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Review

Foundations of Engineering Mathematics Applied for Fluid Flows

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Abstract: Based on a brief historical excursion, a list of principles is formulated which substantiates the choice of axioms and methods for studying nature. The axiomatics of fluid flows are based on conservation laws in the frames of engineering mathematics and technical physics. In the theory of fluid flows within the continuous medium model, a key role for the total energy is distinguished. To describe a fluid flow, a system of fundamental equations is chosen, supplemented by the equations of the state for the Gibbs potential and the medium density. The system is supplemented by the physically based initial and boundary conditions and analyzed, taking into account the compatibility condition. The complete solutions constructed describe both the structure and dynamics of non-stationary flows. The classification of structural components, including waves, ligaments, and vortices, is given on the basis of the complete solutions of the linearized system. The results of compatible theoretical and experimental studies are compared for the cases of potential and actual homogeneous and stratified fluid flow past an arbitrarily oriented plate. The importance of studying the transfer and transformation processes of energy components is illustrated by the description of the fine structures of flows formed by a free-falling drop coalescing with a target fluid at rest.

Keywords: fluid; flows; dynamic; structure; axiomatics; fundamental equations; dissipation; complete solution; ligaments; waves; vortices; plate; wake; drop; impact

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1. Introduction

By recognizing mathematics as "the language in which the book of the Universe is written. . . " in the polemical treatise "Assaying Master", G. Galilei ([1], 1623) opened a new epoch in the development of natural sciences. Unification of mathematics with applied disciplines enables introducing the concept of "accuracy" as an instrument for estimation of the conformity degree of conclusions to the basic axioms. Simultaneously, the dual concept of "error" value was introduced in practice as a variability measure for physical quantities and their differences from the "exact values" either calculated theoretically or prescribed. The concept of accuracy includes the estimation of adequacy, which is the mutual correspondence of basic ideas in various fields of knowledge. With the explanation of the physical meaning of "acceleration" and "inertia", introducing analogues of such quantities as "pressure" and "temperature", the new epoch came for physical quantity definitions which most fully characterized the physical state of matter and the measure of its variability. At the same time, attempts were made to find a measure for mechanical motion of macroscopic bodies (solid, liquid, or gaseous).

A crucial contribution to the development of the mathematical and physical sciences belongs to R. Descartes. His remarkable achievements in the context of this topic were the introduction of a coordinate system [2], which included the implicit definition of a dimension as belonging to a certain type of sets with given properties, in this case with a measure of the length and the numeric value for a quantity (i.e., the dimensionless ratio of the value to the unit of measure). Considering conservation laws as the basis for the

definition of a measurement process, R. Descartes selected momentum, being the product of the body mass multiplied by the velocity, as a measure for body motion ([3], 1641). J.C. Maxwell recognized the vector nature of the momentum and velocity two centuries later [4] based on Hamilton's theory of quaternions [5]. By introducing a coordinate system, Descartes not only unified algebra and geometry but also created a basis for searching for connections between the dynamic and geometric parameters of physical phenomena, the properties of which are conserved with the transition to a new coordinate system.

In a short polemic note, G.W. Leibniz introduced "vis viva" ("live force"), equal to double the kinetic energy of a body's movement as a new measure of motion ([6], 1686). "Vis viva" is a part of the energy as a scalar parameter with a more general nature, which includes both potential and other types of energy in general cases. The universality degree of the concept of energy goes far beyond the framework of mechanical processes. Energy is used to describe all physical processes in the widest range of time and space scales.

Almost synchronously with Leibniz's note, I. Newton published his fundamental treatise ([7], 1687) with an alternative approach to mechanics based on the concept of force and the postulation of "laws of motion". The introduction of the "material point", force, and acceleration concepts exerted and still continue to have a crucial influence on the development of theoretical mechanics [8] and many related sciences. The works by Descartes, Leibniz, and Newton contributed to the successful joint development of both hydrodynamics and mathematics as consensual tools for describing complex phenomena within the framework of the "continuous medium" concept.

Based on the concept of "infinitesimal quantities", the theory of fluid flows began to take its modern form in the middle of the 18th century, when J.-le R. d'Alembert was the first to construct a solution to the partial differential equation describing string oscillations. Later, J.-le R. d'Alembert formulated the continuity equation which is the differential form of the matter conservation law for both incompressible fluids and compressible gases [9]. He thereby established the most general form of the local conservation law as a connection of the volume temporal variation value with flux through the covering's surface. A series of experiments on the body motion's drag in fluids carried out by d'Alembert together with de Condorcet and l'abbe Bossut [10] showed the effectiveness of the coordinated efforts of mathematicians and experimenters in solving practical important problems that ensure the protection of public finances from "indomitable inventors".

Based on the Newton's laws of motion [7], L. Euler applied "solidification of a liquid particle" and obtained the first closed system of equations describing the motion of "ideal" compressible and incompressible fluids. In the current interpretation, Euler's system written for density, velocity, pressure, and gravity acceleration [11] is a form of the representation of mass (continuity of the medium) and momentum conservation equations. The conclusion in [11] is as follows: "Everything that the Theory of liquids is contained is held the two above equations (§ 34), so that for continuation of these studies, we lack not the laws of mechanics, but only the Analysis, which is not yet sufficiently developed for this purpose", stimulated the search for solutions to the Euler equations. This search still continues successfully and brings new approximate and exact solutions to particular problems, including the description of traveling gravitational waves in water [12].

The tool needed to correct the deficiency of the Euler equations—the absence of viscous friction, which became more and more apparent—was developed by J. J. Fourier [13] at the beginning of the 19th century. The value of the new differential equation of a parabolic type found by J.J. Fourier in the analysis of heat conduction processes and the method of its solution can hardly be overestimated. In particular, the developed operator was included by his follower C. Navier into the equation to describe a viscous dissipation [14]. It is interesting to note that C. Navier's explanation of the equation's derivation for a viscous continuous medium was based on P.-S. Laplace's idea of the discrete (atomic) structure of matter. A. Fick also used Fourier's representations in deriving the equations of simple diffusion [15].

Working on improvement of the accuracy of pendulum gravimeters, G.G. Stokes reinterpreted Navier's equations within the framework of the theory of continuous medium motion, making a number of reasonable assumptions, particularly stating the independence of viscous forces on pressure, and giving them their modern form [16]. Stokes's works for many years remained invisible in the shadow of the active, practically important research of linear and nonlinear waves (Russell, Rayleigh, Boussinesq, Thomson, Airy, and many others) and vortices (Helmholtz, Thomson, Kirchhoff, and others), while H. Lamb, in his extensive treatise [17], did not emphasize their fundamental nature. Basically, Stokes, like most scientists in the 19th century, studied the flows of a homogeneous and two-layer (multilayer) fluid, although the very fact of density variability was well known since the 18th century [18].

Russian scientists traditionally paid attention to the study of the density variability and its impact on fluid flows. In famous articles, M.V. Lomonosov described the atomic nature of air elasticity [19] and indicated the influence of density inhomogeneity on the air flow in mines [20]. The extended article by M.V. Lomonosov on the significance of the Arctic Ocean [21], which was presented in 1763 and first published a hundred years later, has been analyzed and widely cited until now.

The initiator of the development of a heavy apparatus for flights in the air and the creator of the scientific foundations of the theory of aeronautics [22], the great encyclopedist D.I. Mendeleev investigated the state equation for gases [23], pure liquids [24], and solutions [25]. The title of one of the Mendeleev's fundamental monographs [23] reproduced the name of the article [20], emphasizing the continuation and relations of the ideas.

However, active accounting for the continuous density variability in fluid flows was not realized due to the supposed small effect of its insignificant relative changes both in natural conditions and in many industrial technologies. Actually, the impurities in the fluid itself were presented as certain "passive substances", with the density given by an empirical expression [26] independent on the condition of fluid and gas motions. Furthermore, in fact, the density was really excluded from most parts of the theoretical investigations by the assumption of its constancy, omitting it as an overall multiplier from dynamic equations together with an equation of state closing the system. In this approximation, it was sufficient for describing the flows to calculate the fields of the velocity components and pressure only [27].

The situation began to change significantly at the end of the 19th century, when J. Gibbs discovered the relations between the thermodynamic potentials and the physical properties of fluids or gases, which are density, pressure, entropy, and temperature, among others [28]. Moreover, Gibbs revealed available potential surface energy, which is an additional form of internal energy in fluids with the surface tension's specified existence of the free surface of a fluid. Now, when the thermodynamic parameters of fluids and gases are defined as derivatives of the Gibbs potential [29,30], the energy concept is used for description of both the static properties of fluids and the dynamics of their variations in a flow. However, practical implementation of the dual nature of fluid parameters, such as density, pressure, and enthalpy, which have mechanical and thermodynamic senses in the description of fluid flows, still remained very limited.

Thus, by the end of the 19th century, the equations representing all the conservation laws, which were necessary for the description of a fluid flow, were written by D'Alembert (continuity) as well as Navier, Stokes, Fourier, and Fick for the transport of momentum, temperature, and dissolved matter. Moreover, the energetic basis for descriptions of the medium state (introduced thermodynamic potentials and their derivatives) was constructed and later implemented in the form of the state equations. However, the idea of considering all governing equations together as a self-consistent system and performing analysis while taking into account the compatibility condition was not expressed in general and was not practically implemented in the form of particular examples. As one of the reasons, one can indicate the "subconscious" influence of the smallness of the ratios between the variations in energy (and density) and the value of the total energy, as well as

the lack of scrutinizing analysis of the energy and density distributions in motionless and flowing fluid.

At the same time, due to the insufficient development of methods for analyzing complex systems of nonlinear differential equations for solving practical problems, a variety of semi-empirical and purely constitutive theories were created. There were proposed theories of linear and nonlinear waves [31,32], turbulence, which was being actively developed in the works of W. Thomson, J. Boussinesq, and especially O. Reynolds [33,34], boundary layers [35,36], vortices [37], and others.

In the middle of the last century, stratification effects, which ensure the existence of internal waves [38] and multilayer convective flows [39], began to be studied. However, rare and separate works did not change the essence of the general approach, although the scientific community as a whole recognized the need to use the fundamental equations, which were differential analogs of the conservation laws of matter, momentum, and energy all together [27,40,41], and the importance of the density variability effects. However, complete solutions were not constructed, and only partial solutions describing different structural components of flows such as separate waves, vortices, jets, and wakes were practically studied. For the most part, large-scale (wave) components were investigated, and only a few fine components were studied (e.g., Stokes's solution of flow produced by an oscillating plate along its surface [27]).

Since even the linearized system of fundamental equations has a high order, the complete solution includes several functions of different types [42]. However, in practice, following the work of Stokes and Rayleigh, only one "main solution" is searched for. The existence of additional functions is discussed quite rarely. One of the few examples of an exception is the solution to the problem of sound reflection from a solid wall [27].

Another undiscussed feature of modern hydrodynamics is the freedom of choice of incompatible model equations. Special reduced models and families of constitutive ones remain the main tools to study linear and non-linear waves, boundary layers, jets, wakes, vortices, and vortex systems. For some models, the results of their calculations are of good consistency between each other and with the experiments, albeit in a rather narrow range of parameters.

As the calculations of the continuous groups show, every system of equations within common models of fluid flows is characterized by its own set of infinitesimal symmetries [43] with a corresponding limited number of conserved quantities. The differences in the physical sense of the quantities used, denoted by the same symbols in different systems of equations, make it difficult and even impossible to compare the results and bring them to their general forms, which is necessary to unify the data. Differences in the properties of the included quantities do not allow for creating common requirements for the numerical and experimental techniques or the rules for comparing the data obtained. As a result, demands for indicating experimental measurement error and the temporal and spatial resolutions of the instruments have practically disappeared.

The main object of studies in hydrodynamics is still the limited system of equations, including continuity and momentum transfer in a fluid with a constant density, which are Euler equations (EEs) for an ideal fluid and Navier–Stokes equations (NSEs) for a viscous one. The solvability of a 3D NSE in the constant density approximation has not been proven yet ("6th Millennium Problem" [44]).

Calculations of the flow velocity in a homogeneous fluid within the EE and the NSE cannot be directly compared with the experimental data due to the impossibility of identifying a "liquid particle", which has no distinguishable boundaries. All the indirect and associative methods for the fluid velocity measurements are based on explicit or implicit assumptions, including "passivity" of impurities, the applicability of the Bernoulli equation, and independence of the parameters of the diffusion and heat transfer processes of the experimental conditions, making the implementation degree for real flows difficult to estimate.

Extensive observations show that all flows of real fluids are characterized by a fine structure. The structure is expressed more or less clearly depending on the conditions of the experiment, the quality, and the completeness of the measuring systems. The structure parameters depend on a large number of impacting factors characterizing the properties of the medium, boundary conditions, and external perturbations. The structures of a flow include high-gradient boundaries separating the regions with a more uniform distribution of the parameters of flow pattern change under the impact of non-equilibrium processes, chemical reactions, and the transformation of matter (e.g., ionization, radiation, and absorption of radiation energy).

When studying flows, the impact of the internal energy transfers and conversion processes was not practically analyzed. Transformation of internal energy from its latent potential form into active perturbations of pressure and temperature has not been explored yet, nor has the impact of the form complexity of the state equation, which in modern physics is determined on the basis of thermodynamic potential. The traditional thermodynamic quantities, such as density, entropy, pressure, and concentration, which are defined as derivatives of free enthalpy (Gibbs potential [28]), can change very quickly in a flow.

The introduction of thermodynamic potentials allows for expanding the number of energy transfer mechanisms in hydrodynamics. Thus, the traditional energy transfer by a flow with a velocity U is supplemented with transport by waves of various types with a group velocity c_g, the impact of fast thermodynamically non-equilibrium processes of fast local energy release or absorption, and slow dissipative processes. Fast processes of internal energy conversion manifest themselves in flows induced by a freely falling drop in a liquid at rest [45].

At the same time, the need to improve the theory of flows is growing. The density and total amount of energy in natural and technological processes are increasing due to a number of factors, and as a sequence, is growing the value of the damage caused by natural disasters both local (e.g., fires, floods, and heavy storms) and global ones are associated with natural weather variability and climate change, supplemented by the uncontrolled anthropogenic impact on the environment.

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The highlighted role of mathematics in the description of hydrodynamic processes forces us to return once again to the analysis of the interaction of two key branches of natural science, such as mathematics and hydrodynamics, in order to clarify the content of the terms used, the rules for choosing systems of equations, the methods for their analytical or numerical solutions, as well as the requirements for experimental techniques following from the theory.

In the absence of a canonical definition of mathematics, in practice, different representations are used. These include definite ones, such as "Mathematics is a science of quantity: discrete quantities were studied by arithmetic, continuous ones were done by geometry" by Aristotle, "Mathematics includes only those sciences in which either order or measure is considered" by R. Descartes, "Mathematics... is a science of quantitative relations and spatial forms of the real world" (F. Engels—A.N. Kolmogorov), figurative ones, such as "Mathematics is a set of abstract forms—mathematical structures" by N. Bourbaki, "Mathematics is a language" by G. Galilei and J. Gibbs, "Mathematics is a millstone that grinds what is poured into them" by T.G. Huxley, and pragmatic ones, such as "The science on indirect measurements" by O. Comte.

The abundance of definitions reflects the variety of methods and tools for this branch of science, which is applied to an increasing number of disciplines. Among them, engineering sciences occupy a special place, being focused on the practically used description of a research object (natural system or technology) and the prognosis of its future behavior.

Engineering sciences, which are focused on optimizing the description of the physical object, predicting its long-term variability, defining criteria for identify catastrophic scenarios of events and signs (precursors) for their realization, and assessing the object's response to anthropogenic factors, including directed efforts for controlling the state, contain such disciplines as engineering mathematics and technical mechanics.

Applied to descriptions of fluid and gas flows, *Engineering mathematics* is defined as

> "*An axiomatic science about the principles of choosing the content of symbols, rules of operations and criteria for assessing accuracy*".

The goals of engineering mathematics are to describe the current physical status of fluids and gases as well as the dynamics and structure of a flow and to predict its natural evolution and responses to additional external influences, including targeted control.

Connected by the general principles with engineering mathematics, Technical mechanics is defined as *"An empirio-axiomatic science on the rules for choosing physical quantities, and methods for measuring and evaluating errors in the determination of their values"*.

Technical mechanics is aimed at the selection of physical quantities corresponding to the principles of engineering mathematics which can be measured with a guaranteed estimation of accuracy in the frame of current metrology sciences or by introducing a new procedure, which gives room to measure the physical quantities characterizing fluid flow dynamics and structure. The selected measurable physical parameters of flows must allow for performing a qualitative and quantitative comparison with the results of engineering mathematics applied for the description of a phenomenon under study.

A special place in the definition of dual disciplines belongs to the concept of "accuracy" and "errors or inaccuracy" in the theoretical and practical description of phenomena, respectively.

In mathematics, the internal criteria for assessing accuracy are naturally formulated in the arithmetic and algebra of number fields on the basis of the distinguished properties of two numbers: "zero" and "one". In the mathematical analysis of continuous quantities, the analogues procedure for the comparison of infinitely and regularly decreasing variations of the primary variable and functions (i.e., the Cauchy–Weierstrass algorithm) is applied. In modern applied mathematics, where calculations with the use of conditionally converging series and diverging (singular) functions are widely used, the introduction of a universal criterion at this stage is difficult and requires individual analysis of the problem under study. One of the tools for determining the accuracy is the procedure for comparing the calculations with the experimental data, which needs to include the identity proof of the compared quantities defined in different branches of sciences. In a number of countries, for determining the measurement error, the standards of physical quantities and the procedures for their application are used, being recommended by the relevant international or national organizations.

The opportunity for directly comparing these dual engineering disciplines of mathematics and mechanics is ensured by the implementation of the general scientific (logical and philosophical) principles underlying modern natural science, as well as by the unity in defining the content of the concepts used, which characterize the physical quantities and laws of their changes.

In physics, as the basis for describing the dynamics of nature, a set of conservation laws for the basic physical quantities or their differential analogues is chosen. Their implementation takes into account the conventional logic principles, forming a basement for formal rules of scientific studies. The modern set of these principles is given below.

2. General Principles ("Laws", Demands, and Regulations) of the Science Philosophy

In developing the theory of knowledge, Aristotle [46] formulated the first group of philosophy laws, which included the following principles:

- **Identities**: the concept should be used in the same meaning in the course of reasoning;

- **Internal consistency**: "it is impossible to exist and not to exist at the same time", or "it is impossible to speak correctly, simultaneously affirming and denying something" (i.e., binary logic);
- **Excluded third**: "A or not-A is true, there is no third".

When analyzing the nature of objectivity in describing nature, G.V. Leibniz, who believed that "The great foundation of mathematics is the principle of consistency, that is the statement that a judgment cannot be true and false at the same time", following Aristotle, Descartes, and a number of other predecessors, expanded a number of principles of the philosophy of science. Later, Leibniz's list was repeatedly supplemented, shortened, modified, extended, and today includes the following principles:

- **Meaningfulness**: definability of the essence of a studied subject, object, concept, method, or position in the considered and independent categories. The basis of mechanics is formed by the axiomatically defined mathematical and physical concepts of number, set, space and time, motion, matter (constituents of fluid or gas and their properties), and flow;
- **Identities**: complete conservation of the meaning of an object at operation;
- **Consistency**: two opposed properties cannot be simultaneously true or false;
- **Uniqueness**: excluded third. From the conflicting judgments, one is true, the other is false, and the third is not given. At the same time, this permits describing independent properties in different categories, like the dualism of a "point particle" and a wave distributed in space;
- **Sufficient reason (raison d'etre)**: the presence of history and confirmation of meaning;
- **Minimum sufficiency**: "You should not multiply things unnecessarily", "Do not multiply the number of entities beyond measure", "It is useless to do less with more", "Blessed is the Lord who made everything difficult unnecessary and everything necessary easy!" (G. Skovoroda), and "Of all the explanations, the best is the simplest";
- **Causality**: changes are a consequence of the previous and the cause of the future;
- **Completeness**: description of the known properties of an object with an error estimation and the potential to include newly discovered properties.

In accordance with the outlined principles, the main problem of fluid mechanics is description of the self-consistent temporal change of the fluid status, spatial position, and interaction of the studied medium with outer solids, fluids, and gases.

For solving the main problem of fluid mechanics, which is describing the structure and dynamics of a fluid or gas flow, the following is necessary:

- Indicate the principles for the research object's definition and select the physical quantities characterizing the object;
- Choose the methods for studying the properties of the studied object and processes in the course of their change;
- Give examples of studying the selected phenomena by various independent methods and show the consistency of the results obtained with the estimation of the calculation accuracy and the measurement errors. The general basis for the construction of the theory and methods of experimental studies of a fluid and gas flow in engineering mechanics is the laws of physical quantity conservation within the continuous medium model, which admits infinitesimal representations of physical quantities.

In mathematics, the condition for the conservation of distance defines one of the types of space transformation into itself, which is the transformation of motion [47], which coincides with the concept of an ideal fluid flow [48,49].

In mechanics, the conservation laws have generalized the historical experience of describing phenomena, reflecting the fundamental properties of the existence and immutability of matter (its possible transformations, such as radioactive transformations in nuclear physics, will not be considered here), as well as the parameters of its motion. They are based on the concept of homogeneity of space and time and the isotropy of space.

Traditionally, the fluid mechanics are developed in the "continuous medium" approximation. However, the "continuous medium" methodology, with continuous values of the physical quantities themselves and their derivatives on arbitrary scales, does not match with the concepts of discrete structure of matter. Atomic–molecular properties are expressed in scales to the order of $10^{-8}\ldots10^{-7}$ cm, and nuclear properties are expressed in scales to the order of 10^{-13} cm.

At intermediate scales to the order of $10^{-7}\ldots10^{-6}$ cm (i.e., the size of an atomic-molecular cluster [50]), the influence of both the atomic–molecular interactions [51,52] and macroscopic properties, for example, in the form of a latent potential part of the internal energy's corresponding surface tension [53], are significant.

The distribution of internal energy and pressure in the near-surface layer determines the state of the medium that can be liquid or gaseous and actively influences the structure and dynamics of the ongoing processes, particularly the dynamics of the ocean and atmosphere interaction [40,41]. The choice of a description based on a scale-invariant set of conservation laws allows for the passing of a discrete medium from the model to a continuous one while preserving the meaning of the characterizing quantities. The sizes of the microstructural components establish the natural limits of applicability of the continuous medium model. Namely, the minimal sizes of the studied macroscopic phenomena should exceed the scale of the molecular cluster such that $l_f > \delta_c \sim 10^{-6}\ldots10^{-5}$ cm.

The engineering sciences under consideration are based on the universal conservation laws of matter (total mass or density), as well as the measures of motion, which are momentum and total energy. The mobility of atoms, molecules, and their associations—clusters or macroscopic "liquid particles" with larger structural components—leads to a continuous change of the distribution of matter and the tensor of inertia in space. Due to the independent mobility of small components changing in the moment of inertia and the inhomogeneous dissipation rate of the momentum or realizing latent internal energy, the angular momentum cannot be used as an invariant parameter of the flow and thus is not considered further.

The main parameter of the state and dynamics of fluid and gas is the total energy E_t, including the mechanical part (kinetic and potential) and internal energy, which is determined by the equilibrium thermodynamic parameters (Gibbs potential) [28–30]. The internal energy contains the available potential surface, chemical, electromagnetic, and other types of energy.

Taking into account the total energy gives room to consider all the mechanisms for its transfer in a flow, including transfer with a local flow velocity and a group wave one, slow diffusion processes with a characteristic rate, and rather fast ones in the course of localized direct atomic–molecular interactions (for example, at the size of a molecular cluster upon free surface elimination in merging fluids [45,53]).

The slow transfer of invariant quantities by atomic–molecular processes is described by its own laws, including the corresponding dissipative coefficients, such as the kinematic viscosity ν, thermal diffusivity κ_T, and diffusion κ_S for momentum, heat, and substance transfer, respectively. Although the potentials characterize thermodynamically equilibrium states, their application to the description of the characteristics of non-equilibrium processes is justified by small deviations of the state of the systems from the equilibrium one. Large deviations from the equilibrium values are taken into account by introducing fast localized sources, which determine the energy changes in the course of direct atomic–molecular processes, such as in the fast release of the available surface potential energy contained in the eliminated free surface of merging fluids [45].

The rules for choosing the quantities and methods for comparing their values in relation to the description of a fluid or gas flow will be considered below.

3. Elementary Mathematics in the Theory of a Fluid Flow

The mathematical basis for the theory of a fluid flow, describing changes in the position, dynamic state, and physical properties of a medium under study, includes *real*

numbers, with the properties defined here a priori (Note: In hydrodynamics, *real numbers* are used both to mark points in space when introducing a coordinate system and to describe physical quantities. The use of *complex numbers*, which facilitate calculations in the study of dissipating media (i.e., the immersion of the configuration space into the algebra of complex numbers) leads to an extension of the feasible solution space and requires performing special analysis to select among the solutions the physically justified ones). The rules for classifying the constituent elements of sets, being "a collection of certain and distinguishable objects, conceived as a single whole", for performing operations with them are given in a number of monographs and reference books [47]. In mechanics, one of the main types of sets, which is a mathematically defined vector space, was selected as the basis for describing physical phenomena.

The concepts of "space" and "time" in classical mechanics have been introduced axiomatically as primary quantities and are considered as two independent continuous sets existing independently for matter and material processes.

By introducing a coordinate frame, each point in space is associated with an element from the set of real numbers. The minimum quantity of the set elements, which are coordinates characterizing a selected point in the space, is called its dimension.

The axiomatic of a vector space allows the operations of summation and multiplication; internal composition (summation of vectors); associativity; commutativity with the rule of algebraic summation (subtraction) of vectors; associativity of the product of factors; multiplication by one; and distributivity. An important property of the available operations is an outer composition, which is the conservation of a scalar by a vector product in the initial vector space [47].

A space admitting the introduction of a distance $\rho(1,\ 2) = \sqrt{\left(x_i^1 - x_i^2\right)^2}$, between elements $x_i^1,\ x_i^2$, with the properties $\rho \geq 0,\ \rho(1,\ 2) + \rho(2,\ 3) \geq \rho(1,\ 3),\ \rho(1,1) = 0,$ is called metric [47]. The space of basic variables expressed with length coordinates is named the configuration space.

The space permits deformations and transformations into itself, both discrete (affine or projective) and continuous, with a transformation parameter, which can have its own dimension or be reduced to the dimension of the base space by introducing a dimensional coefficient (in the case of classical space–time using the world constant (i.e., the light velocity)). By introducing a new variable, the transformation parameter, the configuration space is extended up to a four-dimensional space.

Observable invariants of the spaces, which are the distances between the points or time intervals between the events, are used to define the coordinate values. In mechanics, the space is considered to be a three-dimensional metric (Euclidean) with a standard basis given by orts $e_1,\ e_2,\ e_3$ or $(x,\ y,\ z)$.

The mathematical definition of motion is based on the introduction of an absolute coordinate system centered at the point where the radius vector specifies the position of a material point with mass M in the configuration space $\mathbb{R}^3$ at the initial and subsequent time instants, which is in a 4D unified configuration and time space.

In the aggregate of the configuration space $(x,\ y,\ z)$ transformations, the orthogonal mapping into themselves (into space $(x',\ y',\ z')$) with the conservation of the distance between the elements is distinguished. This transformation in the Cartesian coordinate system is given by the formulas $x'_i = a_{ik}x_k$, $a_{ji}a_{jk} = 0$ at $i \neq k$ and $a_{ji}a_{jk} = 1$ at $i = k$.

The *motion* in a geometric sense is defined as a continuous orthogonal transformation of metric space into itself, with the time t as an independent continuous parameter of transformation, which conserves the distances between points and the relative positions of objects [47]. In this case, the determinant composed of the coefficients of the matrix a_{ik} equals $\|a_{ik}\| = +1$. An orthogonal transformation with the determinant $\|a_{ik}\| = -1$, which does not conserve the orientation of the figures, specifies a reflection about some axis.

The introduction of the concept of motion, which is characterized by its own laws (functions) of changing the positions of objects, leads to a further expansion of the space and the introduction of the functional space of the problem.

A motion in Euclidean space $\mathbb{R}^3$ is characterized by a group of transformations, which includes independent subgroups of rectilinear shifts $\delta\mathbf{r} = \mathbf{v}_t\delta t$ with a velocity $\mathbf{v}_{\boldsymbol{\tau}}$ and rotations around the instantaneous center $\mathbf{r} = 0$ with an angular velocity Ω:

$$\delta\mathbf{r} = (\mathbf{v}_t + \Omega \times \delta\mathbf{r})\delta t. \tag{1}$$

The transformations given by the group of motions are studied by elementary geometry.

The motion in the prescribed coordinate frame with the center at a point $\mathbf{r} = 0$ is characterized by the trajectory $S_t(x,\ y,\ z)$ and the velocity $\mathbf{v} = \left(\mathrm{v}_x, \mathrm{v}_y, \mathrm{v}_z\right) = \frac{d}{d}\frac{\mathbf{R}}{S}\frac{dS_t}{dt} = \boldsymbol{\tau}\frac{dS_t}{dt}$, where $\mathbf{R}$ is the radius vector to the body and $\boldsymbol{\tau}$ is the unit vector specifying the local direction of the tangent to the trajectory of the body.

The transition from geometry to mechanics includes the introduction of the concept of mass M, which is an additional independent quantity with its own dimension. Mass is a positively defined scalar quantity which, hereinafter, is considered constant for a mechanical body with a finite size and infinitesimal material points. In mechanics, the mass M is a measure of inertia and the gravitational interaction of bodies. In a general case, a body mass M is a variable, and the nature of its change must be determined independently. The introduction of a new quantity (i.e., mass) leads to a further extension of the dimension of the problem's functional space, which becomes five-dimensional.

The **physical definition** of mechanical motion is based on recording the distances between bodies, each of them being characterized by its own mass and changes in distance over time. In this case, some bodies with fixed distances between them form a basis in which the position of a moving body (material point) is recorded. Distances are invariants, as their values do not depend on a choice of the coordinate system, and their values when passing from one system to another are transformed in accordance with the systems of measurement.

The length, time, and mass standards are accurate enough to enable the functioning of such sophisticated instruments as global positioning systems. Based on the external composition rule in the list of properties of vector space, it follows the equivalence of the vector spaces of the momentum $\mathbf{p}$ and the velocity of the body $\mathbf{v}$. Consequently, the definition of the physical motion of a material point relative to the system of bodies, which form the coordinate frame, is equivalent to the operation of transforming space into itself (i.e., the geometric definition of motion). It is the unity of the different forms of motion definition which connects the invariant properties of space (homogeneity and isotropy) with the conservation laws (Noether's theorem). The invariants of motion of a material point are the momentum $\mathbf{p}$, kinetic energy $E_a^k = \frac{M_a\mathbf{v}_a^2}{2} = \frac{\mathbf{p}^2}{2M_a}$, and angular momentum $\mathbf{M}_a = [\mathbf{r}_a\mathbf{p}_a]$.

The description of body motions is carried out on the basis of Newton's laws [7] in algebras of real and complex numbers or quaternions (the latter representation is preferable for symbolic programming in navigation). All parameters of solid motion of a mathematical (kinematic) nature, which are based on definitions of the coordinates, velocities, and accelerations, or a physical nature (momentum and energy) are observable (i.e., their value can be determined (measured) by various independent methods with objective control of the error). The methods of their definitions ultimately come down to measurements of the distances, time intervals, and mass. Precision samples and procedures (international standards) were developed for that purpose.

Descriptions of the dynamics of bodies are also carried out in phase space (formed by the components of velocities, momentums, or wave number vectors). An extended six-dimensional configuration space, which unites the spaces of coordinates and velocities, is used for a complete description of the motion as well. The complete functional space of the problem of body movement is eight-dimensional and includes three spatial coordinates, three speeds, time, and mass.

Since the physical state of solids does not change in the course of motion, the set of such parameters as mass, velocity or momentum, angular momentum, and energy turns out to be complete and sufficient to describe all types of motion.

4. Parameters of a Fluid Flow

The description of substance flows in liquid, gaseous, and plasma states is based on the concept of a "continuous medium", which allows the use of continuous functions over the entire range of scales. The key property of a fluid is fluidity (i.e., the ability to move under the action of any small perturbations). This property manifests itself in the decomposition of the flow velocity which, according to the definition of Cauchy–Helmholtz [54], has the following form:

$$\mathrm{v}_i(r_r + \delta r_k) = \mathrm{v}_i(r_k) + \varepsilon_{ijk}\Omega_j \delta r_k + \frac{\partial \mathrm{v}_i}{\partial x_l}\delta r_l, \tag{2}$$

where ε_{ijk} is a unit antisymmetric tensor of the third rank. In addition to the terms describing displacement and rotation in Equation (1), Equation (2) takes into account the change in the "liquid particle" shape. An additional term describing the shear of the velocity destroys the independence of the action of the rectilinear shift and rotation operators and changes the group properties of the motion operator as a whole.

The difference between the representations of "motion" (Equation (1)) and "decomposition" (Equation (2)) reflects the existence of two independent continuums, those being a metric space (whose motions (Equation (1)) are characterized by the group of motion) and a medium immersed in it (decomposition (Equation (2)) includes the shift operator). To describe the body motion, it is sufficient to use a three-dimensional Euclidean space $\mathbb{R}^3$, supplemented by a point body with constant mass M which acts as a parameter connecting the kinematically determined velocity $\mathbf{v}$, with the momentum $\mathbf{p} = M\,\mathbf{v}$ serving the measure of motion.

The demand for a uniqueness rule requires identification of the difference between the properties of the metric space and the physical space of the continuous medium submerged in it. For an independent description of the flows of a continuous medium, the dimension of the physical space of a problem must have a higher dimension than that of the configuration space in order to save their independent identities. A natural extension of the problem space is introduction into the analysis of the density inhomogeneity in the initial state and its further variability $\rho = \rho(x, y, z, t)$, which corresponds to the properties of real fluids.

The transition to a space with a higher dimension significantly changes the technique of mathematical description of flows and the physical content of the mathematical quantities, particularly the kinematic vorticity $\omega = \mathrm{rot}\mathbf{v}$. Now, the rate of vorticity generation is defined not only by the spatiotemporal variability of the flow velocity but also by the gradients of the thermodynamic quantities $\frac{d\omega}{dt} = \nabla P \times \nabla \rho^{-1}$, which are gradients of the pressure P and density (Bjerknes's theorem).

In the Helmholtz interpretation, the fluid vorticity is identified through rotation of the elements of the medium [54]. The difference between the concepts of "rotation of a part of a continuous medium" and vorticity as a measure of deformation of a fluid particle was noted by J. Bertrand [55–57] and S. Lee [58] in the 19th century. However, due to the insufficient development of some branches of mathematics and the technique of hydrodynamic experiments, their ideas and objections were not supplemented by a constructive development.

One of the hidden difficulties in describing flows is associated with the assumptions of homogeneity, continuity, and deformability of a continuous medium, which give no room for identifying an individual "particle" [11] having no physically distinguishable boundaries. The mass of a "particle" decreases indefinitely when its size tends toward zero, and the object of study disappears.

In the experiment, to measure the flow velocity, the tracking of markers is used, including solid particles, gas bubbles, and droplets of immiscible liquids, which have stable individual characteristics. However, the introduction of a marker, which is an

additional physical object, complicates the behavior of a carrying fluid, as a new and more complex multi-component medium is formed. It was Descartes who first noted that a free macroscopic solid is not only carried by a flow but also twisted by the flow around its own axis [3]. The rotating marker additionally perturbs the fluid environment. A small marker becomes involved in the Brownian motion. The transport of soluble impurities is influenced by diffusion effects. The dynamics of droplets of immiscible liquids are perturbed by the effects of non-uniform surface tension.

The combined action of many factors leads to an uncontrolled difference between the motion of markers and the flow of the carrier fluid in which they are immersed. The generally accepted hypothesis of "passivity" of impurities is not confirmed in fine-resolution experimental studies on the redistribution of impurities both in waves, where an initially homogeneous solution or suspension is redistributed and forms a fine structure [53], or in vortices, where a patch of dyed fluid transforms into spiral arms and helical lines and splits into individual fibers.

The absence of criteria for identifying a "liquid particle" means that the "fluid velocity", having a mathematical meaning, is not a physically observable flow parameter, for which one can specify a method for evaluating the error in the course of an experiment and a procedure for reducing it to a given value. The need to meet the criterion of observability of physical quantities for the fluid flow parameters was noted by Stokes, Maxwell, Reynolds, and many others leading hydrodynamics researchers, but analysis of the conditions for observability of a "liquid particle" has not previously been carried out. From the conditions for the conservation laws' application, it follows that the observable (measured with the error control) flow parameters are the specific momentum $\mathbf{p}$, which can be determined by measuring the interaction of the flow impact on a standard body, or the flow rate, as well as the complete energy E_t. Both universal and specific methods to determine their values for concrete experimental conditions have not been developed yet.

At the beginning of the last century, it was found that the property of liquid fluidity (i.e., the ability to move under arbitrary small external influences), was due to the mobility of atoms and molecules, their associations (i.e., clusters [50–52]), and individual structural components [53], each of which is characterized by its own energy. Large-scale components (waves, flows, vortices, and ligaments) interact with each other and all other multi-scale components [59] and provide action of various mechanisms for energy transfer. The transitions of kinetic, potential (gravitational, chemical, and concentration), and internal energy into other forms complicate the descriptions of flows.

In modern fluid mechanics, the main parameter of an equilibrium continuous medium at rest is the internal energy, which is described by the Gibbs potential G [30]. Derivatives of the thermodynamic potential define the traditional parameters of a continuous medium, such as the density $\rho(x_1,\ x_2,\ x_3)$, pressure $P(x_1,\ x_2,\ x_3)$, temperature $T(x_1,\ x_2,\ x_3)$, and concentration of dissolved or suspended particles $S_i(x_1,\ x_2,\ x_3)$, which have a clear physical meaning and are available for observation. In modern fluid mechanics, the parameters of the medium, such as the density and pressure, are considered to be quantities with a double nature, namely mechanical and thermodynamic.

The main flow parameters are the total specific energy E_t and momentum $\mathbf{p}$, which manifests itself in the forceful action in dynamics and continuous variations in the flow structure (i.e., in the evolution of distinguished spatial patterns of different physical quantities).

Within the classical hydrodynamics, in the systems of equations of motion for ideal (Euler equations (EEs) [11]) and viscous fluids (Navier–Stokes equations (NSEs) [27,40,41]), the constant density, as a coefficient in all terms of the equations, can be omitted. Then, the systems of equations for fluid motion [27,40,41] are transformed into algorithms for a special transformation of the Euclidean space into itself. In this case, the difference between the two various continuums, which are metric space and the functional space of submerged fluid, is lost. In the three-dimensional formulation, the sets of EE and NSE equations degenerate on singular components and become insoluble [42].

The axiomatic definition of the medium state (fluid and gas) and its flow is further carried out, taking into account the need to preserve the meaning of the physical quantities used and the condition for resolving the governing system of equations. A unified description of the dynamics and structure of flows of fluids and gases is carried out in the absolute Cartesian coordinate system of the metric (Euclidean) space based on the fundamental conservation laws, following the basic methodology of physics [27,40,41].

5. Definition of Fluid Flow

A medium which has the property of fluidity (i.e., the ability to change position in space under the influence of arbitrary small perturbations of physical quantities), characterized by thermodynamic potentials and their derivatives such as thermodynamic quantities and kinetic as well as other physical coefficients (in particular, those determining the propagation of electromagnetic or acoustic waves), is named weakly compressible fluid if occupying a finite volume (or gas or plasma) if it fills all available space.

The definition of a fluid flow is as follows:

The transfer of momentum, energy, and matter, accompanied by self-consistent changes in physical quantities which determine the state of a continuous medium.

The redistribution of matter and energy without a transfer of momentum is called a process (for example, the diffusion transport of matter). Flows are characterized by dynamics (changes in values of physical quantities and the magnitude of the fluid impact on solids) and structure (spatial pattern of the physical quantities' distribution).

The measurable quantities describing a fluid flow are the density ρ, scalar total energy E_t, including the specific kinetic $E_M = \frac{\rho v^2}{2}$, potential E_p and internal E_i energy described by the Gibbs potential G ($E_t = E_M + E_p + E_i$), and vector momentum $\mathbf{p} = (p_x,\ p_y,\ p_z) = \rho\mathbf{v}$, which are supplemented with the parameters of the fluid state. The fluid velocity is defined as the instantaneous ratio of two invariant quantities, which are the momentum and density ($\mathbf{v} = \mathbf{p}/\rho$). Furthermore, it is assumed that the fluid velocity $\mathbf{v}$ is identical to the velocity of transformation of the Euclidean space $\mathbb{R}^3$ into itself.

Taking into account different mechanisms of energy transfer with intrinsic spatiotemporal parameters as the base value, characterizing the medium equilibrium state, the Gibbs potential is chosen, as well as its derivatives, determining the density, pressure, temperature, entropy, and other parameters [28–30]. The set of the state quantities includes

$$\begin{gathered} G(x.\ y,\ z,\ t) = G(\rho(x.\ y,\ z,\ t),\ P(x.\ y,\ z,\ t),\ T(x.\ y,\ z,\ t),\ S_i)(x.\ y,\ z,\ t), \\ \rho(x,\ y,\ z,\ t)\ = \rho(P((x.\ y,\ z,\ t),\ T(x.\ y,\ z,\ t),\ S_i(x.\ y,\ z,\ t)). \end{gathered} \tag{3}$$

To improve the accuracy of determining the values of physical quantities in practice, additional functional relations between the physical quantities characterizing a fluid are also used as equations of state. Among them are the dependences of the velocity of sound, electrical conductivity, the optical refraction index on pressure, temperature, and salinity [26].

The *axiomatically* introduced system of equations for fluid motion, taking into account the general principles of the choice of physical quantities and fundamental conservation laws [27,40,41], has the following form:

$$\begin{cases} \rho = \rho(P,T,S_n),\ \ G = G(x,y,z), \\ \frac{\partial \rho}{\partial t} + \nabla\cdot(\mathbf{p}) = Q_m\,, \\ \frac{\partial S_i}{\partial t} + \nabla\cdot(S_i\mathbf{v} + \mathbf{I}_i) = Q(S_i), \\ \frac{\partial (p^i)}{\partial t} + \nabla_j \Pi^{ij} = \rho g^i + 2\rho\ \varepsilon^{ijk} v_j \Omega_k + Q^i(f), \\ \frac{\partial E}{\partial t} + \nabla_i (E\ v u^i) + \nabla_i \left(q^i + P v^i - \sigma^{ij} v_j + \frac{\partial w}{\partial S_n} I_n^i \right) = Q(e), \end{cases} \tag{4}$$

where ρ is the density, the ratio of two invariant quantities $\mathbf{v} = \mathbf{p}/\rho$ is the fluid velocity, $\mathbf{I}_n$ is the concentration and density of the diffusion flux of the nth impurity, $\Pi^{ij} = \rho\ u^i u^j + P\delta^{ij} - \sigma^{ij}$ is the momentum flux density tensor, $\sigma^{ij} = \mu\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3}\delta^{ij}\frac{\partial v^k}{\partial x^k}\right) + \zeta\delta^{ij}\frac{\partial v^k}{\partial x^k}$ is the symmetric

viscous stress tensor, δ^{ij} is the fundamental metric tensor μ, ζ is the first and second dynamic viscosities, Ω is the global rotation angular velocity, $E = \rho\left(\frac{v^2}{2} + \varepsilon + U\right)$ is the total energy density, ε is the specific internal energy, $w = \varepsilon + \frac{P}{\rho}$ is the specific enthalpy, U is the specific gravity potential, $\mathbf{g} = -\nabla U$ is the acceleration of gravity, and $\mathbf{q}$ is the vector of the heat flux density. The acceded sources of mass for the nth impurity, force, and energy Q_m, $Q(S_n)$, $Q^i(f)$, $Q(e)$, respectively, describe the impact of non-hydrodynamic processes, which take place in fluid flows and can be equal to zero. The introduction of sources reflects the possible impact on the fluid flow structure and dynamics of external factors with non-hydrodynamic natures and the uncertainties of phenomena on the smallest length scale of the order of atomic sizes. Sources in the basic system (Equation (4)) give room to analyze the transformation of latent potential energy into its active form and estimate its impact on the flow dynamics and structure. In the absence of external disturbances and internal sources, they vanish. To the author's knowledge, the set of Equation (1) has not been analyzed up to now, taking into consideration the compatibility conditions necessary for construction of the complete set of solutions.

Under the assumption that the gradients of potentials, physical quantities, and the intensity of external sources are small, the system in Equation (1) is transformed into a system of equations for describing the transfer of matter, concentration of individual components, temperature, and momentum [27], which is widely used in environmental and technical fluid mechanics [40,41]:

$$\begin{cases} G = G(P,S,T) = G(\mathbf{x},t),\ \rho = \rho(P,S,T) = \rho(\mathbf{x},t), \\ \frac{\partial \rho}{dt} + \nabla_j(p^j) = Q_\rho, \\ \frac{\partial(p^i)}{\partial t} + \left(\nabla_j \frac{p^j}{\rho}\right) p^i = -\nabla^i P + \rho\, g^i + \nu\Delta(p^i) + 2\varepsilon^{ijk} p_j \Omega_k + Q^i,\ . \\ \frac{\partial \rho T}{\partial t} + \nabla_j \cdot (p^j T) = \Delta(\kappa_T \rho T) + Q_T, \\ \frac{\partial \rho S_i}{\partial t} + \nabla_j \cdot (p^j S_i) = \Delta(\kappa_S \rho S_i) + Q_{Si}. \end{cases} \quad (5)$$

The system in Equation (5), which includes the quantities of mechanical, thermodynamic, and kinematic natures, takes into account the dissipation of momentum and the influence of internal or external sources of matter, temperature, and substances on a flow structure and dynamics.

The system in Equation (5) is supplemented with the initial and boundary conditions, including impermeability for the density and the components of the substance (impurity concentration) in the fluid, the values of the temperature or its flux, the no-slip condition for momentum or velocity on solid boundaries, the equality of momentum fluxes on the contact surfaces of two fluids, and the damping of all disturbances at infinity. It should be noted the high dimension of the physical problem space, which is determined by all independent quantities of the system (i.e., the dimensions of the coordinate's space, time, density, pressure, temperature, concentration of solutes or suspended particles, as well as kinetic coefficients) included in the equations.

Connecting Equation (5) forms a system of coupled algebraic differential equations, with the solution being constructed by taking into account the compatibility condition [42]. The rank of the system (the order of the highest derivative, if it is possible to reduce the system to one equation), as well as the order of its linearized version and the degree of the characteristic (dispersion) equation, determines the minimal number of eigenfunctions which constitutes a complete solution. The complete system with the diffusion equation for one impurity has the tenth rank [42]. Accordingly, the flow pattern for this set is formed by composing ten functions with intrinsic spatiotemporal scales. The abundance of flow components differing in scales and structure is manifested in continuous changes in the observed flow pattern. Over time, the number of structural components can increase due to the processes of nonlinear interaction of the flow components [59].

Due to the independence (individual behavior) of physical quantities, the fields for each of them, which are characterized by their own geometry, spatial, and temporal scales,

must be simultaneously determined in the experiment. The accuracy of the state equations, which is the difference in the values of the density, speed of sound, refractive index, and other reliably determined quantities calculated by solving the system in Equation (5) and obtained experimentally by measuring the values of the temperature, pressure, salinity, and other quantities, determines the error of analytical and numerical calculations.

Consistency of the infinitesimal symmetries of the system in Equation (5) with the basic principles of physics [43] testifies to the validity of its choice as a basis for studying fluid flows. The practical recommendations for the selection of parameters for numerical simulation and experimental techniques are based on the analysis of the intrinsic spatial-time scales of the system in Equation (5) with the initial and boundary conditions of the problem.

The unity of the content of the physical parameters—in theory, numerical modeling and experiments based on the system Equation (5)—allows one to estimate the accuracy of the solutions based on the physical properties of fluids and experimental conditions. Equation (5) contains no additional parameters and does not require their introduction for the development of numerical codes.

Equation (5) is parametrically and length-scale invariant [42,43]. The emergence of "new flow regimes" is usually explained by the influence of ignored energy conversion from its potential form to an active one and back, a change in forms of the state equation, or limitations of techniques, particularly insufficient or excessive sensitivity, temporal or spatial resolution, limitations of the dynamic range of an instrument, or a calculation method necessary to identify all structural components.

Since the symmetries of the fundamental system and equations of other flow models (numerous versions of turbulence theories, theories of waves, vortices, jets, wakes, and others) differ significantly [42,43], the same symbols in different systems have various physical meanings. To check the consistency of the results, it is necessary to calculate the identically defined fields of parameters or to find, for example, the forces and torques acting on a selected obstacle in its own coordinate frame or the flow rate in a selected section.

Until now, the general properties of Equation (2) and even its linearized version have hardly been studied. Periodic solutions of the linearized system have been constructed by methods of the theory of singular perturbations [60] in the approximation of weak dissipation [42].

6. Classification of Infinitesimal Periodic Flow Components

The fluid in natural and industrial conditions, where the density $\rho(z)$ is specified by the distributions of pressure, temperature, and concentration of impurities, under the action of buoyancy effects becomes stably stratified. The total density variability in a stratified fluid and partial contributions due to temperature and salinity variations are described by the buoyancy length scales $\Lambda = \left|\frac{1}{\rho}\frac{d\rho}{dz}\right|^{-1}$, $\Lambda_T = \left|\frac{1}{\rho(T)}\frac{d\rho(T)}{dz}\right|^{-1}$, and $\Lambda_S = \left|\frac{1}{\rho(S)}\frac{d\rho(S)}{dz}\right|^{-1}$, frequency $N_\rho = \sqrt{g/\Lambda}$, $N_T = \sqrt{g/\Lambda_T}$, and $N_S = \sqrt{g/\Lambda_S}$, and period $T_b = 2\pi/N$, T_T, T_S (the axis z is directed vertically upward, g is the gravity acceleration, and the effect of compressibility is neglected). In strongly stratified fluid, which is typical for laboratory conditions, $N \sim 1\ \mathrm{s}^{-1}$ in the environment, $N \sim 0.01\ \mathrm{s}^{-1}$ in potentially homogeneous fluid, and $N \sim 10^{-5}\mathrm{s}^{-1}$ for actually homogeneous fluid, which is generally used in theory ($N \equiv 0$). Furthermore, values for the buoyancy frequency, where N_ρ, N_T, and N_S are supposed to be constant at all depths.

Taking into account that the stratification effects allows one to construct complete solutions of the linearized system in Equation (5) using the compatibility condition and give a classification of the flow structural components, the set of Equation (5) for stratified fluids contains small dissipative coefficients and can be treated by singular perturbation theory methods [60]. Substitution of the solution in the form of plane waves with a positive frequency $\omega > 0$ and a complex wave number $\mathbf{k} = \mathbf{k}_1 + i\mathbf{k}_2$ for the density $\rho' = A_\rho \exp i(\mathbf{k}x - \omega t)$ and similar treatment for perturbations of other physical quan-

tities into the linearized system in Equation (5) gives a dispersion equation of the tenth degree [42]:

$$\begin{gathered} D_\nu(k,\omega)\cdot F(k,\omega)=0 \\ F(k,\omega)=-D_\nu(k,\omega)D_{\kappa_T}(k,\omega)D_{\kappa_S}(k,\omega)\left(k^2+i\frac{k_z(\Lambda_T+\Lambda_S)}{\Lambda_T\Lambda_S}\right)+ \\ D_{\kappa_T}(k,\omega)\left(\frac{\omega\,k_z}{\Lambda_S}D_\nu(k,\omega)-N_S^2k_\perp^2\right)+D_{\kappa_S}(k,\omega)\left(\frac{\omega\,k_z}{\Lambda_T}D_\nu(k,\omega)-N_T^2k_\perp^2\right) \\ D_\nu(k,\omega)=-i\omega+\nu\,k^2 D_{\kappa_T}(k,\omega)=-i\omega+\kappa_T\,k^2,\ D_{\kappa_S}(k,\omega)=-i\omega+\kappa_S\,k^2, \\ k_\perp^2=k_x^2+k_y^2. \end{gathered} \tag{6}$$

The multiplicative structure of Equation (6) is composed of three singular operators of the viscous boundary layer type $D_\nu(k,\omega)=-i\omega+\nu\,k^2$ and is similar for the temperature and concentration components. The main wave operator reflecting the action of all factors, which are buoyancy, compressibility, and dissipation, clearly demonstrate a multiscale structure of a periodic flow, simultaneously containing both large wave and fine components [42].

The regular roots of Equation (6), with the imaginary part being small compared with the real one, describe infinitesimal waves of various types, which are inertial, gravitational, acoustic, or hybrid ones. A rich family of roots with a singularly perturbed type characterizes *ligaments*. The real and imaginary parts of these roots are in the same order.

Waves are defined as components of a flow where the parameters of local temporal variability (frequency ω) and an instantaneous spatial structure (wavenumber $\mathbf{k}$ or wavelength λ) are related by a dispersion relation $\omega=\omega(\mathbf{k},\ \mathbf{kA},\ldots)$, where $\mathbf{A}$ is the amplitude.

Ligaments are thin and extended components of flows described by the set of singular solutions of the complete and linearized system in Equation (5) and the corresponding algebraic dispersion relation in Equation (6) for fluid with density, temperature, and salinity stratification. The number of ligaments depends on the system rank (Equation (5)). The transverse scales of periodic ligaments $\delta_\omega^\nu=\sqrt{\nu/\omega}$, $\delta_\omega^{\kappa_T}=\sqrt{\kappa_T/\omega}$, and $\delta_\omega^{\kappa_S}=\sqrt{\kappa_S/\omega}$, as well as for transient ligaments $\delta_{\boldsymbol{\tau}}^\nu=\sqrt{\nu\cdot\boldsymbol{\tau}}$, $\delta_{\boldsymbol{\tau}}^{\kappa_T}=\sqrt{\kappa_T\cdot\boldsymbol{\tau}}$, and $\delta_{\boldsymbol{\tau}}^{\kappa_S}=\sqrt{\kappa_S\cdot\boldsymbol{\tau}}$, where $\boldsymbol{\tau}$ is the duration of the flow formation, or ligaments in a stationary flow with velocity U are $\delta_U^\nu=\nu/U$, $\delta_U^{\kappa_T}=\kappa_T/U$, and $\delta_U^{\kappa_S}=\kappa_S/U$. The length scales are defined by the kinematic coefficients, ν, κ_T, and κ_S, as well as the wave frequency ω, the time interval $\boldsymbol{\tau}$ of the flow formation duration, and the velocity U. The length of the ligaments depends on the lifetime of the process under study. Ligaments are distinguished by a high level for the vorticity and mechanical energy dissipation rate. All the flow components co-exit, transfer, and disappear simultaneously, despite the difference in characteristic scales. Each of the flow components provides the transfer of energy, matter, and vorticity.

The total numbers and type of ligaments are determined by the rank of the set in Equation (5) and the order of its linearized version. The minimum number of ligaments is four. Their existence is provided by the variability in density and action of the fluid viscosity. There are six ligaments when heat conduction effects are included. The eight ligaments exist for the case of including the equation for salinity diffusion's presence into the set of governing equations [27]. With time, the number of ligaments and their locations are changed as a result of the non-linear interactions of all the flow components, both for large waves and fine ligaments [59].

It is unsteady ligaments with transverse scales $\delta_{\boldsymbol{\tau}}^\nu=\sqrt{\nu\cdot\boldsymbol{\tau}}$, $\delta_{\boldsymbol{\tau}}^{\kappa_T}=\sqrt{\kappa_T\cdot\boldsymbol{\tau}}$, and $\delta_{\boldsymbol{\tau}}^{\kappa_S}=\sqrt{\kappa_S\cdot\boldsymbol{\tau}}$, and a length $l_l=u\cdot\boldsymbol{\tau}$ (u is the local velocity) which provide the connection of atomic–molecular processes with macroscopic structural components, such as waves (intrinsic time is the inverse period ω), vortices, and others.

As the kinetic coefficients tend toward zero, the thickness of the ligaments decreases uniformly. The vanishing of the coefficient lowers the rank of the system in Equation (5) and discretely reduces the number of ligaments. In the three-dimensional formulation, there are up to six ligaments if diffusion is taken into account, and there are four if only the viscosity is kept.

In the case of a really homogeneous fluid in infinite space, different roots of the dispersion equation become identical for an incompressible fluid:

$$\mathbf{k}^2\left(\omega + i\nu\mathbf{k}^2\right)^2 = 0 \tag{7}$$

This is also the case for a compressible gas [42]:

$$\left(\mathbf{k}^2\left(1 - \frac{i\omega\,\widetilde{\nu}}{c_s^2}\right) - \frac{\omega^2}{c_s^2}\right)\left(\omega + i\nu\,\mathbf{k}^2\right)^2 = 0. \tag{8}$$

where $\widetilde{\nu} = \zeta + 4\nu/3$, the shear (first) and convergence (second) kinematic viscosity are ν and ζ, respectively, and the sound velocity is c_s. The multiplicity of the roots in Equations (7) and (8) indicates the degeneration of the Navier–Stokes equations for a homogeneous and barotropic fluid [26] in singular components. In this case, two different spaces, which are the metric $\mathbb{R}^3$ that permits motion with the operator in Equation (1) and the space of a submerged homogeneous fluid with the flow decomposition in Equation (2), become indistinguishable. The violation of the uniqueness principle in the determination of the same object manifests here in the degeneration of the problem.

A consequence of the high rank of the system of fundamental equations, solutions of which include several superposed regular and singular functions with incommensurable spatial and temporal properties, is the non-stationarity of all types of flows. The patterns of such flows are constantly being self-transformed.

In some experiments and numerical simulations, ligaments cannot be identified due to insufficient sensitivity or resolution of the instruments, as well as shadowing by high-level perturbations generated by other flow components.

Vortices are complex unsteady flow components with relatively high vorticities $\boldsymbol{\omega} = \mathrm{rot}\mathbf{u}$, which are composed of a set of ligaments. In a vortex, free solids are transported by the flow and simultaneously twist around their own axis, which was noted by Descartes [7] and confirmed later in many laboratory experiments. The uniform miscible fluid volume is split by ligaments, dividing the vortex into individual fibers. The solutions of the Navier–Stokes system characterizing the velocity and pressure fields do not only admit experimental verification with control of the accuracy in an actually homogeneous incompressible fluid where "Eulerian liquid particles" become unidentifiable, and the fluid velocity is a non-observable quantity.

Using all solutions for the system in Equation (5) and the dispersion in Equation (6) allows for solving the linear problem of periodic internal wave generation by an oscillating body in complete 2D and 3D formulations with physically justified initial and boundary conditions [61].

Calculating the ligaments generated by periodic internal waves incident on an inclined wall or a critical level, at which the wave and buoyancy frequencies coincide in a medium with a variable density gradient, gives room to completely solve the wave problem [62]. The differences between the calculations of evanescent waves infiltrating into the supercritical region where their frequency exceeds the local buoyancy frequency, which was published in 1998 [62], from the results of experiments [63] published in 2012 do not exceed a few percent.

Since both waves and ligaments are described by functions of the same type in a linear formulation, they all directly interact with each other, despite differences in their own length scales. Thin high-gradient interfaces (ligaments) are formed in the regions of intersections of internal wave beams [42,64], and as a result of ligament interactions, new ligaments and internal waves are generated [59].

7. Theoretical and Laboratory Studies of Flows around an Obstacle Based on a Reduced Fundamental System

The atmosphere and hydrosphere of the Earth with their densities set by the distributions of all thermodynamic quantities, such as the pressure, temperature, concentration of dissolved substances, and suspended fine particles, are generally stably stratified due to

gravity action. The transfer of matter by diffusion processes is being studied quite actively. Less attention is paid to the study of the accompanying energy transfer, although it is a reserve of available potential energy that gives room to produce the mechanical motion of a stratified fluid, particularly diffusion-induced flows on topography [65,66].

The stratification effects on the dynamics, structure, and geometry of flows are noticeable, despite the smallness of the relative changes in density values and are actively studied in environmental and laboratory conditions [38–42]. Due to the limited vertical dimensions of the working volumes in the laboratory and the sensitivity of the density measurement methods, differential heating and regulation of the concentration of the stratifying additive are used to simulate stratification. Differentially heated atmospheric air and water, various gases, and aqueous solutions of metal salts are used as working media. Due to the smallness of the ratio of kinetic coefficients (both for seawater and an aqueous solution of common table salt, where the Lewis number is $\mathrm{Le} = \kappa_T/\kappa_S \sim 100$ [26]), the salt stratification decays more slowly than the temperature one and is used more widely.

Due to the high heat capacity of aqueous solutions, the dissipation effects practically do not heat the fluid, and their influence remains unnoticed when studying the flow patterns around bodies in laboratory conditions. In this regard, when performing calculations and laboratory modeling of flows in a stratified basin, the fluid is considered isothermal and incompressible, with the state equation defined by the distribution of the dissolved salt concentration. The initial stable density distribution is often chosen to be $\rho_0(z) = \rho_{00} s(z) = \rho_{00} \exp(-z/\Lambda)$, where ρ_{00} is the density on the reference level, with constant values for the buoyancy scale Λ, frequency N, and period T_b.

The reduced system of fundamental Equation (6) for a one-component incompressible stratified medium in the Boussinesq approximation, when density variations are neglected everywhere except for the term with gravity, takes the following form:

$$\begin{cases} \rho = \rho_0 + \rho_{00} \cdot s, \qquad \operatorname{div} \mathbf{v} = 0, \\ \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} \ \mathbf{v}) = -\frac{1}{\rho_{00}} \nabla P + \nabla \cdot (\nu \nabla \mathbf{v}) - s \cdot \mathbf{g} \ , \\ \frac{\partial s}{\partial t} + \nabla \cdot (\ s \, \mathbf{v} \ ) = \nabla \cdot (\kappa_S \nabla s) + \frac{\mathrm{v}_z}{\Lambda}, \end{cases} \tag{9}$$

where the fluid velocity is $\mathbf{v} = \mathbf{p}/\rho$, $P(x, \ z, \ t)$ is the pressure except for the hydrostatic one, and s is the salinity perturbation including the salt contraction coefficient.

Physically valid no-slip and no-flux boundary conditions on the surface of a solid impermeable body with geometric dimensions of a length L, width W, and height h which can move with a constant velocity U starting at $t = 0$ have the following form:

$$\begin{gathered} \mathbf{u}|_{\,t \le 0} = 0, \quad s|_{\,t \le 0} = 0, \qquad P|_{\,t \le 0} = 0, \\ \mathrm{u}_x|_\Sigma = \ \mathrm{u}_z|_\Sigma = 0, \qquad \left[\frac{\partial s}{\partial \mathbf{n}}\right]\Big|_\Sigma = \frac{1}{\Lambda}\frac{\partial z}{\partial \mathbf{n}}, \\ \mathrm{u}_x|_{x, \ z \to \infty} = \ U, \quad \mathrm{u}_z|_{x, \ z \to \ \infty} = 0 \, . \end{gathered} \tag{10}$$

The governing system of Equation (9), together with the initial and boundary conditions in Equation (10), is characterized by a set of temporal and spatial scales having significantly different values. Among the large linear scales are the buoyancy scale $\Lambda = |d \ln \rho / dz|^{-1}$, which characterizes the manifestation level of the initial stratification, the geometric dimensions of the body h, L, and W, the attached internal wave length $\lambda_a = UT_b$, and the viscous wave scale. The small scales characterizing basic ligaments accompanying stratification in a viscous fluid and the basic flow with velocity U are $\delta_U^\nu = \sqrt{\nu/U}$, $\delta_N^{\kappa_S} = \sqrt{\kappa_S/N}$, and $\delta_U^\nu = \nu/U$, $\delta_U^{\kappa_S} = \kappa_S/U$, respectively. The system in Equations (9) and (10) was selected as a basis for design of the stands and development of a technique for laboratory studies of flows around an obstacle.

The laboratory experiments based on the system in Equation (9) were conducted at the stands of the Unique Science Facility's "Hydrophysical Complex for modeling hydrodynamic processes in the environment and their impact on underwater technical objects, as well as distribution of impurities in the ocean and atmosphere (USF HPC IPMech

RAS)" [67]. The laboratory set-up included transparent tanks with different dimensions, additional devices for filling with a stratified fluid, towing models, generated surface and internal waves, placing and moving sensors, sonar, hydrophones and microphones, schlieren and optic flow visualization instruments, an experiment control, and data processing.

Numerical simulation for the formulated mathematical problem is constructed on the basis of the finite volume method using the computational utility OpenFOAM [68]. The open source of this computational package enabled the creation of original program codes in the C++ environment for modifying and improving the existing approaches and standard solvers for numerical simulation of stratified flows in order to be able to perform direct numerical simulations in a wide range of flow parameters. More detailed information on the modifications made to the standard OpenFOAM solvers can be found in other papers by the authors indicated in the references in [68].

7.1. Slow Diffusion-Induced Flow on a Sloping Plate

Stratified fluids in the field of mass forces (in particular, gravity) are examples of thermodynamically non-equilibrium systems. They have a reserve of available potential gravitational energy, since the center of mass lies below the geometric center of the fluid volume. At the impermeable boundaries, the molecular flux of a stratified substance is interrupted. The violation of the flux leads to an accumulation of diffusing material in some places and a deficit in others. An arising hydrostatic pressure gradient in the field of mass forces (gravitation) accelerates the fluids and forms a flow, which exists even in the absence of destabilizing external factors [65,66]. Due to the easy attainability of the formation conditions, such flows are formed in a stratified fluid or gas near any inclined boundaries. They are common in the atmosphere ("mountain and valley winds"), the world ocean, where their formation is associated with the stratification and global rotation effects, and in thermally inhomogeneous lakes.

First, a stationary solution of the linearized system in Equation (9) with the boundary conditions of Equation (10) describing the flow on an inclined plane was constructed. Similar salinity and velocity profiles of the flow were characterized by a single combination scale $\delta = \sqrt[4]{\nu\kappa_S/N^2\sin^2\alpha}$, where α was the inclination angle of the plane to the horizon [65,66]. The thickness of the induced flow tended toward infinity as the angle α tended toward zero, and the solution itself became divergent.

The profiles of the salinity perturbations along the normal to the plate surface in the asymptotic solution for the transient flow formation problem and the small-time approximation of the exact solution of the creeping flow formation problem are characterized by the length scale $\delta_N^{\kappa_S} = \sqrt{\kappa_S/N}$, where the time t is normalized by the buoyancy period, $\boldsymbol{\tau} = t/T_b$, $(\xi,\ \zeta)$ is the local coordinate frame, and the axis ζ is normal to the plane:

$$s' = -2\frac{\delta_N^{\kappa_S}\sqrt{\boldsymbol{\tau}}}{\Lambda}\mathrm{ierfc}\left(\frac{\zeta}{2\delta_N^{\kappa_S}\sqrt{\boldsymbol{\tau}}}\right) \tag{11}$$

Meanwhile, the induced velocity is described by both diffusion scales (i.e., for the density $\delta_N^{\kappa_S}$ and the velocity $\delta_N^{\nu} = \sqrt{\nu/N}$):

$$\begin{gathered} u(\zeta) = \frac{N^2\,\delta_s\,\boldsymbol{\tau}^{3/2}}{\nu-\kappa_s}\left[\mathrm{i}^3\mathrm{erfc}\left(\frac{\zeta}{2\delta_\nu\sqrt{\boldsymbol{\tau}}}\right) - \mathrm{i}^3\mathrm{erfc}\left(\frac{\zeta}{2\delta_s\sqrt{\boldsymbol{\tau}}}\right)\right]\sin 2\alpha, \\ \mathrm{i}^n\mathrm{erfc}(z) = \int_z^\infty \mathrm{i}^{n-1}\mathrm{erfc}(x)dx,\ \mathrm{i}^0\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}dx,\ \mathrm{i}^{-1}\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}e^{-z^2}. \end{gathered} \tag{12}$$

The structural flow components with incommensurable values of the length scale evidence the total unsteadiness of the phenomena under study.

The calculations of the flow pattern in the complete non-linear formulation show a system of cells which was formed near a plate with a length l (Figure 1) within the whole range of the inclination angle of the plate $0 \le \alpha < 90^\circ$. In this case, the flow pattern around

the center of the plate agreed with the exact and asymptotic solutions for the sloping infinite plane.

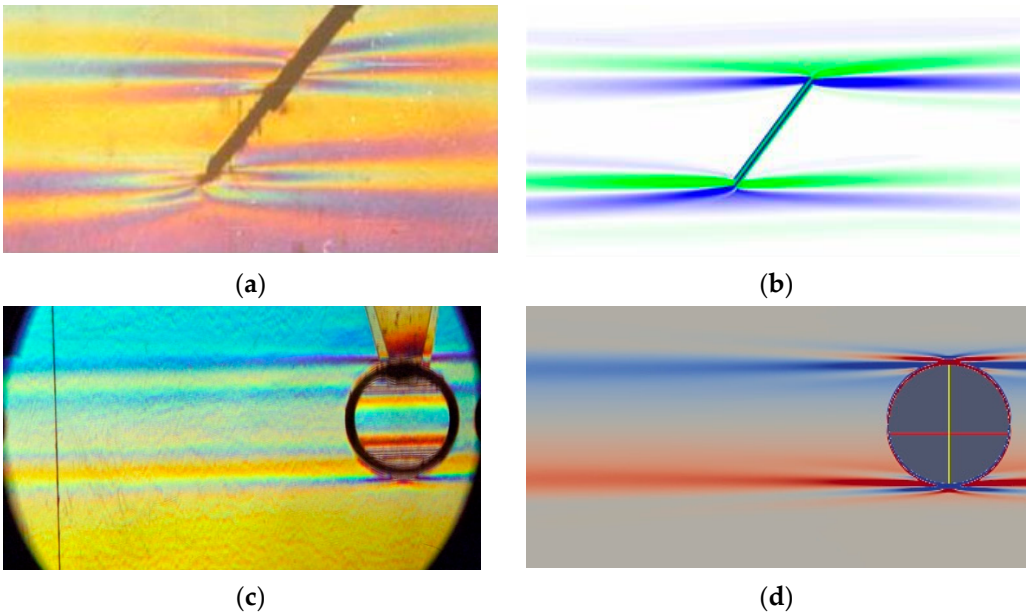


Figure 1. Schlieren and numerical visualization of the diffusion-induced flow: (**a**,**b**) on a fixed inclined plate, $L = 2.5$ cm, $N = 0.84$ s^{-1}, and $T_b = 7.5$ s; (**c**,**d**) cylinder, $D = 5$ cm, and $T_b = 10.5$ s.

Diffusion-induced flows on topography have been actively studied for the past 70 years. However, due to the internal multiscale nature, a complete calculation of such a flow even for the case of bodies with a simple shape, such as a cylinder, a plate of a finite length, or a wedge, is carried out only with the use of supercomputer technologies.

The calculated patterns of the fields of physical quantities, which were consistent with the data of a high-resolution schlieren visualization of diffusion-induced flows, are presented in Figure 1. The fields of the horizontal component of the refractive index gradient, which were visualized by schlieren instruments, were compared with the calculated density gradient fields (these parameters were connected by a linear relation for a solution of sodium chloride). The experiments were conducted in a transparent tank filled with a continuously stratified solution of common table salt [69].

The structure of the visualized perturbation fields shows that some flow components with the parameters from Equation (12) belonged to the class of ligaments. The corresponding attached internal waves (at $U = \sqrt{\nu N} \sim 0.1$ cm/s and $\lambda = UT_b \sim 0.5$ cm) were too small in amplitude and could not be detected by existing instruments. The creeping diffusion-induced flows, which are formed on asymmetric bodies, provide their self-propulsion at the horizons of neutral buoyancy [69].

When calculating the formation of a flow pattern around a body starting to move, the diffusion-induced flow was chosen as the initial condition. The solvability of the problem of flow around a plate in a two-dimensional formulation for both stratified and homogeneous fluids was used for comparison of the new results with the previously obtained ones and to note good agreement in terms of the drag estimate [68].

The complete solution of the system in Equation (6) enables calculating the fields of all physical parameters of the flow, including the vorticity and its baroclinic generation rate, as well as the energy and its dissipation rate. The condition for the solvability of the minimum scales of the flow structural components should be taken into account when choosing the mesh parameters and experimental techniques [68].

7.2. Pattern of Flow around a Moving Plate in Wave and Vortex Flow Regimes

The problem of flow around a moving body with a velocity U based on the system in Equation (9) with the boundary conditions in Equation (10) includes a set of length scales of a geometric and dynamic nature. The set includes the buoyancy scale Λ, body sizes L, W, H, and length of the attached internal wave $\lambda_a = 2\pi U T_b$. The group of fine scales contains the thicknesses of the ligaments associated with the natural oscillation of the medium δ_N^{ν}, $\delta_N^{\kappa_S}$ and those accompanying the internal attached waves δ_U^{ν}, $\delta_U^{\kappa_S}$. For reliable registration of all the flow components, the dimensions of the observation or calculation area should noticeably exceed the macroscale of the problem L, W, H, λ, and the dimensions of the resolution cells should be several times smaller than the microscales δ_N^{ν}, $\delta_N^{\kappa_S}$ and δ_U^{ν}, $\delta_U^{\kappa_S}$. The time for observing the flow should noticeably exceed the buoyancy period T_b. The time step Δt must satisfy the Courant criterion $\mathrm{Co} = |\mathbf{v}|\Delta t/\Delta r \leq 1$, which is determined by the mesh cell size Δr, values of the microscales δ_N^{ν}, $\delta_N^{\kappa_S}$,δ_U^{ν}, $\delta_U^{\kappa_S}$, and local flow velocity $\mathbf{v}$.

The flow parameters depend on the medium's properties, such as the density, stratification, coefficients of viscosity and diffusion of the stratifying component, and the size, shape, position, surface quality, direction, and velocity of movement of a body. We consider the motion of a thin rectangular sharp-edged plate with its plane oriented at an arbitrary angle α to the horizon (with an angle of attack) which starts a uniform motion with a velocity U in the horizontal direction.

When studying the flow pattern formed during the motion of a plate with a length L and velocity U oriented along the trajectory, the system of equations for incompressible stratified fluid motion in the two-dimensional formulation and local coordinate system associated with the body ξ, ζ is transformed into a single equation of internal waves for the stream function Ψ (fluid velocity $v_x = \frac{\partial \Psi}{\partial z}$, $v_z = \frac{\partial \Psi}{\partial x}$):

$$\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}\right) + N^2\left(\cos\varphi\frac{\partial}{\partial \xi} - \sin\varphi\frac{\partial}{\partial \zeta}\right)^2 - \nu\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}\right)^2\right]\Psi = 0 \tag{13}$$

where the boundary conditions for the stream function on the plate surface are

$$\left.\frac{\partial \Psi}{\partial \zeta}\right|_{\zeta=0} = U\,\vartheta\left(\xi + \frac{L}{2} - Ut\right)\vartheta\left(\frac{L}{2} + Ut - \xi\right), \quad \left.\frac{\partial \Psi}{\partial \xi}\right|_{\zeta=0} = 0, \tag{14}$$

This includes the attenuation of all disturbances at infinity, where φ is the inclination angle of the trajectory to the horizon and ϑ is the Heaviside function.

Substitution of the solution in the form of plane wave transforms (Equation (13)) into the dispersion equation is expressed as

$$\omega^2\left(k^2 + k_z^2\right) - N^2(k\cos\varphi - k_z\sin\varphi)^2 + i\omega\nu\left(k^2 + k_z^2\right)^2 = 0, \tag{15}$$

This takes the simplest form when the plate moves along a horizontal surface:

$$\omega^2\left(k^2 + k_z^2\right) - N^2k^2 + i\omega\nu\left(k^2 + k_z^2\right)^2 = 0 \tag{16}$$

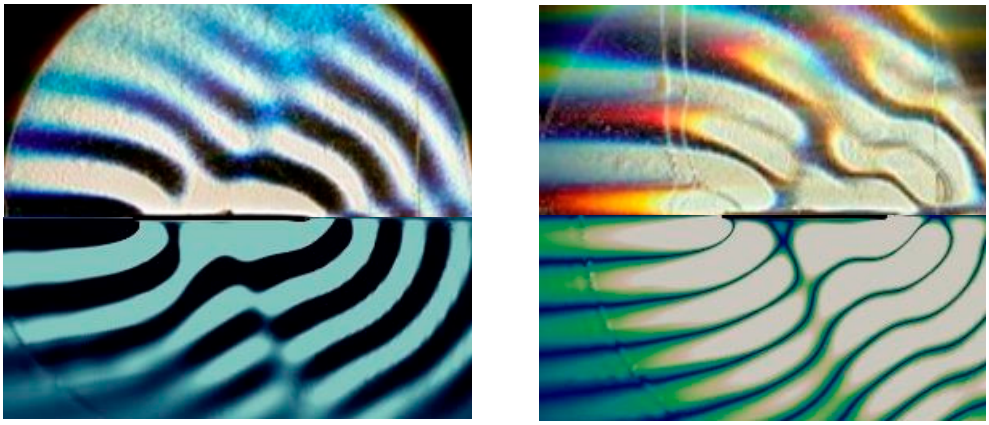
The roots of Equation (16), which significantly differ in the relations between the real and imaginary parts, are denoted by the indices (w) and (l). They characterize the internal waves and thin ligaments:

$$k_{l,w}(\omega, k) = \pm\sqrt{-k^2 + \frac{i\omega}{2\nu}\left[1 \pm \sqrt{1 + \frac{4i\nu k^2N^2}{\omega^3}}\right]}. \tag{17}$$

By substituting Equation (17) into the boundary conditions and solving the linear system of equations, one can find the stream function and then calculate the velocity

components and all other physical quantities of the problem. However, even in the simplest case, the integrals cannot be calculated analytically, and in order to illustrate the flow properties, they are calculated numerically, as in more complex cases, including direct numerical solving of Equation (9) in a complete nonlinear formulation. Below, we present the results of numerical visualization of the analytical solutions of the linearized system in Equation (9), as well as the problem in the complete formulation.

At relatively low Reynolds numbers, a typical stratified flow pattern around a moving horizontal plate, which is shown in Figure 2, consists of specifically arranged groups of upstream perturbations, attached internal waves, and a thin density wake formed by extended ligaments in the form of thin and long interfaces.



(**a**) (**b**)

Figure 2. Schlieren (upper part of the images) and calculated (lower one) stratified flow patterns around a uniformly moving horizontal plate: (**a**,**b**) $T_b = 7.6$ s, $L = 7.5$ cm, $U = 0.27,\ 0.39$ cm/s, where the thin wavy lines in the Schlieren images are density markers which visualize the profile of the horizontal velocity component in the fields of upstream perturbations.

The phase surfaces of the internal waves separate the half-waves of crests and troughs which bind toward horizons in front of the body and close up behind it. For all the cases considered, perturbations were pronounced at the edges of a uniformly moving plate.

In the case of the long plate with respect to the attached internal wavelength $\lambda_a < L$, the phase surfaces of the internal waves were broken above and beneath the plate. The comparisons in Figure 2 show that the numerical and experimental data were in good qualitative agreement for the calculated and visualized internal wave fields [68].

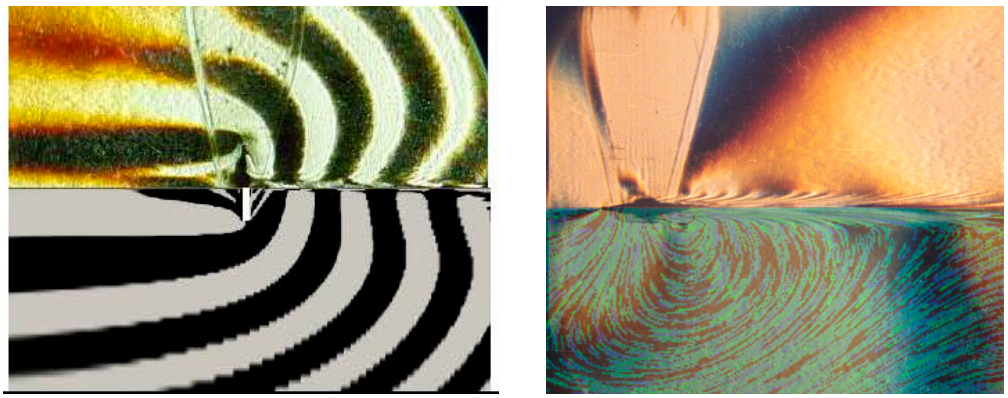
With the increase in the velocity, the general flow structure undergoes some modifications, such as decreased declination of the internal wave phase surfaces toward the direction of the body motion, a change in geometry of the fine-structural interfaces, and the degree of manifestation of separate flow components (Figure 3a). The strongest structural changes were revealed in the wake past the plate, where a system of short transverse interfaces in the form of tilted ligaments (streaky structure) was observed (Figure 3b).

With a further increase in the velocity, the ligaments in the wake became more and more pronounced and actively interacted. Short interfaces were gradually elongated and transformed into a vortex system where typical vortex elements, such as vortex dipoles outlined by thin interfaces, were observed (Figure 4).

In the vortex flow regime, when the vortex elements become a dominant flow component, with the internal wavelength being comparable to the observation area size, the most contrasting structural changes are manifested in the wake flow past the plate (Figure 5).

Both the laboratory and numerical simulations show that the wake flow structure past the tilted plate consisted of a typical vortex street in the form of a sequence of mushroom-like elements. In the strongly stratified medium (Figure 5a), the wake vortices gradually collapsed downstream, being broken up into a set of fine structural elements, while in the homogeneous fluid (Figure 5b), the vortex street expanded while evolving downstream in

the vertical direction within the observation area. A variety of multilayer fine structural flow elements was formed on the vortex shells and in the regions of interaction of the vortex flow multiscale components between themselves and with the plate surface.



(**a**) (**b**)

Figure 3. Internal wave field patterns around a uniformly moving horizontal plate in a stratified fluid, with schlieren visualization by the "vertical slit–filament" method and numerical visualization of the complete solution of the linearized problem $T_b = 7.6$ s: (**a**) $L = 7.5$ cm and $U = 0.40$ cm/s; (**b**) $L = 2.5$ cm and $U = 2.3$ cm/s. The upper parts of the images correspond to the schlieren visualization, and the lower one shows the calculation results of the density gradient field and streamline patterns.

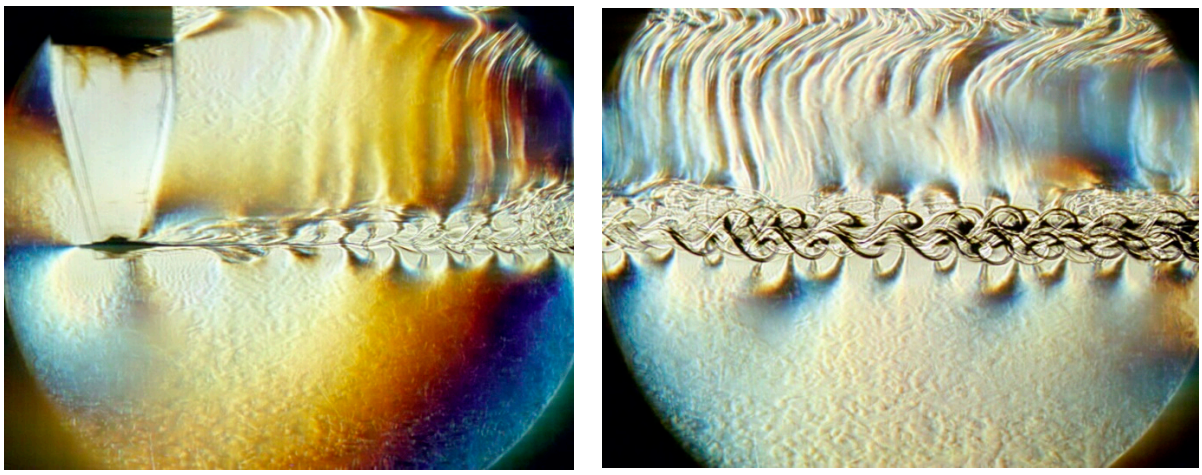
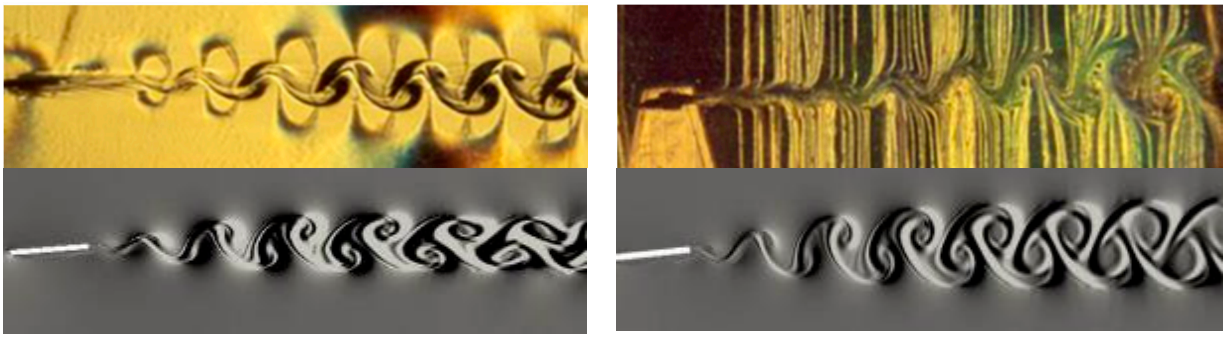


Figure 4. Schlieren images of the flow past a uniformly moving plate, with gradual transformation of the ligaments into a sequence of vortex dipoles inside the density wake; $T_b = 7.5$ s, $L = 2.5$cm, $U = 5.25$ cm/s.

All the flow components evolved and actively interacted with each other and with the free stream. In the unsteady flow regime, one can distinguish slowly evolving components, such as upstream and attached wave fields, rapidly changing ones, including fine-structured layers or ligaments, and their sets, which are vortices. The calculations and observations of the flow patterns were in good qualitative agreement with each other in all the flow regions, including the upstream perturbations, a system of internal waves, the wake with fine structures, and the vortices.



(**a**) (**b**)

Figure 5. Schlieren (upper part of the images) and calculated (lower one) vortex flow patterns past a uniformly moving tilted plate ($L = 2.5$cm, $U = 4.3$ cm/s): (**a**,**b**) $T_b = 7.6$; $6.3 \cdot 10^5$ s, $N = 0.83$; 10^{-5} s^{-1}.

The fine structure of the flow patterns, which is a consequence of atomic–molecular interactions in moving matter and the intense action of energy transformations in the atomic–molecular processes at the boundary of fluid (gas) with a submerged solid body, as well as inside of an inhomogeneous fluid (or gas), exists in all flows at any phase states of matter. As an illustration, Figure 6 presents the schlieren visualization data of the flow pattern around the wing in the wind tunnel (photographs were kindly provided by Professor V.G. Sudakov, TsAGI) and the plate towed in the stratified basin (Figure 6b,d). As can be seen in the photographs presented, the families of transversely located fine interfaces or ligaments are visualized both in compressible gas flows with transonic velocities and near a slowly moving body in a weakly compressible stratified fluid. The structural similarity of the gas and fluid flow images indicates the parametric invariance of the fact of the ligaments' existence.

The presented results of theoretical and experimental studies of the flow pattern around the strip show that the reduced system of fundamental Equation (9) with physically justified boundary conditions described all the details of the observed flow pattern, which were upstream disturbances, internal waves, wakes, vortices, and ligaments in a wide range of flow parameters. Complete solutions of the system in Equation (9) in a unified formulation described both the creeping diffusion-induced flows on a fixed obstacle and the wave and vortex structures at sufficiently large Reynolds numbers to the order of $10^4 \div 10^5$ without involving additional parameters [68].

At high velocities, the flow pattern becomes more complicated, and the flow, which contains a large number of structural components with its own parameters, is continuously transformed. The algorithms and criteria for constructing the complete solutions of Equations (9) and (10) allow for calculating all the physical quantities of the flows, which are the fields of density, velocity, pressure, vorticity, its baroclinic generation rate, mechanical energy dissipation rate, forces, and torques acting on a 2D body in a flow without involving additional hypotheses and constants in either stratified or homogeneous (potential or actual) environments [68]. The transition to a 3D formulation requires the development of new algorithms and numerical codes, which take into account the dimension expansion of the complete space of a problem and the appearance of new groups of ligaments. In this case, solving problems in a 2D formulation can be used to control the quality of the developed algorithms and methods for comparing the theoretical and experimental results.

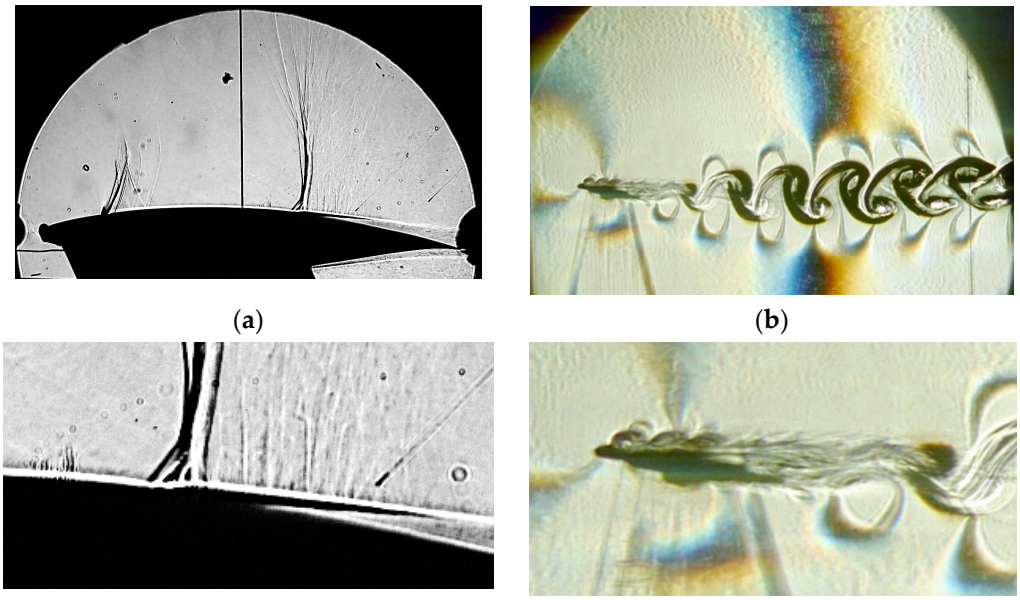


Figure 6. Schlieren images of flow around bodies. (**a**,**c**) Wing in the TsAGI wind tunnel at Ma $=$ 0.77 and plate in the stratified basin of the LMT IPMech RAS (T_b $=$ 7.55 s, L $=$ 2.5 cm, angle of attack α $=$ 12.5°, U $=$ 3.6 cm/s, Re = 900, Fr = 1.73). (**b**,**d**) Enlarged sections of the figures with fine structures.

Returning to the definition of mathematics, in the conclusion of this section, one can say that the physical variables of the problem are the density, momentum (in the Boussinesq approximation, the velocity $\mathbf{v} = \mathbf{p}/\rho_{00}$ differs from the momentum by a constant coefficient ρ_{00}^{-1}), pressure, and concentration of the stratifying component, which are chosen in accordance with the conservation laws. In this work, the principles of accuracy (error) control can be implemented by comparing the fields of the selected physical quantities obtained by independent methods, including analytical (in this case, non-uniform asymptotic expansions do not allow an estimation of accuracy), numerical (obtained while taking into account the solvability condition of the ligaments), and experimental methods, with a high spatial resolution. Here, the objects of comparison are the components of the density gradient fields, calculated theoretically and reconstructed from the schlieren visualization data of flows of an aqueous solution of table salt. The density and refractive index of the working fluid are related by an almost constant coefficient.

8. Influence of Fast Energy Transfer Processes on the Dynamics and Structure of Impact Flows Produced in a Motionless Target Fluid of a Coalescing Free-Falling Drop

Attention has been paid in recent decades to the study of the hydrodynamics and acoustics of a drop's impact and is explained by the fundamental nature of the topic, the growth of technical applications, the improvement of instruments, and programs of data collection and processing. The compactness of the process allows for carrying out research even in small laboratories, and the diversity and reproducibility of the ongoing processes (in general) provide the potential to obtain new experimental data, clarifying and expanding the existing representations on the physical nature of flows. The complexity and outlines of elements as well as the reproducibility of the fast changeable flow pattern allow one to trace the action of various mechanisms for energy transfer, including both small-scale fast and slow diffusion mechanisms. Attention is paid to the study of the relationships between hydrodynamic and acoustic processes, and the search for mechanisms for the excitation of gravitational capillary waves or soundwaves.

In experiments, new groups of capillary waves appeared sequentially when the flow structure changed. Short waves appeared first at the boundary of the parch of a droplet coalescing with the target fluid [45] and even on the surface of the coalescing drop, followed

by around the growing crown, later at the edge of the descending crown and inside the cavity, and further around the basement of the growing splash and at its top around a detachment zone of an escaping drop. The formation of a streamer and cavity after its submerging was accompanied by the generation of new groups of circular capillary waves [70,71].

At the initial contact of a freely falling drop with the target fluid, a high-frequency sound packet was formed, and with some delay, a new lower-frequency packet or group of packets was observed [72]. The main source of sound in the droplet impact flow was considered to be an oscillating gas volume, excited by the compression during the primary contact of the merging liquids and by the rapid retraction of the remainder of the air bridge during the pinch-off of a large bubble [73].

The emission processes of waves during the coalescence of a freely falling drop visualize the action of different mechanisms for transfer of the total energy, including the kinetic energy of motion, potential, and internal energy. To describe the internal energy, the free enthalpy (the Gibbs potential G) was chosen, with its derivatives determining the traditional thermodynamic quantities [29,30,71].

The thermodynamic and kinetic quantities of a drop's impact flow included in the equations are the density of the air ρ_a and water ρ_d (further $\rho_{a,d}$); kinematic $\nu_{a,d}$ and dynamic $\mu_{a,d}$ viscosities of the media; conventional σ_d^a and normalized $\gamma = \sigma_d^a/\rho_d$ $\mathrm{cm}^3/\mathrm{s}^2$ coefficients of the surface tension; the acceleration of gravity g; the diameter D, surface area S_d, volume V_d, mass M, and velocity U of the droplet's contact with a target fluid, and the duration of its complete coalescence $\boldsymbol{\tau}_D = D/U \sim 10^{-3}$ s. The ratios of these parameters set the characteristic dimensionless parameters, such as the numbers of Reynolds $\mathrm{Re} = UD/\nu$; Froude $\mathrm{Fr} = U^2/gD$; Bond $\mathrm{Bo} = gD^2/\gamma$; Onezorge $\mathrm{Oh} = \nu/\sqrt{\gamma D}$; and Weber $\mathrm{We} = U^2D/\gamma$. The kinetic energy of a freely falling liquid droplet is equal to $E_k = \frac{MU^2}{2}$, and the available surface potential energy (ASPE) is $E_\sigma = \sigma S_d$. The potential energy E_p is determined by the position of the fluid in the gravity field and particularly by the shape and area of the free surface.

The Gibbs potential is distributed non-uniformly inside a fluid with a free surface. Far from the boundaries inside of a homogeneous fluid, the value of the potential is determined by the entropy s, temperature T, specific volume $V = 1/\rho$, and pressure P [28,29]:

$$G_f = -sT + VP \tag{18}$$

The differential of the Gibbs potential dG_S depends on the concentration of the impurity components S_i and the chemical potential μ_i:

$$dG_i = -sdT + VdP + \mu_i dS_i. \tag{19}$$

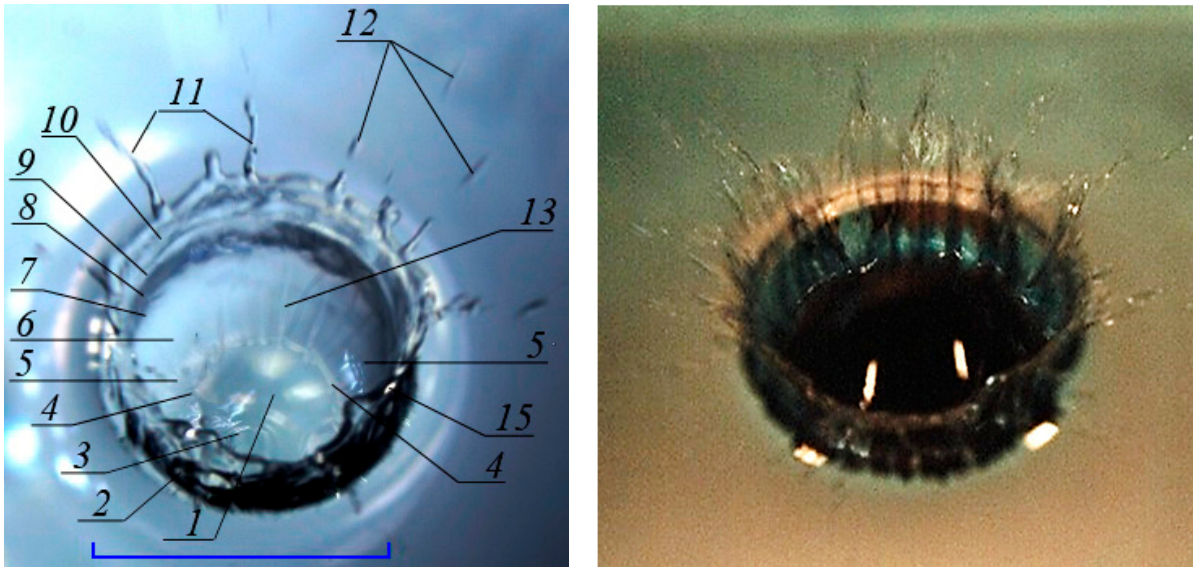
The anisotropy of the atomic–molecular interactions near the free surface forms large gradients of physical quantities. It was found by optical and X-ray reflectometry and atomic force microscopy that the density, dielectric constant, and dipole moment in the bulk of the fluid and in the structurally distinguished near-surface layer with a thickness of the order of the molecular cluster size ($\delta_\sigma \sim 10^{-6}$ cm) differed markedly [50–53]. Taking into account the differences of the physical properties near the free surface of a fluid, the additional term $\Delta G = -S_\sigma d\sigma$ is introduced in the Gibbs potential, which has the meaning of the ASPE:

$$dG_\sigma = -sdT + VdP - S_\sigma d\sigma. \tag{20}$$

For drop with diameter $D \sim 0.5$ cm falling with a velocity $U \sim 1$ m/s, the ratio of the ASPE E_σ to the kinetic energy of the falling drop E_k does not exceed several percent, but the ASPE density $W_\sigma = \frac{E_\sigma}{V_\sigma}$ is three orders of magnitude higher than the kinetic energy density $W_k = \frac{E_k}{V}$. When fluids coalesce, free surfaces are eliminated, and potential energy is quickly transformed into an active form, including pressure, thermal, chemical, and mechanical energy perturbations.

The ASPE transformation process plays a significant role in the formation of the fine structure of the impurity distribution in the impact flows of a freely falling drop. The elimination of the near-surface layers of a droplet and target fluid with a thickness of the order of the size of a molecular cluster ($\delta_\sigma \sim 10^{-6}$ cm) for the falling drop velocity U of the order of several meters per second occurs rather rapidly ($\tau_\sigma = \delta_\sigma / U \sim 10^{-8}$ s), while the entire droplet coalesces out in a time of the order of $\tau_D = D/U \sim 10^{-3}$ s. The released energy is stored in a thin layer on the outer contour of the confluence region and accelerates thin jets of coalesced fluids, which form typical line structures at the bottom of the cavity and the walls of the crown [45]. In turn, the energy of mechanical motion is partially converted into ASPE with an increase in the area of the deformed free surface.

A photograph of the flow pattern of a coalescing drop of a ferric chloride solution and an ink drop decaying into separate fibers is shown in Figure 7. Upon reaching the surface of the fluid, the jets going out from the drop–target fluid confluence region form thin spikes. Small droplets (splashes) fly out from the tops of the spikes, with the velocity being an order higher in magnitude than the contact velocity of the droplet. Over time, as the line of contact of the fluids advances, due to the effects of molecular diffusion of momentum (viscous damping of flows), the thickness of the jets increases, and the velocity decreases. At the same time, the size of the droplets, flying more and more slowly, grows as well.



(**a**) (**b**)

Figure 7. Decay of a freely falling droplet into individual jets at the boundary of the confluence region with the target fluid. (**a**) A drop of a saturated solution of ferrous sulfate merges with water; $D = 0.5$ cm, $U = 3.5$ m/s. (**b**) A drop of ink merges with water; $D = 0.5$ cm, $U = 3.2$ m/s, with the weak color background of the cavity and crown wall and dark fast jets (ligaments) reaching the tips of the spikes. Denotation: 1 = drop, 2 = edge of a crown with teeth, 3 = a wake of an emitted droplet's impact on the surface of a coalescing drop, 4 = the boundary of the region of the drop and target fluid confluence, 5 = annular capillary waves at the bottom of the cavity covering the region of confluence, 6 = bottom of the cavity, 7 = the border of the cavity and the crown, 8 and 9 = the wall and the upper edge of the crown, respectively 10 = veil, 11 = spikes, 12 = small droplets (splashes), 13 = fast jets (ligaments) at the bottom of the cavity, 14 = 3D texture of the crown wall, and 15 = teeth on the crown edge; the length of the tag is 1 cm.

As the flow pattern further evolves, the kinetic energy of the formed flow transforms into potential energy and the ASPE of the distorted surface of the fluid, and this is backwards when the crown and splashes sink and the ejected drops return. The reverse transformation of the energy of motion in the ASPE of the newly formed free surface occurs much more slowly ($\tau \sim 10^{-3} \ldots 10^{-1}$ s).

After the end of the active phase of evolution, the heavier colored fluid forms a toroidal vortex in the body of the target fluid, consisting of colored fibers separated by transparent

sections. The vortex slowly sinks, breaking up into new rings and forming an expanding cascade [74]. The movement of the entire structure and its constituent individual elements gradually slows down, as well as the diffuse spreading of the fibers and the decrease in the density difference between the fibers and the receiving fluid as well as the viscous dissipation of the remnants of the initial momentum of the drop. In the final phase, it is the dissipative processes which determine the geometry and dynamics of the flows.

The dynamics of the processes of total energy conversion and their influence on the evolution and structure of a fluid flow requires further study.

9. Discussion of Results

Based on the results of a brief excursion into the history of the development of nature research, the following principles for constructing the methodology of a scientific investigation are formulated:

- **Meaningfulness**: definability of the essence of a subject under study;
- **Identities**: immutability of the content of a subject;
- **Consistency**: internal unity of operations;
- **Uniqueness**: binary logic;
- **Sufficient reason**: prehistory and a consistent environment;
- **Minimum sufficiency**: as simple as possible;
- **Causality**: the directed temporal evolution of events;
- **Completeness**: description of the known properties without involving additional concepts and the openness to accept new facts.

The implementation of the listed principles is carried out within the framework of engineering mathematics, which is defined as "Axiomatic science for the principles of choosing the content of symbols, rules of operations and criteria for assessing accuracy", as well as supplementing technical mechanics, or "Empirio-axiomatic science for the criteria of choosing physical quantities, measurement techniques and procedures for assessing the error".

The unity of the scientific basis allows direct comparison of the results of theoretical and experimental studies with independent and mutual control of the accuracy of calculations and measurement errors.

The general basis for consistent engineering mathematics and experimental mechanics of fluid flows within the framework of a continuous medium model is the axiomatically introduced system of fundamental equations representing differential analogues of the conservation laws for matter, concentration of dissolved components, momentum, and energy (including latent energy and the mechanisms for its conversion into active forms). The system of equations determines the physical quantities characterizing the equilibria and flows of fluids or gases.

The energy basis for describing the state and flows of a fluid involves all forms of energy, which are mechanical kinetic, potential, and internal, including chemical, surface, and electromagnetic energies. The mechanisms for reciprocal energy transfer include direct and reverse transition processes from one form to another (for example, available potential surface or chemical energy due to fluid stratification can be converted into mechanical energy of fluid flows and vice versa). The energy is transferred with the flow velocity, group velocity of waves, and dissipative processes and is rapidly transformed from one form to another as the result of direct atomic–molecular interactions (for example, during the conversion of ASPE into other forms).

Taking into account the distinguished role of energy in the structure formation and the impact on the dynamics of flows, the internal energy was selected as the first physical quantity used for describing the equilibrium state of fluids and gases in the form of a thermodynamic potential (the Gibbs potential was nominated as the basic one). The derivatives of the Gibbs potential and their combinations define traditional thermodynamic quantities, such as the density, pressure, temperature, concentration of components, and surface tension coefficient. The inescapable molecular processes of matter, momentum,

heat, and transfer are characterized by kinetic coefficients. Additional coefficients describe the processes of propagation of electrical current and acoustic and electromagnetic waves with various lengths. All the coefficients really are functions of thermodynamic variables and can be involved in complementary equations of state.

Fluid or gas flow is defined as an inseparable transfer of the independent measures of fluid motion, such as momentum, energy, and matter, accompanied by self-consistent changes in the thermodynamic (density, pressure, temperature, and concentration of components), kinetic (dissipative coefficients), and additional coefficients characterizing the propagation of acoustic or electromagnetic waves, electric current, and other phenomena.

Within the framework of the "continuous medium" model, the description of fluid flows is carried out by continuous functions based on the solutions of the system of axiomatically introduced fundamental equations describing the transfer of momentum, energy, and matter with physically substantiated initial and boundary conditions.

The analysis of the fundamental system of equations was carried out while taking into account the compatibility condition, which determined the rank of the complete nonlinear system, the order of its linearized version, and the degree of the characteristic (dispersion) equation. The rank of the system, which defines the minimum number of independent functions making up the complete solution, is 6 for a one-component medium without a thermodynamic state equation (NSE for stratified and homogeneous fluids), 8 when energy (temperature) transfer is taken into account, and 10 for the case of introducing the additional equation of matter transfer into the system. Due to the nonlinearity of the equations, all components of the flows interact with each other. The superposition of a large number of inseparable eigenfunctions with independent space–time properties, which form the fields of registered physical quantities, is manifested in the unsteadiness and evolution of the flow structure.

The classification of the flow structural components for a fluid with weak dissipation, carried out on the basis of the complete solution of the linearized system of fundamental equations, includes large-scale regular components, such as waves or vortices, jets or wakes, as well as singular components, including families of fine ligaments. The transverse scales of ligaments are determined by the dissipative properties of the medium and the characteristic time of the process (i.e., by the duration of flow formation, frequency of periodic flow, or velocity of a uniform free stream). Due to the dual nature of ligaments, they reflect the impact of the atomic–molecular properties which are presented by the kinetic and mechanical coefficients defined by the flow type. Ligaments connect processes on micro- and macroscales. The nonlinear interactions of the structural components of all solutions of the linearized system of fundamental equations generate new components, including vortices which are unsteady localized perturbations with a high level of vorticity formed by embedded ligaments.

The construction of programs for numerical simulations and experimental techniques, taking into account the fluid properties based on the system of fundamental equations, enables carrying out coordinated theoretical and experimental studies of flows and estimating the accuracy of calculations and the error of experiments without involving additional hypotheses, equations, and parameters.

As an illustration, the 2D problem on the uniform flow of (strongly and weakly) stratified and (potentially and actually) homogeneous fluids around an arbitrarily oriented plate was considered within the framework of a reduced system of fundamental equations when the heat conductivity and compressibility effects were neglected. The fields of various physical variables were calculated for a wide range of flow parameters, including diffusion-induced creeping flows and pronounced wave and unsteady vortex regimes, within a unified formulation without involving additional equations or constants. The computational and experimental results were in good agreement as a whole and in their individual details.

In the observations of the patterns of droplet flows, a number of effects have been identified which show the influence of the processes of rapid conversion of the available

potential surface energy on the pattern of substance distribution. These include the disintegration of a drop into separated filaments in the vicinity of the confluence line, the formation of fast, small droplets moving faster than falling droplets, the formation of linear and mesh structures of filaments containing drop matter on the walls of the cavity and crown, and the filament structure of vortex elements at all the stages of subsequent flow evolution.

The development of numerical simulation codes based on the system of fundamental equations in a full 3D formulation, together with consistent experimental research techniques which allow registering the large-scale flow components and resolving all the fine-structural ones with the estimation of measurement errors, will help to significantly improve the accuracy of describing the dynamics and structure of a fluid and gas flow and develop substantiated estimates for predicting their evolution, effective flow control methods, as well as motion control techniques for a free vehicle in both gaseous and liquid media.

10. Conclusions

In the context of compatible axiomatic engineering mathematics and empirio-axiomatic technical mechanics as branches of sciences having in-definition demands to control the theoretical accuracy and experimental errors, the description of the fluid or gas status and flows is based on the conserved mathematical and physical quantities. Motion is defined as the transformation of metric space into itself with the conservation of distances. A fluid flow is defined as a transfer of momentum, matter, and energy.

The state and flow of a fluid or gas are defined by the complete set of fundamental equations, describing the transport of matter, constituents, momentum, and complete energy, including kinetic, potential, and internal energy. The thermodynamic parameters of fluid are defined as the derivatives of thermodynamic potentials. The transport of physical quantities is characterized by kinetic and material coefficients. The conventional system of fundamental equations supplementing empirical equations of state for thermodynamic potentials (the Gibbs potential is selected as basic) and density, as well as the boundary and initial conditions, is closed, well-posed, and resolvable. The system, which is characterized by a high rank and high dimension of the functional space, becomes degenerated on singular components in an approximation of a constant density.

The complete solutions of the fundamental equation system, which was analyzed while taking into account the owed modern version of conventional rules by Archimedes–Leibnitz (Meaningfulness, Identities, Consistency, Uniqueness, Sufficient reason, Minimum sufficiency, Causality, and Completeness) and the compatibility conditions, describe large-scale flow components (e.g., waves, vortices, jets, and wakes) and a rich family of ligaments (singularly perturbed components in an approximation of weak dissipation). The fluid flows, as a result of the superposition of many functions with different spatiotemporal parameters, are unsteady.

The nonlinear interactions of flow components result in the permanent evolution of the dynamical and structural parameters of a flow.

A common fundamental basis allows for providing a direct comparison of experimental and theoretical results with an estimation of the accuracy and errors. To increase the accuracy of the system state description and the prognostic potential of solutions, new technical instruments and codes allowing the definition of all flow parameters have to be developed.

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