

Article

Geometric Properties of Some Generalized Mathieu Power Series inside the Unit Disk

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† Živorad Tomovski was supported by DAAD, during his visit to the Department of Physics at the University of Potsdam in Germany from 15 June 2021 to 15 September 2021 to collaborate with Ralf Metzler.

Abstract: We consider two parametric families of special functions: One is defined by a power series generalizing the classical Mathieu series, and the other one is a generalized Mathieu type power series involving factorials in its coefficients. Using criteria due to Fejér and Ozaki, we provide sufficient conditions for these functions to be close-to-convex or starlike inside the unit disk, and thus univalent. One of our proofs is assisted by symbolic computation.

Keywords: univalent function; starlike function; close-to-convex function; generalized Mathieu-type series

MSC: 33E20; 40A10; 30C45



Citation: Tomovski, Ž.; Gerhold, S.; Bansal, D.; Soni, A. Geometric Properties of Some Generalized Mathieu Power Series inside the Unit Disk. *Axioms* **2022**, *11*, 568. <https://doi.org/10.3390/axioms11100568>

Academic Editor: Mircea Merca

Received: 2 September 2022

Accepted: 8 October 2022

Published: 19 October 2022

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1. Introduction and Preliminaries

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and $\mathcal{A}$ denote the class of all analytic functions inside the unit disk $\mathbb{U}$, normalized by the conditions $f(0) = 0$, $f'(0) = 1$. We denote by $\mathcal{S}$ the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$, i.e.,

$$\mathcal{S} = \{f \in \mathcal{A} \mid f \text{ is one-to-one in } \mathbb{U}\}.$$

A set $\Omega \subseteq \mathbb{C}$ containing the origin is called starlike with respect to the origin, if, for any point $z \in \Omega$, the line segment from the origin to z lies in the interior of Ω . A function $f \in \mathcal{A}$ that maps the unit disk $\mathbb{U}$ onto a starlike domain is called starlike function, and the class of such functions is denoted by $\mathcal{S}^*$. Analytically, starlike functions are characterized by the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{U}.$$

An analytic function $f \in \mathcal{A}$ is called close-to-convex if $\Re\{e^{i\theta}zf'(z)/g(z)\} > 0$, $z \in \mathbb{U}$, for some $\theta \in \mathbb{R}$ and for some starlike function $g \in \mathcal{S}^*$. By taking $\theta = 0$ and $g = f$, it is clear that every starlike function is close-to-convex. However, the converse is not true. The Noshiro–Warschawski theorem implies that close-to-convex functions are univalent in $\mathbb{U}$, but the converse is not true in general. Thus, it is convenient to show that f is close-to-convex in order to check the univalence of f . There is also a geometric version of the definition: An analytic function f is called close-to-convex in $\mathbb{U}$, if the complement of $f(\mathbb{U})$ can be written as the union of non-intersecting half-lines. These classes are studied in detail in the literature (see the books of Duren [1] and Goodman [2]). Verifying these geometric

properties for various special functions is an active research area, we mention [3–5] and the references therein.

The following infinite series was named after Émile Leonard Mathieu (1835–1890), who investigated it in his 1890 monograph [6] on elasticity of solid bodies:

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2}, \quad r \in \mathbb{R}_+. \quad (1)$$

An integral representation of the series $S(r)$ is given by (see [7])

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{t \sin(rt)}{e^t - 1} dt.$$

The generalized Mathieu type power series or generalized Mathieu type power series of fractional order μ is defined by (see [8]):

$$F_{\mu}(r; z) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^{\mu+1}} z^n, \quad \mu, r \in \mathbb{R}_+, |z| < 1.$$

In 1998, Alzer et al. [9] obtained the following bounds for Mathieu’s series (1):

$$\frac{1}{r^2 + \frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2 + 1/6},$$

where ζ denotes the zeta function. One can refer to [9–16] about the study of Mathieu’s series and its generalizations. In particular, series of the form

$$S_{\mu}^{(\alpha, \beta)}(r; \{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{2a_n^{\beta}}{(a_n^{\alpha} + r^2)^{\mu}},$$

for certain parametric choices of the sequence $\{a_n\}$, have been studied extensively. Here, the range of the real parameters α, β depends on the growth order of the sequence $\{a_n\}$. Such series give rise to a large number of different integral representations, some of which involve the Fox–Wright function, as well as the Riemann zeta function (see [17–19]). While a large amount of research has been devoted to integral representations, inequalities and asymptotics, results on geometric properties of such series are scarce. In the present investigation, our aim is to study geometric properties of generalized Mathieu type power series. As mentioned above, both starlikeness and close-to-convexity are characterized by geometric properties of the range of $\mathbb{U}$. It is obvious that $z \mapsto F_{\mu}(r; z) \notin \mathcal{A}$, so we use the following normalization:

$$\begin{aligned} \mathbb{F}_{\mu}(r; z) &= \frac{(r^2 + 1)^{\mu+1}}{2} \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^{\mu+1}} z^n \\ &= z + \sum_{n=2}^{\infty} \frac{n(r^2 + 1)^{\mu+1}}{(n^2 + r^2)^{\mu+1}} z^n. \end{aligned} \quad (2)$$

For $\mu = 1$, $\mathbb{F}_{\mu}(r; z)$ becomes a Mathieu type power series (see [8]). Geometric properties of the series $\mathbb{F}_1(r; z)$ have already been discussed in [20]; there, the notation $\mathbb{S}(r; z)$ is used for $\mathbb{F}_1(r; z)$. Recently, Gerhold et al. [21] considered the following generalized Mathieu type power series:

$$Q_{\mu}(r; z) = \sum_{n=1}^{\infty} \frac{2n!}{((n!)^2 + r^2)^{\mu+1}} z^n, \quad \mu, r \in \mathbb{R}^+, |z| < 1.$$

Since $Q_\mu(r; z) \notin \mathcal{A}$, we define the normalization

$$\begin{aligned} Q_\mu(r; z) &= \frac{(r^2 + 1)^{\mu+1}}{2} \sum_{n=1}^{\infty} \frac{2n!}{((n!)^2 + r^2)^{\mu+1}} z^n \\ &= z + \sum_{n=2}^{\infty} \frac{n!(r^2 + 1)^{\mu+1}}{((n!)^2 + r^2)^{\mu+1}} z^n. \end{aligned} \quad (3)$$

We are interested in starlikeness and close-to-convexity of $\mathbb{F}_\mu(r; z)$ and $Q_\mu(r; z)$. Thus, we generalize the results of [20] on $\mathbb{F}_1(r; z)$ from $\mu = 1$ to more general values of μ , and additionally provide the first results on geometric properties of $Q_\mu(r; z)$.

Recall that a sequence of real numbers $\{a_n\}_{n \geq 1}$ satisfying the condition

$$2a_{n+1} \leq a_n + a_{n+2}, \quad n \geq 1$$

is called a convex sequence. It is clear that, if $f(x)$ is a convex function (of a real variable) for $x \geq 1$, then the sequence $a_n = f(n)$, $n = 1, 2, \dots$, is convex. We need the following lemmas to prove our main results.

Lemma 1 (Ozaki [22], Corollary 7). Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Suppose

$$1 \geq 2a_2 \geq \dots \geq (n+1)a_{n+1} \geq \dots \geq 0$$

or

$$1 \leq 2a_2 \leq \dots \leq (n+1)a_{n+1} \leq \dots \leq 2.$$

Then, f is close-to-convex with respect to the starlike function $z/(1-z)$.

When comparing the preceding lemma with Lemma 1.1 of [20], note the slightly different notation used there (see the definition of close-to-convex on p. 912 of [20]).

Lemma 2 (Fejér [23], Satz IX). If $\{a_n\}$ is a non-negative real sequence with $a_1 = 1$ and such that $\{na_n\}_{n \geq 1}$ and $\{na_n - (n+1)a_{n+1}\}_{n \geq 1}$ are non-increasing, then the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{S}^*$.

Lemma 3. Let $\{a_n\}_{n \geq 1}$ be a non-increasing sequence of non-negative real numbers with $a_1 = 1$, which is convex, i.e.,

$$a_1 - a_2 \geq \dots \geq a_k - a_{k+1} \geq \dots \geq 0.$$

Then

$$\Re \left(\sum_{n=1}^{\infty} a_n z^{n-1} \right) > \frac{1}{2}, \quad z \in \mathbb{U}.$$

Proof. According to Lemma 3.4 in [24] and Lemma 1.3 in [20], this is due to Fejér [23]. As we could not find the result in this reference, we give a proof for the reader's convenience, which is a simple variation of a proof found in another paper by Fejér ([25], Theorem 1), and without claiming originality. Let $z = re^{i\theta}$ with $r \in (0, 1)$ and $\theta \in (0, 2\pi)$. (The cases $r = 0$ and $\theta = 0$ are both trivial.) Define $\tilde{a}_n = r^{n-1}a_n$. Since a_n is non-increasing and non-negative, we have $\tilde{a}_n = o(1)$. The sequence $\tilde{a}_n$ is decreasing and convex (see p. 98 in [23]), and it is easy to see that it is actually strictly convex. We define $s_n = 1/2 + \sum_{k=1}^n \cos k\theta$ and

$$\sigma_n = \sum_{k=0}^n s_k = \frac{1}{2} \left(\frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \right)^2,$$

which satisfies $0 \leq \sigma_n = O(1)$. Using summation by parts twice, we find, for $N \geq 3$,

$$\begin{aligned} \Re \left(\sum_{n=1}^{N+1} a_n z^{n-1} \right) &= 1 + \sum_{n=1}^N \tilde{a}_{n+1} \cos n\theta = 1 + \sum_{n=1}^N \tilde{a}_{n+1} (s_n - s_{n-1}) \\ &= 1 + \tilde{a}_{N+1} (\sigma_N - \sigma_{N-1}) - \tilde{a}_2 s_0 - \left((\tilde{a}_{N+1} - \tilde{a}_N) \sigma_{N-1} - (\tilde{a}_3 - \tilde{a}_2) \sigma_0 \right. \\ &\quad \left. - \sum_{n=3}^N \sigma_{n-2} (\tilde{a}_{n+1} - 2\tilde{a}_n + \tilde{a}_{n-1}) \right). \end{aligned}$$

Then, $N \rightarrow \infty$ yields

$$\begin{aligned} \Re \left(\sum_{n=1}^{\infty} a_n z^{n-1} \right) &= 1 - \frac{1}{2} \tilde{a}_2 + \frac{1}{2} (\tilde{a}_3 - \tilde{a}_2) + \sum_{n=3}^{\infty} \sigma_{n-2} (\tilde{a}_{n+1} - 2\tilde{a}_n + \tilde{a}_{n-1}) \\ &> 1 - \tilde{a}_2 + \frac{1}{2} \tilde{a}_3 = \frac{1}{2} + \frac{\tilde{a}_1 - 2\tilde{a}_2 + \tilde{a}_3}{2} \geq \frac{1}{2}. \end{aligned}$$

□

2. Close-to-Convexity and Starlikeness of $\mathbb{F}_\mu(r; z)$

Generalizing Theorem 2.2 of [20] to arbitrary $\mu > 0$, we can state:

Theorem 1. *If $\mu > 0$ and $0 < r \leq \sqrt{\mu}$, or $\mu \geq 1$ and $0 < r < \sqrt{\mu+1}$, then $\mathbb{F}_\mu(r; z)$ is close-to-convex with respect to the starlike function $z/(1-z)$.*

Proof. In view of Lemma 1, it is sufficient to prove that the sequence $\{na_n\}_{n \geq 1}$ is non-increasing. Here, the a_n are the coefficients in the series expansion of $\mathbb{F}_\mu(r; z)$ given by (2). Let

$$f(n) = na_n = \frac{n^2(r^2+1)^{\mu+1}}{(n^2+r^2)^{\mu+1}}.$$

Now, it is sufficient to show that the function $f(x) = \frac{x^2(r^2+1)^{\mu+1}}{(x^2+r^2)^{\mu+1}}$ decreases. Differentiating $f(x)$, we have

$$f'(x) = \frac{2x(r^2+1)^{\mu+1}}{(x^2+r^2)^{\mu+2}} (r^2 - x^2\mu), \quad x \geq 1, \mu > 0.$$

Under the first stated condition on μ and r , this is obviously negative for $x \geq 1$. In the second case, it is negative for $x \geq 2$, and it remains to show that $1 \geq f(2)$. This follows from

$$f(2) = \frac{4(r^2+1)^{\mu+1}}{(r^2+4)^{\mu+1}} \leq 4 \left(\frac{\mu+2}{\mu+5} \right)^{\mu+1} \leq 1.$$

Here, we have used that $(r^2+1)/(r^2+4)$ increases w.r.t. r , and that $(\frac{\mu+2}{\mu+5})^{\mu+1}$ decreases w.r.t. μ . □

Alternative Proof. Using the Bernoulli inequality, we can directly prove that the sequence $\{na_n\}$ is non-increasing under the first stated condition. Namely, we have

$$\begin{aligned} na_n - (n+1)a_{n+1} &= \frac{n^2(r^2+1)^{\mu+1}}{(n^2+r^2)^{\mu+1}} - \frac{(n+1)^2(r^2+1)^{\mu+1}}{((n+1)^2+r^2)^{\mu+1}} \\ &= \left[\frac{r^2+1}{(n^2+r^2)((n+1)^2+r^2)} \right]^{\mu+1} b_n, \end{aligned}$$

where

$$\begin{aligned}
 b_n &= n^2((n+1)^2 + r^2)^{\mu+1} - (n+1)^2(n^2 + r^2)^{\mu+1} \\
 &= n^2(n^2 + 2n + 1 + r^2)^{\mu+1} - (n^2 + 2n + 1)(n^2 + r^2)^{\mu+1} \\
 &= n^2(n^2 + r^2)^{\mu+1} \left(1 + \frac{2n+1}{n^2 + r^2}\right)^{\mu+1} - n^2(n^2 + r^2)^{\mu+1} - (2n+1)(n^2 + r^2)^{\mu+1} \\
 &= n^2(n^2 + r^2)^{\mu+1} \left[\left(1 + \frac{2n+1}{n^2 + r^2}\right)^{\mu+1} - 1\right] - (2n+1)(n^2 + r^2)^{\mu+1} \\
 &\geq n^2(n^2 + r^2)^{\mu+1} \frac{(2n+1)(\mu+1)}{n^2 + r^2} - (2n+1)(n^2 + r^2)^{\mu+1} \\
 &= (2n+1)(n^2 + r^2)^{\mu} [n^2(\mu+1) - (n^2 + r^2)] \\
 &= (2n+1)(n^2 + r^2)^{\mu} [n^2\mu - r^2] \geq 0,
 \end{aligned}$$

provided $\mu > 0$ and $0 < r \leq \sqrt{\mu}$. $\square$

The following result generalizes Theorem 2.3 of [20] to arbitrary $\mu > 0$.

Theorem 2. If $\mu > 0$ and $0 < r \leq \sqrt{\frac{(3+5\mu) - \sqrt{17\mu^2 + 26\mu + 9}}{2}}$, then $\mathbb{F}_{\mu}(r; z)$ is starlike in $\mathbb{U}$.

Proof. We already proved in Theorem 1 that $\{na_n\}$ is non-increasing for all $0 < r \leq \sqrt{\mu}$. To show that $\mathbb{F}_{\mu}(r; z)$ is starlike in $\mathbb{U}$, using Lemma 2, it is sufficient to show that the sequence $\{na_n - (n+1)a_{n+1}\}$ is also non-increasing. That is,

$$\begin{aligned}
 na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} &\geq 0 \\
 \iff (r^2 + 1)^{\mu+1} \left[\frac{n^2}{(n^2 + r^2)^{\mu+1}} - 2 \frac{(n+1)^2}{((n+1)^2 + r^2)^{\mu+1}} + \frac{(n+2)^2}{((n+2)^2 + r^2)^{\mu+1}} \right] &\geq 0 \\
 \iff f(n) - 2f(n+1) + f(n+2) &\geq 0,
 \end{aligned}$$

where

$$f(x) = \frac{x^2}{(x^2 + r^2)^{\mu+1}}, \quad x \geq 1, \mu > 0.$$

To show $f(n) - 2f(n+1) + f(n+2) \geq 0$, $n = 1, 2, 3, 4, \dots$, it is sufficient to prove that $f(x)$ is a convex function, i.e., $f''(x) \geq 0$, $x \geq 1$. Differentiating twice, we have

$$f''(x) = \frac{2[x^4(\mu + 2\mu^2) - x^2(3 + 5\mu)r^2 + r^4]}{(x^2 + r^2)^{\mu+3}}, \quad x \geq 1, \mu > 0.$$

The denominator is positive for all $x \geq 1$ and $r > 0$. Let $\phi(x) = x^4(\mu + 2\mu^2) - x^2(3 + 5\mu)r^2 + r^4$. Obviously, $\phi'(x) = 4x^3(2\mu^2 + \mu) - 2(3 + 5\mu)r^2x \geq 0$ for all $x \geq 1, \mu > 0$ and $0 < r \leq \sqrt{\frac{2(2\mu^2 + \mu)}{5\mu + 3}}$. Thus, $f''(x) \geq 0$ provided $\phi(1) \geq 0$, which in turn gives $0 < r \leq \sqrt{\frac{(5\mu + 3) - \sqrt{17\mu^2 + 26\mu + 9}}{2}}$. This completes the proof. $\square$

We can also generalize Theorem 2.4 of [20]:

Theorem 3. For $\mu > 0$ and $0 < r \leq \sqrt{\frac{2\mu+1}{3}}$, we have

$$\Re\left(\frac{\mathbb{F}_\mu(r; z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{U}.$$

Proof. First, we prove that

$$\{a_n\}_{n=1}^\infty = \left\{ \frac{n(r^2+1)^{\mu+1}}{(n^2+r^2)^{\mu+1}} \right\}_{n=1}^\infty$$

is a decreasing sequence, i.e.,

$$a_n - a_{n+1} \geq 0, \quad n \in \mathbb{N}.$$

Note that

$$\begin{aligned} a_n - a_{n+1} &\geq 0 \\ \iff (r^2+1)^{\mu+1} \left[\frac{n}{(n^2+r^2)^{\mu+1}} - \frac{n+1}{((n+1)^2+r^2)^{\mu+1}} \right] &\geq 0 \\ \iff (r^2+1)^{\mu+1} [f(n) - f(n+1)] &\geq 0, \end{aligned}$$

where

$$f(x) = \frac{x}{(x^2+r^2)^{\mu+1}}, \quad x \geq 1, \mu > 0. \quad (4)$$

To show $f(n) - f(n+1) \geq 0$, $n = 1, 2, 3, \dots$, it is sufficient to prove that $f(x)$ is a decreasing function, i.e., that $f'(x) < 0$, $x \geq 1$. We have

$$f'(x) = \frac{r^2 - (1+2\mu)x^2}{(x^2+r^2)^{\mu+2}} \leq 0, \quad x \geq 1, \mu > 0 \text{ and } 0 < r \leq \sqrt{1+2\mu}.$$

Next, we prove that $\{a_n\}_{n=1}^\infty$ is a convex decreasing sequence. For this, we show

$$a_{n+2} - a_{n+1} \geq a_{n+1} - a_n, \quad n \in \mathbb{N}.$$

Now,

$$\begin{aligned} a_n - 2a_{n+1} + a_{n+2} &\geq 0 \\ \iff (r^2+1)^{\mu+1} \left[\frac{n}{(n^2+r^2)^{\mu+1}} - 2\frac{n+1}{((n+1)^2+r^2)^{\mu+1}} + \frac{n+2}{((n+2)^2+r^2)^{\mu+1}} \right] &\geq 0 \\ \iff (r^2+1)^{\mu+1} [f(n) - 2f(n+1) + f(n+2)] &\geq 0, \end{aligned}$$

where $f(x)$ is given by (4). To show $f(n) + f(n+2) - 2f(n+1) \geq 0$, $n = 1, 2, 3, 4, \dots$, it suffices to prove that $f(x)$ is a convex function or that $f''(x) \geq 0$, $x \geq 1$. It can be easily verified that

$$f''(x) = \frac{2x[(2\mu^2+3\mu+1)x^2-3(\mu+1)r^2]}{(x^2+r^2)^{\mu+3}} \geq 0$$

for all

$$x \geq 1, \mu > 0 \text{ and } 0 < r \leq \sqrt{\frac{2\mu+1}{3}}.$$

Thus, $\{a_n\}_{n=1}^\infty$ is a convex decreasing sequence. Applying Lemma 3 to $\{a_n\}_{n=1}^\infty$, we obtain

$$\Re\left(\sum_{n=1}^\infty a_n z^{n-1}\right) > \frac{1}{2}, \quad z \in \mathbb{U},$$

which is equivalent to

$$\Re\left(\frac{\mathbb{F}_\mu(r; z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{U}.$$

□

We conclude this section with a generalization of Theorem 2.5 of [20].

Theorem 4. For $\mu > 0$ and $0 < r \leq \sqrt{\frac{(5\mu+3)-\sqrt{17\mu^2+26\mu+9}}{2}}$, we have

$$\Re(\mathbb{F}'_\mu(r; z)) > \frac{1}{2}, \quad z \in \mathbb{U}.$$

Proof. From (2), we infer

$$\mathbb{F}'_\mu(r; z) = 1 + \sum_{n=2}^{\infty} \frac{n^2(r^2+1)^{\mu+1}}{(n^2+r^2)^{\mu+1}} z^{n-1}.$$

We have shown in Theorems 1 and 2 that the sequence

$$\frac{n^2(r^2+1)^{\mu+1}}{(n^2+r^2)^{\mu+1}}$$

is convex and non-increasing, and so the result follows from Lemma 3. □

3. Close-to-Convexity and Starlikeness of $\mathbb{Q}_\mu(r; z)$

It turns out that close-to-convexity of the series $\mathbb{Q}_\mu(r; z)$, which has not been studied before, can be inferred from Ozaki's criterion too.

Theorem 5. If $\mu > 0$ and $0 < r \leq \sqrt{\mu}$, then $\mathbb{Q}_\mu(r; z)$ is close-to-convex w.r.t. the starlike function $z/(1-z)$.

Proof. Again, we apply Lemma 1. Let

$$C_n = \frac{n!(r^2+1)^{\mu+1}}{((n!)^2+r^2)^{\mu+1}}.$$

Using the Bernoulli inequality, we can directly prove that the sequence $\{nC_n\}$ is non-increasing under the stated condition. We have

$$\begin{aligned} nC_n - (n+1)C_{n+1} &= \frac{n n!(r^2+1)^{\mu+1}}{[(n!)^2+r^2]^{\mu+1}} - \frac{(n+1)(n+1)!(r^2+1)^{\mu+1}}{[((n+1)!)^2+r^2]^{\mu+1}} \\ &= \left[\frac{(r^2+1)}{((n!)^2+r^2)((n+1)!)^2+r^2} \right]^{\mu+1} B_n, \end{aligned}$$

where

$$\begin{aligned}
 B_n &= n![(n+1)!^2 + r^2]^{\mu+1} - (n+1)(n+1)!((n!)^2 + r^2)^{\mu+1} \\
 &= n![(n^2 + 2n + 1)(n!)^2 + r^2]^{\mu+1} - (n^2 + 2n + 1)n!((n!)^2 + r^2)^{\mu+1} \\
 &= n![(n^2 + 2n)((n!)^2 + ((n!)^2 + r^2)^{\mu+1} - (n^2 + 2n + 1)n!((n!)^2 + r^2)^{\mu+1} \\
 &= n![(n!)^2 + r^2]^{\mu+1} \left[1 + \frac{(n^2 + 2n)((n!)^2)}{(n!)^2 + r^2} \right]^{\mu+1} - (n^2 + 2n + 1)n!((n!)^2 + r^2)^{\mu+1} \\
 &\geq n![(n!)^2 + r^2]^{\mu+1} \left[1 + \frac{(\mu+1)(n^2 + 2n)((n!)^2)}{(n!)^2 + r^2} \right] - (n^2 + 2n + 1)n!((n!)^2 + r^2)^{\mu+1} \\
 &= n![(n!)^2 + r^2]^{\mu+1} + n(n^2 + 2n)(\mu+1)((n!)^3[(n!)^2 + r^2]^{\mu} \\
 &\quad - (n^2 + 2n + 1)n!((n!)^2 + r^2)^{\mu+1} \\
 &\geq n![(n!)^2 + r^2]^{\mu+1} + (n^2 + 2n)(\mu+1)((n!)^3[(n!)^2 + r^2]^{\mu} \\
 &\quad - (n^2 + 2n + 1)n!((n!)^2 + r^2)^{\mu+1} \\
 &= (n^2 + 2n)(\mu+1)((n!)^3[(n!)^2 + r^2]^{\mu} - (n^2 + 2n)n!((n!)^2 + r^2)^{\mu+1} \\
 &= (n!)^3(n^2 + 2n)((n!)^2 + r^2)^{\mu}(\mu(n!)^2 - r^2) \geq 0,
 \end{aligned}$$

provided that $\mu - r^2 \geq 0$ or $0 < r \leq \sqrt{\mu}$. $\square$

The application of Fejér's criterion (Lemma 2) carries over to $\mathbb{Q}_{\mu}(r; z)$ as well. In a part of the following proof, we use a symbolic computation technique that seems not to have been used in this context before.

Theorem 6. If $\mu \geq 2$ and $0 < r \leq \sqrt{\mu}$, then $\mathbb{Q}_{\mu}(r; z)$ is starlike in $\mathbb{U}$.

Proof. We have shown in the preceding proof that $\{nC_n\}_{n \geq 1}$ is non-increasing. Now, we prove that it is convex, provided that $\mu \geq 2$, in order to apply Lemma 2. We have $nC_n = g(n)$, where

$$g(x) = \frac{x\Gamma(x+1)(1+r^2)^{\mu+1}}{(\Gamma(x+1)^2 + r^2)^{\mu+1}}, \quad x \geq 0.$$

Define

$$h(x) = g(x) - g(x+1).$$

Our goal is to show that $\{h(n)\}_{n \geq 1}$ is non-increasing. We have (cf. the proof of Theorem 1)

$$\begin{aligned}
 h(1) - h(2) &= 1 - 8\left(\frac{r^2 + 1}{r^2 + 4}\right)^{\mu+1} + 18\left(\frac{r^2 + 1}{r^2 + 36}\right)^{\mu+1} \\
 &> 1 - 8\left(\frac{r^2 + 1}{r^2 + 4}\right)^{\mu+1} \\
 &\geq 1 - 8\left(\frac{\mu + 1}{\mu + 4}\right)^{\mu+1} \geq 0,
 \end{aligned}$$

where the last estimate follows from $\mu \geq 2$ and the fact that $\left(\frac{\mu+1}{\mu+4}\right)^{\mu+1}$ decreases w.r.t. μ . (This is the only step where $\mu \geq 2$ is used; the rest of the proof works for $\mu > 0$.) Very similarly, it can be shown that

$$h(2) \geq h(3) \geq h(4).$$

To complete the proof, we show that $g(x)$ is a convex function for $x \geq 4$. We have

$$g''(x) = \Gamma(x+1)(\Gamma(x+1)^2 + r^2)^{-\mu-3}(1+r^2)^{\mu+1}A(x),$$

where

$$A(x) = 2(r^2 + \Gamma(x+1)^2)[r^2 - (2\mu+1)\Gamma(x+1)^2]\psi(x+1) \\ + x[r^4 - 2r^2(4\mu+3)\Gamma(x+1)^2 + (2\mu+1)^2\Gamma(x+1)^4]\psi(x+1)^2 \\ + x(r^2 + \Gamma(x+1)^2)(r^2 - (2\mu+1)\Gamma(x+1)^2)\psi'(x+1), \quad (5)$$

and $\psi = \Gamma'/\Gamma$ denotes the digamma function. We will apply the following estimates:

$$-2r^2(4\mu+3)c + (2\mu+1)^2c^2 \geq 0, \quad \mu > 0, 0 \leq r \leq \sqrt{\mu}, c \geq 2, \quad (6)$$

$$\psi(x) < \log x - \frac{1}{2x}, \quad x > 1, \quad (7)$$

$$\psi(x) > \log x - \frac{1}{x}, \quad x > 1, \quad (8)$$

$$\psi'(x) < \frac{1}{x} + \frac{1}{x^2}, \quad x > 0, \quad (9)$$

$$\log(x+1) - \frac{1}{2(x+1)} \leq \sqrt{x}, \quad x \geq 1, \quad (10)$$

$$\left(\log(x+1) - \frac{1}{x+1}\right)^2 \geq \frac{19}{10}, \quad x \geq 4, \quad (11)$$

$$2(r^2 + c)(r^2 - (2\mu+1)c)\sqrt{x} + x(r^4 - 2r^2(4\mu+3)c + (2\mu+1)^2c^2)\frac{19}{10} \\ + x(r^2 + c)(r^2 - (2\mu+1)c)\left(\frac{1}{x+1} + \frac{1}{(x+1)^2}\right) \geq 0, \\ x \geq 4, c \geq \Gamma(5)^2, \mu > 0, 0 \leq r \leq \sqrt{\mu}. \quad (12)$$

Note that (6) and (12) are polynomial inequalities with polynomial constraints, which can be proven by cylindrical algebraic decomposition (CAD), using a computer algebra system. The inequalities (10) and (11) are very easy to show. The estimates (7) and (8) are found on p. 288 of [26], and (9) is an easy consequence of (26) in [27].

For the rest of the proof, we may assume $x \geq 4$. The factors in front of $\psi(x+1)$ and $\psi'(x+1)$ in (5) are clearly negative, whereas the factor in front of $\psi(x+1)^2$ is non-negative, by (6). Thus, (7)–(9) imply

$$A(x) \geq 2(r^2 + \Gamma(x+1)^2)(r^2 - (2\mu+1)\Gamma(x+1)^2)\left(\log(x+1) - \frac{1}{2(x+1)}\right) \\ + x(r^4 - 2r^2(4\mu+3)\Gamma(x+1)^2 + (2\mu+1)^2\Gamma(x+1)^4)\left(\log(x+1) - \frac{1}{x+1}\right)^2 \\ + x(r^2 + \Gamma(x+1)^2)(r^2 - (2\mu+1)\Gamma(x+1)^2)\left(\frac{1}{x+1} + \frac{1}{(x+1)^2}\right).$$

Using (10) and (11), we thus obtain

$$A(x) \geq 2(r^2 + \Gamma(x+1)^2)(r^2 - (2\mu+1)\Gamma(x+1)^2)\sqrt{x} \\ + x(r^4 - 2r^2(4\mu+3)\Gamma(x+1)^2 + (2\mu+1)^2\Gamma(x+1)^4)\frac{19}{10} \\ + x(r^2 + \Gamma(x+1)^2)(r^2 - (2\mu+1)\Gamma(x+1)^2)\left(\frac{1}{x+1} + \frac{1}{(x+1)^2}\right).$$

This is non-negative by (12). The convexity of $g(x)$ for $x \geq 4$ is established, which completes the proof. $\square$

Theorem 3 also has a variant for the series $\mathbb{Q}_\mu(r; z)$:

Theorem 7. For $\mu > 0$ and $0 < r \leq \sqrt{\mu}$, the following inequality holds:

$$\Re\left(\frac{\mathbb{Q}_\mu(r; z)}{z}\right) > \frac{1}{2}, \quad z \in \mathbb{U}.$$

Proof. We use Lemma 3. To show that

$$\{C_n\}_{n=1}^\infty = \left\{ \frac{n!(r^2 + 1)^{\mu+1}}{((n!)^2 + r^2)^{\mu+1}} \right\}_{n=1}^\infty$$

is a decreasing sequence, we define

$$\tilde{g}(x) = \frac{\Gamma(x+1)}{((\Gamma(x+1))^2 + r^2)^{\mu+1}}, \quad x \geq 1, \mu > 0.$$

We have

$$\tilde{g}'(x) = \frac{\Gamma'(x+1)[r^2 - (2\mu+1)(\Gamma(x+1))^2]}{[(\Gamma(x+1))^2 + r^2]^{\mu+2}} \leq 0$$

when $x \geq 1, \mu > 0$ and $0 < r \leq \sqrt{2\mu+1}$. Next, we prove that $\{C_n\}$ is a convex sequence. Define

$$\tilde{h}(x) = \tilde{g}(x) - \tilde{g}(x+1).$$

We first show that $\tilde{h}(1) \geq \tilde{h}(2) \geq \tilde{h}(3)$. We have

$$\tilde{h}(1) - \tilde{h}(2) > \frac{1}{(r^2 + 1)^{\mu+1}} - \frac{4}{(r^2 + 4)^{\mu+1}} \geq 0$$

because the map $x \mapsto x/(r^2 + x)^{\mu+1}$ decreases for $x \geq r^2/\mu$. For the same reason,

$$\begin{aligned} \tilde{h}(2) - \tilde{h}(3) &> \frac{1}{2} \left(\frac{4}{(r^2 + 4)^{\mu+1}} - \frac{24}{(r^2 + 36)^{\mu+1}} \right) \\ &> \frac{1}{2} \left(\frac{4}{(r^2 + 4)^{\mu+1}} - \frac{36}{(r^2 + 36)^{\mu+1}} \right) > 0. \end{aligned}$$

We complete the proof by showing that the function $\tilde{g}(x)$ is convex for $x \geq 3$, similarly as in the proof of Theorem 6. We have

$$\tilde{g}''(x) = \Gamma(x+1)(r^2 + \Gamma(x+1)^2)^{-\mu-3} \tilde{A}(x),$$

where

$$\begin{aligned} \tilde{A}(x) &:= \left(r^4 - 2r^2(4\mu+3)\Gamma(x+1)^2 + (2\mu+1)^2\Gamma(x+1)^4 \right) \psi(x+1)^2 \\ &\quad + (r^2 + \Gamma(x+1)^2)(r^2 - (2\mu+1)\Gamma(x+1)^2) \psi'(x+1). \end{aligned}$$

We state the following inequalities:

$$-2r^2(4\mu+3) + (2\mu+1)^2c \geq 0, \quad \mu > 0, 0 \leq r \leq \sqrt{\mu}, c \geq 2, \quad (13)$$

$$\left(\log(x+1) - \frac{1}{x+1} \right)^2 \geq 1, \quad x \geq 3, \quad (14)$$

$$\frac{1}{x+1} + \frac{1}{(x+1)^2} \leq \frac{1}{2}, \quad x \geq 3, \quad (15)$$

$$(2\mu+1)^2 - \frac{2r^2(4\mu+3)}{c} - \frac{2\mu+1}{2} \left(1 + \frac{r^2}{c} \right) > 0, \quad \mu > 0, 0 \leq r \leq \sqrt{\mu}, c \geq 5. \quad (16)$$

It is very easy to show (14). The estimate (15) is obvious, and (13) and (16) can be proven by computer algebra (see above). From now on, we assume $x \geq 3$. Since $\psi^2(x+1)$ and $\psi'(x+1)$ are non-negative, we have

$$\tilde{A}(x) \geq [-2r^2(4\mu+3)\Gamma(x+1)^2 + (2\mu+1)^2\Gamma(x+1)^4]\psi(x+1)^2 - (r^2 + \Gamma(x+1)^2)(2\mu+1)\Gamma(x+1)^2\psi'(x+1).$$

By (13), the term $[\dots]$ is ≥ 0 . We then use (8) and (9) and obtain

$$\tilde{A}(x) \geq [-2r^2(4\mu+3)\Gamma(x+1)^2 + (2\mu+1)^2\Gamma(x+1)^4] \left(\log(x+1) - \frac{1}{x+1} \right)^2 - (r^2 + \Gamma(x+1)^2)(2\mu+1)\Gamma(x+1)^2 \left(\frac{1}{x+1} + \frac{1}{(x+1)^2} \right).$$

By (14) and (15), we further obtain

$$\begin{aligned} \tilde{A}(x) &\geq [-2r^2(4\mu+3)\Gamma(x+1)^2 + (2\mu+1)^2\Gamma(x+1)^4] \\ &\quad - \frac{1}{2}(r^2 + \Gamma(x+1)^2)(2\mu+1)\Gamma(x+1)^2 \\ &= \Gamma(x+1)^4 \left(\left[(2\mu+1)^2 - \frac{2r^2(4\mu+3)}{\Gamma(x+1)^2} \right] - \frac{2\mu+1}{2} \left[1 + \frac{r^2}{\Gamma(x+1)^2} \right] \right). \end{aligned}$$

This is positive by (16), for $x \geq 3$. $\square$

Theorem 8. For $\mu \geq 2$ and $0 < r \leq \sqrt{\mu}$, we have

$$\Re(\mathbb{Q}'_{\mu}(r; z)) > \frac{1}{2}, \quad z \in \mathbb{U}.$$

Proof. From (3),

$$\mathbb{Q}'_{\mu}(r; z) = 1 + \sum_{n=2}^{\infty} \frac{n n! (r^2 + 1)^{\mu+1}}{((n!)^2 + r^2)^{\mu+1}} z^{n-1}.$$

We have shown in Theorems 5 and 6 that the sequence

$$\frac{n n! (r^2 + 1)^{\mu+1}}{((n!)^2 + r^2)^{\mu+1}}$$

is convex and non-increasing, and so the result follows from Lemma 3. $\square$

4. Two Further Examples

We conclude the paper with two more examples of starlike Mathieu-type power series. Here, we apply the following criterion; see Goodman [28] and the references given there.

Proposition 1. The function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1,$$

is starlike if $\sum_{n=2}^{\infty} n|a_n| \leq 1$.

Example 1. Consider the series

$$z + 4 \sum_{n=2}^{\infty} \frac{2z^n}{(n^2 + 1)^3} = \int_0^z \frac{\mathbb{F}_2(1; t)}{t} dt, \quad (17)$$

which is related to

$$\mathbb{F}_2(1; z) = z + \sum_{n=2}^{\infty} \frac{8n}{(n^2 + 1)^3} z^n. \quad (18)$$

While Theorem 2 is not applicable for $r = 1$ and $\mu = 2$, which are the parameter values chosen in (18), we can show that (17) is starlike, using Proposition 1 and an inequality due to Diananda. Indeed, by Theorem 1 in [11],

$$4 \sum_{n=2}^{\infty} \frac{2n}{(n^2 + 1)^3} = 4 \left(\sum_{n=1}^{\infty} \frac{2n}{(n^2 + 1)^3} - \frac{1}{4} \right) < 4 \left(\frac{1}{2} - \frac{1}{4} \right) = 1.$$

Example 2. The power series

$$z + 4 \sum_{n=2}^{\infty} \frac{(2n-1)!!}{[(2n+1)!! + 1]^2} z^n, \quad |z| < 1,$$

is starlike. Recall that the double factorial is defined by $(2n+1)!! = 1 \times 3 \times \cdots \times (2n+1)$. Starlikeness follows from Proposition 1 and the estimate

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{4n(2n-1)!!}{[(2n+1)!! + 1]^2} &< 2 \sum_{n=1}^{\infty} \frac{(2n+1)(2n-1)!! - (2n-1)!!}{[(2n+1)!! + 1][(2n-1)!! + 1]} \\ &= 2 \sum_{n=1}^{\infty} \left[\frac{1}{(2n-1)!! + 1} - \frac{1}{(2n+1)!! + 1} \right] \\ &= 2 \cdot \frac{1}{(2 \cdot 1 - 1)!! + 1} = 1, \end{aligned}$$

i.e.,

$$\sum_{n=2}^{\infty} \frac{4n(2n-1)!!}{[(2n+1)!! + 1]^2} < 1 - \frac{1}{4} < 1.$$

5. Conclusions

We have provided conditions under which two generalized Mathieu power series are close-to-convex resp. starlike. One of them, involving factorials in its coefficients, was introduced in [21]. The aim of that paper was at asymptotic expansions for large r , and it turned out that the series with the factorials was more difficult to handle than the classical Mathieu series and its immediate extensions. This is not the case for the problems studied in the present paper, which highlights the versatility of the classical criteria we apply, and suggests that geometric results for even more general Mathieu power series might lie behind what we have proved.

Author Contributions: Conceptualization, Ž.T.; methodology, all four authors; writing—original draft preparation, Ž.T. and S.G.; writing—review and editing, Ž.T. and S.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded in part by DAAD.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank some anonymous referees for helpful remarks.

Conflicts of Interest: The authors declare no conflict of interest.

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