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Mild Solutions for Impulsive Integro-Differential Equations Involving Hilfer Fractional Derivative with almost Sectorial Operators

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Abstract: In this manuscript, we establish the mild solutions for Hilfer fractional derivative integro-differential equations involving jump conditions and almost sectorial operator. For this purpose, we identify the suitable definition of a mild solution for this evolution equations and obtain the existence results. In addition, an application is also considered.

Keywords: Hilfer fractional derivative; mild solutions; impulsive conditions; almost sectorial operators; measure of non-compactness

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1. Introduction

Fractional differential equations are a type of mathematical equation that are used to describe the behaviour of a number of complicated and nonlocal systems with memory. Because of the fractional derivative's effective memory function, it has been widely used to describe many physical phenomena, such as flow in porous media and fluid dynamic traffic models; more precisely, fractional differential equations have been widely used in engineering, physics, chemistry, biology, and other fields. One can refer to the references in [1–6].

The theory of impulsive differential equations describes the processes which experience a suddenly change of their state at certain moments. There has been notable developments in the field of impulsive theory, especially in the area of impulsive differential equations with fixed moments. In recent years, the mathematical models of phenomena in physical, engineering, and biomedical sciences focus on the impulsive differential equations. The condition (2) includes such a kind of dynamics.

In [7], Anjali et al. discussed the analysis of Hilfer fractional differential equations with almost sectorial operators, and Abdo et al. [8] proved the existence of soutions for Hilfer fractional differential equations with boundary conditions. Boundary value problems for Hilfer fractional differential inclusions with nonlocal integral boundary conditions is investigated by Wongcharoen et al. [9]. In [10], the authors Yong et al. discussed the multi point boundary value problem for Hilfer fractional differential equation at Resonance. For some recent works of the mild solution, see [11–13].

Motivated by the above-cited works, we consider the impulsive initial Hilfer fractional derivative integro-differential equations involving jump conditions with almost sectorial operator in Banach space $\mathscr{Y}$ of the following form:

$$\mathfrak{D}^{\alpha,\nu}\wp(t)+\mathscr{A}\wp(t)=\mathscr{E}\left(t,\wp(t),\int_0^t\varsigma(t,s)\vartheta(s,\wp(s))ds\right),\quad t\in(0,T]=\mathcal{J}\tag{1}$$

$$\Delta\wp|_{t=t_k} = \mathscr{I}_k(\wp(t_k^-)),\ k = 1, 2, 3, \ldots m \tag{2}$$

$$I_{0+}^{(1-\alpha)(1-\nu)}\wp(0) = \sum_{k=1}^{m} c_k\wp(t_k), \tag{3}$$

where $\mathfrak{D}_{0+}^{\alpha,\nu}$ is the Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\nu \in [0,1]$. $\mathscr{A}$ is an almost sectorial operator in $\mathscr{Y}$ having norm $\|\cdot\|$, $\mathscr{E} : \mathcal{J} \times \mathscr{Y} \times \mathscr{Y} \to \mathscr{Y}$ is a function which is defined later, and $0 < t_1 < t_2 < \ldots < t_m < b, m \in \mathbb{N}$, and c_k are real numbers such that $c_k \neq 0$. For brevity, we will take the following:

$$\mathcal{B}\wp(t) = \int_0^t \varsigma(t,s)\vartheta(s, u(s))ds.$$

In [7], Anjali Jaiswal and Bahuguna studied the equations of the Hilfer fractional deritaive with almost sectorial operator in the abstract sense as follows:

$$\mathfrak{D}^{\alpha,\nu}(t) + \mathscr{A}u(t) = \mathscr{E}(t, u(t)),\ \ t \in (o, T]$$
$$I_{0+}^{(1-\alpha)(1-\nu)}u(0) = u_0.$$

We also refer to the work in [3], where Hamdy M. Ahmed et al. studied the existence for nonlinear Hilfer fractional derivative differential equations with control. Sufficient conditions were established where the time fractional derivative is the Hilfer derivative. In [14], Yong Zhoy et al. studied the factional Cauchy problems with almost sectorial operators of the following form:

$$\mathfrak{D}^{\alpha}u(t) = \mathscr{A}x(t) + \mathscr{F}(t, x(t)),\ \ t \in (0, T]$$
$$I_{0+}^{(1-\alpha)}x(0) = x_0$$

where $\mathfrak{D}^{\alpha}$ is the Riemann–Liouville derivative of order α, $I^{(1-\alpha)}$ is the Riemann–Liouville integral of order $1-\alpha$,$0 < \alpha < 1$, A is an almost sectorial operator on a complex Banach space, and $\mathscr{F}$ is a given function.

The following sections describes the supporting results of the given problem and also generalize the results in [14].

2. Preliminaries

Definition 1 ([15])**.** *For $\alpha > 0$, the fractional integral of order α of a function $f(t)$ is defined by:*

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t f(r)(t-r)^{\alpha-1}dr.$$

Definition 2 ([15])**.** *For $0 < \alpha < 1$, the Riemann–Liouville (R–L) fractional derivative with order α of a function $f(t)$ is defined by the following:*

$$\mathfrak{D}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(r)}{(t-r)^{\alpha}}dr.$$

Definition 3 ([15])**.** *For $0 < \alpha < 1$, the Caputo fractional derivative with order α of a function $f(t)$ is defined by the following:*

$${}^{c}\mathfrak{D}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(r)}{(t-r)^{\alpha}}dr.$$

Definition 4 ([16])**.** *Let $0 < \alpha < 1$ and $0 \leq v \leq 1$. The Hilfer fractional derivative of order α and type v is defined by the following:*

$$\mathfrak{D}_{0+}^{\alpha,v}u(t) = I_{0+}^{v(1-\alpha)}\frac{d}{dt}I_{0+}^{(1-v)(1-\alpha)}u(t).$$

Measure of Non-compactness:

Let $\mathscr{L} \subset \mathscr{Y}$ and be bounded. The Hausdorff measure of non-compactness Φ is defined by the following:

$$\Psi(\mathscr{L}) = \inf\Big\{\zeta > 0 \quad \text{such that} \quad \mathscr{L} \subset \bigcup_{j=1}^{m} B_\zeta(x_j) \quad \text{where} \quad x_j \in \mathscr{Y}, m \in N\Big\}. \tag{4}$$

The Kurtawoski measure of non-compactness Φ on a bounded set $\mathcal{B} \subset \mathscr{Y}$ is defined by the following:

$$\Phi(\mathscr{L}) = \inf\Big\{\epsilon > 0 \quad \text{such that} \quad \mathscr{L} \subset \bigcup_{j=1}^{m} M_j \quad \text{and} \quad \text{diam}(M_j) \le \epsilon\Big\} \tag{5}$$

with the following properties:

1. $\mathscr{L}_1 \subset \mathscr{L}_2$ gives $\Psi(\mathscr{L}_1) \le \Psi(\mathscr{L}_2)$ where $\mathscr{L}_1, \mathscr{L}_2$ are bounded subsets of $\mathscr{Y}$;
2. $\Psi(\mathscr{L}) = 0$ iff $\mathscr{L}$ is relatively compact in $\mathscr{Y}$;
3. $\Psi(\{z\} \bigcup \mathscr{L}) = \Psi(\mathscr{L})$ for all $z \in \mathscr{Y}$ $\mathscr{L} \subseteq \mathscr{Y}$;
4. $\Psi(\mathscr{L}_1 \bigcup \mathscr{L}_2) \le \max\{\Psi(\mathscr{L}_1), \Psi(\mathscr{L}_2)\}$;
5. $\Psi(\mathscr{L}_1 + \mathscr{L}_2) \le \Psi(\mathscr{L}_1) + \Psi(\mathscr{L}_2)$;
6. $\Psi(r\mathscr{L}) \le |r|\Psi(\mathscr{L})$ for every $r \in R$.

Let $\mathfrak{W} \subset C(I, \mathscr{Y})$ and $\mathfrak{W}(r) = \Big\{v(r) \in \mathscr{Y} | v \in \mathfrak{W}\}$. We define

$$\int_0^t \mathfrak{W}(r)dr = \Big\{\int_0^t v(r)dr | v \in \mathfrak{W}\Big\}, \quad \text{for} \quad t \in \mathcal{J}.$$

Proposition 1 ([17]). *If $\mathfrak{W} \subset C(\mathcal{J}, \mathscr{Y})$ is equicontinuous and bounded, then $t \to \Psi(\mathfrak{W}(t))$ is continuous on I and*

$$\Psi(\mathfrak{W}) = max\Psi(\mathfrak{W}(t)), \Psi(\int_0^t v(r)dr) \le \int_0^t \Psi(v(r))dr, \quad for \quad t \in I.$$

Proposition 2 ([18]). *Let $\{v_n : I \to \mathscr{Y}, n \in N\}$ be the Bochner integrable functions such that, for $n \in N$, $\|v_n\| \le m(t)$ a.e $m \in L^1(I, R^+)$. Then, $\xi(t) = \Psi(\{v_n(t)\}_{n=1}^{\infty}) \in L^1(I, R^+)$ and satisfies the following:*

$$\Psi(\{\int_0^t v_n(r)dr : n \in N\}) \le 2\int_0^t \xi(r)dr.$$

Proposition 3 ([19]). *Let $\mathfrak{W}$ be a bounded set. Then, for any $\zeta > 0$, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset \mathfrak{W}$, such that:*

$$\Psi(\mathfrak{W}) \le 2\Psi\{v_n\}_{n=1}^{\infty} + \zeta.$$

Almost Sectorial Operators:

Let $0 < \mu < \pi$ and $-1 < \beta < 0$. We define $S_\mu^0 = \{v \in C \setminus \{0\}$ that is $|argv| < \mu\}$ and its closure by S_μ, that is $S_\mu = \{v \in C \setminus \{0\}|\text{arg}v| < \mu\} \bigcup \{0\}$.

Definition 5 ([20]). *For $-1 < \beta < 0, 0 < \omega < \frac{\pi}{2}$, we define $\{\odot_\omega^\beta\}$ as a family of all closed and linear operators $\mathscr{A} : D(\mathscr{A}) \subset X \to \mathscr{Y}$ this implies the following:*

1. *$\sigma(\mathscr{A})$ is contained in the S_ω;*
2. *For all $\mu \in (\omega, \pi)$, there exists M_μ such that*

$$\|\mathfrak{R}(z, \mathscr{A})\|_{L(X)} \le M_\mu |z|^\beta$$

where $\mathfrak{R}(z,\mathscr{A}) = (zI - \mathscr{A})^{-1}$ is the resolvent operator of $\mathscr{A}$ for $z \in \rho(\mathscr{A})$ and $\mathscr{A} \in \odot_{\omega}^{\beta}$ is called an almost sectorial operator on X.

Proposition 4 ([20]). *Let $\mathscr{A} \in \odot_{\omega}^{\beta}$ for $-1 < \beta < 0$ and $0 < \omega < \frac{\pi}{2}$. Then, the below properties are completed:*

1. $\Im(t)$ *is analytic and* $\frac{d^n}{dt^n}\Im(t) = (-\mathscr{A}^n\Im(t)(t \in S^0_{\frac{\pi}{2}})$;
2. $\Im(t+s) = \Im(t)\Im(s) \ \ \forall \ t,s \in S^0_{\frac{\pi}{2}}$;
3. $\|\Im(t)\|_{L(\mathscr{Y})} \leq C_0 t^{-\beta-1}(t > 0)$; *where* $C_0 = C_0(\beta) > 0$ *is a constant;*
4. *Let* $\sum_{\Im} = \{x \in \mathscr{Y} \ \ : lim_{t\to 0_+}\Im(t)x = x\}$. *Then* $\mathfrak{D}(\mathscr{A}^{\theta}) \subset \sum_{\Im}$ *if* $\theta > 1+\beta$;
5. $\mathfrak{R}(r,-\mathscr{A}) = \int_0^{\infty} e^{-rs}\Im(s)ds$ *for* $r \in \mathbb{C}$ *with* $Re(r) > 0$.

We consider the following Wright-type function [15]:

$$\mathfrak{M}_{\alpha}(\theta) = \sum_{n\in\mathbb{N}} \frac{(-\theta)^{n-1}}{\Gamma(1-\alpha n)(n-1)!}, \quad \theta \in \mathbb{C}.$$

For $-1 < \sigma < \infty$, $r > 0$, the following are satisfied:

(A1) $\mathfrak{M}_{\alpha}(\theta) \geq 0, \ t > 0$;
(A2) $\int_0^{\infty} \theta^{\sigma}\mathfrak{M}_{\alpha} d\theta = \frac{\Gamma(1+\sigma)}{\Gamma(1+\alpha\sigma)}$;
(A3) $\int_0^{\infty} \frac{\alpha}{\theta^{\alpha+1}} e^{-r\theta}\mathfrak{M}_{\alpha}(\frac{1}{\theta^{\alpha}})d\theta = e^{-r^{\alpha}}$.

Define the operator families $\{\mathfrak{S}_{\alpha}(t)\}|_{t\in S^0_{\frac{\pi}{2}-w}}$ and $\{\mathfrak{Q}_{\alpha}(t)\}|_{t\in S^0_{\frac{\pi}{2}-w}}$ as follows:

$$\mathfrak{S}_{\alpha}(t) = \int_0^{\infty} \mathfrak{M}_{\alpha}(\zeta)\varrho(t^{\alpha}\zeta)d\zeta$$

$$\mathfrak{Q}_{\alpha}(t) = \int_0^{\infty} \alpha\zeta\mathfrak{M}_{\alpha}(\zeta)\varrho(t^{\alpha}\zeta)d\zeta.$$

Theorem 1 (Theorem 4.6.1 [15]). *For each fixed $t \in S^0_{\frac{\pi}{2}-\omega}$, $\mathfrak{S}_{\alpha}(t)$ and $\mathfrak{Q}_{\alpha}(t)$ are bounded and linear operators on $\mathscr{Y}$. In addition:*

$$\|\mathfrak{S}_{\alpha}(t)\| \leq \mathfrak{C}_s t^{-\alpha(1+\beta)}, \ \ \|\mathfrak{Q}_{\alpha}(t)\| \leq \mathfrak{C}_p t^{-\alpha(1+\beta)}, \ \ t > 0$$

where $\mathfrak{C}_s$ and $\mathfrak{C}_p$ are constants.

Theorem 2 (Theorem 4.6.2 [15]). *$\mathfrak{S}_{\alpha}(t)$ and $\mathfrak{Q}_{\alpha}(t)$ are continuous in the uniform operator topology for $t > 0$. Moreover, for every $s > 0$, the continuity is uniform on $[s,\infty]$.*

For $T > 0$, we set $\mathcal{J} = [0,T]$ and $\mathcal{J}' = (0,T]$. We introduce $\mathbb{C}(\mathcal{J},\mathscr{Y})$ as the space of continuous functions from $\mathcal{J}$ to $\mathscr{Y}$. Define $\mathfrak{Y} = \Big\{\wp \in \mathbb{C}(\mathcal{J}',\mathscr{Y}) : \lim_{t\to 0} t^{1+\alpha\beta(1-v)}\wp(t)$ exists and is finite$\Big\}$, and $\|\wp\|_{\mathfrak{Y}} = sup'_{t\in\mathcal{J}}\{t^{1+\alpha\beta(1-v)}\|\wp(t)\|\}$.

Then, $\mathfrak{Y}$ is a Banach space. Then:

(a) *For* $v = 1$, $\mathfrak{Y} = \mathbb{C}(\mathcal{J},\mathscr{Y})$ *and* $\|\wp\|_{\mathfrak{Y}} = sup_{t\in\mathcal{J}}\|\wp(t)\|$;
(b) *For* $v = 0$, $\|u\|_{\mathfrak{Y}} = sup_{t\in\mathcal{J}'}\|t^{(1+\alpha\beta)}u(t)\|$;
(c) *Let* $\wp(t) = t^{1+\alpha\beta(1-v)}y(t)$, $t \in \mathcal{J}'$. *Then* $\wp \in \mathfrak{Y}$ *if and only if* $y \in \mathbb{C}(\mathcal{J},\mathscr{Y})$ *and* $\|\wp\|_{\mathfrak{Y}} = \|y\|$.

We define $\mathfrak{B}_r(\mathcal{J}) = \{y \in \mathbb{C}(\mathcal{J},\mathscr{Y})$ such that $\|y\| \leq r\}$ and $\mathfrak{B}_r^{\mathfrak{Y}}(\mathcal{J}') = \{u \in \mathfrak{Y}$ such that$\|\wp\| \leq r\}$.

We assume the following hypotheses to prove our results.

Hypothesis 1 (H1). *For each fixed $t \in \mathcal{J}'$, $\mathscr{E}(t,.,.) : X \times \mathscr{Y} \to \mathscr{Y}$ is a continuous function and for each $\wp \in \mathfrak{C}(\mathcal{J}',\mathscr{Y})$, $\mathscr{E}(.,\wp,\mathcal{B}\wp) : \mathcal{J}' \to \mathscr{Y}$ is strongly measurable.*

Hypothesis 2 (H2). *$\exists$ a function $k \in L^1(\mathcal{J}', \mathbb{R}^+)$ satisfying the following:*

$$I_{0+}^{-\alpha\beta} k \in \mathfrak{C}(\mathcal{J}', \mathbb{R}),\ lim\ t^{(1+\alpha\beta)(1-\nu)} I_{0+}^{-\alpha\beta} k(t) = 0.$$

Hypothesis 3 (H3).

$$sup_{[0,T]}(t^{(1+\alpha\beta)(1-nu)} \|\mathfrak{S}_{\alpha,\nu(t) u_0}\| + t^{(1+\alpha\beta)(1-\nu)} \int_0^t (t-r)^{-\alpha\beta-1} k(r) dr) \leq r$$

for a constant $r > 0$ and $u_0 \in D(\mathscr{A}^\theta), \theta > 1+\beta$, where $\mathfrak{S}_{\alpha,\nu(t)} = I_{0+}^{\nu(1-\alpha)} t^{-1} \mathfrak{Q}_\alpha(t)$.

Hypothesis 4 (H4). *$\exists$ constants γ_k such that $\|\mathscr{I}_k(\wp)\| \leq \gamma_k$, $k = 1, 2, \ldots, m$ for each $\wp \in \mathscr{Y}$.*

Definition 6. *By a mild solution of the Cauchy problem (1.1)–(1.3), we mean a function $\wp \in \mathbb{C}(\mathcal{J}', X)$ that satisfies the following:*

$$\wp(t) = \mathfrak{S}_{\alpha,\nu}(t)\widehat{v} + \int_0^t \mathfrak{K}_\alpha(t-r)\mathscr{E}(r, \wp(r), (\mathcal{B}\wp)r)dr + \sum_{0<t_k<t} \mathfrak{S}_{\alpha,\nu}(t-t_k)\mathscr{I}_k(\wp(t_k^-)),\ \ t \in \mathcal{J}' \tag{6}$$

where

$$\widehat{v} = \sum_{k=1}^m c_k \wp(t_k),\ \ \mathfrak{S}_{\alpha,\nu}(t) = I_{0+}^{\nu(1-\alpha)} \mathfrak{K}_\alpha(t),\ \ \mathfrak{K}_\alpha = t^{\alpha-1}\mathfrak{Q}_\alpha(t).$$

Now, we define an operator $\mathfrak{P} : \mathfrak{B}_r(\mathcal{J}') \to \mathfrak{B}_r(\mathcal{J}')$ as follows:

$$\begin{aligned}(\mathfrak{P}\wp)(t) =& \mathfrak{S}_{\alpha,\nu}(t)\widehat{v} + \int_0^t (t-r)^{\alpha-1}\mathfrak{Q}_\alpha(t-r)\mathscr{E}(r, \wp(r), (\mathcal{B}\wp)r)dr \\ &+ \sum_{0<t_k<t} \mathfrak{S}_{\alpha,\nu}(t-t_k)\mathscr{I}_k(\wp(t_k^-)).\end{aligned} \tag{7}$$

Lemma 1 ([7]). *$\mathfrak{K}_\alpha(t)$ and $\mathfrak{S}_{\alpha,\nu}(t)$ are bounded linear operators on $\mathscr{Y}$, for every fixed $t \in S^0_{\frac{\pi}{2}-\omega}$. Furthermore for $t > 0$:*

$$\|\mathfrak{K}_\alpha(t)x\| \leq \mathfrak{C}_p t^{-1-\alpha\beta}\|x\|,\ \ \ \|\mathfrak{S}_{\alpha,\nu(t)x}\| \leq \frac{\Gamma(-\alpha\beta)}{\Gamma(\nu(1-\alpha)-\alpha\beta)} \mathfrak{C}_p t^{\nu(1-\alpha)-\alpha\beta-1}\|x\|.$$

Proposition 5 ([7]). *$\mathfrak{K}_\alpha(t)$ and $\mathfrak{S}_{\alpha,\nu(t)}$ are strongly continuous, for $t > 0$.*

Let $M_s = \sup_{t\in\mathcal{J}} \left\|\mathfrak{S}_{\alpha,\nu(t)}\right\|$. Assume that $\sum_{c_k}^m \leq \frac{1}{M_s}$.
We have:

$$\left\|\sum_{c_k}^m \mathfrak{S}_{\alpha,\nu}(t_k)\right\| \leq M_s.\frac{1}{M_s} < 1.$$

3. Main Results

Theorem 3. *Let $\mathscr{A} \in \odot_\omega^\beta$ for $-1 < \beta < 0$ and $0 < \omega < \frac{\pi}{2}$. Assuming (H1)–(H4) are satisfied, the operators $\{\mathfrak{F}y\ :\ y \in \mathfrak{B}_r(\mathcal{J})\}$ are equicontinuous provided $\wp_0 \in \mathcal{D}(\mathscr{A}^\theta)$ with $\theta > 1+\beta$.*

Proof. For $y \in \mathfrak{B}_r(\mathcal{J})$ and $t_1 = 0 < t_2 \leq T$, we gave the following:

$$\begin{aligned}\left\|\mathfrak{F}y(t_2) - \mathfrak{F}y(0)\right\| =& \left\|t_2^{(1+\alpha\mu)(1-\nu)}\Big(\mathfrak{S}_{\alpha,\nu(t_2)}\widehat{v} + \int_0^{t_2}(t_2-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_2-r)\mathscr{E}(r, \wp(r), (\mathcal{B}\wp)r)dr \right. \\ &\left. + \sum_{0<t_k<t_2} \mathfrak{S}_{\alpha,\nu}(t_2-t_k)\mathscr{I}_k(\wp(t_k^-))\Big)\right\| \\ &\leq \left\|t_2^{(1+\alpha\mu)(1-\nu)}\mathfrak{S}_{\alpha,\nu(t_2)}\widehat{v}\right\|\end{aligned}$$

$$+\left\|t_2^{(1+\alpha\mu)(1-\nu)}\int_0^{t_2}(t_2-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_2-r)\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\right\|$$
$$+\left\|t_2^{(1+\alpha\mu)(1-\nu)}\sum_{0<t_k<t_2}\mathfrak{S}_{\alpha,\nu}(t_2-t_k)\mathscr{I}_k(\wp(t_k^-))\right\|$$
$$\to 0, \text{ as } t_2\to 0.$$

Now, let $0<t_1<t_2\le T$:

$$\begin{aligned}\left\|\mathfrak{F}y(t_2)-\mathfrak{F}y(t_1)\right\|\le&\left\|t_2^{(1+\alpha\mu)(1-\nu)}\mathfrak{S}_{\alpha,\nu(t_2)}\widehat{v}-t_1^{(1+\alpha\mu)(1-v)}\mathfrak{S}_{\alpha,\nu(t_1)}\widehat{v}\right\|\\
&+\left\|t_2^{(1+\alpha\mu)(1-v)}\int_0^{t_2}(t_2-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_2-r)\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\right.\\
&\left.-t_1^{(1+\alpha\mu)(1-\nu)}\int_0^{t_1}(t_1-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_1-r)\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\right\|\\
&+\left\|t_2^{(1+\alpha\mu)(1-\nu)}\sum_{0<t_k<t_2}\mathfrak{S}_{\alpha,\nu}(t_2-t_k)\mathscr{I}_k(\wp(t_k^-))\right.\\
&\left.-t_1^{(1+\alpha\mu)(1-\nu)}\sum_{0<t_k<t_1}\mathfrak{S}_{\alpha,\nu}(t_1-t_k)\mathscr{I}_k(\wp(t_k^-))\right\|.\end{aligned}$$

Here, using the triangle inequality, we have the following:

$$\begin{aligned}\left\|\mathfrak{F}y(t_2)-\mathfrak{F}y(t_1)\right\|\le&\left\|t_2^{(1+\alpha\mu)(1-\nu)}\mathfrak{S}_{\alpha,\nu(t_2)}\widehat{v}-t_1^{(1+\alpha\mu)(1-\nu)}\mathfrak{S}_{\alpha,\nu(t_1)}\widehat{v}\right\|\\
&+\left\|t_2^{(1+\alpha\mu)(1-\nu)}\int_{t_1}^{t_2}(t_2-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_2-r)\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\right\|\\
&+\left\|t_2^{(1+\alpha\mu)(1-\nu)}\int_0^{t_1}(t_2-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_2-r)\mathscr{E}(r,\wp(r),(Bu)r)dr\right.\\
&\left.-t_1^{(1+\alpha\mu)(1-\nu)}\int_0^{t_1}(t_1-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_2-r)\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\right\|\\
&+\left\|t_1^{(1+\alpha\mu)(1-\nu)}\int_0^{t_1}(t_1-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_2-r)\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\right.\\
&\left.-t_1^{(1+\alpha\mu)(1-\nu)}\int_0^{t_1}(t_1-r)^{\alpha-1}\mathfrak{Q}_\alpha(t_1-r)\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\right\|\\
&+\left\|t_2^{(1+\alpha\mu)(1-v)}\sum_{0<t_k<t_2}\mathfrak{S}_{\alpha,\nu}(t_2-t_k)\mathscr{I}_k(\wp(t_k^-))\right.\\
&\left.-t_1^{(1+\alpha\mu)(1-\nu)}\sum_{0<t_k<t_1}\mathfrak{S}_{\alpha,\nu}(t_1-t_k)\mathscr{I}_k(\wp(t_k^-))\right\|\\
=&\mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3+\mathfrak{I}_4+\mathfrak{I}_5.\end{aligned}$$

By the strong continuity of $\mathfrak{S}_{\alpha,\nu}(t)$, we obtain $\mathfrak{I}_1\to 0$ as $t_2\to t_1$. In addition:

$$\begin{aligned}\mathfrak{I}_2\le&\mathfrak{C}_p t_2^{(1+\alpha\beta)(1-\nu)}\int_{t_1}^{t_2}(t_2-r)^{-\alpha\beta-1}\kappa(r)dr\\
\le&\mathfrak{C}_p\left|t_2^{(1+\alpha\beta)(1-\nu)}\int_0^{t_2}(t_2-r)^{-\alpha\beta-1}\kappa(r)dr-t_2^{(1+\alpha\beta)(1-\nu)}\int_0^{t_1}(t_1-r)^{-\alpha\beta-1}\kappa(r)dr\right|\\
\le&\mathfrak{C}_p\int_0^{t_1}\left|t_1^{(1+\alpha\beta)(1-\nu)}(t_1-r)^{-\alpha\beta-1}-t_2^{(1+\alpha\beta)(1-\nu)}(t_2-r)^{-\alpha\beta-1}\right|\kappa(r)dr.\end{aligned}$$

Then, $\mathfrak{I}_2\to 0$ as $t_2\to t_1$, by using (H2) and the dominated convergence theorem. Since

$$\mathfrak{I}_3\le\mathfrak{C}_p\int_0^{t_1}(t_2-r)^{-\alpha-\alpha\beta}\left|t_2^{(1+\alpha\beta)(1-\nu)}(t_2-r)^{\alpha-1}-t_1^{(1+\alpha\beta)(1-\nu)}(t_1-r)^{\alpha-1}\right|\kappa(r)dr$$

and

$$(t_2-r)^{-\alpha-\alpha\beta}\Big|t_2^{(1+\alpha\beta)(1-\nu)}(t_2-r)^{\alpha-1}-t_1^{(1+\alpha\beta)(1-\nu)}(t_1-r)^{\alpha-1}\Big|\kappa(r)$$

$$\begin{aligned}&\le t_2^{(1+\alpha\beta)(1-\nu)}(t_2-r)^{\alpha-1}\kappa(r)+t_1^{(1+\alpha\beta)(1-\nu)}(t_1-r)^{\alpha-1}\kappa(r)\\&\le 2t_1^{(1+\alpha\beta)(1-\nu)}(t_1-r)^{\alpha-1}\kappa(r)\end{aligned}$$

and $\int_0^{t_1}2t_1^{(1+\alpha\beta)(1-\nu)}(t_1-r)^{\alpha-1}\kappa(r)$ exists, i.e., $\mathfrak{I}_3\to 0$ as $t_2\to t_1$.

For $\epsilon>0$, we have the following:

$$\begin{aligned}\mathfrak{I}_4=&\Big\|\int_0^{t_1}t_1^{(1+\alpha\beta)(1-\nu)}[\mathfrak{Q}_\alpha(t_2-r)-\mathfrak{Q}_\alpha(t_1-r)](t_1-r)^{\alpha-1}\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\Big\|\\\le&\int_0^{t_1-\epsilon}t_1^{(1+\alpha\beta)(1-\nu)}\Big\|\mathfrak{Q}_\alpha(t_2-r)-\mathfrak{Q}_\alpha(t_1-r)\Big\|_{L(X)}(t_1-r)^{\alpha-1}\kappa(r)\\&+\int_{t_1}^{t_1}t_1^{(1+\alpha\beta)(1-\nu)}\Big\|\mathfrak{Q}_\alpha(t_2-r)-\mathfrak{Q}_\alpha(t_1-r)\Big\|_{L(X)}(t_1-r)^{\alpha-1}\kappa(r)\\\le& t_1^{(1+\alpha\beta)(1-\nu)}\int_0^{t_1}(t_1-r)^{\alpha-1}\kappa(r)dr\ \sup_{s\in[0,t_1-\epsilon]}\ \Big\|\mathfrak{Q}_\alpha(t_2-r)-\mathfrak{Q}_\alpha(t_1-r)\Big\|_{L(X)}\\&+\mathfrak{C}_p\int_{t_1}^{t_1}t_1^{(1+\alpha\beta)(1-\nu)}((t_2-r)^{-\alpha-\alpha\beta}+(t_1-r)^{-\alpha-\alpha\beta})(t_1-r)^{\alpha-1}\kappa(r)dr\\\le& t_1^{(1+\alpha\beta)(1-\nu)+\alpha(1+\beta)}\int_0^{t_1}(t_1-r)^{-\alpha\beta-1}\kappa(r)dr\sup_{s\in[0,t_1-\epsilon]}\Big\|\mathfrak{Q}_\alpha(t_2-r)-\mathfrak{Q}_\alpha(t_1-r)\Big\|_{L(X)}\\&+2\mathfrak{C}_p\int_{t_1-\epsilon}^{t_1}t_1^{(1+\alpha\beta)(1-\nu)}(t_1-r)^{-\alpha\beta-1}\kappa(r)dr.\end{aligned}$$

Since $\mathfrak{Q}_\alpha(t)$ is uniformly continuous and $\lim_{t_2\to t_1}\mathfrak{I}_2=0$, then $\mathfrak{I}_4\to 0$ as $t_2\to t_1$, i.e., independent of $y\in\mathfrak{B}_r(\mathcal{J})$.

Clearly, by the strong continuity of $\mathfrak{S}_{\alpha,\nu}(t)$, we obtain the following:

$$\begin{aligned}\mathfrak{I}_5=&\Big\|t_2^{(1+\alpha\mu)(1-v)}\sum_{0<t_k<t_2}\mathfrak{S}_{\alpha,\nu}(t_2-t_k)\mathscr{I}_k(\wp(t_k^-))\\&-t_1^{(1+\alpha\mu)(1-\nu)}\sum_{0<t_k<t_1}\mathfrak{S}_{\alpha,\nu}(t_1-t_k)\mathscr{I}_k(\wp(t_k^-))\Big\|\to 0\ \text{ as }\ t_2\to t_1.\end{aligned}$$

Hence, $\Big\|\mathfrak{F}y(t_2)-\mathfrak{F}y(t_1)\Big\|\to 0$ independently of $y\in\mathfrak{B}_r(\mathcal{J})$ as $t_2\to t_1$; therefore, $\{\mathfrak{F}y\ :\ y\in\mathfrak{B}_r(\mathcal{J})\}$ is equicontinuous. □

Theorem 4. *Let $-1<\beta<0$ and $0<\omega<\frac{\pi}{2}$ and $\mathscr{A}\in\odot_\omega^\beta$. Then, under Assumptions (H1)–(H3) the operator $\{\mathfrak{F}y\ :\ y\in\mathfrak{B}_r(\mathcal{J})\}$ is continuous and bounded provided $\wp_0\in\mathcal{D}(\mathscr{A}^\theta)$ with $\theta>1+\beta$.*

Proof. Firstly, we prove that $\mathfrak{F}$ maps $\mathfrak{B}_r(\mathcal{J})$. Taking $y\in\mathfrak{B}_r(\mathcal{J})$ and define $\wp(t)=t^{-(1+\alpha\beta)(1-\nu)}y(t)$, we have $\wp\in\mathfrak{B}_r^{\mathfrak{Y}}(\mathcal{J}')$. Let $t\in[0,T]$:

$$\|\mathfrak{F}\|\le\|t^{(1+\alpha\beta)(1-\nu)}\mathfrak{S}_{\alpha,\nu}(t)\widehat{v}\|+t^{(1+\alpha\beta)(1-\nu)}\Big\|\int_0^t(t-r)^{\alpha-1}\mathfrak{Q}_\alpha(t-r)\mathscr{E}(r,\wp(r),(\mathcal{B}\wp)r)dr\Big\|$$

From (H2) and (H3), we obtain the following:

$$\|\mathfrak{F}y(t)\|\le t^{(1+\alpha\beta)(1-\nu)}\|\mathfrak{S}_{\alpha,\nu}(t)\widehat{v}\|+t^{(1+\alpha\beta)(1-\nu)}\int_0^t(t-r)^{-\alpha\beta-1}\kappa(r)dr$$

$$\leq \sup_{[0,T]} \ t^{(1+\alpha\beta)(1-\nu)} \int_0^t (t-r)^{-\alpha\beta-1}\kappa(r)dr$$
$$\leq r.$$

Hence, $\|\mathfrak{F}y\| \leq r$, for any $y \in \mathcal{B}_r(I)$.

Now, to verify $\mathfrak{F}$ is continuous in $\mathcal{B}_r(I)$, let $y_n, y \in \mathcal{B}(I), n = 1, 2, \cdots$, with $\lim_{n\to\infty} y_n = y$. Hence, $\lim_{n\to\infty} y_n(t) = y(t)$ and $\lim_{n\to\infty} t^{-(1+\alpha\beta)(1-v)} y_n(t) = t^{-(1+\alpha\beta)(1-v)} y(t)$ and $\lim_{n\to\infty} t^{-(1+\alpha\beta)(1-\nu)}y_n(t) = t^{-(1+\alpha\beta)(1-\nu)}y(t)$, on $\mathcal{J}'$ (H1) implies the following:

$$\begin{aligned}\mathscr{E}(t, \wp_n(t), \mathcal{B}(\wp_n(t))) =& \mathscr{E}(t, t^{-(1+\alpha\beta)(1-\nu)}y_n(t), t^{-(1+\alpha\beta)(1-\nu)}\mathcal{B}(y_n(t))) \\ &\to \mathscr{E}(t, t^{-(1+\alpha\beta)(1-\nu)}y(t), t^{-(1+\alpha\beta)(1-\nu)}\mathcal{B}(y(t))),\end{aligned}$$

as $n \to \infty$.

We use (H2) to obtain the inequality $(t-r)^{-\alpha\beta-1}|\mathscr{E}(r, \wp_n(r), \mathcal{B}(\wp_n(r)))| \leq 2(t-r)^{-(\alpha\beta)(1-\nu)}\kappa(r)$, i.e.,

$$\int_0^t (t-r)^{-\alpha\beta-1}\|\mathscr{E}(r, \wp_n(r), \mathcal{B}(\wp_n(r))) - \mathscr{E}(r, \wp(r), \mathcal{B}(\wp(r))\|dr \ \to \ 0, \text{ as } n \to \ \infty.$$

Let $t \in [0, T]$. Now:

$$\|\mathfrak{F}y_n(t) - \mathfrak{F}y(t)\| \leq t^{(1+\alpha\beta)(1-\nu)}\left\| \int_0^t (t-r)^{\alpha-1}\mathfrak{Q}_\alpha(t-r)(\mathscr{E}(r, \wp_n(r), \mathcal{B}(\wp_n(r)) - \mathscr{E}(r, \wp(r), \mathcal{B}(\wp(r))) dr\right\|.$$

Applying Theorem (1), we have the following:

$$\|\mathfrak{F}y_n(t) - \mathfrak{F}y(t)\| \leq \mathfrak{C}_p t^{(1+\alpha\beta)(1-\nu)} \int_0^t (t-r)^{-\alpha\beta-1}\|\mathscr{E}(r, \wp_n(r), \mathcal{B}(\wp_n(r)) - \mathscr{E}(r, \wp(r), \mathcal{B}(\wp(r))\|dr$$
$$\to 0 \text{ as } n \to \infty.$$

that is, $\mathfrak{F}y_n \to \mathfrak{F}y$ pointwise on $\mathcal{J}$. In addition, Theorem (3) implies that $\mathfrak{F}y_n \to \mathfrak{F}y$ uniformly on $\mathcal{J}$ as $n \to \infty$. Hence, $\mathfrak{F}$ is continuous. □

4. $\Im(t)$ Is Compact

We can assume that, for $t > 0$, the semigroup $\mathfrak{T}(t)$ is compact on $\mathscr{Y}$. Hence, the compactness of $\mathcal{Q}_\alpha(t)$ is as follows:

Theorem 5. *Let $-1 < \beta < 0, 0 < \omega < \frac{\pi}{2}$ and $\mathscr{A} \in \Theta_\omega^\beta$. If $\mathfrak{T}(t)(t > 0)$ is compact and (H1)–(H4) hold, then $\exists$ a mild solution of (1.1)–(1.3) in $\mathcal{B}_r^{\mathcal{Y}}(I')$ for every $\wp_0 \in D(\mathscr{A}^\theta)$ with $\theta > 1 + \beta$.*

Proof. Since we have assumed $\Im(t)$ is compact, it gives the equicontinuity of $\Im(t)(t > 0)$. Moreover, from Theorems (3) and (4), we know that $\mathfrak{F} : \mathfrak{B}_r^{\mathcal{Y}}(J') \to \mathfrak{B}_r^{Y}(J')$ is continuous and bounded and $\varepsilon : \mathfrak{B}_r(J) \to \mathfrak{B}_r(J)$ is bounded, continuous, and $\{\varepsilon y : y \in \mathfrak{B}_r(J)\}$ equicontinuous. We can write $\epsilon : \mathfrak{B}_r(J) \to \mathfrak{B}_r(J)$ as follows:

$$(\epsilon y)(t) = (\epsilon^1 y)(t) + (\epsilon^2 y)(t)$$

where:

$$\begin{aligned}(\epsilon^1 y)(t) &= t^{(1+\alpha\beta)(1-\nu)}\mathfrak{S}_{\alpha,\nu}(t)\wp_0 = t^{1+\alpha\beta)(1-\nu)}\mathcal{I}_{0+}^{\nu(1-\alpha)}t^{\alpha-1}\mathfrak{Q}_\alpha(t)\overline{v} \\ &= \frac{t^{(1+\alpha\beta)(1-\nu)}}{\Gamma(\nu(1-\alpha))}\int_0^t (t-r)^{\nu(1-\alpha)-1}r^{\alpha-1}\int_0^\infty \alpha\theta M_\alpha(\theta)\Im(r^\alpha\theta)\overline{v}d\theta\, dr \\ &= \frac{\alpha t^{(1+\alpha\beta)(1-\nu)}}{\Gamma(\nu(1-\alpha))}\int_0^t\int_0^\infty (t-r)^{\nu(1-\alpha)-1}r^{\alpha-1}\theta M_\alpha(\theta)\Im(r^\alpha\theta)\overline{v}d\theta dr\end{aligned}$$

and

$$(\varepsilon^2 y)(t) = t^{(1+\alpha\beta)(1-\nu)} \int_0^t (t-r)^{\alpha-1} \mathfrak{Q}_\alpha(t-r) \mathscr{E}(r, \wp(r)(\mathcal{B}\wp)r) dr + \sum_{0<t_k<t} \mathfrak{S}_{\alpha,\nu}(t-t_k) \mathscr{I}_k(\wp(t_k^-)).$$

For $\sigma > 0$ and $\zeta \in (0,t)$, we define an operator $\varepsilon^1_{\zeta,\sigma}$ on $\mathfrak{B}_r(J)$ by

$$\begin{aligned}(\varepsilon^1_{\zeta,\sigma} y)(t) &= \frac{t^{(1+\alpha\beta)(1-\nu)}}{\Gamma(\nu(1-\alpha))} \int_\zeta^t \int_\sigma^\infty (t-r)^{(1-\alpha)\nu-1} r^{\alpha-1} \theta M_\alpha(\theta) \Im(r^\alpha\theta) \overline{v} d\theta \, dr \\ &= \frac{\alpha t^{(1+\alpha\beta)(1-\nu)}}{\Gamma(\nu(1-\alpha))} \mathfrak{T}(\zeta^\alpha\sigma) \int_\zeta^t \int_\sigma^\infty (t-r)^{(1-\alpha)\nu-1} r^{\alpha-1} \theta M_\alpha(\theta) \Im(r^\alpha\theta - \zeta^\alpha\sigma) \overline{v} d\theta dr.\end{aligned}$$

Since $\mathfrak{T}(\epsilon^\alpha\delta)$ is compact $\mathcal{V}^1_{\zeta,\sigma}(t) = \{\varepsilon^1_{\zeta,\sigma} y)(t), y \in \mathfrak{B}_r(J)\}$ is precompact in X $\forall \zeta \in (0,t)$ and $\delta > 0$. Moreover, for any $y \in \mathfrak{B}_r(J)$

$$\begin{aligned}\|(\varepsilon^1 y)(t) - (\varepsilon^1_{\zeta,\sigma} y)(t)\| &\le \mathscr{K}(\alpha,\nu) \left\| t^{(1+\alpha\beta)(1-\nu)} \int_0^t \int_0^\sigma (t-r)^{\nu(1-\alpha)-1} r^{\alpha-1} \theta M_\alpha(\theta) \Im(r^\alpha\theta) \overline{v} d\theta dr \right\| \\ &+ \mathscr{K}(\alpha,\nu) \left\| t^{(1+\alpha\beta)(1-\nu)} \int_0^\zeta \int_\sigma^\infty (t-r)^{\nu(1-\alpha)-1} r^{\alpha-1} \theta M_\alpha(\theta) \Im(r^\alpha\theta) \overline{v} d\theta dr \right\| \\ &\le \mathscr{K}(\alpha,\nu) t^{(1+\alpha\beta)(1-\nu)} \int_0^t \int_0^\sigma (t-r)^{\nu(1-\alpha)-1} r^{\alpha-1} \theta M_\alpha(\theta) r^{-\alpha\gamma-\alpha} \|\overline{v}\| \theta^{-\beta-1} d\theta \, dr \\ &+ \mathscr{K}(\alpha,\nu) t^{(1+\alpha\beta)(1-\nu)} \int_0^\zeta \int_\sigma^\infty (t-r)^{\nu(1-\alpha)-1} r^{\alpha-1} \theta M_\alpha(\theta) r^{-\alpha\beta-\alpha} \theta^{-\beta-1} \|\overline{v}\| d\theta \, dr \\ &= \mathscr{K}(\alpha,\nu) t^{(1+\alpha\beta)(1-\nu)} \int_0^t (t-r)^{\nu(1-\alpha)-1} r^{-\alpha\beta-1} \|\overline{v}\| dr \int_0^\sigma \theta^{-\beta} M_\alpha(\theta) d\theta \\ &+ \mathscr{K}(\alpha,\nu) t^{(1+\alpha\beta)(1-\nu)} \int_0^\zeta (t-r)^{\nu(1-\alpha)-1} r^{-\alpha\beta-1} \|\overline{v}\| dr \int_\eta^\infty \theta^{-\beta} M_\alpha(\theta) d\theta \\ &\le \mathscr{K} t^{-\alpha\nu(1+\beta)} \|\overline{v}\| \int_0^\eta \theta^{-\beta} M_\alpha(\theta) d\theta \\ &+ \mathscr{K} t^{-\alpha\nu(1+\beta)} \|\overline{v}\| \int_0^\zeta (1-s)^{\nu(1-\alpha)-1} r^{-\alpha\beta-1} dr \int_\eta^\infty \theta^{-\beta} M_\alpha(\theta) d\theta, \\ &\to 0, \text{ as } \zeta \to 0, \ \sigma \to 0,\end{aligned}$$

where, $\mathscr{K}(\alpha,\nu) = \frac{\alpha}{\Gamma(\nu(1-\alpha))}$.

Therefore, $\mathcal{V}^1_{\zeta,\sigma}(t) = \{\varepsilon^1_{\zeta,\sigma} y)(t), y \in \mathfrak{B}_r(J)\}$ are arbitrarily close to $\mathcal{V}^1(t) = \{\varepsilon^1 y)(t), y \in \mathfrak{B}_r(I)\}$, for $t > 0$. Hence, $\mathcal{V}^1(t)$, for $t > 0$, is precompact in $\mathscr{Y}$.

For $\zeta \in (0,t)$ and $\sigma > 0$, we can present an operator $\varepsilon^2_{\zeta,\sigma}$ on $\mathfrak{B}_r(I)$ by

$$\begin{aligned}(\varepsilon^2_{\zeta,\sigma} y)(t) =& \alpha t^{(1+\alpha\beta)(1-\nu)} \int_0^{t-\zeta} \int_\sigma^\infty \theta M_\alpha(\theta)(t-r)^{\alpha-1} \Im((t-r)^\alpha\theta) \mathscr{E}(r, \wp(r), (\mathcal{B}\wp)r) d\theta dr \\ &+ \sum_{0<t_k<t} \mathfrak{S}_{\alpha,v}(t-t_k) \mathscr{I}_k(\wp(t_k^-)) \\ =& \alpha t^{(1+\alpha\beta)(1-\nu)} \mathfrak{T}(\zeta^\alpha\sigma) \int_0^{t-\zeta} \int_\sigma^\infty \theta M_\alpha(\theta)(t-r)^{\alpha-1} \Im((t-r)^\alpha\theta - \zeta^\alpha\sigma) \mathscr{E}(r, \wp(r), (\mathcal{B}\wp)r) d\theta dr \\ &+ \sum_{0<t_k<t} \mathfrak{S}_{\alpha,\nu}(t-t_k) \mathscr{I}_k(\wp(t_k^-)).\end{aligned}$$

Hence, $\mathcal{V}^2_{\zeta,\sigma}(t) = \{\varepsilon^2_{\zeta,\sigma} y)(t), y \in \mathfrak{B}_r(J)\}$ is precompact in X $\forall \zeta \in (0,t)$ and $\sigma > 0$ due to the compactness of $\Im(\zeta^\alpha\sigma)$. For every $y \in \mathfrak{B}_r(J)$, we obtain the following:

$$
\begin{aligned}
\|(\varepsilon^2 y)(t) - (\varepsilon^2_{\zeta,\sigma} y)(t)\| \leq & \Big\| \alpha t^{(1+\alpha\beta)(1-\nu)} \Big(\int_0^t \int_0^\sigma \theta M\alpha(\theta)(t-r)^{\alpha-1} \Im((t-r)^\alpha\theta) \mathscr{E}(r, \wp(r), (\mathcal{B}\wp)r) d\theta dr \\
& + \sum_{0<t_k<t} \mathfrak{S}_{\alpha,\nu}(t+\sigma-t_k) \mathscr{I}_k(\wp(t_k^-)) \Big) \Big\| \\
& + \Big\| \alpha t^{(1+\alpha\beta)(1-\nu)} \Big(\int_{t-\zeta}^t \int_\sigma^\infty (t-r)^{\alpha-1} \theta M_\alpha(\theta) \Im((t-r)^\alpha\theta) \mathscr{E}(r, \wp(r), (\mathcal{B}\wp)r) d\theta dr \\
& + \sum_{0<t_k<t} \mathfrak{S}_{\alpha,\nu}(t+\zeta-t_k) \mathscr{I}_k(\wp(t_k^-)) \Big) \Big\| \\
\leq & \alpha \mathfrak{C}_0 t^{(1+\alpha\beta)(1-\nu)} \Big(\int_0^t (t-r)^{-\alpha\beta-1} k(r) dr \int_0^\sigma \theta^{-\beta} M_\alpha(\theta) d\theta + \sum_{0<t_k<\sigma} \gamma_k \Big) \\
& + \alpha \mathfrak{C}_0 t^{(1+\alpha\beta)(1-\nu)} \Big(\int_{t-\zeta}^t (t-r)^{-\alpha\beta-1} k(r) dr \int_0^\infty \theta^{-\beta} M_\alpha(\theta) d\theta + \sum_{0<t_k<\zeta} \gamma_k \Big) \\
\leq & \alpha \mathfrak{C}_0 t^{(1+\alpha\beta)(1-\nu)} \Big(\int_0^t (t-r)^{-\alpha\beta-1} k(r) dr \int_0^\sigma \theta^{-\beta} M_\alpha(\theta) d\theta + \sum_{0<t_k<\eta} \gamma_k \Big) \\
& + \frac{\alpha \mathfrak{C}_0 \Gamma(1-\beta)}{\Gamma(1-\alpha\beta)} t^{(1+\alpha\beta)(1-\nu)} \Big(\int_{t-\zeta}^t (t-r)^{-\alpha\beta-1} k(r) dr + \sum_{0<t_k<\zeta} \gamma_k \Big)
\end{aligned}
$$

$\to 0$ as $\sigma \to 0$.

Therefore, $\mathcal{V}^2_{\zeta,\sigma}(t) = \{\varepsilon^2_{\zeta,\sigma} y)(t), y \in \mathfrak{B}_r(J)\}$ are arbitrarily close to $\mathcal{V}^2(t) = \{\varepsilon^2 y)(t), y \in \mathfrak{B}_r(J)\}, t > 0$. This gives the relative compactness of $\mathcal{V}^2(t), t > 0$ in $\mathscr{Y}$. Moreover, $\mathcal{V}(t) = \{\varepsilon y)(t), y \in \mathfrak{B}_r(J)\}$ is relatively compact in $\mathscr{Y}$ $\forall t \in [0, T]$. Hence, $\{\varepsilon y, y \in \mathfrak{B}_r(J)\}$ is relatively compact by using the Arzela–Ascoli Theorem.

Now ε is continuous and $\{\varepsilon y, y \in \mathfrak{B}_r(J)\}$ is relatively compact. Hence, by the Schauder fixed point theorem, $\exists$ a fixed point $y^* \in \mathfrak{B}_r(J)$ of ε. Let $\wp^*(t) = t^{(1+\alpha\beta)(\nu-1)} y^*(t)$. Then, $\wp^*$ is a mild solution of (1.1)–(1.3). $\square$

5. $\Im(t)$ Is Noncompact

We consider as follows,

Hypothesis 5 (H5). *$\exists$ a constant $k > 0$ satisfying the following*

$$\psi(\mathscr{E}(t, \mathfrak{E}_1, \mathfrak{E}_2)) \leq k\psi(\mathfrak{E}_1, \mathfrak{E}_2) \text{ for a.e } t \in [0, T]$$

for every bounded subset $\mathfrak{E}_1, \mathfrak{E}_2 \subset \mathscr{Y}$.

Theorem 6. *Let $-1 < \beta < 0, 0 < \omega < \frac{\pi}{2}$ and $\mathscr{A} \in \Theta^\beta_\omega$. Assume that (H1)–(H5) hold. Then the Cauchy problem (1.1)–(1.3) has a mild solution in $\mathfrak{B}^{\mathcal{Y}}_r(I')$ for every $u_0 \in D(\mathscr{E}^\theta)$ with $\theta > 1 + \beta$.*

Proof. By Theorems (3) and (4), we obtain $\varepsilon : \mathfrak{B}_r(I) \to \mathfrak{B}_r(I)$ as continuous, boundedm and $\{\varepsilon y : y \in \mathfrak{B}_r(I)\}$ is equicontinuous. Now, we verify that there is a subset of $\mathfrak{B}_r(I)$ such that ε is compact in it.

For any bounded set $\mathbb{P}_0 \subset \mathfrak{B}_r(I)$, set the following:

$$\varepsilon^{(1)}(\mathbb{P}_0) = \varepsilon(\mathbb{P}_0), \varepsilon^{(n)}(\mathbb{P}_0) = \varepsilon(\bar{co}(\varepsilon^{(n-1)}(\mathbb{P}_0))), n = 2, 3, \ldots$$

For any $\epsilon > 0$, we can obtain from Propositions (1)–(3), a subsequence $\{y^{(1)}_n\}^\infty_{n=1} \subset \mathbb{P}_0$ satisfying:

$$\begin{aligned}
\psi(\varepsilon^{(1)}(\mathbb{P}_0(t))) \leq & 2\psi\bigg(t^{(1+\alpha\beta)(1-\nu)}\int_0^t (t-r)^{\alpha-1}\mathfrak{Q}_\alpha(t-r)\mathscr{E}(r,\{r^{-(1+\alpha\beta)(1-\nu)}(y_n^{(1)}(r),\mathcal{B}y_n^{(1)}(r))\}_{n=1}^\infty)dr \\
& + \sum_{0<t_k<t}\gamma_k\bigg) \\
\leq & 4\mathfrak{C}_p t^{(1+\alpha\beta)(1-\nu)}\Big(\int_0^t (t-r)^{-\alpha\beta-1}\psi(\mathscr{E}(r,\{r^{-(1+\alpha\beta)(1-\nu)}(y_n^{(1)}(r),\mathcal{B}y_n^{(1)}(r))\}_{n=1}^\infty))dr \\
& + \sum_{0<t_k<t}\gamma_k\Big) \\
\leq & 4\mathfrak{C}_p k t^{(1+\alpha\beta)(1-\nu)}\psi(\mathfrak{P}_0)\Big(\int_0^t (t-r)^{-\alpha\beta-1}r^{-(1+\alpha\beta)(1-\nu)}dr \\
& + \sum_{0<t_k<t}\gamma_k\Big) \\
& = 4\mathfrak{C}_p k t^{-\alpha\beta}\psi(\mathfrak{P}_0)\Big(\frac{\Gamma(-\alpha\beta)\Gamma((-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma(-2\alpha\beta+\nu(1+\alpha\beta))} + \sum_{0<t_k<t}\gamma_k\Big).
\end{aligned}$$

From ϵ is arbitrary, we obtain the following:

$$\psi(\varepsilon^{(1)}(\mathbb{P}_0(t))) \leq 4\mathfrak{C}_p k t^{-\alpha\beta}\psi(\mathbb{P}_0)\Big(\frac{\Gamma(-\alpha\beta)\Gamma((-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma(-2\alpha\beta+\nu(1+\alpha\beta))} + \sum_{0<t_k<t}\gamma_k\Big).$$

Again, for any $\epsilon > 0$, we can obtain from Propositions (1)–(3) a subsequence $\{y_n^{(2)},\mathcal{B}y_n^{(2)}\}_{n=1}^\infty \subset \bar{co}(\varepsilon^{(1)}(\mathbb{P}_0))$ that implies the following:

$$\begin{aligned}
\psi(\varepsilon^{(2)}(\mathbb{P}_0(t))) &= \psi(\varepsilon(\bar{co}(\varepsilon^{(1)}(\mathbb{P}_0(t))))) \\
&\leq 2\psi\bigg(t^{(1+\alpha\beta)(1-\nu)}\int_0^t (t-r)^{\alpha-1}\mathcal{Q}_\alpha(t-r)\mathscr{E}(r,\{r^{-(1+\alpha\beta)(1-\nu)}(y_n^{(2)}(r),\mathcal{B}y_n^{(2)}(r))\}_{n=1}^\infty)dr \\
&+ \sum_{0<t_k<t}\gamma_k\bigg) \\
&\leq 4\mathfrak{C}_p t^{(1+\alpha\beta)(1-\nu)}\Big(\int_0^t (t-r)^{-\alpha\beta-1}\psi(\mathscr{E}(r,\{r^{-(1+\alpha\beta)(1-\nu)}(y_n^{(2)}(r),\mathcal{B}y_n^{(2)}(r))\}_{n=1}^\infty)dr \\
&+ \sum_{0<t_k<t}\gamma_k\Big) \\
&\leq 4\mathfrak{C}_p k t^{(1+\alpha\beta)(1-\nu)}\Big(\int_0^t (t-r)^{-\alpha\beta-1}\psi(r^{-(1+\alpha\beta)(1-\nu)}(\{y_n^{(2)}(r),\mathcal{B}y_n^{(2)}(r)\}_{n=1}^\infty))dr \\
&+ \sum_{0<t_k<t}\gamma_k\Big) \\
&\leq 4\mathfrak{C}_p k t^{(1+\alpha\beta)(1-\nu)}\Big(\int_0^t (t-r)^{-\alpha\beta-1}r^{-(1+\alpha\beta)(1-\nu)}\psi(\{y_n^{(2)}(r),\mathcal{B}y_n^{(2)}\}_{n=1}^\infty)dr + \sum_{0<t_k<t}\gamma_k\Big) \\
&\leq \frac{(4\mathfrak{C}_p k)^2 t^{(1+\alpha\beta)(1-\nu)}\Gamma(-\alpha\beta)\Gamma(-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma(-2\alpha\beta+\nu(1+\alpha\beta))}\psi(\mathbb{P}_0) \\
&\times\Big(\int_0^t (t-r)^{-\alpha\beta-1}r^{-(1+\alpha\beta)(1-\nu)-\alpha\beta}dr + \sum_{0<t_k<t}\gamma_k\Big) \\
&= \Big(\frac{(4\mathfrak{C}_p k)^2 t^{-2\alpha\beta}\Gamma^2(-\alpha\beta)\Gamma(-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma(-3\alpha\beta+\nu(1+\alpha\beta))} + \sum_{0<t_k<t}\gamma_k\Big)\psi(\mathbb{P}_0).
\end{aligned}$$

We can verify the following by the mathematical induction:

$$\psi(\varepsilon^{(n)}(\mathbb{P}_0(t))) \leq \frac{(4\mathfrak{C}_p k)^n t^{-n\alpha\beta}\Gamma^n(-\alpha\beta)\Gamma(-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma(-(n+1)\alpha\beta+\nu(1+\alpha\beta))}\psi(\mathbb{P}_0),\ n \in \mathbb{N}.$$

Let $M = 4\mathfrak{C}_p k T^{-\alpha\beta}\Gamma(-\alpha\beta)$. We can find $m, k \in \mathbb{N}$ big enough such that $\frac{1}{k} < \alpha\beta < \frac{1}{k-1}$ and $\frac{n+1}{k} > 2$ for $n > m$. $\Gamma(-(n+1)\alpha\beta+\nu(1+\alpha\beta)) > \Gamma(\frac{n+1}{k})$. That is:

$$\frac{(4\mathfrak{C}_p k)^n T^{-n\alpha\beta}\Gamma^n(-\alpha\beta)\Gamma(-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma(-(n+1)\alpha\beta+\nu(1+\alpha\beta))} < \frac{(4\mathfrak{C}_p k)^n T^{-n\alpha\beta}\Gamma^n(-\alpha\beta)\Gamma(-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma\left(\frac{n+1}{k}\right)}.$$

Replace $(n+1)$ by $(j+1)k$. Then, the R.H.S of the inequality given above becomes the following:

$$\frac{M^{(j+1)k-1}\Gamma(-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma(j+1)} = \frac{(M^k)^j M^{k-1}\Gamma(-\alpha\beta+\nu(1+\alpha\beta))}{j!} \to 0 \text{ as } j \to \infty.$$

Therefore, there exists a constant $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\frac{(4\mathfrak{C}_p k^n t^{-n\alpha\beta}\Gamma^n(-\alpha\beta)\Gamma(-\alpha\beta+\nu(1+\alpha\beta)}{\Gamma(-(n+1)\alpha\beta+\nu(1+\alpha\beta))} \leq & \frac{(4\mathfrak{C}_p k)^{n_0} T^{-n_0\alpha\beta}\Gamma^{n_0}(-\alpha\beta)\Gamma(-\alpha\beta+\nu(1+\alpha\beta))}{\Gamma(-(n_0+1)\alpha\beta+\nu(1+\alpha\beta))} \\ = & p < 1.\end{aligned}$$

Now:

$$\psi(\varepsilon^{(n_0)}(\mathbb{P}_0(t))) \leq p\psi(\mathbb{P}_0).$$

From $\varepsilon^{(n_0)}(\mathbb{P}_0(t))$ is bounded and equicontinuous, applying Proposition (1), we obtain the following:

$$\psi(\varepsilon^{(n_0)}(\mathbb{P}_0)) = \max_{t\in[0,T]} \psi(\varepsilon^{n_0}(\mathbb{P}_0(t))).$$

Hence:

$$\psi(\varepsilon^{n_0}(\mathbb{P}_0)) \leq p\psi(\mathbb{P}_0),$$

where $p < 1$. Now applying a similar technique as applied in Theorem 4.2 [14], we obtain a nonempty, convex, and compact subset $\mathfrak{C}$ in $\mathfrak{B}_r(J)$ with $\varepsilon(\mathfrak{C}) \subset \mathfrak{C}$ and $\varepsilon(\mathfrak{C})$ is compact. By applying the Schauder fixed point theorem, we obtain a fixed point y^* in $\mathfrak{B}_r(J)$ of ε. Let $\wp^*(t) = t^{(1+\alpha\beta)(\nu-1)}y^*(t)$. Then, $\wp^*(t)$ is a mild solution of (1.1)–(1.3). □

6. Example

We consider the following impulsive system:

$$\begin{aligned}\mathfrak{D}_{0+}^{\alpha,\nu}\wp(t,x) - \partial_x^2\wp(t,x) =& \mathscr{E}(t,\wp(t,x),\mathcal{B}\wp(t,x)) \ \ t \in [0,T],\ \ x \in [0,a] \\ \wp(t,0) = \wp(t,a) =& 0 \ \text{ on } \ t \in [0,T] \\ \mathcal{I}_{0+}^{(1-\alpha)(1-\nu)}\wp(0,x) =& \sum_{k=1}^{m} c_k\wp(t_k,x),\ \ x \in [0,a] \\ \Delta\wp|_{t=1/2} =& \mathscr{I}_1(\wp(\frac{1}{2}^-)),\end{aligned} \tag{8}$$

in Banach space $\mathscr{Y} = C^{\alpha}([0,a])(0 < \alpha < 1)$ of all Holder continuous functions, where $\alpha = \frac{1}{4}$, $\nu = \frac{1}{2}$, $\mathscr{E}(t,\wp,\mathcal{B}\wp) = t^{-\frac{1}{5}}\cos^2\wp$, $c_k \in \mathbb{R}, k = 1,2,...,m$, such that $\sum_{k=1}^{m}|c_k| < \frac{1}{M_s}$. Here, we can convert the above problem (1.1–1.3) in abstract form as follows:

$$\begin{aligned} \mathfrak{D}^{\alpha,\nu}\wp(t) + \mathscr{A}\wp(t) =& \mathscr{E}(t,\wp(t),\int_0^t \varsigma(t,s)\vartheta(s,\wp(s))ds) \quad t \in (0,T] = \mathcal{J} \\ \Delta\wp|_{t=t_k} =& \mathscr{I}_k(\wp(t_k^-)), k = 1,2,3,...m \\ I_{0+}^{(1-\alpha)(1-\nu)}\wp(0) =& \sum_{k=1}^{m} c_k\wp(t_k,x). \end{aligned} \tag{9}$$

Here, $\mathscr{A} = -\partial_x^2$ with $\mathfrak{D}(\mathscr{A}) = \{\wp \in C^{2+\alpha}([0,a])$ such that $\wp(t,0) = 0 = \wp(t,a)\}$. It follows from the work in [20] $\exists$ constants $\delta, \epsilon > 0$, such that $\mathscr{A} + \delta \in \odot_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}-1}(\mathscr{Y})$. To verify the compactness of semigroup $\Im(t)$, it is enough to prove that $\Re(\alpha, -(\mathscr{A}+\delta)$ is compact for every $\alpha > 0$ (see Lemma 4.66 [15]). Since $\mathfrak{D}(\mathscr{A} \subset C^{2+\alpha}([0,a]))$ and $C^{2+\alpha}([0,a])$ are compactly embedded in $C^{\alpha}([0,a])$, the compactness of the resolvent operator follows for every $\alpha > 0$. We choose $l(t) = t^{-\frac{1}{5}}$:

$$r = \sup_{[0,T]}(t^{(1+\alpha\beta)(1-\nu)}\|\mathfrak{S}_{\alpha,\nu}(t)u_0\|) + \frac{T^{\frac{17}{20}}\Gamma(-\frac{\beta}{4})\Gamma(\frac{4}{5})}{\Gamma(\frac{4}{5}-\frac{\beta}{4})}.$$

Then, the Hypotheses (H1)–(H5) are satisfied. According to Theorem 5, the Problem (6.1) has a mild solution in $\mathfrak{B}_r^{\mathfrak{Y}}((0,T])$.

7. Conclusions

In this paper, we proved the mild solutions of Hilfer fractional integro-differential equation with impulsive almost sectorial operator, by applying the fixed point theory. We will find to investigate stability of similar problem in our future research work.

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