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Applying Set Theory

Saharon Shelah [1,2]

[1] Einstein Institute of Mathematics Edmond J. Safra Campus, Givat Ram The Hebrew University of Jerusalem, Jerusalem 9190401, Israel; shelah@math.huji.ac.il
[2] Department of Mathematics Hill Center—Busch Campus Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

Abstract: We prove some results in set theory as applied to general topology and model theory. In particular, we study $\aleph_1$-collectionwise Hausdorff, Chang Conjecture for logics with Malitz-Magidor quantifiers and monadic logic of the real line by odd/even Cantor sets.

Keywords: set theory; model theory; general topology; monadic logic

MSC: 03C55; 03C85; 03E75; 03E35

1. Introduction

In §2 we prove a result in general topology saying: if $\diamondsuit^*_{\aleph_1}$ then any normal space is $\aleph_1 - \mathrm{CWH}$ (= collectionwise Hausdorff); done independently of and in parallel to Fleisner and Alan D. Taylor.

In §3 we prove the Chang Conjecture for Magidor-Malitz Quantifiers. A recent related piece of work is [1].

In §4 we prove the Monadic Theory of the tree ${}^{\omega>}2$ is complicated under a quite weak set theoretic assumption. Earlier [2] proved this (i.e., the result on the monadic logic) assuming CH or at least a consequence of it.

The present note was circulated in the Spring of 1979 in a collection; it include each of the sections (as well as other preprints) but those three were not published. However, Ref. [3]. have results related to Section 3; in particular it was conjectured there (in Remark 2.15) that there are two non-principal ultrafilters of ω with no common lower bound in the Rudin Keisler order; a conjecure which had been refuted in [4].

Later, Gurevich-Shelah [5] proved undecidability in ZFC, with further developments then more in Shelah [6], still the older proof gives information not covered by them. For more see [4,7,8].

The results are old, still in particular, §2 gives a direct proof of the result compared to others and §4 gives a considerably more transparent easier proof of the later result of [5] but with an extra weak hypothesis.

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2. A Note in General Topology If $\diamondsuit^*_{\aleph_1}$ Then Any Normal Space is $\aleph_1 - \mathrm{CWH}$ (=Collectionwise HAUSDORFF)

The normal Moore space problem has been a major theme in general topology, see the recent survey Dow-Tall [9]. In this connection, Fleissner ([10], p. 6) proved: $(\mathbf{V} = \mathbf{L})$ every normal first countable (topological) space is CWH (CWH means collectionwise Hausdorff). He used a strengthening of diamond. The author proved Fleissner strengthening (for $\aleph_1$) does not follow from $\mathrm{ZFC} + \diamondsuit^+_{\aleph_1}$ (see [11], Th. 5, p. 31). Here we prove nevertheless $\diamondsuit^*_{\aleph_1}$ implies every normal first countable space is $\aleph_1 - \mathrm{CWH}$.

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The central idea of the proof in §2 is inspired by one key idea in Fleissner [10]. Fleissner implicitly used a stronger combinatorial principle $\diamondsuit_{SS}$. In 1979, the author and independently both Fleissner and Alan D. Taylor all saw (as mentioned in [12,13] that a weaker principle, $\diamondsuit^*_{\omega_1}$, would suffice. Later Smith and Szeptycki [12] derive better results. On more recent results on diamond and strong negation, see [14] and references there.

Definition 1. *Below δ always denotes a limit ordinal $(< \omega_1)$. For transparency, below we refer to the following equivalent form of $\diamondsuit^*_{\omega_1}$.*

Definition 2. *Let $\diamondsuit^*_{\aleph_1}$ mean that there exist a sequence $\langle \mathbf{g}_\delta : \delta < \omega_1 \rangle$ where $\mathbf{g}_\delta = \langle \bar{g}^{\delta,k} : k < \omega \rangle$ is of the form $\bar{g}^{\delta,k} = \langle g_n^{\delta,k} : n < \omega \rangle$, where $g_n^{\delta,k} : \delta \to \omega$ has the property that, for any sequence $\bar{g} = \langle g_n : n < \omega \rangle$ with $g_n : \{\delta : \delta < \omega_1\} \to \omega$, there is a club (closed unbounded) set $C \subseteq \omega_1$ such that, for each $\gamma \in C$, there is $k = k(\gamma) \in \omega$ with*

$$\bar{g} \restriction \gamma := \langle g_n \restriction \gamma : n < \omega \rangle = \bar{g}^{\gamma,k} = \langle g_n^{\gamma,k} : n < \omega \rangle.$$

Theorem 1. *Assume $\diamondsuit^*_{\aleph_1}$. If X is Hausdorff first countable normal and $|X| = \aleph_1$ then X is CWH.*

Proof. Let $\langle \mathbf{g}_\delta : \delta < \omega_1 \rangle$ be as in Definition 2.

Without loss of generality $X_* = \{\delta : \delta < \omega_1\} \subseteq X$ and X_* is closed discrete in the space X. Let $U^\delta_n (n < \omega)$ be a basis of open neighborhoods of δ (for $\delta < \omega_1$). We shall define by induction on $\alpha < \omega_1$ a limit ordinal $\gamma_\alpha < \omega_1$ and $\langle f_n(\gamma) : n < \omega, \gamma < \gamma_\alpha \rangle$ such that γ_α is increasing continuous with α and $\gamma_0 = 0$. For $\alpha = 0$ choose $\gamma_\alpha = \omega; f_n(\gamma) = 0$. For limit α let γ_α be $\cup\{\gamma_\beta : \beta < \alpha\}$. For $\alpha = \beta + 1$ if $\gamma_\alpha > \alpha$ then we let $\gamma_\alpha = \gamma_\beta + \omega$ and let $f_n(\gamma) = 0$ for $\gamma \in [\gamma_\beta, \gamma_\alpha)$. Finally assume that $\alpha = \delta^*, \gamma_{\delta^*} = \delta^*$ so $\delta^* \in X_*$.

We have chosen above the functions $\langle g_n^{\delta^*,k} : n < \omega, k < \omega \rangle$ with $g_n^{\delta^*,k} : \delta^* \to \omega$; now for each $n, k < \omega$ let $A_\ell^{\delta^*,n,k} = \cup\{U^\delta_{g_n^{\delta^*,k}(\delta)} : \delta < \delta^*, f_n(\delta) = \ell\}$ (for $n < \omega, \ell < 2$). Call $k < \omega$ <u>good</u> for δ^* when for infinitely many (pairs) n, ℓ we have

$$B_\ell^{\delta^*,n,k} := \mathrm{cl}(A_\ell^{\delta^*,n,k}) \cap (X_* \setminus \delta^*) \neq \emptyset.$$

We let $\gamma_\alpha = \gamma_{\delta^*+1} = \min\{\delta : \delta > \delta^*$ and if $\ell < 2$ and $n, k < \omega$ and $B_\ell^{\delta^*,n,k} \neq \emptyset$ then $(\delta^*, \delta) \cap B_\ell^{\delta^*,n,k} \neq \emptyset\}$.

Now we choose $f_n \restriction [\delta^*, \gamma_\alpha)$ such that for any k good for δ^*, for some $n, \ell, \delta \geq \delta^*$ we have

$$f_n(\delta) = 1 - \ell(\text{ for } \delta \in \mathrm{cl}(A_\ell^{\delta^*,n,k})).$$

Then we complete arbitrarily the f_n so that its domain is γ_α.

Thus we have defined $f_n(n < \omega)$ with $f_n : \omega_1 \to 2$. For each n the sets $f_n^{-1}\{1\} \cap X_*, f_n^{-1}\{0\} \cap X_*$ form a partition of X_*, both are closed and discrete subsets of X. But X is normal. So there are functions $g_n : X_* \to \omega$ for $n < \omega$ so that letting for $\ell = 0, 1$

$$A^n_\ell = \cup\{U^\delta_{g_n(\delta)} : \delta \in X_*, f_n(\delta) = \ell\}$$

we have

$$A^n_0 \cap A^n_1 = \emptyset.$$

Let g_n^+ be any function from ω_1 to ω extending g_n. For some closed unbounded set $C \subseteq X_*$ we have: $\delta^* \in C \Rightarrow (\exists k)(\langle g_n^+ \restriction \delta^* : n < \omega \rangle = \langle g_n^{\delta^*,k} : n < \omega \rangle)$. Let the first such k be denoted $k(\delta^*)$. Without loss of generality every $\delta^* \in C$ satisfy $\gamma_{\delta^*} = \gamma$ hence if $\delta^* \in C \wedge n < \omega \wedge k < \omega \wedge \ell < 2$ and $B_\ell^{\delta^*,n,k} = \mathrm{cl}(A_\ell^{\delta^*,n,k}) \cap (X_* \setminus \delta^*) \neq \emptyset$ then $\min(B_\ell^{\delta^*,n,k}) < \min(C \setminus (\delta^* + 1))$.

For $\delta^* \in C$ now $k(\delta^*)$ cannot be good for δ^*, (by the definition).

Now for at least one n (in fact, for infinitely many n-s) we have $\mathrm{cl}(A^n_\ell | \delta^*) \cap (X_* \setminus \delta^*) = \emptyset$ for $\ell \in \{0,1\}$, let $n(\delta^*)$ be the first such n.

Define

$$B_n = \{\delta : \text{ for some } \delta^* \in C \cup \{0\} \text{ we have } \delta^* \leq \delta < \min(C \backslash \delta) \text{ and } n = \max\{n(\delta^*), n(\delta)\}$$

Now $\bigcup\limits_n (g_n \restriction B_n)$ almost exhibits X_* has the right sequence of neighborhoods. Now we can deal with each B_n separately (just choose $\mathscr{U}_n$ by induction on n such that $\mathscr{U}_n$ is open, $\mathscr{U}_n \cap X_* = B_n$ and $\mathscr{U}_n \subseteq X \backslash c\ell(\bigcup\limits_{\ell<n} \mathscr{U}_\ell)$, possible by normality).

By dealing as follows with each interval $[\delta^*, \min(C \backslash (\delta^* + 1))$ for $\delta^* \in C \cup \{0\}$ we have $U^\delta_{g_n(\delta)}(\delta \in B_n)$ as required.

That is, for $\gamma \in C \cup \{0\}$ with γ^+ its successor in C, choose a (countable) family of pairwise disjoint open sets $\mathscr{U}_\gamma(\beta)$ for $\beta \in X_* \wedge \gamma \leq \beta < \gamma^+$, with $\beta \in \mathscr{U}_\gamma(\beta)$, this is possible as in the choice of the $\mathscr{U}_n$'s.

Now for $\beta \in X_*$ we let $W_\beta = \mathscr{U}_{n(\beta)} \cap \mathscr{U}_{\gamma(\beta)}(\beta) \cap \mathscr{U}^\beta_{g_{n(\beta)}(\beta)}$ where:

- $\gamma(\gamma) = \max(C \cap (\beta + 1))$
- $m(\beta) = \max\{n(\delta^*), n((\delta^*)^+) : \delta^* = \max(C \cap \beta) \leq \beta < (\delta^*)^+\}$

Finally $\langle W_\beta : \beta \in X_* \rangle$ is a sequence of pairwise disjoint open sets of X with $\beta \in X_* \Rightarrow \beta \in W_\beta$, so we are done. □

Remark 1. *As in [10] it suffices to assume every point in the space has a neighborhood basis of cardinality* $\aleph_1$.

3. Chang Conjecture for Magidor-Malitz Quantifiers

Silver (see [15]) had proved the consistency of Chang conjecture, i.e.,

$\oplus$ any model M with universe $\aleph_2$ (and countable signature = vocabulary) τ, has an elementary submodel N, $(\|N\| = \aleph_1, |N \cap \omega_1| = \aleph_0)$

Silver did this by starting with a model $\mathbf{V}$ with κ Ramsey (in fact, something weaker suffices), forcing MA and then collapsing κ to $\aleph_2$ by $\mathbb{P}^\kappa_{\mathrm{Set}} = \{f : \mathrm{Dom}(f) \subseteq \{\mu : \aleph_1 < \mu < \kappa, \mu \text{ a cardinal}\}$ has cardinality $\leq \aleph_1$, and for some $\alpha < \omega_1$, $(\forall \mu \in \mathrm{Dom} f)(f(\mu)$ is a function from α to $\mu)\}$. See also Koszmider [16] for a topological application.

We can ask whether this submodel N can inherit more properties from M.

Definition 3. *Let us define a (technical variant of) Magidor-Malitz quantifiers.* $M \models (Q^n \bar{x})\varphi(x_1, \ldots, x_n)$ *means that there is a set* $A \subseteq M$, A is of cardinality $\|M\|$ *such that* $(\forall a_1 \ldots a_n \in A)\varphi(a_1 \ldots a_n)$.

The result is that:

Claim 1. *In* $\oplus$ *above, we can have* N *an elementary submodel of* M *even for the logic* $\mathbb{L}(Q^0, Q^1, \ldots)_{n<\omega}$. *So e.g., Suslinity of trees is preserved.*

For this we need the following.

Definition 4. *Call a forcing* $\mathbb{P}$ *suitable* <u>*when*</u> *for any sequence* $\langle p_i : i < \omega_1 \rangle$ *of members of* $\mathbb{P}$ *there is a set* $\mathscr{U} \subseteq \omega_1$ *of cardinality* $\aleph_1$ *such that: for any finite* $u \subseteq \mathscr{U}$ *there is* $q \in \mathbb{P}$ *such that* $\bigwedge\limits_{i \in u} q \geq p_i$.

Claim 2. *Forcing by suitable forcing preserves satisfaction of sentences of Magidor-Malitz quantifiers for models of power* $\aleph_1$.

See ([17], 1.5–13, p. 34).

Claim 3. *There is a suitable forcing $\mathbb{P}$, $|\mathbb{P}| = 2^{\aleph_1}$, such that in $\mathbf{V}^{\mathbb{P}}$: if $\mathbb{Q}$ is a suitable forcing of power $\aleph_1$, $\underset{\sim}{M}$ a $\mathbb{Q}$-name of a model of power $\aleph_1$, in a language $L \in \mathbf{V}$, universe $\aleph_1$, then there is a directed $\mathbf{G} \subseteq \mathbb{P}$, which determines $\underset{\sim}{M}$ as M and such that for any sentence ψ from the $\mathbb{L}(Q^0, Q^1, \ldots)$ (the variant of Magidor-Malitz logic from Definition 3)*

$$\Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{M} \models \psi\text{” implies } M \models \psi.$$

Proof. Just iterate the required forcing notions, with direct limit (i.e., finite support) and remember it is known that suitability is preserved under iteration, i.e., Claim 2.

Proof of Main result Claim 1:

Do as Silver, start with $\mathbf{V} \models$ “κ Ramsey”, force by $\mathbb{P}$ from Claim 3, and then use $\mathbb{P}^{\kappa}_{\text{Set}}$. The rest is as in his proof.

But we have to choose G as in Definition 3, and notice that more is reflected to the submodel he uses, (just check the definition carefully) and work a little, and remember that $\aleph_1$-complete forcing preserves satisfaction of sentences in $\mathbb{L}(Q^0, \ldots)$ (and $\mathbb{P}^{\kappa}_{\text{Set}}$ is $\aleph_1$-complete). □

4. A Remark on the Monadic Theory of Order

In [2] we prove the undecidability of the monadic theory of (the order) R, assuming CH, or the weaker Baire-like hypothesis that $\mathbb{R}$ is not the union of fewer than continuum sets of first category sets. This condition is weaken below to ‘‘not (St) at least for T where a closely related theory is the monadic theory T of $M = ({}^{\omega\geq}2, \lhd)$ where ${}^{\omega\geq}2$ is the set of sequences of zeros and ones of length $\leq \omega$, $\lhd$ is the (partial) order of being initial segment. T is closely related to Rabin’s monadic theory of $({}^{\omega>}2, \lhd)$ which he proved decidable [18]. We prove here that the statement “$\neg$(St)” implies the undecidability of T (and all results on its complexity, see [2] and the paper of Gurevich on the subject) but it was not clear (at that time) whether (St) is consistent with ZFC.

Definition 5. *A Cantor [set] C is a non-empty subset of ${}^{\omega\geq}2$ with the properties*

(a) *C is closed under initial segments,*

(b) *if η has length ω then $\eta \in C \equiv (\forall n)(\eta{\restriction}n \in C)$,*

(c) *$\eta \in C \cap {}^{\omega>}2$ implies $\eta \frown \langle 0\rangle \in C$ or $\eta \frown \langle 1\rangle \in C$,*

(d) *for every $\eta \in C \cap {}^{\omega>}2$, there is $\nu \in C \cap {}^{\omega>}2$ $\eta \lhd \nu$ and $\nu \frown \langle 0\rangle \in C, \nu \frown \langle 1\rangle \in C$.*

Definition 6. *(1) For a Cantor C, the set of its splitting points is* $\mathrm{Sp}(C) = \{\eta \in {}^{\omega>}2 : \eta \frown \langle 0\rangle \in C \text{ and } \eta \frown \langle 1\rangle \in C\}$.
(2) For a set $A \subseteq {}^{\omega>}2$, C is an A-Cantor, if $\mathrm{Sp}(C) \subseteq A$.
(3) For a set $S \subseteq \omega$, C is called an S-Cantor, if

$$\mathrm{Sp}(C) \subseteq \bigcup_{n\in S} {}^{n}2.$$

(4) An odd Cantor is one that is an $\{2n+1 : n < \omega\}$-Cantor. An even Cantor is one that is an $\{2n : n < \omega\}$-Cantor.

Now the statement we speak about is

Definition 7. *Let* (St) *mean: the set ${}^{\omega}2$ is the union of less than $2^{\aleph_0}$ Cantors each of them odd or even.*

Problem 1. *Is* (St) *consistent with* ZFC*?; solved in [4].*

Claim 4. *Let* $\{C_i : i < \alpha\}$ *be a family of odd and even Cantors,* ${}^{\omega\geq}2 = \bigcup\limits_{i<\alpha} C_i$. *Then* $2^{\aleph_0} \leq |\alpha|^+$.

Proof. Let for $\eta, \nu \in {}^{\omega}2, \rho = p(\eta, \nu)$ be defined by $\rho(2n) = \eta(n), \rho(2n+1) = \nu(n)$, and then let $\eta = \mathrm{pr}_1(\rho), \nu = \mathrm{pr}_2(\rho)$.

Now for any even C, and η there is at most one ν such that $p(\eta, \nu) \in C$; why? if ν_0, ν_1 are such ν's, $\rho_\ell = p(\eta, \nu_\ell)$, then, by the definition of $p(-,-)$, for some $m < \omega, \rho_0 \restriction m = \rho_1 \restriction m, \rho_0(m) \neq \rho_1(m)$. If $m = 2n$ then $\rho_\ell(m) = \rho_\ell(2n) = \eta(n)$ so they are equal, contradiction. If $m = 2n+1$, then ($\rho_0(m) \neq \rho_1(m)$ and) $\rho_0 \restriction m = \rho_1 \restriction m$ is a splitting point of C, however m is odd and C is an even Cantor, a contradiction. So really there is at most one ν, and let $\varrho(\eta, C)$ be the unique ν $p(\eta, \nu) \in C$ if there is one and η otherwise.

Similarly if C is odd and $\eta \in {}^{\omega}2$, then for at most one $\nu, p(\nu, \eta) \in C$ and let $\varrho(\eta, C)$ be ν for this η, and let $\varrho(\eta, C) = \eta$ otherwise. Our definition of the function ϱ does not contradict, because no Cantor is odd and even.

Let for $\eta \in {}^{\omega}2, \mathrm{Dp}(\eta) = \{\varrho(\eta, C_i) : i < \alpha\}$. So clearly $\mathrm{Dp}(\eta)$ is a subset of ${}^{\omega}2$ of cardinality $\leq |\alpha|$.

Now if $\eta, \nu \in {}^{\omega}2$, by hypothesis $\rho = p(\eta, \nu)$ belongs to some C_i. If C_i is odd this implies $\nu = \varrho(\eta, C_i) \in \mathrm{Dp}(\eta)$ and if C_i is even this implies $\eta = \varrho(C_i, \nu) \in \mathrm{Dp}(\nu)$.

If $|\alpha|^+ < 2^{\aleph_0}$ we can easily find a counterexample. □

Definition 8. *Assume* $\neg(\mathrm{St})$.

If $S_n \subseteq \omega$ *are infinite pairwise almost disjoint (for* $n \in \{0,1,2\}$*),* $C_i (i < \alpha < 2^{\aleph_0})$ *are Cantors, each an* S_n*-Cantor for some* n *(or just an* $S_n \cup S_2$*-Contor for some* n*),* C *is a Cantor for every* $\eta \in C \cap {}^{\omega>}2, \ell \in \{0,1\}$*, there is* ν*, such that* $\eta \lhd \nu \in \mathrm{Sp}(C), \nu \in \bigcup\limits_{k \in S_\ell} {}^{k}2$.

Then there is $\eta \in C \setminus \bigcup\limits_{i<\alpha} C_i \setminus {}^{\omega>}2$.

(2) Similarly for $S_n \subseteq {}^{\omega>}2$

Proof. (1) We can find a Cantor $C' \subseteq C$, and $0 = k(0) < k(1) < \ldots < k(n) < \ldots < \omega$:

$(*)$ if $\eta \in {}^{k(n)}2$, then there are exactly two $\nu \in {}^{k(n+1)}2 \cap C', \eta \lhd \nu$, and if they are ν_1, ν_2 and $m := \min\{m : \nu_1(m) \neq \nu_2(m)\}$ then $m \in S_0 \cup S_1$ but $\notin S_2 \cup (S_0, \cap S_1)$. Moreover $m \in S_0$ if n is even.

Let $A = \{\eta \restriction k(n) : n < \omega, \eta \in C'\}$, so $A \subseteq C'$. Clearly there is an isomorphism f, of the models $({}^{\omega\geq}2, \lhd), (C', \lhd)$.

Let $C'_i = \{f(\eta) : \eta \in C', \eta \in C_i\}$, it is easy to check that each C'_i is countable, or the union of a countable set and a Cantor which is odd or is even.

We can find odd Cantor $C'_i (\alpha \leq i < \alpha\omega)$ all countable sets we mentioned are covered by them. Now by—"$\neg(\mathrm{St})$" there is $\eta \in {}^{\omega\geq}2$ $\eta \notin \bigcup\limits_{i<\alpha\omega} C'_i$ (as $\alpha\omega < 2^{\aleph_0}$) and $f^{-1}(\eta)$ is the required elements.

(2) Similarly. □

(Now we have added Claim 5 and Definition 9 in 2019).

Claim 5. *Assume* $\neg(\mathrm{St})$. *(1) The monadic theory* T *is undecidable.*

Proof. Below let P vary on Cantors and note that we can repeat the proof of [2] with small adaptation (and prove T is undecidable). That is, the change needed is in ([2], (7.4)) which has a set-theoretic hypothesis (CH or the Baire-like hypothesis mentioned above), so we repeat it with the needed changes below. □

Definition 9. *Assume* $\neg(\mathrm{St})$ *and let* J *be an index-set of cardinality at most* $2^{\aleph_0}$*, (1) Assume the* $D_i (i \in J)$ *countable dense subsets of* ${}^{\omega>}2$ *and* $D = \bigcup\limits_{i \in J} D_i$ *and* $\bar{D} = \langle D_i : i \in J \rangle$ *(The main case*

is that the D_i-s are pairwise almost disjoint). Then there is $Q \subseteq {}^{\omega}2 \setminus D$, $Q = Q[\bar{D}]$ such that for every Cantor P :

(A) if $P \cap D \subseteq D_i (i \in J)$ and D_i is dense in P then $|P \cap Q| < 2^{\aleph_0}$

(B) if for some $i \in J$ the sets $P \cap D_i, P \setminus D_i$ are dense in P then $P \cap Q \neq \emptyset$.

(2) For some such $\bar{D}$ we can strengthen clause (B) above to

(B) if P is a Cantor and for every $i \in J$ the set $D_i \cap P$ is nowhere-dense in P then for every, dense subsets D_1^*, D_2^* of $P \cap D$ we can find $D_1^\bullet \subseteq D_1^*, D_2^\bullet \subseteq D_2^*$ satisfying for any P we have: is $P \cap D_1^\bullet, P \cap D_2^\bullet$ are dense in P then $P \cap Q \neq \emptyset$.

Proof. (1) Let $\{P_\alpha : 0 < \alpha < 2^{\aleph_0}\}$ be any enumeration of the Cantor sets. We define $x_\alpha, \alpha < 2^{\aleph_0}$ by induction on α.

For $\alpha = 0, x_\alpha \in \mathbb{R}$ is arbitrary.

For any $\alpha > 0$, if P_α does not satisfy the assumptions of (B) then let $x_\alpha = x_0$ and if P satisfies the assumptions of (B) (hence in particular D is dense in P) let $x_\alpha \in P_\alpha - \bigcup\{P_\beta : \beta < \alpha, (\exists i \in J)(P_\beta \cap D \subseteq D_i$ and D is dense in $P_\beta)\} = D$.

This is possible; to prove this let $\mathscr{U} = \{\beta < \alpha :$ there is $i \in J$ such that $P_\beta \cap D \subseteq D_i\}$ and for $\beta \in \mathscr{U}$ let $i_\beta \in J$ be such that $P_\beta \subseteq D_{i_\beta}$ Let $i(*) \in J$ be such that $P \cap D_{i(*)}, P \setminus D_{i(*)}$ are dense in P. Now we apply Definition 8(2), (or Definition 8(1) if we restrict the D_i-s, does not matter).

So by (St) and the hypothesis $|P_\alpha \cap D| < 2^{\aleph_0}$ there exists such x_α.

Now let $Q = \{x_\alpha : \alpha < 2^{\aleph_0}\}$. If P satisfies the assumptions of (A), then $P \in \{P_\alpha : 0 < \alpha < 2^{\aleph_0}\}$. So for some $\alpha, P = P_\alpha$, hence $P \cap D \subseteq \{x_\beta : \beta < \alpha\}$, so $|P \cap D| < 2^{\aleph_0}$. If $P = P_\alpha$ satisfies the assumption of (B) then $x_\alpha \in P_\alpha, x_\alpha \in Q$, hence $P_\alpha \cap Q \neq \emptyset$.

(2) Similarly.

So we have proved the lemma. □

Definition 10. *We can interpret the monadic theory of $(R, <)$ in T, but the converse was not clear at the time, but looking at it again probably we can carry the proof for $\mathbb{R}$.*

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