thus,*

$$\left|\frac{2}{\mathrm{p}+\mathrm{q}}\mathcal{A}(\mu_1^n,\mathrm{p}((1-\mathrm{p})\mu_1+\mathrm{p}\mu_2)^n)-\frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})}\mathcal{L}_n^n(\mu_1,(1-\mathrm{p}^2)\mu_1+\mathrm{p}^2\mu_2)\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2(\mu_2-\mu_1)^2}{(\mathrm{p}+\mathrm{q})}\left(\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{s+1}-\mathrm{q}^{s+1}}\right)^{\frac{1}{s}}\left(n(n-1)\Delta_1\mu_1^{r(n-2)}+\Delta_2|\mathcal{M}^\diamond|^r\right)^{\frac{1}{r}}. \tag{17}$$

Proof. The proof directly follows from Theorem 6, applying $\Upsilon(x)=x^n,\psi=1$ and $\zeta(\mu_2,\mu_1)=\mu_2-\mu_1$. $\square$

Proposition 7. *Let* $0<\mu_1<\mu_2$, $n>1$, $r>1$, $\frac{1}{s}+\frac{1}{r}=1$ *and* $0<\mathrm{q}<\mathrm{p}\leq 1$. *Thus,*

$$\left| \frac{2}{\mathrm{p}+\mathrm{q}} \mathcal{A}(\mu_1^n, \mathrm{p}((1-\mathrm{p})\mu_1 + \mathrm{p}\mu_2)^n) - \frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})} \mathcal{L}_n^n(\mu_1, (1-\mathrm{p}^2)\mu_1 + \mathrm{p}^2\mu_2) \right|$$
$$\leq \frac{\mathrm{p}\mathrm{q}^2(\mu_2-\mu_1)^2}{(\mathrm{p}+\mathrm{q})} \gamma^{\frac{1}{s}} \left(\left(\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+1}-\mathrm{q}^{r+1}} - \frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}} \right) \mu_1^{r(n-2)} + \frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}} |\mathcal{M}^{\diamond}|^r \right)^{\frac{1}{r}}. \tag{18}$$

Proof. The proof directly follows from Theorem 7, applying $\Upsilon(x) = x^n, \psi = 1$ and $\zeta(\mu_2, \mu_1) = \mu_2 - \mu_1$. $\square$

Proposition 8. *Let* $0 < \mu_1 < \mu_2$, $n > 1$, $r \geq 1$ *and* $0 < \mathrm{q} < \mathrm{p} \leq 1$. *Thus,*

$$\left| \frac{2}{\mathrm{p}+\mathrm{q}} \mathcal{A}(\mu_1^n, \mathrm{p}((1-\mathrm{p})\mu_1 + \mathrm{p}\mu_2)^n) - \frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})} \mathcal{L}_n^n(\mu_1, (1-\mathrm{p}^2)\mu_1 + \mathrm{p}^2\mu_2) \right|$$
$$\leq \frac{\mathrm{p}^{2-\frac{1}{r}}\mathrm{q}^2(\mu_2-\mu_1)^2}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}} \left(\frac{\mathrm{p}^2}{\mathrm{q}^2+\mathrm{p}\mathrm{q}+\mathrm{p}^2} \right)^{1-\frac{1}{r}} \left(\frac{n(n-1)(\mathrm{p}^4-\mathrm{p}^3+\mathrm{p}^2\mathrm{q}^2)\mu_1^{r(n-2)} + \mathrm{p}^3|\mathcal{M}^{\diamond}|^r}{(\mathrm{p}+\mathrm{q})(\mathrm{p}^2+\mathrm{q}^2)(\mathrm{q}^2+\mathrm{p}\mathrm{q}+\mathrm{p}^2)} \right)^{\frac{1}{r}}. \tag{19}$$

Proof. The proof directly follows from Theorem 8, applying $\Upsilon(x) = x^n, \psi = 1$ and $\zeta(\mu_2, \mu_1) = \mu_2 - \mu_1$. $\square$

Proposition 9. *Let* $0 < \mu_1 < \mu_2$, $n > 1$, $r \geq 1$ *and* $0 < \mathrm{q} < \mathrm{p} \leq 1$. *Thus,*

$$\left| \frac{2}{\mathrm{p}+\mathrm{q}} \mathcal{A}(\mu_1^n, \mathrm{p}((1-\mathrm{p})\mu_1 + \mathrm{p}\mu_2)^n) - \frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})} \mathcal{L}_n^n(\mu_1, (1-\mathrm{p}^2)\mu_1 + \mathrm{p}^2\mu_2) \right|$$
$$\leq \frac{\mathrm{p}^{2-\frac{1}{r}}\mathrm{q}^2(\mu_2-\mu_1)^2}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}} \left(\left[\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+1}-\mathrm{q}^{r+1}} - \frac{(\mathrm{p}-\mathrm{q})(1+\mathrm{q})}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}} + \frac{\mathrm{q}(\mathrm{p}-\mathrm{q})}{\mathrm{p}^{r+3}-\mathrm{q}^{r+3}} \right] n(n-1)\mu_1^{r(n-2)} \right.$$
$$\left. + \left[\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{r+2}-\mathrm{q}^{r+2}} - \frac{\mathrm{q}(\mathrm{p}-\mathrm{q})}{\mathrm{p}^{r+3}-\mathrm{q}^{r+3}} \right] |\mathcal{M}^{\diamond}|^r \right)^{\frac{1}{r}}. \tag{20}$$

Proof. The proof directly follows from Theorem 9, applying $\Upsilon(x) = x^n, \psi = 1$ and $\zeta(\mu_2, \mu_1) = \mu_2 - \mu_1$. $\square$

Proposition 10. *Let* $0 < \mu_1 < \mu_2$, $n > 1$, $r > 1$, $\frac{1}{s} + \frac{1}{r} = 1$ *and* $0 < \mathrm{q} < \mathrm{p} \leq 1$. *Thus,*

$$\left| \frac{2}{\mathrm{p}+\mathrm{q}} \mathcal{A}(\mu_1^n, \mathrm{p}((1-\mathrm{p})\mu_1 + \mathrm{p}\mu_2)^n) - \frac{(n+1)(\mathrm{p}-\mathrm{q})}{\mathrm{p}^2(\mathrm{p}^{n+1}-\mathrm{q}^{n+1})} \mathcal{L}_n^n(\mu_1, (1-\mathrm{p}^2)\mu_1 + \mathrm{p}^2\mu_2) \right|$$
$$\leq \frac{\mathrm{p}\mathrm{q}^2(\mu_2-\mu_1)^2}{(\mathrm{p}+\mathrm{q})} \lambda^{\frac{1}{s}} \left(\frac{n(n-1)(\mathrm{p}^3-\mathrm{p}^2+\mathrm{p}\mathrm{q}^2+\mathrm{p}^2\mathrm{q})\mu_1^{r(n-2)} + \mathrm{p}^2|\mathcal{M}^{\diamond}|^r}{(\mathrm{p}+\mathrm{q})(\mathrm{q}^2+\mathrm{p}\mathrm{q}+\mathrm{p}^2)} \right)^{\frac{1}{r}}, \tag{21}$$

Proof. The proof directly follows from Theorem 10, applying $\Upsilon(x) = x^n, \psi = 1$ and $\zeta(\mu_2, \mu_1) = \mu_2 - \mu_1$. $\square$

3.2. Application to Bounded Functions

We suppose that the following condition is satisfied:

$$|\,{}_{\psi\mu_1}\mathcal{D}^2_{\mathrm{p},\mathrm{q}}\Upsilon| \leq \digamma, \tag{22}$$

which means that the twice (p, q)-differentiable function Υ is in absolute value bounded from the positive real number $\digamma$. Applying the above condition, we are in a position to derive some new interesting inequalities using our main results.

Proposition 11. *Under the conditions of Theorem 1, the following inequality holds:*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int\limits_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\;{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)(\psi(\mathrm{p}^4-\mathrm{p}^3+\mathrm{p}^2\mathrm{q}^2)+\mathrm{p}^3)F}{(\mathrm{p}+\mathrm{q})^2(\mathrm{p}^2+\mathrm{q}^2)(\mathrm{q}^2+\mathrm{p}\mathrm{q}+\mathrm{p}^2)}. \tag{23}$$

Proposition 12. *Under the conditions of Theorem 2, the following inequality holds:*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int\limits_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\;{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})^{2-\frac{1}{r}}}(\psi d_1+d_2)^{\frac{1}{r}}F. \tag{24}$$

Proposition 13. *Under the conditions of Theorem 3, the following inequality holds:*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int\limits_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\;{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}h^{\frac{1}{s}}\left(\frac{\psi(\mathrm{q}^2+\mathrm{p}^2+\mathrm{p}\mathrm{q}-\mathrm{p}-\mathrm{q})+\mathrm{p}+\mathrm{q}}{(\mathrm{p}+\mathrm{q})(\mathrm{q}^2+\mathrm{p}\mathrm{q}+\mathrm{p}^2)}\right)^{\frac{1}{r}}F. \tag{25}$$

Proposition 14. *Under the conditions of Theorem 4, the following inequality holds:*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int\limits_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\;{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}(\psi k_1+k_2)^{\frac{1}{r}}F. \tag{26}$$

Proposition 15. *Under the conditions of Theorem 5, the following inequality holds:*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int\limits_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\;{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\omega^{\frac{1}{s}}\left(\frac{\psi(\mathrm{p}+\mathrm{q}-1)+1}{(\mathrm{p}+\mathrm{q})}\right)^{\frac{1}{r}}F. \tag{27}$$

Proposition 16. *Under the conditions of Theorem 6, the following inequality holds:*

$$\left|\frac{\mathrm{q}\Upsilon(\psi\mu_1)+\mathrm{p}\Upsilon(\psi\mu_1+\mathrm{p}\zeta(\mu_2,\mu_1,\psi)}{\mathrm{p}+\mathrm{q}}-\frac{1}{\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\int\limits_{\psi\mu_1}^{\psi\mu_1+\mathrm{p}^2\zeta(\mu_2,\mu_1,\psi)}\Upsilon(x)\;{}_{\psi\mu_1}\mathrm{d}_{\mathrm{p},\mathrm{q}}x\right|$$
$$\leq\frac{\mathrm{p}\mathrm{q}^2\zeta^2(\mu_2,\mu_1,\psi)}{(\mathrm{p}+\mathrm{q})}\left(\frac{\mathrm{p}-\mathrm{q}}{\mathrm{p}^{s+1}-\mathrm{q}^{s+1}}\right)^{\frac{1}{s}}(\psi\Delta_1+\Delta_2)^{\frac{1}{r}}F. \tag{28}$$

Proposition 17. *Under the conditions of Theorem 7, the following inequality holds:*

$$\left|\frac{qY(\psi\mu_1)+pY(\psi\mu_1+p\zeta(\mu_2,\mu_1,\psi)}{p+q}-\frac{1}{p^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+p^2\zeta(\mu_2,\mu_1,\psi)} Y(x)\,{}_{\psi\mu_1}d_{p,q}x\right|$$
$$\leq \frac{pq^2\zeta^2(\mu_2,\mu_1,\psi)}{(p+q)}\gamma^{\frac{1}{s}}\left(\psi\left(\frac{p-q}{p^{r+1}-q^{r+1}}-\frac{p-q}{p^{r+2}-q^{r+2}}\right)+\frac{p-q}{p^{r+2}-q^{r+2}}\right)^{\frac{1}{r}}\digamma. \tag{29}$$

Proposition 18. *Under the conditions of Theorem 8, the following inequality holds:*

$$\left|\frac{qY(\psi\mu_1)+pY(\psi\mu_1+p\zeta(\mu_2,\mu_1,\psi)}{p+q}-\frac{1}{p^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+p^2\zeta(\mu_2,\mu_1,\psi)} Y(x)\,{}_{\psi\mu_1}d_{p,q}x\right|$$
$$\leq \frac{p^{2-\frac{1}{r}}q^2\zeta^2(\mu_2,\mu_1,\psi)}{(p+q)^{2-\frac{1}{r}}}\left(\frac{p^2}{q^2+pq+p^2}\right)^{1-\frac{1}{r}}\left(\frac{\psi(p^4-p^3+p^2q^2)+p^3}{(p+q)(p^2+q^2)(q^2+pq+p^2)}\right)^{\frac{1}{r}}\digamma. \tag{30}$$

Proposition 19. *Under the conditions of Theorem 9, the following inequality holds:*

$$\left|\frac{qY(\psi\mu_1)+pY(\psi\mu_1+p\zeta(\mu_2,\mu_1,\psi)}{p+q}-\frac{1}{p^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+p^2\zeta(\mu_2,\mu_1,\psi)} Y(x)\,{}_{\psi\mu_1}d_{p,q}x\right|$$
$$\leq \frac{p^{2-\frac{1}{r}}q^2\zeta^2(\mu_2,\mu_1,\psi)}{(p+q)^{2-\frac{1}{r}}}\left(\psi\left[\frac{p-q}{p^{r+1}-q^{r+1}}-\frac{(p-q)(1+q)}{p^{r+2}-q^{r+2}}+\frac{q(p-q)}{p^{r+3}-q^{r+3}}\right]\right.$$
$$\left.+\left[\frac{p-q}{p^{r+2}-q^{r+2}}-\frac{q(p-q)}{p^{r+3}-q^{r+3}}\right]\right)^{\frac{1}{r}}\digamma. \tag{31}$$

Proposition 20. *Under the conditions of Theorem 10, the following inequality holds:*

$$\left|\frac{qY(\psi\mu_1)+pY(\psi\mu_1+p\zeta(\mu_2,\mu_1,\psi)}{p+q}-\frac{1}{p^2\zeta(\mu_2,\mu_1,\psi)}\int_{\psi\mu_1}^{\psi\mu_1+p^2\zeta(\mu_2,\mu_1,\psi)} Y(x)\,{}_{\psi\mu_1}d_{p,q}x\right|$$
$$\leq \frac{pq^2\zeta^2(\mu_2,\mu_1,\psi)}{(p+q)}\lambda^{\frac{1}{s}}\left(\frac{\psi(p^3-p^2+pq^2+p^2q)+p^2}{(p+q)(q^2+pq+p^2)}\right)^{\frac{1}{r}}\digamma. \tag{32}$$

Remark 2. *Since the class of ψ-preinvex functions have large applications in many mathematical areas, they can be applied to obtain several new results in convex analysis, special functions, quantum mechanics, related optimization theory, and mathematical inequalities and may stimulate further research in different areas of pure and applied sciences. For more details, please see [17–24].*

4. Conclusions

In this paper, we have established a new post-quantum integral identity using twice (p,q)-differentiable functions. From the applied identity as an auxiliary result, we have obtained some new variants of Hermite–Hadamard's inequality essentially pertaining to the class of ψ-preinvex functions. In order to illustrate the efficiency of our main results, some applications regarding special means of positive real numbers and twice (p,q)-differentiable functions that are bounded are provided as well. To the best of our knowledge, these results are new in the literature. Since the class of ψ-preinvex functions have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, and mathematical inequalities and may stimulate further research in different areas of pure and applied sciences. Studies relating convexity, partial convexity, and preinvex functions (as contractive operators) may have useful applications in complex interdisciplinary studies, such as maximizing the likelihood from multiple linear regressions involving Gauss–Laplace distribution. For more details, please see [25–33].

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