Article

An Index for Graphs and Graph Groupoids

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Abstract: In this paper, we consider certain quantities that arise in the images of the so-called graph-tree indexes of graph groupoids. In text, the graph groupoids are induced by connected finite-directed graphs with more than one vertex. If a graph groupoid $\mathbb{G}$ G contains at least one loop-reduced finite path, then the order of $\mathbb{G}$ is infinity; hence, the canonical groupoid index $[\mathbb{G} : \mathbb{K}]$ of the inclusion $\mathbb{K} \subseteq \mathbb{G}$ is either ∞ or 1 (under the definition and a natural axiomatization) for the graph groupoids $\mathbb{K}$ of all "parts" K of G. A loop-reduced finite path generates a semicircular element in graph groupoid algebra. Thus, the existence of semicircular systems acting on the free-probabilistic structure of a given graph G is guaranteed by the existence of loop-reduced finite paths in $\mathbb{G}$. The non-semicircularity induced by graphs yields a new index-like notion called the graph-tree index Γ of $\mathbb{G}$. We study the connections between our graph-tree index and non-semicircular cases. Hence, non-semicircularity also yields the classification of our graphs in terms of a certain type of trees. As an application, we construct towers of graph-groupoid-inclusions which preserve the graph-tree index. We further show that such classification applies to monoidal operads.

Keywords: graph groupoids; graph-trees; index; the semicircular law; operads

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1. Introduction

A directed *graph* $G = (V(G), E(G))$ is a combinatorial object consisting of the vertex set $V(G)$ of all *vertices* and the edge set $E(G)$ of all directed *edges* (or oriented edges up to the direction in G). In the text, we assume that all given graphs are connected, finite, and have more than one vertex. Such a directed graph is depicted in a diagrammatic form as a set of dots (for vertices) jointed by arrowed curves (for directed edges), where the arrows of the curves indicate the direction on the graph (e.g., [1–4]).

Graphs are the main objects not only in pure mathematical fields, but also in related applied areas (e.g., [5–14]).

Free probability is one of the main areas of operator algebra theory studying "noncommutative" measure-theoretic and corresponding statistical analysis on operator-theoretic structures (e.g., [15–18]). In free probability, semicircular elements whose free distributions obey the semicircular law play key roles, as the semicircular law is the noncommutative-analytic counterpart of the Gaussian distribution (or the normal distribution) of classical (commutative) functional analysis by the (free) central limit theorem(s) (e.g., [16–19]).

The main results of this paper include (i) characterizing the semicircularity on (M_G, τ) by the loop-ness on $\mathbb{G}$; (ii) considering a certain measure on $\mathbb{G}$ called the non-loop index of $\mathbb{G}$, providing the information of groupoidal elements in $\mathbb{G}$ that are not loop-reduced finite paths; (iii) showing how our measuring tool of (ii) implies the non-semicircularity on (M_G, τ) with respect to (i); and (iv) constructing and studying a tower of C^*-probability spaces that are free homomorphic from the base to the top, preserving our non-loop index.

1.1. Motivation

Amalgamated free-probabilistic operator-algebraic structures induced by directed graphs (which are necessarily neither connected nor finite) have been studied in [3,20–31]. In particular, algebraic structures called graph groupoids are constructed by directed graphs in [20]. These are algebraically pure groupoids equipped with multiple units, vertices generated by the generators, and edges (e.g., [32,33]). We introduced von Neumann algebra generated by graph groupoids preserving the graph-theoretic properties of given graphs, and such combinatorial properties were measured and analyzed in an amalgamated free probabilistic manner. The amalgamated freeness of the von Neumann algebra was characterized in [21], and, under a natural representation, graph groupoids were used to generate groupoid W^*-dynamical systems in operator algebra in [24]. As an application, graph groupoids satisfying *fractality* were considered in [22,23]. Recently, certain free-probabilistic structures "over $\mathbb{C}$" induced by connected finite-directed graphs were studied in [34], without considering amalgamation, differently from the above-mentioned earlier works. The main reason why we need a new type of free-probabilistic structure of [34] is for studying the semicircular law induced by graphs canonically.

Recently, semicircular elements were generated by orthogonal projections in [35–38]. These studies not only showed how to construct semicircular elements from mutually orthogonal projections (different from the usual free probabilistic methods), but also illustrated how the semicircular law is preserved or distorted by operator-algebraic actions (e.g., [19,37,38]).

In [34], motivated by the main results of [19,20,22–24,34–38], the relations between graphs and semicircular elements were studied. It is explained there that, instead of applying the amalgamated free structures of [21], it is better to consider a different type of (non-amalgamated) free-probabilistic structure induced by directed graphs, especially where they are connected, finite, and have more than one vertex. In this new model, the analysis and application of the semicircularity works well without ignoring the combinatorial properties of given graphs (equivalently, the algebraic properties of the corresponding graph groupoids). In particular, a certain algebraic object of a graph groupoid induced by the combinatorial property of a graph implies the semicircularity under the free-probabilistic language. In this paper, we show that the semicircularity in our setting implies the combinatorial property of such an object conversely; thus, we provide a characterization of the semicircularity in terms of this combinatorial property.

All other studies of this paper are based on the above characterization. Since the semicircularity of our free-probabilistic structures has been characterized, it is natural to consider how such semicircularity is preserved in bigger or smaller free-probabilistic structures than the original. Moreover, it is important to ask how we can characterize such semicircularity-preserving conditions, and how, if possible, we can quantize such conditions. In algebra, it seems natural to consider some kind of "index", such as the group index. Because of the technical difficulties, instead of a semicircularity-preserving index, we introduce a "non-semicircularity"-preserving index in this paper. From the study of the index preserving non-semicircularity, we establish an abstract theory in terms of operad theory.

1.2. Overview

In the first part of this paper, Sections 1–4, we construct a C^*-probability space (M_G, τ) generated by the graph groupoid $\mathbb{G}$ of a connected finite-directed graph G with more than one vertex and characterize the free distributions of free random variables induced by the generating operators of (M_G, τ). In [34], we showed that if a loop-reduced finite path w exists in $\mathbb{G}$, it will induce infinitely many semicircular elements in (M_G, τ). Motivated by this, here we fully characterize the semicircularity on (M_G, τ) in terms of the "loop-ness" on $\mathbb{G}$. It is important to note that lots of graphs do exist; these are the "trees" whose graph groupoids do not contain loop-reduced finite paths. Based on our semicircularity characterization in terms of the loop-ness, we introduce the so-called graph-tree index of

graph groupoids and show that these index quantities give information of elements of $\mathbb{G}$, which are "not" semicircular in (M_G, τ).

In the second part of this paper, Sections 6–8, we consider how such "non-semicircularity" in (M_G, τ), determined by the "non-loop-ness" of $\mathbb{G}$ which is characterized by the "tree-ness" of G, classifies the family $\mathcal{G}$ of all our connected finite graphs (including the single-vertex graph "up to graph isomorphisms") in terms of the tree-ness of graphs, equivalently; the non-loop-ness on graph groupoids; and, hence, the non-semicircularity of the corresponding C^*-probability spaces. By studying these, we obtain similar but different properties to those of the Jones index theory. Jones index theory starts from a subfactor-inclusion of II_1-factors, but ours starts from an inclusion of graphs (under some additional conditions).

In the final third part of the paper, Sections 9–11, we consider operad-theoretic properties from the main results of Sections 6–8 (see Section 9). As applications, certain discrete statistical models are studied in Sections 10 and 11.

In summary, in Section 12 we consider the main results of the paper and explain their connections.

1.3. Why Connected Finite Graphs with More than One Vertex?

We finish this section by explaining why we consider "connected" "finite" graphs with "more than one vertex" as the main objects of this paper (remark that, to consider our monoidal and operadic structures (e.g., [39–43]), we later consider a single-vertex graph consisting of only a single vertex with no edges. However, this single-vertex graph is added just for algebraic and categorial convenience. However, our main objects are connected finite graphs with more than one vertex, and their "classification" needs to include a single-vertex graph.)

First of all, we restrict our interests to connected graphs because all (connected or disconnected) graphs have their connected components, which are connected graphs. Thus, combinatorially, studying graphs means considering their connected components; algebraically, studying corresponding graph groupoids means investigating the direct summands of subgroupoids, which are the graph groupoids of connected components; and operator-algebraically, studying corresponding graph groupoid algebra means considering the direct-product summands of subalgebra generated by the subgroupoids of connected components. Thus, without a great loss of generality, we focus on connected graphs (e.g., [22,23,34]).

Secondly, we restrict our interests to finite (connected) graphs. In analyses, sometimes "infinite" graphs play important roles in characterizing, visualizing, or explaining operator-algebraic structures and corresponding sub-structures (e.g., [6,8,24,37,41,44–48]). However, here, we do not focus on infinite graphs as our object. The main reason why we concentrate on finite graphs is to follow pure-combinatorial graph-theoretic properties. For instance, in the text, we characterize the semicircularity of free-probabilistic structures generated by graph groupoids in terms of the "loop-ness" of (shadowed) graphs. This characterization may not hold if a given graph is an infinite graph. Even though we have infinite graphs, the loop-ness implies their semicircularity, but we cannot conclude that the inverse is true. In other words, the combinatorial characterization of the semicircularity may not be obtained in an infinite-graph setting. Additionally, our graph groupoid index and graph-tree index, which will be considered in rgw text, are ∞, meaning the infinity in general. This means that the quantization techniques used for our classification of graphs do not work well in an infinite-graph setting. That is why we concentrate on finite graphs (e.g., [34]).

Finally, the reason why we focus on connected finite graphs with more than one vertex is simple; if a finite-connected graph G has a single vertex $\{v\}$, then it is either a single-vertex graph $(\{v\}, \phi)$ with its edge set ϕ, an empty set, or a graph $\left(\{v\}, \{l_i\}_{i=1}^N\right)$, for some $N \in \mathbb{N}$, where $l_1, \dots, l_N$ are the loop edges connecting v to itself. In particular, if G is a single-vertex-N-many-loop-edge graph, then its graph groupoid $\mathbb{G}$ is a group that is isomorphic to the free group F_N with N-generators. Additionally, the corresponding graph groupoid C^*-algebra is $*$-isomorphic to $C^*(F_N)$, which is studied in [22–24]. Furthermore,

under our approach, the graph-tree of such a graph G becomes a single-vertex graph, which is not so interesting. Thus, to avoid the difficulties of handling $\{C^*(F_N)\}_{N \in \mathbb{N}}$ and the triviality of our classification, we focus on graphs with more than one vertex.

2. Preliminaries

In this section, we introduce basic definitions and concepts from proceeding works. For more details, see [20–24,34].

2.1. Graph Groupoids

Throughout this paper, we automatically assume that all given graphs have more than one vertex,

$$|V(G)| > 1, \text{ for all given graphs } G,$$

where $|X|$ is the cardinality of a set X. Even though a graph has only one vertex, an interesting analytic and algebraic structure is constructed (e.g., [16,20–23]). However, for our main purposes we assume that all given graphs have more than a single vertex.

Let $G = (V(G), E(G))$ be a directed graph with the vertex set $V(G)$ and the edge set $E(G)$. If $e \in E(G)$ is an edge connecting the initial vertex v_1 to the terminal vertex v_2 in the direction of G, then we write $e = v_1 e$ or $e = e v_2$, or $e = v_1 e v_2$ to indicate that an edge e has an initial vertex v_1 and terminal vertex v_2. We can say that "v_1 and e" and "e and v_2" are admissible, respectively. Note that the admissibility depends on the direction of G.

For a graph G,, one can define the oppositely directed graph G^{-1} of G with the vertex set

$$V(G^{-1}) = \left\{ v^{-1} = v : v \in V(G) \right\} = V(G),$$

and the edge set

$$E(G^{-1}) = \left\{ e^{-1} : e \in E(G) \right\},$$

where e^{-1} means an edge and $e^{-1} = v_2 e^{-1} v_1$ in $E(G^{-1})$ if $e = v_1 e v_2$ in $E(G)$, with $v_1, v_2 \in V(G) = V(G^{-1})$. This oppositely directed edge $e^{-1} \in E(G^{-1})$ is called the *shadow of* $e \in E(G)$, and the graph G^{-1} is called the *shadow of* G. This shadow-ness satisfies

$$(G^{-1})^{-1} = G, \text{ as graphs,}$$

with

$$\left(e^{-1} \right)^{-1} = e, \ \forall e \in E(G).$$

We define the *shadowed graph* $\widehat{G}$ of G using a new graph with the vertex set,

$$V(\widehat{G}) = V(G) \cup V(G^{-1}) = V(G) = V\left(G^{-1}\right),$$

and edge set,

$$E(\widehat{G}) = E(G) \cup E(G^{-1}),$$

where G^{-1} is the *shadow* of G. In other words, $\widehat{G}$ is the graph union of G and G^{-1}. Recall that if K_1 and K_2 are directed graphs, then the graph union $K = K_1 \cup K_2$ is a new directed graph with

$$V(K) = V(K_1) \cup V(K_2) \text{ and } E(K) = E(K_1) \cup E(K_2).$$

Note the difference between the graph union K and the "disjoint" graph union $K' = K_1 \sqcup K_2$ with

$$V(K') = V(K_1) \sqcup V(K_2) \text{ and } E(K') = E(K_1) \sqcup E(K_2),$$

where $\sqcup$ is the disjoint union (e.g., [21–23,34]).

Two edges, $e_1 = v_1 e_1 v_1'$ and $e_2 = v_2 e_2 v_2'$, of the shadowed graph $\widehat{G}$ are said to be *admissible* if $v_1' = v_2$, equivalently, a finite path $e_1 e_2$ is well-defined on $\widehat{G}$. If w is a finite path

on $\widehat{G}$ with the initial vertex v_1 and rgw terminal vertex v_2, then we write $w = v_1 w$, $w = w v_2$, or $w = v_1 w v_2$; finite paths w_1 and w_2 are *admissible* if a new finite path $w_1 w_2$ is well-defined on $\widehat{G}$. By construction, every finite path can be expressed by a word in $E(\widehat{G})$. Denote the set of all finite paths by $FP(\widehat{G})$. If $w = e_1 e_2, \ldots, e_n \in FP\left(\widehat{G}\right)$ with $e_1, \ldots, e_n \in E\left(\widehat{G}\right)$, then one can find the shadow w^{-1} of $w \in FP\left(\widehat{G}\right)$ by:

$$w^{-1} = e_n^{-1} \ldots e_2^{-1} e_1^{-1} \in FP_r\left(\widehat{G}\right),$$

by the shadows $e_k^{-1} \in E\left(\widehat{G}\right)$ of e_k for all $n \in \mathbb{N}$.

Define the *free semigroupoid* $\mathbb{F}^+(\widehat{G})$ of $\widehat{G}$, using an algebraic structure:

$$\mathbb{F}^+\left(\widehat{G}\right) \overset{\text{denote}}{=} \left(\mathbb{F}^+\left(\widehat{G}\right), \cdot\right),$$

where:

$$\mathbb{F}^+\left(\widehat{G}\right) = \{\phi\} \cup V\left(\widehat{G}\right) \cup FP\left(\widehat{G}\right),$$

The binary operation $(\cdot)$ is the admissibility of $\widehat{G}$, where the additional element ϕ of $\mathbb{F}^+\left(\widehat{G}\right)$ is axiomatized to be the empty word in $V\left(\widehat{G}\right) \cup E\left(\widehat{G}\right)$. This empty word ϕ represents the cases where two elements of $\mathbb{F}^+\left(\widehat{G}\right)$ are "not admissible" or "undefined on $\widehat{G}$ up to direction"

$$w_1 w_2 = \phi \iff w_1 \text{ and } w_2 \text{ are not admissible,}$$

for $w_1, w_2 \in \mathbb{F}^+\left(\widehat{G}\right)$. Canonically, the empty word ϕ satisfies:

$$\phi w = \phi = w \phi, \ \forall w \in \mathbb{F}^+\left(\widehat{G}\right).$$

Now, we define the *reduction* (RR) in the admissibility $(\cdot)$ of $\mathbb{F}^+(\widehat{G})$ with the rule:

$$\text{(RR)} \ \ w = v_1 w v_2 \in \mathbb{F}^+\left(\widehat{G}\right) \Rightarrow w w^{-1} = v_1, \text{ and } w^{-1} w = v_2,$$

on $\mathbb{F}^+\left(\widehat{G}\right)$, where $v_1, v_2 \in V\left(\widehat{G}\right)$, including the case where $v = vvv$ in $V\left(\widehat{G}\right)$. The admissibility of $\mathbb{F}^+\left(\widehat{G}\right)$ under (RR) is called the "reduced admissibility".

Definition 1. *The algebraic pair* $\left(\mathbb{F}^+\left(\widehat{G}\right)/(RR), \bullet\right)$ *of the quotient set* $\mathbb{F}^+\left(\widehat{G}\right)/(RR)$ *and the reduced admissibility* $(\bullet)$ *is called the graph groupoid. We denote it by* $\mathbb{G}$.

Graph groupoids are indeed algebraic groupoids with a single binary operation with multiple units (e.g., [3,20–23,49]).

Notation. If there is no confusion, we denote the reduced admissibility $(\bullet)$ by $(\cdot)$:

$$w_1 \bullet w_2 \overset{\text{denote}}{=} w_1 w_2, \text{ in } \mathbb{G}, \ \forall w_1, w_2 \in \mathbb{G}.$$

With $FP_r\left(\widehat{G}\right)$, we denote the set of all "reduced" finite paths of $\mathbb{G}$, giving:

$$\mathbb{G} = \{\phi\} \cup V\left(\widehat{G}\right) \cup FP_r\left(\widehat{G}\right),$$

which is set theoretically.

2.2. Graph Groupoid C* Algebra

Consider a representation of the graph groupoid $\mathbb{G}$ of a graph G and a corresponding operator algebra. We define the graph Hilbert space H_G of G by:

$$H_G \overset{def}{=} \left(\underset{v \in V(\widehat{G})}{\oplus} \mathbb{C}\, \xi_v \right) \oplus \left(\underset{w \in FP_r(\widehat{G})}{\oplus} \mathbb{C}\, \xi_w \right),$$

with the orthonormal basis

$$\mathcal{B}_G = \{\xi_w : w \in \mathbb{G} \setminus \{\phi\}\},$$

and the zero vector $\xi_\phi = 0_{H_G}$. By definition, there is a natural vector multiplication on H_G:

$$\xi_{w_1} \xi_{w_2} = \begin{cases} \xi_{w_1 w_2} & \text{if } w_1 w_2 \neq \phi \\ \xi_\phi = 0_{H_G} & \text{if } w_1 w_2 = \phi, \end{cases}$$

for all $w_1, w_2 \in \mathbb{G}$. We define a *canonical left action*:

$$L : \mathbb{G} \to B(H_G),$$

of $\mathbb{G}$ by:

$$L(w) \overset{def}{=} L_w \in B(H_G), \text{ for all } w \in \mathbb{G},$$

where $B(H_G)$ is the operator algebra (which is a C^* algebra under its operator norm) of all (Hilbert-space) operators on H_G, while L_ws are the (left-)multiplication operators with their symbols ξ_w, i.e.,

$$L_w(\xi_{w'}) = \xi_w \xi_{w'} = \xi_{ww'}, \ \forall w, w' \in \mathbb{G},$$

with their adjoints, $L_w^* = L_{w^{-1}}$, for all $w \in \mathbb{G}$.

If $v \in V\left(\widehat{G}\right)$, then L_v is a projection on H_G, since:

$$L_v^* = L_{v^{-1}} = L_v = L_{vv} = L_v L_v = L_v^2,$$

in $B(H_G)$. Hence, if $w \in FP_r\left(\widehat{G}\right)$, then L_w is a partial isometry on H_G because:

$$L_w^* L_w = L_{w^{-1}} L_w = L_{w^{-1} w}$$

is a projection in $B(H_G)$ since $w^{-1} w \in V\left(\widehat{G}\right)$ by (RR). Trivially, the operator $L_\phi = 0_G$, the zero operator in $B(H_G)$, which is a projection. Thus, the operators $\{L_w\}_{w \in \mathbb{G}}$ are either projections or partial isometries in $B(H_G)$ (also, see [20–24,34]).

On the graph Hilbert space H_G, if $w_1, w_2 \in \mathbb{G}$, then:

$$L_{w_1} L_{w_2} = L_{w_1 w_2}, \text{ and } L_{w_1}^* = L_{w_1^{-1}},$$

Hence, the pair (H_G, L) forms a well-defined Hilbert-space representation of $\mathbb{G}$.

Definition 2. *Let (H_G, L) be the representation of the graph groupoid $\mathbb{G}$ of a graph G. Define the C^* algebra M_G by:*

$$M_G \overset{denote}{=} C^*(L(\mathbb{G})) \overset{def}{=} \overline{\mathbb{C}[L(\mathbb{G})]},$$

in $B(H_G)$, where $C^*(X)$ means the C^*-subalgebra of $B(H_G)$ generated by $X \cup X^*$ of a subset $X \subseteq B(H_G)$, where $X^* = \{x^* : x \in X\}$, $\mathbb{C}[Y]$ is the polynomial algebra in a set Y, and $\overline{Z}$ is the closure of a subset Z of $B(H_G)$. This groupoid C^* algebra M_G is called the graph groupoid $(C^*\text{-})$ algebra of G (or of $\mathbb{G}$). Define a C^*-subalgebra D_G of M_G by:

$$D_G \overset{def}{=} \underset{v \in V(\widehat{G})}{\oplus} (\mathbb{C} \cdot L_v),$$

which is generated by the projections $\{L_v\}_{v \in V(\widehat{G})}$, where $\oplus$ is the direct product of C^* algebra. We call D_G the diagonal subalgebra of M_G.

Every element x of the graph groupoid algebra M_G is expressed by:

$$x = \sum_{w \in \mathbb{G}} t_w L_w \text{ with } t_w \in \mathbb{C}.$$

The unity (or the multiplication-identity operator) 1_G of M_G is determined to be:

$$1_G = \sum_{v \in V(\widehat{G})} L_v \in D_G \text{ in } M_G,$$

since

$$1_G L_w = L_{v_1} L_w = L_{v_1 w} = L_w = L_{w v_2} = L_w L_{v_2} = L_w 1_G,$$

for all $w = v_1 w v_2 \in \mathbb{G}$, with $v_1, v_2 \in V\left(\widehat{G}\right)$, implying that:

$$1_G T = T = T 1_G, \ \forall T \in M_G.$$

Now, we define *a conditional expectation*,

$$E : M_G \to D_G,$$

by:

$$E\left(\sum_{w \in \mathbb{G}} t_w L_w \right) \overset{def}{=} \sum_{v \in V(\widehat{G})} t_v \, L_v,$$

for all $\sum_{w \in \mathbb{G}} t_w \, L_w \in M_G$. For example, if $v_1, v_2 \in V\left(\widehat{G}\right)$ and $w_1, w_2, w_3 \in FP_r\left(\widehat{G}\right)$, then:

$$E\left(2L_{v_1} - L_{v_2} + iL_{w_1} - 3L_{w_2}^* + L_{w_3} \right) = 2L_{v_1} - L_{v_2},$$

in $D_G \subseteq M_G$. This is indeed a well-defined conditional expectation in the sense of [17], since it is a bounded operator from M_G onto D_G, satisfying:

$$E(d) = d, \ \forall d \in D_G,$$

and

$$E(d_1 x d_2) = d_1 E(x) d_2, \ \forall x \in M_G, \text{ and } d_1, d_2 \in D_G,$$

and

$$E(x^*) = E(x)^*, \ \forall x \in M_G.$$

Thus, the pair (M_G, E) forms an amalgamated D_G-valued C^*-probability space with amalgamation over D_G (see [16–18,20,21]).

Recall that two directed graphs G_1 and G_2 are said to be graph-isomorphic. If there exist bijections,

$$g_V : V(G_1) \to V(G_2), \ g_E : E(G_1) \to E(G_2),$$

such that:

$$g_E(e) = g_E(v_1 e v_2) = g_V(v_1) g_E(e) g_V(v_2),$$

in $E(G_2)$ for all $e = v_1 e v_2 \in E(G_1)$, with $v_1, v_2 \in V(G_1)$. The pair (g_V, g_E) is said to be a graph isomorphism from G_1 to G_2. Recall also that two groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be groupoid isomorphic. If there exists a bijection $g : \mathcal{G}_1 \to \mathcal{G}_2$ such that:

$$g(w_1 w_2) = g(w_1) g(w_2) \text{ in } \mathcal{G}_2,$$

for all $w_1, w_2 \in \mathcal{G}_1$.

Proposition 1. *Let G_1 and G_2 be directed graphs. The shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$ are graph-isomorphic if and only if the graph groupoid algebra forms M_{G_1} and M_{G_2} are $*$-isomorphic—i.e.,*

$$\widehat{G_1} \overset{graph}{=} \widehat{G_2} \Longleftrightarrow M_{G_1} \overset{*\text{-}iso}{=} M_{G_2},$$

where "$\overset{graph}{=}$" means "being graph-isomorphic to" and "$\overset{*\text{-}iso}{=}$" means "being $*$-isomorphic to".

Proof. ($\Rightarrow$) If $\widehat{G_1} \overset{graph}{=} \widehat{G_2}$ via a graph isomorphism (g_V, g_E), then the graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomoprhic via the groupoid isomorphism g, satisfying:

$$g(w) = g(v_1 w v_2) = g_V(v_1)(g_E(e_1) \cdots g_E(e_k))g_V(v_2),$$

in $\mathbb{G}_2$ for all $w = v_1(e_1 \dots e_k)v_2 \in \mathbb{G}_1$ with $v_1, v_2 \in V\left(\widehat{G}\right)$ and $e_1, \dots, e_k \in E\left(\widehat{G}\right)$, with the axiomatization: $g(\phi) = \phi$—i.e.,

$$\widehat{G_1} \overset{graph}{=} \widehat{G_2} \Longrightarrow \mathbb{G}_1 \overset{groupoid}{=} \mathbb{G}_2,$$

where "$\overset{groupoid}{=}$" means "being groupoid-isomorphic to". By Definition 2, M_{G_1} and M_{G_2} are $*$-isomorphic as C^*-algebra.

($\Leftarrow$) Assume now that $\widehat{G_1} \overset{graph}{\neq} \widehat{G_2}$. Then, $\mathbb{G}_1 \overset{groupoid}{\neq} \mathbb{G}_2$. Hence, $M_{G_1} \overset{*\text{-}iso}{\neq} M_{G_2}$. □

2.3. From Undirected Graphs to Graph Groupoids

In this section, motivated by Proposition 1, we re-construct graph groupoids $\mathbb{G}$ from "undirected" graphs G. Without a loss of generality, one may understand that undirected graphs represent the shadowed graphs of directed graphs by regarding each undirected edge as two edges with opposite directions (an edge and its shadow):
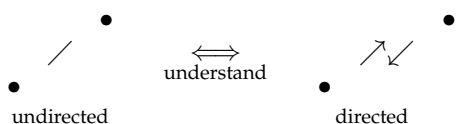
Let $G = (V(G), E(G))$ be an undirected graph (with more than one vertex) with the vertex set $V(G)$ and its "undirected" or non-oriented edge set $E(G)$. If $e \in E(G)$ is a undirected edge connecting the vertices $v_1, v_2 \in V(G)$ (which are not necessarily distinct), then one can assign two directions on e,

$$e_+ = v_1 e_+ v_2, \text{ or } e_- = v_2 e_- v_1.$$

If we carry out such an orientation process for all edges of $E(G)$ and fix the directions for edges, then such a undirected graph G becomes a directed graph $\overrightarrow{G}$ for a fixed direction with the shadow $\left(\overrightarrow{G}\right)^{-1} \overset{denote}{=} \overleftarrow{G}$. Such a directed choice $\overrightarrow{G}$ gives the corresponding shadowed graph $\widehat{\overrightarrow{G}} \overset{denote}{=} \widehat{G}$, inducing the graph groupoid $\mathbb{G}$.

Note here that the construction of the shadowed graph $\widehat{G}$ is free from the choice of directions on G, by Proposition 1. Hence, the construction of the graph groupoid $\mathbb{G}$ is also free from the choice of directions in G. If $\overrightarrow{G}^1$ and $\overrightarrow{G}^2$ are the directed graphs induced by a given undirected graph G, then their shadowed graphs satisfy:

$$\widehat{\overrightarrow{G}^1} \overset{graph}{=} \widehat{\overrightarrow{G}^2} \overset{say}{=} \widehat{G},$$

Hence, the corresponding graph groupoids generated by these are groupoid-isomorphic to $\mathbb{G}$.

In other words, if we regard each undirected edge $e \in E(G)$ with a possible directed edge e_+ or e_- in $E\left(\widehat{G}\right)$, where $\widehat{G}$ is the shadowed graph in the sense of the above paragraphs, then for any choice of directions on all edges one can have the same (or, isomorphic) graph groupoid $\mathbb{G}$ with the reduction (RR):

$$e_+ e_-, \ e_- e_+ \in V(G), \ \forall e \in E(G),$$

satisfying the set equality:

$$\mathbb{G} = \{\phi\} \cup V(G) \cup FP_r\left(\widehat{G}\right).$$

Note again that the construction of $\widehat{G}$ (and, hence, that of $\mathbb{G}$) is free from the choice of directions of all edges of G. Thus, from below, if we fix any undirected graph G, one can identify it in the shadowed graph $\widehat{G}$ in the above sense.

Under our settings, two undirected graphs G_1 and G_2 are said to be graph-isomorphic if there exists a graph isomorphism (g_V, g_E) such that $g_V : V(G_1) \rightarrow V(G_2)$ and $g_E : E(G_1) \rightarrow E(G_2)$ are bijections and:

$$g_E(v_1 e_+ v_2) = g_V(v_1) g_E(e_+) g_V(v_2),$$

Hence, automatically,

$$g_E(v_2 e_- v_1) = g_V(v_2) g_E(e_-) g_V(v_1),$$

in $E\left(\widehat{G}\right)$ whenever $e_+ = v_1 e_+ v_2$ with $v_1, v_2 \in V(G)$, where e_+ (and e_-) is an arbitrarily fixed direction of e, satisfying:

$$g_E(e_+) = g_E(e) = g_E(e_-).$$

Proposition 2. *If two undirected graphs G_1 and G_2 are graph-isomorphic (as undirected graphs), then the corresponding graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic.*

Proof. By definition, if two undirected graphs G_1 and G_2 are isomorphic, then the shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$ are isomorphic as directed graphs. Hence, the graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic. $\square$

By Proposition 2, one can obtain the following result.

Corollary 1. *If two undirected graphs G_1 and G_2 are graph-isomorphic, then the graph groupoid algebra forms M_{G_1} and M_{G_2} are $*$-isomorphic.*

Proof. This is shown by Propositions 1 and 2. $\square$

2.4. Semicircular Elements

Let (A, ψ) be a mathematical pair of a topological (noncommutative) $*$-algebra A (for instance, a C^*-algebra, a von Neumann algebra, or a Banach $*$-algebra) and a (bounded) linear functional ψ on A. Then, this is said to be a (noncommutative) *topological (free) $*$-probability space* (resp., a C^*-probability space; resp., a W^*-probability space; resp., a Banach $*$-probability space; etc.). An operator $a \in A$ is said to be a free random variable if we regard it as an element of (A, ψ). For example, if $a \in (A, \psi)$ is self-adjoint in A as an operator in the sense that $a^* = a$, then a is called a self-adjoint free random variable. It can be found that even though A is a commutative algebra, the corresponding topological $*$-probability space (A, ψ) is determined as a statistical-analytic structure. However, free probability generally applies for cases where A is noncommutative. Such a free-probabilisitic structure (A, ψ) is understood as a noncommutative counterpart of a measure space (X, μ) of a measurable set X and a measure μ in commutative analysis. In particular, if (A, ψ) is unital in the sense

that: (i) A has the unity 1_A and (ii) $\psi(1_A) = 1$, then it is a noncommutative version of a probability space (Y, ρ) with the total measure $\rho(Y) = 1$. Thus, in general, topological "$*$-probability" spaces are the noncommutative analogue of "measure" spaces.

If $a_1, \dots, a_s \in (A, \psi)$ are free random variables for $s \in \mathbb{N}$, then the free distribution of $a_1, \dots, a_s$ is characterized by the joint free moments:

$$\psi\left(\prod_{k=1}^{n} a_{i_k}^{r_k}\right) = \psi\left(a_{i_1}^{r_1} \cdots a_{i_n}^{r_n}\right),$$

Equivalently, the joint free cumulants are:

$$k_n^{\psi}\left(a_{i_1}^{r_1}, \dots, a_{i_n}^{r_n}\right),$$

for all $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ and $(r_1, \dots, r_n) \in \{1, *\}^n$ for all $n \in \mathbb{N}$, where $k_n^{\psi}(\bullet)$ is the free cumulant on A in terms of ψ. For more details, see e.g., [17,18].

Thus, the free distribution of a "self-adjoint" free random variable a is fully characterized by:

$$\text{the free-moment sequence } (\varphi(a^n))_{n=1}^{\infty}, \tag{1}$$

or:

$$\text{the free-cumulant sequence } \left(k_n^{\varphi}(a, \dots, a)\right)_{n=1}^{\infty}.$$

Definition 3. *A self-adjoint free random variable $x \in (A, \psi)$ is said to be semicircular if:*

$$\psi(x^n) = \omega_n c_{\frac{n}{2}}, \ \forall n \in \mathbb{N}, \tag{2}$$

where:

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$ and

$$c_k = \frac{1}{k+1}\binom{2k}{k} = \frac{(2k)!}{k!(k+1)!}$$

are the k-th Catalan numbers for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

By the Möbius inversion of [17], a self-adjoint free random variable x is semicircular in (A, ψ) if and only if:

$$k_n^{\psi}(x, \dots, x) = \delta_{n,2} \tag{3}$$

for all $n \in \mathbb{N}$ by (2), where δ is the Kronecker delta.

Therefore, according to *the semicircular law*, the free distributions of all semicircular elements are characterized by the free-moment sequence:

$$(0, c_1, 0, c_2, 0, c_3, 0, c_4, \dots), \tag{4}$$

and, equivalently, by the free-cumulant sequence:

$$(0, 1, 0, 0, 0, 0, \dots), \tag{5}$$

by (2) and (3), universally.

3. Radial Operators of Graph C^*-Probability Spaces

In the rest of this paper, we assume all given directed graphs are "connected", "finite", and have more than one vertex. Recall that a graph G is disconnected if there exists two distinct vertices:

$$v_1 \neq v_2 \in V\left(\widehat{G}\right) = V(G),$$

in the shadowed graph $\widehat{G}$ of G, such that there are no reduced finite paths $w \in FP_r\left(\widehat{G}\right)$ in the graph groupoid $\mathbb{G}$, such that either:

$$w = v_1 w v_2, \text{ or } w = v_2 w v_1.$$

Additionally, a graph G is finite if:

$$|V(G)| < \infty, \text{ and } |E(G)| < \infty.$$

By regarding the shadowed graph $\widehat{G}$ as a undirected graph of Section 2.3, say G_u, the connectedness and finiteness will be defined similarly without considering the direction in G (up to graph isomorphisms).

Definition 4. *Let M_G be the graph groupoid algebra of a graph G. Define operators $T_w \in M_G$ by:*

$$T_w = L_w + L_w^* = L_w + L_{w^{-1}}, \ \forall w \in FP_r\left(\widehat{G}\right). \tag{6}$$

Such operators T_w of (6) are called the w-radial operators for $w \in FP_r\left(\widehat{G}\right)$.

By (6), every reduced-finite-path-radial operator is self-adjoint in M_G. We define a linear functional φ on the diagonal subalgebra D_G of M_G using a morphism:

$$\varphi\left(\sum_{v \in V(\widehat{G})} t_v L_v\right) = \sum_{v \in (\widehat{G})} t_v.$$

Then, this is not only a well-defined bounded linear functional on D_G because $\left|V\left(\widehat{G}\right)\right| < \infty$, but also a trace satisfying:

$$\varphi(S_1 S_2) = \varphi(S_2 S_1), \ \forall S_1, S_2 \in D_G.$$

We define a linear functional τ on M_G using:

$$\tau \overset{\text{def}}{=} \varphi \circ E \text{ on } M_G. \tag{7}$$

Since φ is a trace on D_G and E is a conditional expectation from M_G to D_G, the morphism τ of (7) is a well-defined bounded linear functional. Thus, a well-defined C^*-probability space (M_G, τ) is constructed by a graph G.

For example, the unity $1_G = \sum\limits_{v \in V(\widehat{G})} L_v \in D_G$ of M_G satisfies:

$$\tau(1_G) = \varphi\left(\sum_{v \in V(\widehat{G})} L_v\right) = \sum_{v \in V(\widehat{G})} 1 = |V(G)|.$$

Since our graph G is assumed to be finite, $|V(G)| < \infty$, implying that τ is indeed bounded on M_G.

Definition 5. *The C^*-probability space (M_G, τ) is called the graph C^*-probability space of G (or of $\mathbb{G}$).*

Two C^*-probability spaces, (A_1, ψ_1) and (A_2, ψ_2), are said to be free-isomorphic if there exists an $*$ isomorphism:

$$\Phi : A_1 \to A_2,$$

such that:

$$\psi_2(\Phi(a)) = \psi_1(a), \ \forall a \in (A_1, \psi_1).$$

In such a case, we call the ∗ isomorphism Φ a free isomorphism. If two C^*-probability spaces are free-isomorphic, then they have the same free-probabilistic structure.

Theorem 1. *If two shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$ are graph-isomorphic, then the graph C^*-probability spaces $\left(M_{G_1}, \tau_1\right)$ and $\left(M_{G_2}, \tau_2\right)$ will be free-isomorphic. Symbolically,*

$$G_1 \overset{graph}{=} G_2 \implies \left(M_{G_1}, \tau_1\right) \overset{free\text{-}iso}{=} \left(M_{G_2}, \tau_2\right),$$

where "$\overset{free\text{-}iso}{=}$" means "being free-isomorphic to".

Proof. In the proofs of Propositions 1 and 2, we have:

$$\widehat{G_1} \overset{graph}{=} \widehat{G_2} \iff \mathbb{G}_1 \overset{groupoid}{=} \mathbb{G}_2 \iff M_{G_1} \overset{*\text{-}iso}{=} M_{G_2}.$$

Indeed, if $g : \mathbb{G}_1 \to \mathbb{G}_2$ is the groupoid-isomorphism induced by a graph isomorphism (g_V, g_E) satisfying:

$$g(w) = \begin{cases} \phi, \text{ the empty word of } \mathbb{G}_2 & \text{if } w = \phi \text{ in } \mathbb{G}_1 \\ g_V(w) & \text{if } w \in V\left(\widehat{G}\right) \\ g_E(e_1)...g_E(e_n) & \text{if } w = e_1 \ ... \ e_n \in FP_r\left(\widehat{G}\right), \end{cases}$$

in $\mathbb{G}_2$, for all $w \in \mathbb{G}_1$, where $e_1, \ldots, e_n \in E\left(\widehat{G}\right)$ for $n \in \mathbb{N}$, we have an ∗ isomorphism:

$$\Phi : M_{G_1} \to M_{G_2},$$

which satisfies:

$$\Phi\left(\sum_{w \in \mathbb{G}_1} t_w L_w^{(1)}\right) = \sum_{g(w) \in \mathbb{G}_2} t_{g(w)} L_{g(w)}^{(2)}, \text{ in } M_{G_2},$$

for all $\sum_{w \in \mathbb{G}_1} t_w L_w^{(1)} \in M_{G_1}$, where $\left(H_{G_k}, L^{(k)}\right)$ are the Hilbert-space representations of $\mathbb{G}_k$ for $k = 1, 2$.

By (7), it is shown that:

$$\tau_2\left(\Phi\left(\sum_{w \in \mathbb{G}_1} t_w L_w^{(1)}\right)\right) = \tau_2\left(\sum_{g(w) \in \mathbb{G}_2} t_{g(w)} L_{g(w)}^{(2)}\right)$$

$$= \sum_{g(v) \in V\left(\widehat{G_2}\right), v \in V(G_1)} t_{g(v)} = \sum_{g_V(v) \in V(G_2)} t_{g_V(v)}$$

$$= \sum_{v \in V\left(\widehat{G_1}\right)} t_v = \tau_1\left(\sum_{w \in \mathbb{G}_1} t_w L_w^{(1)}\right).$$

Therefore, Φ is a free isomorphism. Hence:

$$\left(M_{G_1}, \tau_1\right) \overset{free\text{-}iso}{=} \left(M_{G_2}, \tau_2\right).$$

□

4. Semicircular Elements of (M_G, τ)

In this section, we study semicircular elements in the graph C^*-probability space (M_G, τ) of a given graph G. Let:

$$T_w = L_w + L_w^* = L_w + L_{w^{-1}} \in M_G$$

be the w-radial operator of a reduced finite path $w \in FP_r\left(\widehat{G}\right)$ in $\mathbb{G}$, which is a self-adjoint free random variable of (M_G, τ).

If $w = v_1 w v_2 \in FP_r\left(\widehat{G}\right)$ with $v_1 \neq v_2$ in $V\left(\widehat{G}\right)$, then:

$$w^n = \phi = w^{-n} = \left(w^{-1}\right)^n = (w^n)^{-1} \text{ in } \mathbb{G}, \tag{8}$$

for all $n \in \mathbb{N} \setminus \{1\}$, because v_1 and v_2 are distinct in $\mathbb{G}$. Therefore:

$$T_w^n = \left(L_w + L_{w^{-1}}\right)^n = \sum_{(e_1,\dots,e_n) \in \{\pm 1\}^n} \left(\prod_{l=1}^n L_{w^{e_l}}\right)$$

$$= \begin{cases} L_{\left(ww^{-1}\right)^{\frac{n-1}{2}} w} + L_{\left(w^{-1}w\right)^{\frac{n-1}{2}} w^{-1}} + 0_G & \text{if } n \text{ is odd} \\[2ex] L_{\left(ww^{-1}\right)^{\frac{n}{2}}} + L_{\left(w^{-1}w\right)^{\frac{n}{2}}} + 0_G & \text{if } n \text{ is even} \end{cases}$$

by (8)

$$= \begin{cases} L_w + L_{w^{-1}} = T_w & \text{if } n \text{ is odd} \\[2ex] L_{v_1} + L_{v_2} & \text{if } n \text{ is even,} \end{cases} \tag{9}$$

for all $n \in \mathbb{N}$.

Theorem 2. *Let $w = v_1 w v_2 \in FP_r\left(\widehat{G}\right)$ with $v_1 \neq v_2 \in V\left(\widehat{G}\right)$. If T_w is the w-radial operator of (M_G, τ), then the free distribution of T_w is characterized by the free moments:*

$$\tau(T_w^n) = 2\omega_n, \text{ for all } n \in \mathbb{N}, \text{where: } \omega_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{10}$$

Proof. With (9), one has that:

$$T_w^{2k-1} = T_w \text{ in } (M_G, \tau),$$

Hence:

$$\tau\left(T_w^{2k-1}\right) = \tau(T_w) = \varphi(E(T_w)) = \varphi(0_G) = 0,$$

for all $k \in \mathbb{N}$. Meanwhile:

$$T^{2n} = L_{v_1} + L_{v_2} \text{ in } (M_G, \tau),$$

This implies that:

$$\tau\left(T_w^{2n}\right) = \varphi(E(L_{v_1} + L_{v_2})) = \varphi(L_{v_1} + L_{v_2}) = 2,$$

for all $n \in \mathbb{N}$. Thus, the free-distributional data (10) hold.   □

By (10), if $w = v_1 w v_2$ is a reduced finite path with distinct vertices v_1 and v_2 in $\mathbb{G}$, then the free distribution of the w-radial operator $T_w \in (M_G, \tau)$ is characterized by the free moment sequence:

$$\left(\tau(T_w^n)\right)_{n=1}^\infty = (0, 2, 0, 2, 0, 2, 0, 2, \dots) = (2\omega_n)_{n=1}^\infty.$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (10).

Lemma 1. *Let $w = vwv \in FP_r\left(\widehat{G}\right)$ be a "loop" reduced finite path with the identical initial and terminal vertices $v \in V\left(\widehat{G}\right)$ in $\mathbb{G}$. Additionally, let $T_w \in (M_G, \tau)$ be the w-radial operator. Then:*

$$\tau(T_w^n) = \omega_n c_{\frac{n}{2}}, \; \forall n \in \mathbb{N}, \tag{11}$$

where ω_n are in the sense of (10) for all $n \in \mathbb{N}$ and c_k are the k-th Catalan numbers for all $k \in \mathbb{N}_0$.

Proof. If w is a loop-reduced finite path adjacent to the vertex v in $\mathbb{G}$, then w^n and w^{-n} are "non-empty" loop-reduced finite paths of $\mathbb{G}$ whose adjacent vertices are v for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$:

$$\tau(T_w^n) = \sum_{(e_1,\ldots,e_n)\in\{\pm1\}^n} \tau\left(L_{\prod_{l=1}^n w^{e_l}}\right),$$

and each summand satisfies:

$$\tau\left(L_{\prod_{l=1}^n w_l^{e_l}}\right) = \begin{cases} 1 & \text{if } \sum_{l=1}^n \varepsilon_l = 0 \\ \\ 0 & \text{otherwise,} \end{cases}$$

since $\prod_{l=1}^n w^{e_l} = v$ in $\mathbb{G}$ if and only if $\sum_{l=1}^n e_l = 0$; equivalently, $\prod_{l=1}^n w^{e_l} \in FP_r\left(\widehat{G}\right)$ in $\mathbb{G}$ if and only if $\sum_{l=1}^n e^l \neq 0$. Thus, one has:

$$\tau(T_w^n) = \sum_{(e_1,\ldots,e_n)\in\{\pm1\}^n,\ \sum_{l=1}^n e_l=0} 1,$$

$$\text{Equivalently: } \tau(T_w^n) = \left|\left\{(\varepsilon_1,\ldots,\varepsilon_n) \in \{\pm1\}^n : \sum_{l=1}^n \varepsilon_l = 0\right\}\right|. \tag{12}$$

It is well-known that:

$$\left|\left\{(\varepsilon_1,\ldots,\varepsilon_n) \in \{\pm1\}^n : \sum_{l=1}^n \varepsilon_l = 0\right\}\right| = \begin{cases} c_{\frac{n}{2}} & \text{if } n \text{ is even} \\ \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

where c_k are the k-th Catalan numbers for all $k \in \mathbb{N}$ (e.g., [17–19,35–38]). Therefore, the free-distributional data (11) hold by (12). $\square$

For convenience, we say "w is a loop" if w is a loop-reduced finite path.

Theorem 3. *A w-radial operator T_w is semicircular in (M_G, τ) if and only if $w \in FP_r\left(\widehat{G}\right)$ is a loop in $\mathbb{G}$.*

Proof. ($\Leftarrow$) Since a w-radial operator T_w is self-adjoint in M_G by definition, the free distribution of T_w is characterized by its free-moment sequence in (M_G, τ). If w is a loop, then:

$$(\tau(T_w^n))_{n=1}^\infty = (0, c_1, 0, c_2, 0, c_3, \ldots) = \left(\omega_n c_{\frac{n}{2}}\right)_{n=1}^\infty,$$

by (11). Therefore, it is semicircular in (M_G, τ) by (2) or (4).
($\Rightarrow$) Conversely, if w is not a loop in $\mathbb{G}$, then the free distribution of T_w is characterized by (10), implying that it is not semicircular in (M_G, τ) by (2). $\square$

The above theorem characterizes the semicircular law of "reduced-finite-path-radial" operators in (M_G, τ). The semicircularity of (M_G, τ) induced by the generators $\{L_w\}_{w\in\mathbb{G}}$ of M_G is fully characterized by the combinatorial property, the "loop-ness" on the graph groupoid $\mathbb{G}$. Theorem 3 generalizes the semicircularity necessary condition of [34]. More generally, we can obtain the following result.

Theorem 4. *Let (M_G, τ) be the graph C^*-probability space of a graph G, and let $T = \sum_{w\in\mathbb{G}} t_w L_w$ be an arbitrary non-zero "self-adjoint" free random variable of (M_G, τ). Then, T is semicircular in*

(M_G, τ) *if and only if there exists a loop $w_0 \in \mathbb{G}$, such that $T = T_{w_0}$, where $T_{w_0} = L_{w_0} + L_{w_0^{-1}}$ is the w_0-radial operator of (M_G, τ).*

Proof. ($\Leftarrow$) If $T = T_{w_0} \in (M_G, \tau)$ for a loop $w_0 \in \mathbb{G}$, then it is semicircular in (M_G, τ) according to Theorems 2 and 3.

($\Rightarrow$) Assume that $T \neq T_{w_0}$ for some loops w_0 of $\mathbb{G}$ in (M_G, τ). We already know that if $T = T_w$ with a non-loop-reduced finite path w of $\mathbb{G}$, then T is not semicircular in (M_G, τ) according to (10). Additionally, if $T = L_v$ for $v \in V(G)$, then it is not semicircular either.

Suppose now a given self-adjoint free random variable T that is not a reduced-finite-path-radial operator of M_G. Then, by the self-adjointness of T, this can be re-expressed by:

$$
T = \left(\sum_{v \in V(\widehat{G})} t_v L_v \right) + \left(\sum_{w \in FP_r(\widehat{G})} \left(t_w L_w + \overline{t_w} L_{w^{-1}} \right) \right),
$$

in (M_G, τ), where $t_v \in \mathbb{R}$ for all $v \in V\left(\widehat{G}\right)$ and $t_w \in \mathbb{C}$ for all $w \in FP_r\left(\widehat{G}\right)$. These have conjugates $\overline{t_w}$ in $\mathbb{C}$ (e.g., [23]). For convenience, denote:

$$
T_V = \sum_{v \in V(\widehat{G})} t_v L_v, \text{ and } T_{FP} = \sum_{w \in FP_r(\widehat{G})} \left(t_w L_w + \overline{t_w} L_{w^{-1}} \right),
$$

decomposing

$$
T = T_V + T_{FP} \text{ in } M_G,
$$

as a self-adjoint free random variable of (M_G, τ).

Clearly, if $T_V \neq 0_G$ and $T_{FP} = 0_G$, then T is not semicircular in (M_G, τ) because:

$$
\tau(T) = \varphi(T_V) = \sum_{v \in V(\widehat{G})} t_v \neq 0,
$$

This implies that the first (and, hence, odd) free moment of T is non-zero. Therefore, if $T_V \neq 0_G$ in T, then T is not semicircular in (M_G, τ).

Now, let $T_V = 0_G$ and $T_{FP} \neq 0_G$, which is not a radial operator containing a summand,

$$
S_w \overset{\text{denote}}{=} t_w L_w + \overline{t_w} L_w, \text{ for some } w \in FP_r\left(\widehat{G}\right).
$$

First of all, if $t_w \neq 1$ in $\mathbb{C}$, then the summand S_w is not semicircular. Hence, T is not semicircular in (M_G, τ); secondly, if $t_w = 1$ and w is not a loop, the summand S_w is not semicircular. Hence, T is not semicircular in (M_G, τ) either (see [34]). Finally, if $t_w = 1$ and w is a loop, then the summand S_w is semicircular. However, since T contains other summands according to our assumption that T is not a radial operator, the free moments of T are not identical to those of S_w (see [34]); hence, T cannot be semicircular in (M_G, τ). In conclusion, if T is not a reduced-finite-path-radial operator, then it is not semicircular in (M_G, τ).

Therefore, if an arbitrary self-adjoint free random variable T satisfies $T \neq T_{w_0}$ for some loops $w_0 \in \mathbb{G}$, then it is not semicircular in (M_G, τ). Equivalently, if a self-adjoint free random variable T is semicircular in (M_G, τ) then there exists a loop $w \in \mathbb{G}$, such that $T = T_w$ in (M_G, τ). $\square$

The above theorem fully characterizes the semicircularity in (M_G, τ) according to the "loop-ness" of $\mathbb{G}$!

Now, let K be a graph:

$$
V(K) = \{v_1, v_2, v_3\}, \text{ and } E(K) = \{e_{12}, e_{23}\},
$$

where e_{ij} are the edges connecting vertex v_i to vertex v_j for $i,j \in \{1,2,3\}$. Then, the corresponding graph groupoid $\mathbb{K}$ does not contain a loop adjacent to each vertex, because:

$$\mathbb{K} = \{\phi\} \cup \{v_1, v_2\} \cup \left\{ \begin{array}{c} e_{12}, e_{12}^{-1}, e_{23}, e_{23}^{-1} \\ e_{12}e_{23}, e_{23}^{-1}e_{12}^{-1} \end{array} \right\}.$$

This example demonstrates that there are (connected finite-directed) graphs (with more than one vertex) that do not induce semicircular elements in their graph C^*-probability spaces by the semicircularity characterization, Theorem 4.

Recall that an "undirected" graph K (in the sense of Section 2.3) is a finite-connected tree if it is does not contain (undirected) loops. Equivalently, its graph groupoid $\mathbb{K}$ does not contain loops.
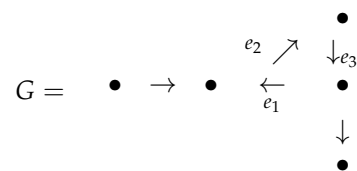
Corollary 2. *Let G_u be a (finite-connected) undirected tree as the shadowed graph $\widehat{G}$ of G. Then, the corresponding graph C^*-probability space (M_G, τ) does not contain semicircular elements.*

Proof. Let G_u be a tree understood as the shadowed graph $\widehat{G}$ of the finite-connected "directed graph" G. Then, the corresponding graph groupoid $\mathbb{G}$ and the graph C^*-probability space (M_G, τ) can be well-determined (see Section 2.3). By Theorem 4, (M_G, τ) does not contain semicircular elements because the graph groupoid $\mathbb{G}$ does not contain loops. $\square$

For our purpose, we finish this section with the following concept.

Definition 6. *Let $\mathbb{G}$ be a graph groupoid of a graph G. A loop $w \in \mathbb{G}$ is called a loop-diagram if no loop $w_o \in \mathbb{G}$ exists, such that $w = w_o^k$ for all $k \in \mathbb{N} \setminus \{1\}$.*

Note that, if $w \in \mathbb{G}$ is a loop, then there will always exist unique loop-diagrams $w_o \in \mathbb{G}$ and $k \in \mathbb{N}$, such that $w = w_o^k$ in $\mathbb{G}$. In particular, if $w = w_o^1$ in $\mathbb{G}$, then w itself is a loop-diagram in $\mathbb{G}$ according to Definition 6. Additionally, if $w_o \in \mathbb{G}$ is a loop-diagram, then one can take infinitely many loops $\left\{ w_o^k \right\}_{k=1}^{\infty}$ in $\mathbb{G}$. For instance, if:

$$G = \quad \bullet \; \rightarrow \; \bullet \; \underset{e_1}{\leftarrow} \; \begin{array}{c} \bullet \\ {}^{e_2}\nearrow \quad \downarrow e_3 \\ \bullet \\ \downarrow \\ \bullet \end{array}$$

then one can take the loop-diagrams of $\mathbb{G}$:

$$d_1 = e_1 e_2 e_3, \; d_2 = e_2 e_3 e_1, \; \text{and } d_3 = e_3 e_1 e_2,$$

and their shadows:

$$d_1^{-1}, \; d_2^{-1}, \; \text{and } d_3^{-1},$$

in the graph groupoid $\mathbb{G}$ of G. Additionally, this verifies that:

$$\left(\bigcup_{k=1}^{3} \left\{ d_k^{\pm n} : n \in \mathbb{N} \right\} \right) \subset \mathbb{G}$$

are the infinitely many loops of $\mathbb{G}$. Note that, if the finiteness assumption exists, there are only finitely many loop-diagrams in a given graph (even though there are infinitely many loops).

As we discussed in [34], even though the graph groupoid $\mathbb{G}$ does not contain loops, a graph G whose undirected graph G_u is a tree can induce certain semicircular elements (artificially but naturally) from its so-called the fractal cover G_{oo}, which is a fractal graph generating the graph fractaloid $\mathbb{G}_{oo}$, a groupoid satisfying the fractality. Every fractal groupoid $\mathbb{G}$ has loop-diagrams adjacent to all its vertices; hence, $\mathbb{G}$ has infinitely many

loops at all vertices. Thus, even though (M_G, τ) of a graph such as G does not contain semicircular elements according to Corollary 2, the graph C^*-probability space $(M_{G_{oo}}, \tau_{oo})$ contains infinitely many semicircular elements. For more details, see [34].

We note that the main results of Section 4 show that the existence of the semicircular elements in (M_G, τ) is determined by the existence of loops in $\mathbb{G}$, which is characterized by the fact that the undirected graph G_u of a graph G is a tree, meaning that the semicircularity in (M_G, τ) is the loop-ness of $\mathbb{G}$, which is characterized by the tree-ness of the undirected graph of G.

Notation and Assumption. From below, if we say "a graph G is a tree", then this means that "the undirected graph G_u of G is a tree".

By the above assumption, one can summarize this section as that the semicircularity of (M_G, τ) is the loop-ness of $\mathbb{G}$, which is the "tree-ness" of G.

5. Graph Groupoid Index and Graph-Tree Index

In this section, we define the graph groupoid index and the graph-tree index on graph groupoids as a function from $\mathcal{G} \times \mathcal{G}$ to $\mathbb{R}_1^\infty$, where $\mathcal{G}$ is the family of all graphs and $\mathbb{R}_1^\infty$ is the set of all positive real numbers greater than or equal to 1. These quantities measure how a certain graph groupoid $\mathbb{K}$ is embedded in a graph groupoid $\mathbb{G}$. For our purposes, we restrict our interests to connected finite-directed graphs with more than one vertex, whose shadowed graphs are understood as undirected graphs up to graph isomorphisms in the sense of Section 2.3. Additionally, for algebraic convenience, we include the single-vertex graph $\mathbb{I}$ in our scope, where $|V(\mathbb{I})| = 1$, and $E(\mathbb{I}) = \phi$, with ϕ being the empty set.

5.1. The Graph Groupoid Index

Let G be a (connected finite-directed) graph (with more than one vertex) with the graph groupoid $\mathbb{G}$, and let G_u be the corresponding undirected graph regarded as the shadowed graph $\widehat{G}$ of G. Recall that J is a subgraph of G if it is a graph with:

$$V(J) \subseteq V(G),$$

and

$$E(J) = \{e \in E(G) : e = v_1 e v_2, \text{ for } v_1, v_2 \in V(J)\}.$$

Recall also that U is a full subgraph of G if it is a graph with:

$$E(U) \subseteq E(G),$$

and

$$V(U) = \{v \in V(G) : e = ve, \text{ or } e = ev \text{ for } e \in E(U)\}.$$

Undirected versions of subgraphs and full subgraphs are canonically defined.

Recall that, according to our assumption, a undirected graph G_u is connected. Even though G_u is connected, it is possible that a subgraph J_u or a full-subgraph K_u is disconnected. More generally, we next define the following general concept.

Definition 7. *Let G_u be the undirected graph of G. A combinatorial structure $K_u = (V(K_u), E(K_u))$ is said to be a undirected part of G_u if K_u is a (connected or disconnected) graph with:*

$$V(K_u) \subseteq V(G_u), \text{ and } E(K_u) \subseteq E(G_u),$$

This includes the case where either:

$$V(K_u) = \phi, \text{ or } E(K_u) = \phi,$$

where ϕ is the empty set. We denote a relation "K_u as part of G_u" by "$K_u \subseteq G_u$." In particular, if $\phi \neq V(K_u) \subseteq V(G_u)$ and $E(K_u) = \phi$, then the part K_u is said to be a vertex part of G_u (or a vertex graph independently). The "directed" version of a part is similarly defined according to the direction.

Notation and Assumption. Note that if G_u is the undirected graph of G, and $K_u \subseteq G_u$ is a part, then there exists a "directed" part K of G whose shadowed graph $\widehat{K}$ induces K_u as in Section 2.3. Therefore, from below, we can simply say that "$K_u \subseteq G_u$" and "$K \subseteq G$" are parts without considering whether they are undirected or directed.

By definition, all subgraphs and full subgraphs of G are parts of G. Note that, according to Definition 7, if a part K_1 of G satisfies that $V(K_1) = \phi$, then $E(K_1) = \phi$ automatically, because the part K_1 must be a graph combinatorially (which means "no vertices, no edges connecting vertices"). In such a case, we axiomatize such a part K_1 to be the empty part of G (or the empty graph independently). However, if K_2 is a part of G satisfying $E(K_2) = \phi$, then $V(K_2)$ is not necessarily empty. It can simply be a subset of $V(G)$ and, hence, K_2 could form either the empty graph or the vertex graph embedded in G.

Now, let G be a given directed graph and $K \subseteq G$ be a part. Suppose that K has n-many connected components $K_1, \dots, K_n$, where each K_i is a connected part of G (such as a connected graph) with:

$$V(K_i) \subseteq V(K), \text{ and } E(K_i) \subseteq E(K),$$

for all $i = 1, \dots, n$, with:

$$V(K) = \bigsqcup_{i=1}^{n} V(K_i) \text{ and } E(K) = \bigsqcup_{i=1}^{n} E(K_i),$$

where $\sqcup$ is the disjoint union. Then, we "collapse" the connected parts $K_1, \dots, K_n$ to the vertices $x_1, \dots, x_n$ by identifying each K_i in the collapsed vertex x_i for all $i = 1, \dots, n$. For instance, if:

and

is a part of G with rgw connected components:

in G, then we collapse $K_{0,1}$ and $K_{0,2}$ to the vertices x_1 and x_2,

$$x_1 = \star \quad \text{and} \quad x_2 = \times.$$

Then, by identifying all the connected components $K_1, \ldots, K_n$ on the collapsed vertices $x_1, \ldots, x_n$, we construct a new graph G_K using a graph with:

$$V(G_K) = (V(G) \setminus V(K)) \cup \{x_1, \ldots, x_n\},$$
$$\text{and } E(G_K) = E(G) \setminus E(K), \tag{13}$$

with the identification rule. If either $e = ev$ or $e = ve$ for $e \in E(G) \setminus E(K)$ with $v \in V(K_i) \subseteq V(K)$, for some $i = 1, \ldots, n$, then we identify e with $e = ex_i$ and $e = x_i e$, respectively, where x_i is the collapsed vertex of K_i for $i = 1, \ldots, n$. For example, if $K_0 \subseteq G_0$ are given as in the above paragraph, then:

$$(G_0)_{K_0} = \quad \star \quad \rightarrow \quad \times \quad .$$
$$\downarrow$$
$$\bullet$$

Proposition 3. *If G is a given graph and $K \subseteq G$ is a part of it, then the corresponding new graph G_K of (13) is a connected finite-directed graph. Equivalently, if G_u is the undirected graph of G and $K_u \subseteq G_u$ is a undirected part, then the undirected graph $G_{K:u}$ of G_K is connected and finite.*

Proof. If K is a part of G with its connected components $K_1, \ldots, K_n$ for $n \in \mathbb{N}$, there always exists a reduced finite path $w = vwx$ or $w = xwv$ for all $v \in V(G) \setminus V(K)$ and $x \in \bigcup\limits_{i=1}^{n} V(K_i) = V(K)$ due to the connectedness of G. This guarantees that, for all $v \in V(G) \setminus V(K)$, there always exists a reduced finite path y of the graph groupoid $\mathbb{G}_{\mathbb{K}}$ of the new graph G_K of (13), such that $y = vyx_i$ or $y = x_i yv$ for any collapsed vertices $\{x_1, \ldots, x_n\}$ in $\mathbb{G}_{\mathbb{K}}$. This implies that the graph G_K of (13) is connected. Clearly, the finiteness is satisfied. $\quad \square$

As we can see in the above example, the graph $(G_0)_{K_0}$ is indeed connected and finite.

Definition 8. *The graph G_K of (13) induced by the part inclusion $K \subseteq G$ of a given graph G is called the quotient graph of G by K. If K is the empty part of G, then the corresponding quotient graph G_K is axiomatized to be G itself. Equivalently, the corresponding undirected graph $G_{K:u}$ of G_K is called the (undirected) quotient graph G_K.*

Note that, if $G_{K:u}$ is the undirected quotient graph of the quotient graph G_K, then it is not difficult to check that $G_{K:u}$ is isomorphic to $(G_u)_{K_u}$, where $K_u \subseteq G_u$ are the undirected counterparts of $K \subseteq G$.

According to Definitions 8 and 9, if a given graph G is finite and connected, then the quotient graphs G_K are finite and connected too for all parts $K \subseteq G$. As an independent graph, the quotient graphs G_K have their graph groupoids $\mathbb{G}_{\mathbb{K}}$ for all parts $K \subseteq G$. It is easy to verify that if K is a vertex part of G then $G_K = G$ by (13); if $K = G$, then G_G is the vertex graph $\{x\}$ with $V(G_G) = \{x\}$, and $E(G_G) = \phi$, where x is the collapsed vertex of G in G.

Definition 9. *Let $\mathbb{G}_{\mathbb{K}}$ be the graph groupoid of the quotient graph G_K of a graph G according to part $K \subseteq G$. Then, the cardinality $|\mathbb{G}_{\mathbb{K}}|$ is called the index of the part inclusion $K \subseteq G$. We denote this by $[G : K]$. Suppose that $K_u \subseteq G_u$ is the undirected part-inclusion induced by the inclusion $K \subseteq G$. Then, the (undirected) index $[G_u : K_u]$ is also defined as the index $[G : K] = |\mathbb{G}_{\mathbb{K}}|$:*

$$[G_u : K_u] \overset{def}{=} [G : K] = |\mathbb{G}_{\mathbb{K}}|.$$

For example, if $K_0 \subseteq G_0$ is given as above in the text, then:

$$[G_0 : K_0] = 1 + 3 + 2 \cdot 2 + 2 \cdot 2 = 12,$$

where $1 = |\{\phi\}|$, $3 = \left| V\left(\widehat{(G_0)_{K_0}}\right) \right|$, $2 \cdot 2 = \left| E\left(\widehat{(G_0)_{K_0}}\right) \right|$, and the next $2 \cdot 2$ are the cardinality of the reduced-length-2 finite paths of $(\mathbb{G}_0)_{\mathbb{K}_0}$.

It must be noted that the single-vertex graph $\mathbb{I}$, with $V(\mathbb{I}) = \{v\}$ and $E(\mathbb{I}) = \phi$, has its graph groupoid, which is also denoted by $\mathbb{I}$, with:

$$|\mathbb{I}| = 1,$$

since the empty word does not exist in $\mathbb{I}$ (because the single vertex x is admissible to itself, meaning that $x^n = x$ for all $n \in \mathbb{Z}$). This implies that:

$$[G : G] = |\mathbb{G}_{\mathbb{G}}| = |\mathbb{I}| = 1,$$

for all graphs G according to Definition 9.

5.2. Graph-Tree Index

Let G be a graph with its graph groupoid $\mathbb{G}$ and G_u be the corresponding undirected graph understood as the shadowed graph $\widehat{G}$ of G. Suppose that $w_o = e_1 \; \ldots \; e_k \in \mathbb{G}$ is a loop-diagram with:

$$
\begin{aligned}
e_1 = v_1 e_1 v_2, \; e_2 = v_2 e_2 v_3, \ldots, \; e_k = v_k e_k v_1 \in E\left(\widehat{G}\right), \\
\text{where } v_1, \ldots, v_k \in V\left(\widehat{G}\right),
\end{aligned}
\tag{14}
$$

for some $k \in \mathbb{N}$, inducing infinitely many loops $\{w_o^n\}_{n=1}^{\infty}$ in $\mathbb{G}$ adjacent to the vertex v_1. For this loop-diagram $w_o = v_1 w_o v_1$, define a part $W_o \subseteq G$ by a graph with:

$$
\begin{aligned}
V(W_o) = \{v_1, \ldots, v_k\}, \\
\text{and } E(W_o) = \{\overline{e_1}, \ldots, \overline{e_k}\},
\end{aligned}
\tag{15}
$$

where:

$$
\overline{e_i} = \left\{
\begin{array}{ll}
e_i & \text{if } e_i \in E(G) \\
\\
e_i^{-1} & \text{if } e_i \in E(G^{-1}),
\end{array}
\right.
$$

Hence, $\overline{e_i} \in E(G)$ for all $i = 1, \ldots, k$, where G^{-1} is the shadow of G. Then, this graph W_o induced by the loop-diagram w_o is a well-defined part of G according to (15), satisfying:
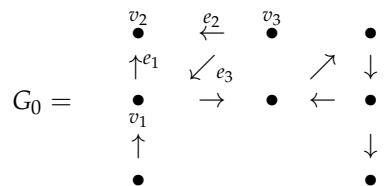
$$
\begin{aligned}
\mathbb{W}_o = \{\phi\} \cup V\left(\widehat{W_o}\right) \cup FP_r\left(\widehat{W_o}\right), \\
\text{with } \{w_o^{\pm n}\}_{n=1}^{\infty} \subset FP_r\left(\widehat{W_o}\right),
\end{aligned}
\tag{16}
$$

where w_o^{-k} are the shadows of w_o^k for all $k \in \mathbb{N}$.

According to (14) and (16), this part W_o of G contains all loops induced by the fixed loop-diagram w_o. Therefore, one can verify that the part $W_o \subseteq G$ of (15) is the maximal part of G containing all loops induced by w_o.

Definition 10. *Let $w_o \in \mathbb{G}$ be a loop-diagram determined by (14), and let $W_o \subseteq G$ be a part (15) of a given graph G. Then, this connected finite-directed graph W_o is called the loop-diagram part of w_o (in short, the w_o part) in the shadowed graph $\widehat{G}$ of G. Clearly, if $W_{o:u} \subseteq G_u$ is the corresponding undirected part of W_o in the undirected graph G_u of G, then it is also called the loop-diagram part of w_o in G_u, which is understood to be the shadowed graph $\widehat{W_o}$ of W_o.*

Suppose that $w_o = v_1 w_o v_1 = e_1 e_2^{-1} e_3 \in \mathbb{G}_0$ is a loop of a graph G_0:

$$
G_0 = \quad
\begin{array}{ccccccc}
\overset{v_2}{\bullet} & \overset{e_2}{\longleftarrow} & \overset{v_3}{\bullet} & & \bullet & & \\
\uparrow e_1 & \swarrow e_3 & & \nearrow & & \downarrow & \\
\underset{v_1}{\bullet} & \longrightarrow & \bullet & \longleftarrow & \bullet & & \\
\uparrow & & & & \downarrow & & \\
\bullet & & & & \bullet & &
\end{array}
\quad .
$$

Then, this loop w_o is a loop-diagram of $\mathbb{G}$ generating infinitely many loops $\{w_o^n\}_{n=1}^{\infty}$ in $\mathbb{G}_0$. From w_o, one can construct the corresponding w_o-part W_o of G_0, where:

$$
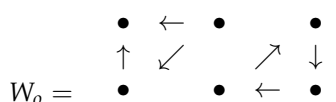V(W_o) = \{v_1, v_2, v_3\} \text{ and } E(W_o) = \{e_1, e_2, e_3\},
$$

according to (15), with its shadowed graph $\widehat{W_o} \subseteq \widehat{G_0}$, being equivalent to $W_{o:u} \subseteq G_{0:u}$, where $G_{0:u}$ is the undirected graph of G_0.

Definition 11. *Let* $w_1, \ldots, w_N \in \mathbb{G}$ *be "all" the loop-diagrams in a graph G for some $N \in \mathbb{N}$. If $W_{k:u} \subseteq G_u$ are the undirected w_k parts of G_u for all $k = 1, \ldots, N$, then we define a new part $W_{o:u} \subseteq G_u$ by the graph union of the loop-diagram parts $W_{1:u}, \ldots, W_{N:u}$, with:*

$$
\begin{aligned}
V(W_{o:u}) &= \overset{N}{\underset{k=1}{\cup}} V(W_{k:u}), \\
\text{and } E(W_{o:u}) &= \overset{N}{\underset{k=1}{\cup}} E(W_{k:u}).
\end{aligned}
\tag{17}
$$

This part, $W_{o:u} \subseteq G_u$, of (17) is called "the" loop-part of G_u. The part $W_o \subseteq G$, in the sense of (15), whose shadowed graph $\widehat{W_o}$ is equivalent to the undirected graph $W_{o:u}$ of (17), is also called the loop-part of G.

Note that, according to (17), the loop-part W_o of a given graph G is the "maximal" part of G whose graph groupoid $\mathbb{W}_o$ contains all loops of the graph groupoid $\mathbb{G}$ of G. For instance, if a given graph G_0 is as in the above paragraph, then the loop part W_o of G_0 is a disconnected graph:

$$
W_o = \quad
\begin{array}{ccccccc}
\bullet & \longleftarrow & \bullet & & \bullet & & \\
\uparrow & \swarrow & & \nearrow & & \downarrow & \\
\bullet & & \bullet & \longleftarrow & \bullet & &
\end{array}
\quad .
$$

Definition 12. *Let $W \subseteq G$ be the loop-part of a given graph G and let G_W be the corresponding quotient graph (13). We call the undirected graph $G_{W:u}$ of the quotient graph G_W the tree of G (or the tree induced by G_u). The graph groupoid $\mathbb{G}_{\mathbb{W}}$ of G_W (or that of $G_{W:u}$) is said to be the tree groupoid of G (or of G_u). If there is no confusion, we also call the quotient (directed) graph G_W the tree of G as well.*

Recall that a undirected tree is a undirected graph whose graph groupoid does not contain loops. Therefore, one can understand "directed trees" to be the directed graphs whose undirected graphs are (undirected) trees. The following result allows us to understand why Definition 12 is meaningful.

Theorem 5. *Let $W \subseteq G$ be the loop-part and G_W be the corresponding quotient graph. Then, G_W is a directed tree and, equivalently, the corresponding undirected graph $G_{W:u}$ of G_W is a tree.*

Proof. Let $W_1, \ldots, W_N$ be the loop-diagram parts of all loop-diagrams $w_1, \ldots, w_N$ of the graph groupoid $\mathbb{G}$ of a given graph G for $N \in \mathbb{N}$, and let the graph union $W \overset{\text{denote}}{=} \overset{N}{\underset{k=1}{\cup}} W_k$ be

the loop-part of G. Then, W has its connected components $K_1, \ldots, K_n$, for some $n \leq N$, and these are identified with the collapsed vertices $x_1, \ldots, x_n$ in the quotient graph G_W, which is connected and finite according to Proposition 3. Since all connected components K_i are collapsed to be the vertices x_i for all $i = 1, \ldots, n$, there are no loops in the graph groupoid $\mathbb{G}_W$ of G_W. Therefore, it becomes a directed tree; hence, the corresponding undirected graph $G_{W:u}$ is a tree. $\square$

The above theorem shows that the quotient graphs G_W of connected finite-directed graphs G by the loop-part W become connected finite-directed trees inducing the trees $G_{W:u}$.

Note that, in the proof of Theorem 5, it is said that the loop-part W, the graph union of all loop-diagram parts $W_1, \ldots, W_N$ of a given graph G, has its connected components $K_1, \ldots, K_n$, for some $n \leq N$. For example, let us assume that a graph G contains the following "part K":

$$K = \quad e_4 \downarrow \quad \begin{array}{ccc} \bullet & \xrightarrow{e_1} & \bullet \\ & \nearrow e_5 & \uparrow \ e_2 \\ \bullet & \xrightarrow[e_3]{} & \bullet \end{array} ,$$

in G. Then, one can find the following loop-diagrams:

$$w_1 = e_1 e_5^{-1} e_4^{-1}, \ w_2 = e_2^{-1} e_3^{-1} e_5,$$

$$w_3 = e_2 e_5^{-1} e_3, \ w_4 = e_5 e_1^{-1} e_4^{-1},$$

and

$$w_5 = e_1 e_2^{-1} e_3^{-1} e_4^{-1}, \ w_6 = e_2^{-1} e_3^{-1} e_4^{-1} e_1,$$

$$w_7 = e_3^{-1} e_4^{-1} e_1 e_2^{-1}, \ w_8 = e_4^{-1} e_1 e_2^{-1} e_3^{-1},$$

"in K" and their shadows $w_1^{-1}, \ldots, w_8^{-1}$, inducing 16 loop-diagram parts (up to graph isomorphism) in G. For instance, the w_1-part is a graph W_1 with:

$$E(W_1) = \{e_1, e_4, e_5\} \text{ in } G,$$

etc. It is not hard to check that, according to Definition 11, $W_k = W_k^{-1}$, where W_k^{-1} are the w_k^{-1}-parts for all $k = 1, \ldots, 8$. Thus, up to the graph isomorphisms, we have a total of eight loop-diagram parts $W_{1:u}, \ldots, W_{8:u}$, as the undirected graphs of W_k, for all $k = 1, \ldots, 8$. Thus, according to (17), one can obtain the part:

$$K_u = \bigcup_{k=1}^{8} W_{k:u},$$

of the undirected graph G_u of G. It is not difficult to check that this very part K_u is connected. Indeed, this part K_u is the corresponding undirected graph of the part $K \subseteq G$. It is connected, and, hence, has only one connected component. Therefore, the graph union of eight loop-diagram parts becomes a single part. Thus, in general, one can verify that if $W_1, \ldots, W_N$ are loop-diagram parts forming the loop-part $W = \bigcup_{k=1}^{N} W_k$, then W may have n-many connected components for some $n \leq N$ in $\mathbb{N}$ in general.

Definition 13. *Let G be a given graph with the loop-part $W \subseteq G$. Then, the quotient graph G_W, or its corresponding undirected graph $G_{W:u}$, is called the graph-tree of G (or, in short, the G-tree). Additionally, the graph grouopoid $\mathbb{G}_W$ of G_W is called the G-tree groupoid. The index,*

$$[G : W] = |\mathbb{G}_W| = |\mathbb{G}_W|,$$

of the part-inclusion $W \subseteq G$ is called the (graph-)tree-index of G (or of the corresponding undirected graph G_u of G). We denote this tree-index of G by $\Gamma(G)$:

$$\Gamma(G) = [G : W] = |\mathbb{G}_{\mathbb{W}}|,$$

where W is the loop-part of G.

The range of tree-indices of given graphs is contained in:

$$\mathbb{R}_1^+ = \{t \in \mathbb{R} : 1 \leq t < \infty\}.$$

Lemma 2. *For a given graph G, the tree-index $\Gamma(G)$ is finite. Moreover:*

$$1 \leq \Gamma(G) < \infty. \tag{18}$$

Proof. In line with our assumption that all given graphs are connected, "finite", and have more than one vertex, the G-tree G_W is connected and "finite", where $W \subseteq G$ is the loop-part. Moreover, since $G_{W:u}$ does not contain any loops, the G-tree groupoid $\mathbb{G}_{\mathbb{W}}$ does not contain loops, implying the finiteness of $\mathbb{G}_{\mathbb{W}}$—i.e., $|\mathbb{G}_{\mathbb{W}}| < \infty$.

Assume now that a graph K is a circulant graph with:

$$V(K) = \{v_1, \ldots, v_N\},$$

and

$$E(K) = \{e_1, \ldots, e_N\},$$

with

$$e_i = v_i e_i v_{i+1}, \text{ for all } i = 1, \ldots, N-1,$$

for any $N \in \{2, 3, 4, \ldots\}$, and

$$e_N = v_N e_N v_1.$$

Then, this graph K has its loop-part that is identified with itself. Thus, the corresponding K-tree is the quotient graph K_K with:

$$V(K_K) = \{x\}, \text{ and } E(K_K) = \phi,$$

This is graph-isomorphic to the single-vertex graph $\mathbb{I}$, satisfying:

$$\Gamma(G) = [G : G] = |\mathbb{G}_{\mathbb{G}}| = |\mathbb{I}| = 1,$$

This implies that, in general:

$$\Gamma(G) \geq 1.$$

Therefore, the G-tree index satisfies:

$$1 \leq \Gamma(G) < \infty.$$

□

Using (18), one can obtain the following range of tree-indices.

Theorem 6. *Let $\Gamma : \mathcal{G} \to \mathbb{R}_1^+$ be the function from:*

$$\mathcal{G} = \{\mathbb{I}\} \cup \left\{ G : \begin{array}{l} G \text{ is a connected, finite graph,} \\ \\ \text{having more than one vertex,} \end{array} \right\}$$

to $\mathbb{R}_1^+$, where every $G \in \mathcal{G}$ is unique up to graph isomorphisms, defined by:

$$\Gamma(G) = \text{ the } G\text{-tree index, } \forall G \in \mathcal{G}.$$
$$\textit{Then, } \Gamma(\mathcal{G}) \subsetneqq \mathbb{N} \text{ in } \mathbb{R}_1^+. \tag{19}$$

Proof. According to (18), the range $\Gamma(\mathcal{G})$ is contained in $\mathbb{R}_1^+$. By definition, for all $G \in \mathcal{G}$,

$$\Gamma(G) = [G : W] = |\mathbb{G}_{\mathbb{W}}|,$$

where $W \subseteq G$ is the loop-part. Since the G-tree groupoid $\mathbb{G}_W$ is a discrete finite set,

$$|\mathbb{G}_{\mathbb{W}}| \in \mathbb{N} \Longleftrightarrow \Gamma(G) \in \mathbb{N},$$

implying that:

$$\Gamma(\mathcal{G}) \subseteq \mathbb{N}.$$

Suppose that we have a single-edge tree,

$$G_e = \quad \underset{v_1}{\bullet} \xrightarrow{\ e\ } \underset{v_2}{\bullet},$$

in $\mathcal{G}$. Note that the G_e-tree is identical to itself because this graph G_e does not contain its loop-part. Equivalently, the loop-part of G_e is the empty graph; hence, $(G_e)_\phi = G_e$ according to Definition 9. It is easy to check that:

$$|\mathbb{G}_e| = \left| \{\phi\} \cup \{v_1, v_2\} \cup \left\{ e, e^{-1} \right\} \right| = 5.$$

This illustrates that:

$$2, 3, 4 \notin \Gamma(\mathcal{G}).$$

Therefore, the relation (19) holds theoretically. $\square$

Note that, by the definition of the family $\mathcal{G}$ in Theorem 6, this contains a graph L_1 with:

$$V(L_1) = \{v\}, \text{ and } E(L_1) = \{e = vev\}.$$

Then, the loop-part W of L_1 is identical to itself. Thus, the quotient graph $(L_1)_W = (L_1)_{L_1}$ is the single-vertex graph $\mathbb{I}$. Hence:

$$\Gamma(L_1) = 1 = \Gamma(\mathbb{I}).$$

All other graphs $G \in \mathcal{G} \setminus \{L_1\}$ containing nonempty loop-parts satisfy:

$$\Gamma(G) = 1, \text{ or } \Gamma(G) \geq 5.$$

Corollary 3. *Let $G \in \mathcal{G}$ be an element containing "nonempty" loop-part W. Then:*

$$\Gamma(G) = 1, \text{ or } \Gamma(G) \geq 5, \tag{20}$$

in $\mathbb{N}$.

Proof. Let $G \in \mathcal{G}$ contain its loop-part W. If $W = G$, then $\Gamma(G) = 1$, as we discussed in the above paragraph. Now, suppose that $W \subsetneqq G$ in $\mathcal{G}$. Then, the minimal possible quotient graph G_W for $W \subset G$ is (graph-isomorphic to) the single-edge tree:

$$G_e = \quad \bullet \longrightarrow \bullet,$$

This satisfies $\Gamma(G) = 5 = \Gamma(G_e)$. Therefore, the relation (20) holds true. $\square$

Now, we define the following concept:

Definition 14. *The function* $\Gamma : \mathcal{G} \to \mathbb{R}_1^+$ *introduced in Theorem 6 is called the graph-tree index.*

Based on (19) and (20), the range of the graph-tree index Γ is properly contained in $\mathbb{N}$.

5.3. Graph-Tree Equivalence

In this section, motivated by the main results of Section 5.2, we consider an equivalence relation in the set:

$$\mathcal{G} = \{\mathbb{I}\} \cup \left\{ G : \begin{array}{c} G \text{ is a connected, finite graph,} \\ \\ \text{having more than one vertex,} \end{array} \right\},$$

These are determined up to graph isomorphisms, where $\mathbb{I}$ is the single-vertex graph. We classify $\mathcal{G}$ under the relation (up to graph isomorphisms).

Consider L_2 be a tree with:

$$V(L_2) = \{v_1, v_2, v_3\} \text{ and } E(L_2) = \{e_{12}, e_{23}\},$$

where $e_{ij} = v_i e_{ij} v_j$,

$$L_2 = \quad \bullet \to \bullet \to \bullet,$$

up to graph isomorphisms, with its corresponding undirected graph $L_{2:u}$ (regarded as the shadowed graph $\widehat{L_2}$). Then, the corresponding L_2-tree groupoid $\mathbb{L}_2$ satisfies:

$$|\mathbb{L}_2| = \left| \left\{ \begin{array}{c} \phi,\ v_1\ v_2,\ v_3, \\ e_{12}, e_{12}^{-1}, e_{23}, e_{23}^{-1}, \\ e_{12}e_{23},\ e_{23}^{-1}e_{12}^{-1} \end{array} \right\} \right| = 10.$$

Observe that if $G_1, G_2 \in \mathcal{G}$ are graphs:

$$G_1 = \quad \bullet \longrightarrow \underset{\circlearrowleft}{\bullet} \longrightarrow \underset{\circlearrowleft}{\bullet},$$

and

$$G_2 = \quad \bullet \longrightarrow \bullet \longrightarrow \bullet \ \overset{\nearrow}{\to} \ \overset{\bullet}{\uparrow}\ \bullet,$$

Then, the corresponding loop parts W_1 and W_2 are:

$$W_1 = \quad \underset{\circlearrowleft}{\bullet} \quad \underset{\circlearrowleft}{\bullet},$$

and

$$W_2 = \quad \overset{\nearrow}{\underset{\bullet \ \to\ \bullet}{}} \overset{\bullet}{\uparrow},$$

respectively. Then, the corresponding G_k-trees $(G_k)_{W_k}$,

$$(G_1)_{W_1} = \quad \bullet \longrightarrow \bullet \longrightarrow \bullet = (G_2)_{W_2},$$

are obtained for $k = 1, 2$ (up to graph isomorphisms), meaning that:

$$(G_k)_{W_k} \overset{\text{graph}}{=} L_2,\ \forall k = 1, 2.$$

Even though G_1 and G_2 are not isomorphic in $\mathcal{G}$, the G_1-tree and the G_2-tree are isomorphic to the tree L_2. This means that:

$$\Gamma(G_1) = |\mathbb{L}_2| = 10 = |\mathbb{L}_2| = \Gamma(G_2).$$

Lemma 3. *In the family $\mathcal{G}$, define a relation $\mathcal{R}$ by:*

$$G_1 \, \mathcal{R} \, G_2 \overset{def}{\Longleftrightarrow} \Gamma(G_1) = \Gamma(G_2).$$

Then, the relation $\mathcal{R}$ is an equivalence relation on $\mathcal{G}$.

Proof. Let $G \in \mathcal{G}$. Then, clearly $\Gamma(G) = \Gamma(G)$ in $\mathbb{N}$. Therefore, $G \, \mathcal{R} \, G$. Suppose now that $G_1 \, \mathcal{R} \, G_2$ in $\mathcal{G}$. Then,

$$\Gamma(G_1) = \Gamma(G_2) \Longleftrightarrow \Gamma(G_2) = \Gamma(G_1),$$

Hence, $G_2 \, \mathcal{R} \, G_1$ in $\mathcal{G}$.
If $G_1 \, \mathcal{R} \, G_2$ and $G_2 \, \mathcal{R} \, G_3$ in $\mathcal{G}$, then:

$$\Gamma(G_1) = \Gamma(G_2) = \Gamma(G_3),$$

implying that $\Gamma(G_1) = \Gamma(G_3)$; hence, $G_1 \, \mathcal{R} \, G_3$ in $\mathcal{G}$.
Therefore, the relation $\mathcal{R}$ is an equivalence relation on $\mathcal{G}$. $\square$

The above lemma shows that the tree-index Γ classifies the family $\mathcal{G}$ by the equivalence relation $\mathcal{R}$ of Lemma 3. One can define the quotient set,

$$\mathcal{G}_\Gamma \overset{def}{=} \mathcal{G}/\mathcal{R} = \{[G] : G \in \mathcal{G}\},$$
$$\text{where } [G] \overset{def}{=} \{K \in \mathcal{G} : K \, \mathcal{R} \, G\} \tag{21}$$

are the $\mathcal{R}$-equivalence classes of G for all $G \in \mathcal{G}$. Additionally, one can define the function:

$$\Gamma_o : \mathcal{G}_\Gamma \to \mathbb{R}_1^+,$$
$$\text{by } \Gamma_o([G]) = |\mathbb{K}_\mathbb{W}|, \tag{22}$$

for all $[G] \in \mathcal{G}_\Gamma$, where $\mathbb{K}_\mathbb{W}$ is the K-tree groupoid of the K-tree K_W for the loop-part inclusion $W \subseteq K$ for some $K \in [G]$ in $\mathcal{G}$.

Theorem 7. *Let $\mathcal{G}_\Gamma$ be the quotient set (21), and Γ_o, the function (22). Then,*

$$\Gamma_o(\mathcal{G}_\Gamma) \subsetneqq \mathbb{N},$$
$$\text{and } \Gamma_o([G]) = 1, \text{ or } \Gamma([G]) \geq 5, \tag{23}$$

for all $[G] \in \mathcal{G}_\Gamma$.

Proof. The proof of (23) is done by (19), (20) and Lemma 3. $\square$

By (23), without the loss of much generality, the graph-tree index Γ on $\mathcal{G}$ is used to find the cardinalities Γ_o of the graph groupoids of the trees representing the elements of $\mathcal{G}_\Gamma$. We define a family $\mathcal{T}$ by:

$$\mathcal{T} \overset{def}{=} \left\{ K : \begin{array}{c} K \text{ is a connected finite tree,} \\ \text{or} \\ K \text{ is the single-vertex graph } \mathbb{I} \end{array} \right\}, \tag{24}$$

where $K \in \mathcal{T}$ is unique up to graph isomorphisms. This family $\mathcal{T}$ of (24) is called the (connected-finite-)tree family.

Theorem 8. *Let $\mathcal{T}$ be the tree family (24) and let $\mathcal{G}_\Gamma$ be the quotient set (22). Then, they are equipotent set-theoretically.*

$$\mathcal{G}_\Gamma \overset{equipotent}{=} \mathcal{T}, \; set\text{-}theoretically. \tag{25}$$

Suppose $\Gamma_0 : \mathcal{T} \to \mathbb{N}$ is the map defined by:

$$\Gamma_0(K) = |\mathbb{K}|, \; \forall K \in \mathcal{T},$$

where $\mathbb{K}$ are the graph groupoids of $K \in \mathcal{T}$. Then, for any $[G] \in \mathcal{G}_\Gamma$, there exists a unique $K \in \mathcal{T}$, such that:

$$\Gamma_o([G]) = \Gamma_0(K), \tag{26}$$

where Γ_o is in the sense of (22). In particular,

$$K = G_W, \; where \; W \; is \; the \; loop\text{-}part \; of \; G,$$

in $\mathcal{T}$ up to graph isomorphisms.

Proof. If $[G] \in \mathcal{G}_\Gamma$, then there exists a connected finite graph $G \in \mathcal{G}$ whose G-tree G_W for the loop-part inclusion $W \subseteq G$ is a tree, satisfying:

$$[G_W] = [G] \text{ in } \mathcal{G}_\Gamma.$$

For any $[G] = [G_W] \in \mathcal{G}_\Gamma$, there exists a tree G_W (up to graph isomorphisms) in $\mathcal{T}$. Thus, one can define a function:

$$\begin{aligned} g : \mathcal{G}_\Gamma &\to \mathcal{T}, \\ \text{by } g([G]) = g([G_W]) &= G_W, \end{aligned} \tag{27}$$

for all $[G] \in \mathcal{G}_\Gamma$. By (21), if $[G] = [K]$ in $\mathcal{G}_\Gamma$, then:

$$g([G]) = g([G_W]) = G_W = g([K]),$$

in $\mathcal{T}$. This implies that the function g of (27) is injective.

Let $\mathcal{T}$ be the tree family (24). Then, for any arbitrary $K \in \mathcal{T}$, there exists a connected finite graph $G \in \mathcal{G}$ whose G-tree G_W for the loop-part inclusion $W \subseteq G$ is graph-isomorphic to a tree K. Equivalently, there exists $[G] \in \mathcal{G}_\Gamma$, such that:

$$g([G]) = g([G_W]) = K,$$

implying the surjectivity of the function g of (27). Thus, the function g is bijective. Hence, two families $\mathcal{G}_\Gamma$ and $\mathcal{T}$ are equipotent set-theoretically. Thus, the relation (25) holds.

By the equipotence (25), we have that:

$$\Gamma_o([G]) = \Gamma_o([G_W]) = |\mathbb{G}_\mathbb{W}| = \Gamma_0(g([G])),$$

Hence, there exists a unique tree $K = g([G]) = G_W \in \mathcal{T}$, such that:

$$\Gamma_o([G]) = \Gamma_0(K).$$

Therefore, the relation (26) holds as well. □

Using the above theorem, one can obtain the following result.

Corollary 4. *If $G \in \mathcal{G}$, then there exists $K \in \mathcal{T}$, such that:*

$$\begin{aligned} K &\overset{graph}{=} G_W, \; the \; G\text{-}tree, \\ and \; \Gamma(G) &= \Gamma_0(K), \; in \; \mathbb{N}, \end{aligned} \tag{28}$$

where Γ is the graph-tree index on $\mathcal{G}$.

Proof. The relation (28) holds by (23), (25) and (26).　□

The above series of results show that our family $\mathcal{G}$ is classified to be the quotient set $\mathcal{G}_\Gamma$ by our graph-tree index Γ, and this classification is fully characterized by the tree family $\mathcal{T}$. Free-probabilistically, recall that since each G-tree, say K, for $G \in \mathcal{G}$, does not have loops, the corresponding graph C^*-probability space (M_K, τ) does not contain semicircular elements. Since

$$\Gamma(K) = |\mathbb{K}| < \infty,$$

the C^*-algebra M_K is a C^*-subalgebra of the matricial algebra,

$$M_{|\mathbb{K}|}(\mathbb{C}) = M_{\Gamma_0(K)}(\mathbb{C}) = M_{\Gamma(K)}(\mathbb{C}),$$

by Definition 2.

6. The Gluing on Graphs

Let $G_1, G_2 \in \mathcal{G}$ and assume that $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$ are parts. Suppose further that the fixed parts K_1 and K_2 are graph-isomorphic (as connected, or disconnected graphs). Then, by gluing or identifying K_1 and K_2 (under graph isomorphism) to the common part, say $K \overset{\text{graph}}{=} K_1 \subseteq G_1$ and $K \overset{\text{graph}}{=} K_2 \subseteq G_2$, one can construct a new graph G as a graph with:

$$\begin{aligned} V(G) &= (V(G_1) \setminus V(K_1)) \cup V(K) \cup (V(G_2) \setminus V(K_2)), \\ &\text{and } E(G) = E(G_1) \cup E(G_2), \end{aligned} \tag{29}$$

with the identification rule: if $e \in E(K_l)$ in $E(G_l)$, then identify e as an edge $e \in E(K)$ in $E(G)$.

For instance, if:

$$G_1 = \quad \bullet \to \star \Longrightarrow \star \to \bullet,$$

and

$$G_2 = \quad \begin{matrix} & & \bullet \\ & \nearrow & \uparrow \\ \star & \Longrightarrow & \star \end{matrix},$$

then, by gluing (or, identifying) the common (or graph-isomorphic) parts,

$$\star \Longrightarrow \star,$$

in G_1 and G_2, one has a new graph G of (29):

$$G = \quad \begin{matrix} & & & & \bullet \\ & & & \nearrow & \uparrow \\ \bullet \to & \star & \Longrightarrow & \star & \to \bullet \end{matrix}.$$

Definition 15. *A new graph G of (29), induced by $G_l \in \mathcal{G}$, by gluing the common (or, graph-isomorphic) parts $K_l \subseteq G_l$, for $l = 1, 2$, is called the glued graph of G_1 and G_2 by gluing K_1 and K_2. We denote this by:*

$$G = G_{1 \; K_1} \#_{K_2} G_2.$$

As a special case of (29), one can have the "vertex-gluing". Let $G_k \in \mathcal{G}$, and fix the vertices $v_k \in V(G_k)$, for $k = 1, 2$. Then, one can construct a new graph, denoted by:

$$G \overset{\text{denote}}{=} G_{1 \; v_1} \#_{v_2} G_2,$$

by gluing (or, by identifying) the two vertices v_1 and v_2 to a new vertex—for instance:

$$v_\# = v_1 \# v_2.$$

Note that, since G_1 and G_2 are taken from $\mathcal{G}$, if they are different "in $\mathcal{G}$" then they are not graph-isomorphic by the very definition of the family $\mathcal{G}$.

Definition 16. *Let $G_1, G_2 \in \mathcal{G}$ (which are not necessarily distinct in $\mathcal{G}$), and let $v_k \in V(G_k)$ be fixed for $k = 1, 2$. Then, we identify two distinct vertices v_1 and v_2 with a new vertex $v_\# = v_1 \# v_2$, called the glued vertex of v_1 and v_2. Then, we define a graph:*

$$G = G_1 \,_{v_1}\#_{v_2} G_2,$$

by a graph with:

$$\begin{aligned} V(G) &= (V(G_1) \setminus \{v_1\}) \cup \{v_\#\} \cup (V(G_2) \setminus \{v_2\}), \\ \text{and } E(G) &= E(G_1) \cup E(G_2), \end{aligned} \tag{30}$$

with the identification rule. If either $e = ev_k$ or $e = v_k e$ in $E(G_k)$, then it is identified with $e = ev_\#$, or $e = v_\# e$, respectively, in $E(G)$. This new graph G of (30), obtained by gluing v_1 and v_2, is called the glued graph of G_1 and G_2 with the glued vertex $v_\#$.

For instance, if:

$$G_1 = \quad \bullet \to \bullet \leftarrow \underset{v_1}{\bullet} \to \bullet,$$

and

$$G_2 = \quad \begin{matrix} & & \overset{v_2}{\bullet} \\ & \swarrow & \downarrow \\ \bullet & \to & \bullet \end{matrix} \ ,$$

then the glued graph $G = G_1 \,_{v_1}\#_{v_2} G_2$ of (30) is a new graph:

$$G = \quad \begin{matrix} \bullet \to \bullet & \longleftarrow & \overset{v_\#}{\bullet} & \to & \bullet \\ & \swarrow & \downarrow & & \\ \bullet & \to & \bullet & & \end{matrix} \ ,$$

by gluing v_1 and v_2 to $v_\#$.

Theorem 9. *Let $G_1, G_2 \in \mathcal{G}$ and $G = G_1 \,_{K_1}\#_{K_2} G_2 \in \mathcal{G}$, the glued graph of G_1 and G_2 by gluing the common parts $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$. Then, the graph C^*-probability spaces (M_{G_k}, τ_k) of G_k are free-probabilistic sub-structures of the graph C^*-probability space (M_G, τ), by regarding $K_l \subseteq G_l$ as $K \subseteq G$, in the sense that:*

$$M_{G_k} \overset{*\text{-sub}}{\subset} M_G, \text{ and } \tau_k = \tau \mid_{M_{G_k}},$$

for all $l = 1, 2$, where "$\overset{*\text{-sub}}{\subset}$" means "being C^*-subalgebra of".

Proof. If we identify $K_l \subseteq G_l$ with $K \subseteq G$ in the glued graph G, then $G_k \subset G$ are parts of G. Hence, the graph groupoids $\mathbb{G}_k$ of G_k are the subgroupoid of the graph groupoid $\mathbb{G}$ of G, implying that the graph groupoid algebra M_{G_k} of $\mathbb{G}_k$ are C^*-subalgebra of the graph groupoid algebra M_G of $\mathbb{G}$. Furthermore, by (7) and (29), one has:

$$\tau_k = \tau \mid_{M_{G_k}}, \text{ for all } k = 1, 2,$$

because $K_1 \overset{\text{graph}}{=} K_2 \overset{\text{graph}}{=} K$ in G. Therefore, the C^*-probability spaces $\left(M_{G_k}, \tau_k\right)$ are the C^*-probability subspaces of (M_G, τ), for all $k = 1, 2$. $\quad\square$

The above theorem shows that free-probabilistic properties of $\left(M_{G_k}, \tau_k\right)$ are preserved in (M_G, τ) whenever:

$$G = G_1 {}_{K_1}\#_{K_2} G_2.$$

7. The Graph-Tree Index Γ and Graph-Tree Towers

In this section, we consider certain towers of part-inclusions induced by a graph $G \in \mathcal{G}$. In fact, we are interested in such towers preserving the graph-tree index $\Gamma(G)$ in each step. Recall first that every graph $G \in \mathcal{G}$ has its G-tree $G_{\mathbb{W}} \in \mathcal{T}$, where $W \subseteq G$ is the loop-part and:

$$\Gamma(G) = |\mathbb{G}_{\mathbb{W}}| < \infty.$$

Certain Quotient Graphs Induced by $G \in \mathcal{G}$

In this section, we fix an arbitrary graph $G \in \mathcal{G}$ with its loop-part $W \subseteq G$, and, hence, the corresponding G-tree $G_{\mathbb{W}}$. Additionally, suppose throughout this section that the loop part W of G induced by the N-many loop-diagram parts $W_1, \ldots, W_N$ induced by the loop-diagrams $w_1, \ldots, w_N \in \mathbb{G}$, for some $N \in \mathbb{N}$. Then, there exist k-many connected components $K_1, \ldots, K_k$ of W, for some $k \leq N$ in $\mathbb{N}$, such that:

$$W = \overset{k}{\underset{l=1}{\cup}} K_l \text{ in } G. \tag{31}$$

For instance, if:

$$G = \begin{array}{ccccccccc} \bullet & \leftarrow & \overset{v_2}{\bullet} & & & & & \bullet \\ \downarrow & \nearrow & \uparrow & & & & \swarrow & \uparrow \\ \underset{v_1}{\bullet} & \rightarrow & \bullet & \rightarrow & \underset{v_3}{\bullet} & \rightarrow & \bullet, \end{array} \tag{32}$$

in $\mathcal{G}$, then it has the disconnected loop-part W in G

$$W = \begin{array}{ccccccccc} \bullet & \leftarrow & \overset{v_2}{\bullet} & & & & & \bullet \\ \downarrow & \nearrow & \uparrow & & & & \swarrow & \uparrow \\ \underset{v_1}{\bullet} & \rightarrow & \bullet & & \underset{v_3}{\bullet} & \rightarrow & \bullet, \end{array}$$

consisting of two connected components: $\tag{33}$

$$K_1 = \begin{array}{ccc} \bullet & \leftarrow & \overset{v_2}{\bullet} \\ \downarrow & \nearrow & \uparrow \\ \underset{v_1}{\bullet} & \rightarrow & \bullet \end{array} \qquad K_2 = \begin{array}{ccc} & & \bullet \\ & \swarrow & \uparrow \\ \underset{v_3}{\bullet} & \rightarrow & \bullet, \end{array}$$

These induce the K-tree $K_{\mathbb{W}}$ of K:

$$K_{\mathbb{W}} = \underset{x_1}{\bullet} \longrightarrow \underset{x_2}{\bullet}, \tag{34}$$

by (31) and (33), where x_l are the collapsed vertices of K_l, for $l = 1, 2$ (see (17)).

Let $G \in \mathcal{G}$, and suppose the loop-part $W \subseteq G$ has its connected components $K_1, \ldots, K_k$ (induced from the diagram-loop parts $W_1, \ldots, W_N$), for some $k \leq N$ in $\mathbb{N}$. Suppose $G_1 \in \mathcal{G}$ is graph-isomorphic to G, whose loop-part $W_1 \subseteq G_1$ is graph-isomorphic to the loop-part W of G, with its connected components $J_1, \ldots, J_k$, where each J_i is graph-isomorphic to K_i, for all $i = 1, \ldots, k$.

Then, by gluing the common (or, graph-isomorphic) connected components $K_l \subseteq G$ and $J_l \subseteq G_1$, we obtain the glued graph:

$$G^2(K_l) = G {}_{K_l}\#_{J_l} G_1, \tag{35}$$

in the sense of (29), for any arbitrarily fixed $l \in \{1, \ldots, k\}$. Without a loss of generality, the graph G^2 of (35) is identified with $G_1^2(J_l)$, since:

$$G^2(K_l) \stackrel{\text{graph}}{=} G_1^2(J_l), \quad \forall l = 1, \ldots, k. \tag{36}$$

By (36), we identify $G^2(K_l)$ and $G_1^2(J_l)$, for $l = 1, \ldots, k$.

For example, if a graph $G \in \mathcal{G}$ is in the sense of (32) with its connected components $K_1, K_2 \subseteq G$ of (33) induced by the loop-diagram parts (32), and if $J \in \mathcal{G}$ is a graph, isomorphic to K, whose loop-part has two connected components J_1 and J_2, isomorphic to K_1 and K_2, respectively, then we obtain a new graph:

$$K^2(K_1) = K_{K_1} \#_{J_1} J,$$

by gluing $K_1 \subseteq K$ and $J_1 \subseteq J$,

$$
K^2(K_1) = \quad
\begin{array}{ccccccccc}
\bullet & & & \bullet & \leftarrow & \overset{v_2}{\bullet} & & & \bullet \\
\uparrow & \searrow & & \downarrow & \nearrow & \uparrow & & \swarrow & \uparrow \\
\bullet & \leftarrow & \bullet \leftarrow & \underset{v_1}{\bullet} & \rightarrow & \bullet & \rightarrow & \underset{v_3}{\bullet} & \rightarrow \bullet
\end{array} \quad ,
$$

by (29) and (35). According to (36), the new graph $K^2(K_1)$ can be identified with $J^2(J_1)$.

As discussed above, from (36), one may understand given two graphs $G \stackrel{\text{graph}}{=} G_1$ as an identical element of $\mathcal{G}$—i.e., two copies of $G \in \mathcal{G}$. For a fixed $G \in \mathcal{G}$ and its connected part $K \subseteq G$, one takes two "distinct" copies of G's and identifies K in the two copies of G, before gluing them to construct a new graph $G^2(K)$ in $\mathcal{G}$, as in (35).

Definition 17. *Let $G \in \mathcal{G}$, and $K \subseteq G$, a connected part. The graph,*

$$G^2(K) \stackrel{denote}{=} G_K \#_K G,$$

induced by (35) of two copies of G (in the sense of (36)), is called the K-fixed 2-copy of G in $\mathcal{G}$. Similarly, one can have the K-fixed 3-copy of G:

$$G^3(K) = G^2(K)_K \#_K G \in \mathcal{G},$$

Inductively, we have the K-fixed $(n + 1)$-copies of G:

$$G^{n+1}(K) = G^n(K)_K \#_K G, \quad \forall n \in \mathbb{N},$$

in $\mathcal{G}$, for all $n \in \mathbb{N}$, with a notational identity: $G^1(K) = G$.

By Definition 17, we can obtain the following result.

Lemma 4. *Let $G \in \mathcal{G}$ and $K \subseteq G$, a connected part, and let $G^n(K) \in \mathcal{G}$ be the K-fixed n-copies of G for all $n \in \mathbb{N}$ with the identity: $G^1(K) = G$. Then, the quotient graph $\left(G^{n+1}(K)\right)_{G^n(K)}$ is isomorphic to the quotient graph G_K for $K \subseteq G$, in the sense of (13), in $\mathcal{G}$, for all $n \in \mathbb{N}$:*

$$\left(G^{n+1}(K)\right)_{G^n(K)} = G_K \text{ in } \mathcal{G}. \tag{37}$$

Proof. First, consider the case where $n = 1$. Suppose

$$G^2(K) = G_K \#_K G \in \mathcal{G}$$

is the K-fixed 2-copy of G in the sense of Definition 17, where $G^1(K) = G$ in $\mathcal{G}$. Then, by identifying, or collapsing the part $G^1(K) = G$ in $G^2(K)$, one has:

$$\left(G^2(K)\right)_{G^1(K)} = \left(G^2(K)\right)_G \stackrel{\text{graph}}{=} G_K,$$

in $\mathcal{G}$. Similarly, if $n = 2$, then, by collapsing $G^2(K)$ in $G^3(K)$, we have:

$$\left(G^3(K)\right)_{G^2(K)} \overset{\text{graph}}{=} G_K,$$

by Lemma 4. Therefore, inductively,

$$\left(G^{n+1}(K)\right)_{G^n(K)} \overset{\text{graph}}{=} G_K,$$

This implies that:

$$\left(G^{n+1}(K)\right)_{G^n(K)} = G_K \text{ in } \mathcal{G},$$

because every element of $\mathcal{G}$ is uniquely determined up to the graph isomorphisms. Thus, the relation (37) holds. $\square$

Using (37), one directly obtains the following corollary.

Corollary 5. *Let $G \in \mathcal{G}$, and $v \in V(G)$, a fixed vertex, and let*

$$G^n(v) \overset{denote}{=} G^n(\{v\}) \in \mathcal{G}$$

be the $\{v\}$-fixed (or, v-fixed) n-copies of Definition 17, where $\{v\} = (\{v\}, \phi)$ is a vertex-part of G. Then:

$$\left(G^{n+1}(v)\right)_{G^n(v)} = G \text{ in } \mathcal{G}. \tag{38}$$

Proof. The relation (38) is immediately shown by (37). Indeed, one has:

$$\left(G^{n+1}(v)\right)_{G^n(v)} = G_{\{v\}} = G,$$

in $\mathcal{G}$, by (37), where $\{v\} = (\{v\}, \phi)$ is a vertex-part of G. $\square$

By (37), one also has the following result.

Theorem 10. *Let $G \in \mathcal{G}$, and $K \subseteq G$, a connected part, and let $G^n(K) \in \mathcal{G}$ be the K-fixed n-copies of G, for $n \in \mathbb{N}$. Then:*

$$\left[G^{n+1}(K) : G^n(K)\right] = [G : K] = |\mathbb{G}_{\mathbb{K}}|, \tag{39}$$

for all $n \in \mathbb{N}$.

Proof. By (37), we have:

$$\left(G^{n+1}(K)\right)_{G^n(K)} = G_K, \text{ in } \mathcal{G},$$
$$\text{implying that } \left[G^{n+1}(K) : G^n(K)\right] = \left|\left(\mathbb{G}^{n+1}(K)\right)_{\mathbb{G}^n(K)}\right|, \tag{40}$$

where $\left(\mathbb{G}^{n+1}(K)\right)_{\mathbb{G}^n(K)}$ are the graph groupoids of the quotient graphs $\left(G^{n+1}(K)\right)_{G^n(K)}$, which are graph-isomorphic to G_K, and, hence, identified with G_K in $\mathcal{G}$ for all $n \in \mathbb{N}$.

Thus, using (40), one has:

$$\left[G^{n+1}(K) : G^n(K)\right] = |\mathbb{G}_{\mathbb{K}}| = [G : K],$$

for all $n \in \mathbb{N}$, where $\mathbb{G}_{\mathbb{K}}$ is the graph groupoid of G_K. Therefore, Formula (39) holds. $\square$

By (39), one obtains the following corollary.

Corollary 6. *Let $G \in \mathcal{G}$ and $v \in V(G)$, a fixed vertex, and let $G^n(v) \in \mathcal{G}$ be the $\{v\}$-fixed n-copies of G, for all $n \in \mathbb{N}$. Then:*

$$\left[G^{n+1}(v) : G^n(v) \right] = |\mathbb{G}| = [G : \{x\}], \tag{41}$$

for all $n \in \mathbb{N}$, for all $x \in V(G)$, inducing the vertex-parts $\{x\} = (\{x\}, \phi)$ of G in $\mathcal{G}$.

Proof. First of all, it must be noted that, if $x \in V(G)$, inducing the vertex-part $\{x\} \subseteq G$, then the corresponding quotient graphs $G_x = G_{\{x\}}$ are graph-isomorphic to G, and, hence, identified with G in $\mathcal{G}$. Thus, by (39), we have:

$$\left[G^{n+1}(v) : G^n(v) \right] = [G : \{v\}] = |\mathbb{G}|,$$

implying the Formula (41). □

Additionally, we have the following result.

Theorem 11. *Let $G \in \mathcal{G}$, and $K \subseteq G$, a connected part, and let $G^n(K) \in \mathcal{G}$ be the K-fixed n-copies of G, for all $n \in \mathbb{N}$. Suppose G_K is the quotient graph for $K \subseteq G$, and $\left(M_{G_K}, \tau \right)$ is the corresponding graph C^*-probability space of G_K. If*

$$G_K(n+1) \overset{denote}{=} \left(G^{n+1}(K) \right)_{G^n(K)}$$

are the quotient graphs for $G^n(K) \subseteq G^{n+1}(K)$, then:

$$\left(M_{G_K(n+1)}, \tau_{n+1} \right) \overset{free\text{-}iso}{=} \left(M_{G_K}, \tau \right), \tag{42}$$

for all $n \in \mathbb{N}$.

Proof. By Theorem 9, if two graphs G_1 and G_2 are graph-isomorphic, then the corresponding graph C^*-probability spaces $\left(M_{G_1}, \tau_{G_1} \right)$ and $\left(M_{G_2}, \tau_{G_2} \right)$ are free-isomorphic. Therefore, by (37), the free-isomorphic relation (42) holds. □

By (42), the following corollary holds.

Corollary 7. *Let $G \in \mathcal{G}$, and $v \in V(G)$, a fixed vertex, and let $G^n(v) \in \mathcal{G}$ be the $\{v\}$-fixed n-copies of G, for all $n \in \mathbb{N}$. If $G_v(n+1) = \left(G^{n+1}(v) \right)_{G^n(v)}$ in $\mathcal{G}$, then:*

$$\left(M_{G_v(n+1)}, \tau_{n+1} \right) \overset{free\text{-}iso}{=} \left(M_G, \tau \right), \ \forall n \in \mathbb{N}. \tag{43}$$

Proof. The free-isomorphic relation (43) holds by (38) and (42). □

Additionally, using Theorem 11 and Corollary 7, one obtains the following free-probabilistic information.

Corollary 8. *Let $G \in \mathcal{T}$ be a tree, and $x \in V(G)$, a fixed vertex, and let $G^n(x) \in \mathcal{T}$ be the $\{x\}$-fixed n-copies of G, for all $n \in \mathbb{N}$. If $G_v(n+1) = \left(G^{n+1}(x) \right)_{G^n(n)}$ in $\mathcal{T}$, then:*

$$\left(M_{G_v(n+1)}, \tau_{n+1} \right) \overset{free\text{-}iso}{=} \left(M_G, \tau \right), \ \forall n \in \mathbb{N}. \tag{44}$$

Proof. The relation (44) is obtained by (42) and (43). □

By Corollary 8, we obtain the following main result of this section.

Theorem 12. *Let $G \in \mathcal{G}$ and $W \subseteq G$, the loop-part, inducing the G-tree $G_W \in \mathcal{T}$. Assume that $K \subseteq W$ is a connected component of the loop-part W in G, generating the collapsed vertex x of G_W. Let*

$$G_{W:x}(n+1) \overset{def}{=} \left(G_W^{n+1}(x) \right)_{G_W^n(x+1)}, \ \forall n \in \mathbb{N},$$

with identity: $G_{W:x}(1) = G_W$, where $G_W^n(x)$ are the $\{x\}$-fixed n-copies of $G_W \in \mathcal{T}$, for all $n \in \mathbb{N}$. Then:

$$\begin{aligned} G_{W:x}(n) &= G_W \text{ in } \mathcal{T}, \\ \text{and hence, } [G_{W:x}(n+1) : G_{W:x}(n)] &= |\mathbb{G}_{\mathbb{W}}| = [G : W] = \Gamma(G), \end{aligned} \tag{45}$$

and

$$\left(M_{G_{W:x}(n)}, \tau_n \right) \overset{free\text{-}iso}{=} \left(M_{G_W}, \tau \right), \tag{46}$$

for all $n \in \mathbb{N}$. In particular, these free-isomorphic C^*-probability spaces of (46) do not contain semicircular elements.

Proof. By (38), the graph-isomorphic relation of (45) hold for all $n \in \mathbb{N}$. Thus, the index relations of (45) holds by (39) and (41) for all $n \in \mathbb{N}$. Therefore, the free-isomorphic relation (46) holds by (42)–(44).

　Since the graph G_W is a tree in $\mathcal{T}$, it does not contain any loops in its graph groupoid $\mathbb{G}_{\mathbb{W}}$. It guarantees that the graph C^*-probability space $\left(M_{G_W}, \tau \right)$ does not contain any semicircular elements by Corollary 2. □

　The above theorem shows that one can construct a tower of trees:

$$G_W = G_W^1(x) \subseteq G_W^2(x) \subseteq G_W^3(x) \subseteq G_W^4(x) \subseteq \cdots \tag{47}$$

"in $\mathcal{T}$," whenever a loop-part inclusion $W \subseteq G$ is given "in $\mathcal{G}$," where $x \in V(G_W)$ is a collapsed vertex of an any connected component of the loop-part W of $G \in \mathcal{G}$, and $\left\{ G_W^n(x) \right\}_{n=1}^{\infty}$ are the collection of $\{x\}$-fixed n-copies of G. This tower (47) satisfies:

$$\left(G_W^{n+1}(x) \right)_{G_W^n(x)} = G_W \text{ in } \mathcal{T},$$

and

$$\begin{aligned} \left[G_W^{n+1}(x) : G_W^n(x) \right] &= |\mathbb{G}_{\mathbb{W}}| = [G : W] = \Gamma(G), \\ \text{and } \left(M_{\left(G_W^{n+1}(x) \right)_{G_W^n(x)}}, \tau_{n+1} \right) &\overset{free\text{-}iso}{=} \left(M_{G_W}, \tau \right), \end{aligned} \tag{48}$$

for all $n \in \mathbb{N}$, by (45) and (46). Furthermore, under (318), all free-isomorphic C^*-probability spaces do not have semicircular elements.

Definition 18. *Let $G \in \mathcal{G}$ and $W \subseteq G$, the loop-part inclusion, inducing the G-tree $G_W \in \mathcal{T}$. A tower (317) of trees, induced by G_W and its $\{x\}$-fixed n-copies for an arbitrary collapsed vertex x of a connected component of W, satisfying the relations of (48), is called the (G, x)-tree tower.*

　By Theorem 12 and Definition 18, one immediately has the following corollary.

Corollary 9. *Let $G \in \mathcal{G}$, and $W \subseteq G$, the loop-part inducing the G-tree $G_W \in \mathcal{T}$. Suppose*

$$G_W = K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$$

is a (G, x)-tree tower (47), where $K_n = G_W^n(x)$ for all $n \in \mathbb{N}$, and $x \in V(G_W)$ is the collapsed vertex of a connected component of W. Then:

$$(K_{n+1})_{K_n} = G_W \text{ in } \mathcal{T}, \ \forall n \in \mathbb{N},$$

and

$$[K_{n+1} : K_n] = |\mathbb{G}_{\mathbb{W}}| = [G : W] = \Gamma(G),$$

$$and \left(M_{(K_{n+1})_{K_n}}, \tau_{n+1} \right) \overset{free\text{-}iso}{=} (M_{G_W}, \tau), \tag{49}$$

for all $n \in \mathbb{N}$*. The* C^**-probability spaces of (49) have no semicircular elements.*

Proof. The relations of (49) are shown by (48) under Definition 18. $\square$

The above corollary shows that, for any $K \in [G] = [G_W] \in \mathcal{G}_\Gamma$, where $\mathcal{G}_\Gamma$ is the graph-tree family (21) equipotent to the tree-family $\mathcal{T}$, one can construct (G, x)-tree towers (47) for all collapsed vertices x of connected components of the loop-part $W \subseteq G$. Note here that "all" (G, x)-tree towers satisfy the relations of (49). The quotient structures for the steps of the tower are all equivalent from each other, combinatorially, algebraically, and free-probabilistically.

This shows that our graph-tree index Γ on $\mathcal{G}$ preserves the non-semicircularity on the (steps of the) towers (47) up to quotient relation (on the steps).

8. The Tree-Monoid ($\mathcal{T}\mathcal{V}$, $\odot$)

Define a new family TV using:

$$TV \overset{def}{=} \{(G, v) : G \in \mathcal{T}, \; v \in V(G)\}. \tag{50}$$

Note that, even though $G \in \mathcal{T}$ is a fixed tree, if $v \neq x$ in $V(G)$, then the elements (G, v) and (G, x) are "distinct" in TV by definition. For a fixed tree $G \in \mathcal{T}$, there are $|V(G)|$-many elements $\{(G, v)\}_{v \in V(G)}$ in TV.

From the set TV of (50), define its partition $\mathcal{T}\mathcal{V}$ by:

$$\mathcal{T}\mathcal{V} = \{\mathcal{T}\mathcal{V}(n) : n \in \mathbb{N}\},$$

$$\text{with } \mathcal{T}\mathcal{V}(n) = \{(G, v) \in \mathcal{T}\mathcal{V} : |V(G)| = n\}, \; \forall n \in \mathbb{N}. \tag{51}$$

The partition $\mathcal{T}\mathcal{V}$ of (51) satisfies:

$$TV = \bigsqcup_{n \in \mathbb{N}} \mathcal{T}\mathcal{V}(n), \text{ set-theoretically,}$$

where $\sqcup$ is the disjoint union. We here consider the partition $\mathcal{T}\mathcal{V}$ of TV as a family (or, a small category) of the sets $\mathcal{T}\mathcal{V}(n)$, for all $n \in \mathbb{N}$.

On this set TV of (50), define an operation $\odot$ by:

$$(G_1, v_1) \odot (G_2, v_2) \overset{def}{=} (G_1 \; {}_{v_1}\#_{v_2} \; G_2, \; v_\#), \tag{52}$$

where $G_1 \; {}_{v_1}\#_{v_2} \; G_2$ is the vertex-glued graph (30) of the trees G_1 and G_2 by gluing $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ to the identified, or glued vertex $v_\# = v_1 \# v_2$.

By (52), one may understand our $\{x\}$-fixed 2-copy $G^2(x)$ of $G \in \mathcal{T}$ as:

$$G^2(x) = G \; {}_x\#_x \; G \overset{regard}{=} (G, x) \odot (G, x);$$

inductively,

$$G^{n+1}(x) \overset{regard}{=} (G^n(x), x) \odot (G, x),$$

$$\text{in } TV, \text{ for } n \geq 2 \text{ in } \mathbb{N}, \; (G^{n+1}(x), x) = (G^n(x), x) \odot (G, x), \tag{53}$$

in $\mathcal{T}\mathcal{V}$ by (52), for all $n \in \mathbb{N}$.

By (49) and (53), "if" the operation $\odot$ is well-defined on TV, then the construction of the graph-tree towers of (47) is also carried out by the operation (52) by (53).

Lemma 5. *The operation* $\odot$ *of (52) on the set* TV *of (50) is well-defined.*

Proof. By the definition (52), for any arbitrary $(G_1, v_1), (G_2, v_2) \in TV$, one can determine a pair (G, v) of the glued graph $G = G_1 \, {}_{v_1}\#_{v_2} G_2$, and the glued vertex $v \in V(G)$ of v_1 and v_2. It is trivial to check that this new graph G forms a new finite connected tree contained in the tree-family $\mathcal{T}$. Thus,

$$(G, v) \in TV,$$

by (50), implying that the operation $\odot$ is closed on TV. $\square$

This well-defined operation $\odot$ on TV satisfies the following properties.

Lemma 6. *The operation $\odot$ of (52) on the set TV of (50) satisfies the following properties:*

$$\odot \text{ is associative on } TV. \tag{54}$$

The trivial element $\mathbb{I} \overset{denote}{=} (\{x\}, x) \in TV$ of the single-vertex graph $\mathbb{I} = \{x\} \in \mathcal{T}$ with its unique vertex x forms the ($\odot$)-identity of TV,

$$\mathbb{I} \odot (G, v) = (G, v) = (G, v) \odot \mathbb{I} \text{ in } TV. \tag{55}$$

$$\odot \text{ is commutative on } TV. \tag{56}$$

Proof. Note that the tree-family $\mathcal{T}$ is defined up to graph isomorphisms. Therefore, if $K \in \mathcal{T}$, and if $K_1 \overset{graph}{=} K$, then $K_1 = K$ in $\mathcal{T}$, and the family TV is determined by $\mathcal{T}$ by (50). Thus, the equalities

$$(T_1, x_1) = (T_2, x_2) \text{ on } TV$$

mean that $T_1 = T_2$ in $\mathcal{T}$. Equivalently, $T_1 \overset{graph}{=} T_2$ and $x_1 = x_2$.

If $K_l = (G_l, v_l) \in TV$, for $l = 1, 2, 3$, then:

$$(K_1 \odot K_2) \odot K_3 = (G_{12}, v_{12}) \odot K_3,$$

where $G_{12} = G_1 \, {}_{v_1}\#_{v_2} G_2$, and $v_{12} = v_1 \# v_2$

$$= ((G_1 \, {}_{v_1}\#_{v_2} G_2) \, {}_{v_{12}}\#_{v_3} G_3, \ (v_1 \# v_2) \# v_3)$$

$$= (G_1 \, {}_{v_1}\#_{v_2} G_2 \, {}_{v_{12}}\#_{v_3} G_3, \ v_1 \# v_2 \# v_3)$$

$$= (G_1 \, {}_{v_1}\#(G_2 \, {}_{v_2}\#_{v_3} G_3), \ v_1 \#(v_2 \# v_3))$$

$$= (G_1, v_1) \odot (G_{23}, v_{23})$$

where $G_{23} = G_2 \, {}_{v_2}\#_{v_3} G_3$ and $v_{23} = v_2 \# v_3$

$$= K_1 \odot (K_2 \odot K_3),$$

implying that:

$$(K_1 \odot K_2) \odot K_3 = K_1 \odot (K_2 \odot K_3),$$

in TV. Therefore, the associativity (54) of $\odot$ holds.

The tree-family $\mathcal{T}$ contains its trivial element $\mathbb{I} = \{x\}$, the single-vertex graph (unique up to graph isomorphisms), and hence, the family TV of (50) contains its trivial element,

$$\mathbb{I} = \{x\} \overset{denote}{=} (\{x\}, x),$$

satisfying that, for all $(G, v) \in TV$,

$$\mathbb{I} \odot (G, v) = (\{x\} \, {}_x\#_v G, \ x \# v) = (G, v),$$

in TV, because

$$\{x\} \, {}_x\#_v G = G, \text{ in } \mathcal{T},$$

and

$$x \# v \text{ is identified with } v \text{ in } G.$$

Similarly, one has:

$$G \odot \mathbb{I} = (G, v).$$

Therefore, the trivial element $\mathbb{I} \in TV$ acts as the ($\odot$)-identity. Thus, the relation (55) holds.

The commutativity (56) of $\odot$ is clear, since:

$$G_1 \,_{v_1}\#_{v_2}\, G_2 = G_2 \,_{v_2}\#_{v_1}\, G_1,$$

in $\mathcal{T}$, and

$$v_1 \# v_2 \text{ and } v_2 \# v_1.$$

Hence,

$$(G_1 \,_{v_1}\#_{v_2}\, G_2, \, v_1 \# v_2) = (G_2 \,_{v_2}\#_{v_1}\, G_1, \, v_2 \# v_1),$$

implying that:

$$(G_1, v_1) \odot (G_2, v_2) = (G_2, v_2) \odot (G_1, v_2),$$

in TV, for all $(G_1, v_1), (G_2, v_2) \in TV$. $\quad\square$

Recall that an algebraic structure $(\Omega, \bullet)$ of a set Ω and an operation $\bullet$ is a monoid if a well-defined operation ($\bullet$) is associative and has its identity. In particular, if the operation $\bullet$ is commutative, then the monoid $(\Omega, \bullet)$ is said to be a commutative monoid.

Theorem 13. *An algebraic pair $(TV, \odot)$ is a commutative monoid.*

Proof. By Lemma 6, the operation $\odot$ is closed on the set $\mathcal{T}\mathcal{V}$. And, by (54) and (12), an algebraic pair $(\mathcal{T}\mathcal{V}, \odot)$ forms a monoid. Moreover, by (56), this monoid is commutative consisting of all mutually commuting elements under $\odot$. $\quad\square$

Notation and Assumption. From below, we understand the set TV of (50) then as a commutative monoid $(TV, \odot)$,

$$TV \stackrel{\text{denote}}{=} (TV, \, \odot).$$

Definition 19. *We call the monoid $TV = (TV, \odot)$, the tree-monoid.*

Now, let $G \in \mathcal{G}$ and $W \subseteq G$, the loop-part inclusion inducing the G-tree G_W in $\mathcal{T}$. For the collapsed vertex $x \in V(G_W)$ of an arbitrary connected component of the loop-part W, one can obtain:

$$(G_W, x) \in TV.$$

As we considered in Section 7, one can have the G-tree tower (47):

$$G_W = G_W^1(x) \subseteq G_W^2(x) \subseteq G_W^3(x) \subseteq \cdots,$$

in $\mathcal{T}$, satisfying (319). By regarding the steps of the tower as:

$$G_W^n(x) \stackrel{\text{denote}}{=} \left(G_W^n(x), x\right) \text{ in } TV, \tag{57}$$

for all $n \in \mathbb{N}$, we naturally obtain the corresponding tower:

$$G(x) \subseteq G_W^2(x) \subseteq G_W^3(x) \subseteq \cdots, \tag{58}$$

in TV by (57) and by fixing the common vertex x. Note the difference between the G-tree tower (47) in $\mathcal{T}$, and the tower (58) induced by $G_W^1(x) = (G_W, x)$ in TV.

Definition 20. *Under the same hypothesis with (57), the tower (58) is called the (G_W, x)-tower in $\mathcal{T}\mathcal{V}$.*

By the monoidal structure we discussed in Theorem 13, one has:

$$G_W^{n+1}(x) = G_W^n(x) \odot G_W^1(x),$$

in the sense of (57) in TV, for all $n \in \mathbb{N}$.

Proposition 4. *Let $G_W^1(x) \subseteq G_W^2(x) \subseteq G_W^3(x) \subseteq \cdots$ be the (G_W, x)-tower (58) in TV. Then:*

$$G_W^n(x) = G_W^{n_1}(x) \odot G_W^{n_2}(x),$$

for all $n = n_1 + n_2$ in $\mathbb{N}$.

Proof. This relation is performed with (57) because the operation $\odot$ is associative (equivalently, TV is a monoid). $\square$

Now, let $G \in \mathcal{G}$ and $G_W^n(x) \in TV$ be in the sense of (57), the steps of the (G_W, x)-tower (58), and fix an arbitrary $k \in \mathbb{N}$. Then, one can construct a sub-tower:

$$G_W^k(x) \subseteq G_W^{2k}(x) \subseteq G_W^{3k}(x) \subseteq \cdots, \tag{59}$$

of the (G_W, x)-tower. One can understand in (59) that:

$$G_W^{(n+1)k}(x) = G_W^{nk}(x) \odot G_W^k(x), \tag{60}$$

in TV, for all $n \in \mathbb{N}$.

Theorem 14. *Let (59) be the sub-tower of the (G_W, x)-tower (58) in the tree-monoid TV. Then:*

$$\left(G_W^{(n+1)k}(x) \right)_{G_W^{nk}(x)} = G_W^k(x) \text{ in } \mathcal{TV},$$

$$\text{and } \left(M_{\left(G_W^{(n+1)k}(x) \right)_{G_W^{nk}(x)}}, \tau_{(n+1)k} \right) \overset{\text{free-iso}}{=} \left(M_{G_W^k(x)}, \tau \right), \tag{61}$$

for all $n \in \mathbb{N}$. All C^*-probability spaces of (61) do not have semicircular elements.

Proof. The combinatorial equivalence in (61) is shown by (57), (58) and (60), and it implies the free-isomorphic relation of (61). Since all free-isomorphic C^*-probability spaces are induced by graph-isomorphic trees, they do not contain semicircular elements. $\square$

9. The Operad $\mathcal{TV}$ Induced by TV

In Sections 5–7, we showed that, if $G \in \mathcal{G}$ and $W \subseteq G$ is the loop-part inclusion inducing the G-tree, the quotient graph G_W, then the graph-tree index $\Gamma(G)$ of G is determined by the graph groupoid index $[G : W] \overset{\text{def}}{=} |\mathbb{G}_\mathbb{W}|$, and there exists a tower of trees:

$$G_W = G_W^1(x) \subseteq G_W^2(x) \subseteq G_W^3(x) \subseteq \cdots, \tag{62}$$

of the $\{x\}$-fixed n-copies $G_W^n(x)$, for all $n \in \mathbb{N}$, where x is the collapsed vertex of G_W of an arbitrary connected component of the loop-part W of G, in the tree-family $\mathcal{T}$, and each step

$$G_W^n(x) \subseteq G_W^{n+1}(x), \text{ for } n \in \mathbb{N},$$

we have a combinatorial equivalence,

$$\left(G_W^{n+1}(x) \right)_{G_W^n(x)} = G_W \text{ in } \mathcal{T}, \tag{63}$$

and an algebraic equivalence,

$$\left[G_W^{n+1}(x) : G_W^n(x) \right] = [G : W] = \Gamma(G), \tag{64}$$

and a free-probabilistic equivalence,

$$\left(M_{\left(G_W^{n+1}\right)_{G_W^n(x)}}, \tau_{n+1}\right) \overset{\text{free-iso}}{=} \left(M_{G_W}, \tau\right). \tag{65}$$

In particular, the free-isomorphic relation (65) shows that the tower (62) constructs the same free-probabilistic structures in every step up to quotient, and they provide equivalent free-probabilistic structures without having semicircular elements.

Additionally, in Section 8, if one has a fixed $(G_W, x) \in TV$, where $G_W \in \mathcal{T}$ and $x \in V(G_W)$ are as above, then:

$$G_W^{n+1}(x) \overset{\text{denote}}{=} \left(G_W^{n+1}(x), x\right) \in TV$$

are well-defined in TV by (57) to be

$$G_W^{n+1}(x) = G_W^n(x) \odot G_W^1(x),$$

for all $n \in \mathbb{N}$, with identity:

$$G_W^1(x) = (G_W, x), \tag{66}$$

where $TV = (TV, \odot)$ is the tree-monoid of Definition 20 and the (G_W, x)-tower,

$$G_W^1(x) \subseteq G_W^2(x) \subseteq G_W^3(x) \subseteq \cdots,$$

of (58) is well-determined whose graph-entries of the steps satisfy (63)–(65).

Motivated by the above results, we here consider how our tree-monoid $TV = (TV, \odot)$ induces an operadic structure (e.g., also see [39,40,42,43]). Such an interesting consideration starts from the fact that:

$$(G_1, v_1) \odot (G_2, v_2) = (G_1\ _{v_1}\#_{v_2} G_2,\ v_1\#v_2) \overset{\text{denote}}{=} (G, v)$$

satisfies the combinatorial relation,

$$|V(G)| = |V(G_1)| + |V(G_2)| - 1,$$

since two vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ are identified with the collapsed vertex v in $V(G)$.

Now, let us decompose the family TV of (50) by:

$$TV = \underset{n \in \mathbb{N}}{\sqcup} \mathcal{TV}(n),$$
$$\text{with } \mathcal{TV}(n) = \{K \in TV : |V(K)| = n\}, \tag{67}$$

for all $n \in \mathbb{N}$, where $\sqcup$ means the disjoint union. By (24), we canonically obtain a family:

$$\mathcal{TV} = \{\mathcal{TV}(n) : n \in \mathbb{N}\} \text{ of } (51),$$

by (67). Moreover, by definitions, one has:

$$\mathcal{TV}(1) = \{(\{x\}, x)\}. \tag{68}$$

By abusing notation, one can understand that:

$$\mathcal{TV} \odot \mathcal{TV} = \{(G, v) \odot (K, x) : (G, v), (K, x) \in \mathcal{TV}\},$$

set-theoretically. Hence,

$$\mathcal{TV} \odot \mathcal{TV} = \mathcal{TV}, \tag{69}$$

as sets. Indeed, the set-equality (69) holds, since

$$\mathcal{TV} \odot \mathcal{TV} \subseteq \mathcal{TV},$$

since $\odot$ is well-defined on TV, and

$$\mathcal{TV} \subseteq \mathcal{TV} \odot \mathcal{TV},$$

since, for all $(G, v) \in \mathcal{TV}$,

$$(G, v) = (G, v) \odot (\{x\}, x),$$

in $\mathcal{TV} \odot \mathcal{TV}$. Additionally, if we understand

$$\mathcal{TV}(n) \odot \mathcal{TV}(m) = \left\{ (G, v) \odot (K, x) : \begin{array}{l} (G, v) \in \mathcal{TV}(n) \\ (K, x) \in \mathcal{TV}(m) \end{array} \right\},$$

set-theoretically, then

$$\mathcal{TV}(n) \odot \mathcal{TV}(m) = \mathcal{TV}(n + m - 1), \tag{70}$$

as subsets of $\mathcal{TV}$, for all $n, m \in \mathbb{N}$.

The notational expression (69) says that the operation, also denoted by $\odot$ on the family $\mathcal{TV} = \{\mathcal{TV}(n)\}_{n \in \mathbb{N}}$, is well-defined in the sense of (70).

Proposition 5. *If $\mathcal{TV}$ is the family (67) and $(\odot)$ is the operation (69) on $\mathcal{TV}$, then the set-equality (70) holds in $\mathcal{TV}$, so that:*

$$\mathcal{TV}(n) \odot \mathcal{TV}(m) = \mathcal{TV}(n + m - 1),$$

as elements of $\mathcal{TV}$, for all $n, m \in \mathbb{N}$.

Proof. By the symbolic definition (69) of $\odot$ on $\mathcal{TV}$, one has:

$$\mathcal{TV}(n) \odot \mathcal{TV}(m) \subseteq \mathcal{TV}(n + m - 1).$$

Indeed, for $(G, v) \in \mathcal{TV}(n)$, and $(K, x) \in \mathcal{TV}(m)$ (equivalently, $|V(G)| = n$, and $|V(K)| = m$),

$$(G, v) \odot (K, x) = (G \, {}_v\#_x \, K, \ v\#x),$$

in TV, with:

$$|V(G \, {}_v\#_x \, K)| = n + m - 1.$$

Additionally, if $(G, v) \in \mathcal{TV}(n + m - 1)$, then, for the fixed vertex v, one can take a tree part K_1 of G with $|V(K_1)| = n$, containing v as its vertex. Then, from the fixed vertex v, by collecting all vertices which are not contained in $V(K_1)$ (except for v), and by collecting all edges which are not contained in $E(K_1)$, we obtain another part K_2 with $|V(K_2)| = m$. One can decide:

$$(K_1, v) \in \mathcal{TV}(n), \text{ and } (K_2, v) \in \mathcal{TV}(m),$$

such that:

$$(K_1, v) \odot (K_2, v) = (G, v),$$

in TV. This shows that:

$$\mathcal{TV}(n + m - 1) \subseteq \mathcal{TV}(n) \odot \mathcal{TV}(m).$$

Therefore, the set-equality (70) holds in the family $\mathcal{TV}$. $\square$

By Proposition 5, we obtain the following result.

Theorem 15. *The pair* $(\mathcal{TV}, \odot)$ *is a monoidal category, where* $\mathcal{TV}$ *is the family (67) and* $(\odot)$ *is the operation (69).*

Proof. By the very construction of

$$\mathcal{TV} = \{\mathcal{TV}(n) : n \in \mathbb{N}\},$$

this forms a small category. If we symbolically define an operation $(\odot)$ on $\mathcal{TV}$ as in (69), then it is well-defined on $\mathcal{TV}$, satisfying (70), by Proposition 5.

By (70), the operation $(\odot)$ is associative in the sense that:

$$(\mathcal{TV}(k_1) \odot \mathcal{TV}(k_2)) \odot \mathcal{TV}(k_3) = \mathcal{TV}(k_1) \odot (\mathcal{TV}(k_2) \odot \mathcal{TV}(k_3)).$$

Indeed, both sides are identified with:

$$\mathcal{TV}(k_1 + k_2 + k_3 - 2) \text{ in } \mathcal{TV},$$

for all $k_1, k_2, k_3 \in \mathbb{N}$.

Furthermore, one can take $\mathcal{TV}(1) = \{\{\{x\}, x\}\} = \{\mathbb{I}\}$ in $\mathcal{TV}$, satisfying:

$$\mathcal{TV}(1) \odot \mathcal{TV}(n) = \mathcal{TV}(n) = \mathcal{TV}(n) \odot \mathcal{TV}(1),$$

for all $n \in \mathbb{N}$ by (70).

Therefore, the category $\mathcal{TV}$ forms a monoidal category $(\mathcal{TV}, \odot)$ under $(\odot)$. $\quad\square$

Remark that this monoidal category $\mathcal{TV} = (\mathcal{TV}, \odot)$ is commutative in the sense that:

$$\mathcal{TV}(k_1) \odot \mathcal{TV}(k_2) = \mathcal{TV}(k_2) \odot \mathcal{TV}(k_1),$$

for all $k_1, k_2 \in \mathbb{N}$, since both sides are identical to:

$$\mathcal{TV}(k_1 + k_2 - 1) \text{ in } \mathcal{TV}.$$

Thus, $\mathcal{TV}$ is a commutative monoidal category.

9.1. Operads

Not only in mathematical analysis, but also in topology and quantum physics, operads are well-known and play important roles (e.g., [50]). In particular, their applications in connection with subfactor theory and knot theory are simply amazing (e.g., [41] and cited papers therein). Here, we introduce a modified definition of Day's original definition of operads (e.g., see Sections 1.2, 1.3 and 1.7 of [50]).

Definition 21. *Let* $\mathcal{P} = \{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ *be a monoidal category with an operation* $\circledast$ *on* $\mathcal{P}$. *This structure* $(\mathcal{P}, \circledast)$ *is an operad, if (i) the well-defined operation* $\circledast$ *satisfies:*

$$\mathcal{P}(n) \circledast \mathcal{P}(m_1) \circledast \dots \circledast \mathcal{P}(m_n) = \mathcal{P}\left(\sum_{k=1}^{n} m_k\right),$$

for all $m_1, \dots, m_n \in \mathbb{N}$, *for all* $n \in \mathbb{N}$; *(ii)* $\circledast$ *is associative in the sense that: if*

$$\overrightarrow{m} = (m_1, \dots, m_t) \in \mathbb{N}^t,$$

$$\overrightarrow{n} = (n_1, \dots, n_s) \in \mathbb{N}^s,$$

$$\overrightarrow{k} = (k_1, \dots, k_u) \in \mathbb{N}^u,$$

for $t, s, u \in \mathbb{N}$, and if

$$\mathcal{P}\left(\overrightarrow{m}\right) = \underset{i=1}{\overset{t}{\circledast}} \mathcal{P}(m_i), \ \mathcal{P}\left(\overrightarrow{n}\right) = \underset{j=1}{\overset{s}{\circledast}} \mathcal{P}(n_j),$$

and

$$\mathcal{P}\left(\overrightarrow{k}\right) = \underset{l=1}{\overset{u}{\circledast}} \mathcal{P}(k_l),$$

then

$$\left(\mathcal{P}\left(\overrightarrow{m}\right) \circledast \mathcal{P}\left(\overrightarrow{n}\right)\right) \circledast \mathcal{P}\left(\overrightarrow{k}\right) = \mathcal{P}\left(\overrightarrow{m}\right) \circledast \left(\mathcal{P}\left(\overrightarrow{m}\right) \circledast \mathcal{P}\left(\overrightarrow{k}\right)\right);$$

(iii) $\circledast$ satisfies the equivalence condition in the sense that: for all

$$\overrightarrow{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n, \ \forall n \in \mathbb{N},$$

one has

$$\mathcal{P}(n) \circledast \mathcal{P}\left(\overrightarrow{m}\right) = \mathcal{P}(n) \circledast \mathcal{P}\left(\sigma \overrightarrow{m}\right) = \mathcal{P}\left(\sum_{k=1}^{n} m_k\right),$$

for all $\sigma \in S_n$, where S_n is the symmetric group over $\{1, \ldots, n\}$, and

$$\sigma \overrightarrow{m} = \left(m_{\sigma(1)}, \ldots, m_{\sigma(n)}\right) \in \mathbb{N}^n;$$

and finally (iv) $\circledast$ has the unit property in the sense that: if

$$\mathbb{U} = \mathcal{P}(1) \text{ and } \mathbb{U}^{\circledast n} = \underbrace{\mathbb{U} \circledast \ldots \cdots \circledast \mathbb{U}}_{n\text{-times}},$$

then

$$\mathcal{P}(n) \circledast \mathbb{U}^{\circledast n} = \mathcal{P}(n) = \mathbb{U}^{\circledast n} \circledast \mathcal{P}(n),$$

in $\mathcal{P}$, for all $n \in \mathbb{N}$.

Interesting examples and applications of operads can be found in e.g., [41,50].

9.2. The Operad $\mathcal{TV}$ Induced by the Tree-Monoid TV

In this section, we prove that our monoidal category $\mathcal{TV} = (\mathcal{TV}, \odot)$, induced by the tree-monoid TV, is a well-determined operad in the sense of Definition 21. Of course, as a commutative monoid, the tree-monoid TV itself is a good algebraic structure. More than that, if TV induces an operad, then it also provides "good" categorial, topological, and quantum-physical properties as in [50], and, hence, the similar applications such as the Jones' operads (or, Temperly–Lieb operads) of planar algebra (e.g., [41]) may/can be possible. The tree-monoid TV provides a new example of operads in connections with graph theory, groupoid theory, representation theory, operator algebra and free probability (especially, "non-semicircularity").

Let $\mathcal{TV} = (\mathcal{TV}, \odot)$ be the monoidal category of Theorem 15.

Theorem 16. *The monoidal category $\mathcal{TV}$ is an operad.*

Proof. Let $\mathcal{TV}$ be the small category (67) equipped with an operation $(\odot)$ of (69). Then, using Theorem 15:

$$\mathcal{TV} \text{ is a monoidal category.} \tag{71}$$

moreover, it is a commutative monoidal category. Recall that the operation $\odot$ on $\mathcal{TV}$ satisfies:

$$\mathcal{TV}(n) \odot \mathcal{TV}(m) = \mathcal{TV}(n + m - 1), \ \forall n, m \in \mathbb{N}, \tag{72}$$

by (70). Observe first that, if

$$m_1, \ldots, m_n \in \mathbb{N},$$

then one has that:

$$\mathcal{TV}(n) \odot \mathcal{TV}(m_1) \odot \ \ldots \ \odot \mathcal{TV}(m_n)$$
$$= (\mathcal{TV}(n) \odot \mathcal{TV}(m_1)) \odot \mathcal{TV}(m_2) \odot \ \ldots \ \odot \mathcal{TV}(m_n)$$

since the category $\mathcal{TV}_o$ is monoidal by (71)

$$= \mathcal{TV}(n + m_1 - 1) \odot \mathcal{TV}(m_2) \odot \ \ldots \ \odot \mathcal{TV}(m_n)$$

by (72)

$$= (\mathcal{TV}(n + m_1 - 1) \odot \mathcal{TV}(m_2)) \odot \mathcal{TV}(m_3) \cdots \odot \mathcal{TV}(m_n)$$

by (71)

$$= \mathcal{TV}(n + m_1 + m_2 - 2) \odot \mathcal{TV}(m_3) \odot \ \ldots \ \odot \mathcal{TV}(m_n)$$
$$= \ldots$$
$$= \mathcal{TV}(n + m_1 + \cdots + m_{n-1} - (n-1)) \odot \mathcal{TV}(m_n)$$
$$= \mathcal{TV}(n + m_1 + \cdots + m_n - n) = \mathcal{TV}(m_1 + \cdots + m_n), \tag{73}$$

for all $n \in \mathbb{N}$. Thus, the operation $\odot$ of $\mathcal{TV}_o$ satisfies the condition (i) of Definition 21.

Now, assume that $\overrightarrow{m}$, $\overrightarrow{n}$, and $\overrightarrow{k}$ are in the sense of the condition (ii) of Definition 21, and suppose:

$$\mathcal{TV}(\overrightarrow{m}) = \overset{t}{\underset{i=1}{\odot}} \mathcal{TV}(m_i), \ \mathcal{TV}(\overrightarrow{n}) = \overset{s}{\underset{j=1}{\odot}} \mathcal{TV}(n_j),$$

and

$$\mathcal{TV}\left(\overrightarrow{k}\right) = \overset{u}{\underset{l=1}{\odot}} \mathcal{TV}(k_l), \ \forall t, s, u \in \mathbb{N},$$

in $\mathcal{TV}_o$. Then, we have that:

$$(\mathcal{TV}(\overrightarrow{m}) \odot \mathcal{TV}(\overrightarrow{n})) \odot \mathcal{TV}\left(\overrightarrow{k}\right) = \mathcal{TV}(N),$$
$$\text{and } \mathcal{TV}(\overrightarrow{m}) \odot \left(\mathcal{TV}(\overrightarrow{n}) \odot \mathcal{TV}\left(\overrightarrow{k}\right)\right) = \mathcal{TV}(N), \tag{74}$$

where

$$N = \left(\sum_{i=1}^{t} m_i - (t-1)\right) + \left(\sum_{j=1}^{s} n_j - (s-1)\right) + \left(\sum_{l=1}^{u} k_l - (u-1)\right),$$

in $\mathbb{N}$ by (71) and (72). This implies that:

$$(\mathcal{TV}(\overrightarrow{m}) \odot \mathcal{TV}(\overrightarrow{n})) \odot \mathcal{TV}\left(\overrightarrow{k}\right) = \mathcal{TV}(\overrightarrow{m}) \odot \left(\mathcal{TV}(\overrightarrow{n}) \odot \mathcal{TV}\left(\overrightarrow{k}\right)\right),$$

in $\mathcal{TV}_o$, implying the associativity (ii) of Definition 21.

Consider now that if:

$$\overrightarrow{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n,$$

then

$$\mathcal{TV}(n) \odot \mathcal{TV}(\overrightarrow{m}) = \mathcal{TV}\left(\sum_{k=1}^{n} m_k\right),$$

by (73). Note that, by (72)

$$\mathcal{TV}(k_1) \odot \mathcal{TV}(k_2) = \mathcal{TV}(k_1 + k_2 - 1),$$

and

$$\mathcal{TV}(k_2) \odot \mathcal{TV}(k_1) = \mathcal{TV}(k_2 + k_1 - 1),$$

implying that,

$$\mathcal{TV}(k_1) \odot \mathcal{TV}(k_2) = \mathcal{TV}(k_2) \odot \mathcal{TV}(k_1), \tag{75}$$

for all $k_1, k_2 \in \mathbb{N}$. Thus,

$$\mathcal{TV}(n) \odot \mathcal{TV}(\overrightarrow{m}) = \mathcal{TV}\left(\sum_{k=1}^{n} m_k\right) = \mathcal{TV}(n) \odot \mathcal{TV}(\sigma \overrightarrow{m}),$$

for all $\sigma \in S_n$, where

$$\sigma \overrightarrow{m} = \left(m_{\sigma(1)}, \ldots, m_{\sigma(n)}\right) \in \mathbb{N}^n, \tag{76}$$

by (75), for all $n \in \mathbb{N}$. Thus, the equivalence condition (iii) of Definition 21 is satisfied for $(\mathcal{TV}_o, \odot)$.

Now, let

$$\mathbb{U} = \mathcal{TV}(1) = \{(\{x\}, x)\},$$

as in (68), and

$$\mathbb{U}^{\odot n} = \underbrace{\mathbb{U} \odot \cdots\cdots\cdots \odot \mathbb{U}}_{n\text{-times}}, \ \forall n \in \mathbb{N}.$$

Recall that

$$\mathcal{TV}(n) \odot \mathbb{U} = \mathcal{TV}(n) = \mathbb{U} \odot \mathcal{TV}(n),$$

in $\mathcal{TV}$, by Theorem 15, for all $n \in \mathbb{N}$. Therefore,

$$\mathcal{TV}(n) \odot \mathbb{U}^{\odot n} = \mathcal{TV}(n) = \mathbb{U}^{\odot n} \odot \mathcal{TV}(n), \tag{77}$$

in $\mathcal{TV}$, for all $n \in \mathbb{N}$. This shows that the operation $\odot$ of $\mathcal{TV}$ satisfies the unit property (iv) of Definition 21.

Therefore, by (73), (74), (76), and (77), our category $\mathcal{TV}$ is an operad. □

The above theorem not only shows our tree-monoid TV induces an operad $\mathcal{TV}$ naturally, but also provides a new example that is different from the free monoids of [50], and the Temperly–Lieb, or Temperly–Lieb-like operads of [41].

10. The Tree-Monoidal Algebra $\mathscr{TV}$

Let $TV = (TV, \odot)$ be the tree-monoid of Definition [14]. In this section, we construct a certain pure-algebraic algebra $\mathscr{TV}$ generated by the monoid TV, and study not only the algebraic properties of $\mathscr{TV}$, but also the natural statistical properties of $\mathscr{TV}$.

Define a (pure-algebraic) vector space $\mathfrak{TV}$ by:

$$\mathfrak{TV} \stackrel{\text{def}}{=} span_{\mathbb{C}}(TV), \tag{78}$$

where $span_{\mathbb{C}}(Y)$ means the vector space spanned by a set Y over $\mathbb{C}$, and, hence, the vector space $\mathfrak{TV}$ is generated by the set TV. By (78), every vector $\xi \in \mathfrak{TV}$ has its expression:

$$\xi = \sum_{(G,v) \in TV} t_{(G,v)} \xi_{(G,v)}, \text{ with } t_{(G,v)} \in \mathbb{C}, \tag{79}$$

where $\sum$ is the finite sum, and $\xi_{(G,v)}$ are the spanning vectors of $\mathfrak{TV}$ induced by $(G,x) \in TV$.

Notation and Assumption 78. (From below, in short, **NA 78**) If there is no confusion, we denote the spanning vectors $\xi_{(G,v)}$ in (79) simply by (G,v). Under this assumption, one can re-write (79) using

$$\xi = \sum_{(G,v) \in TV} t_{(G,v)}(G,v),$$

below. In particular, we write:

$$\xi_G \stackrel{\text{denote}}{=} \sum_{x \in V(G)} 1 \cdot (G,x),$$

in $\mathfrak{TV}$, for all $G \in \mathcal{T}$. □

Now, on this vector space $\mathfrak{TV}$, we define a vector multiplication, also denoted as $\odot$, by:

$$\left(\sum_{T \in TV} t_T T \right) \odot \left(\sum_{S \in TV} s_S S \right) \overset{\text{def}}{=} \sum_{(T,S) \in TV^2} (t_T s_S) T \odot S, \tag{80}$$

where the operation $\odot$ on the right-hand side is the monoidal operation on the tree-monoid $(TV, \odot)$. Of course, in the summands in (80), the notations T, S and $T \odot S$ mean the spanning vectors ξ_T, ξ_S and $\xi_{T \odot S}$ in $\mathfrak{TV}$, respectively, by **NA 78**.

By the very definition (80), the operation $\odot$ is well-defined on the vector space $\mathfrak{TV}$. Moreover, it satisfies:

$$(\xi_1 \odot \xi_2) \odot \xi_3 = \xi_1 \odot (\xi_2 \odot \xi_3), \tag{81}$$

in $\mathfrak{TV}$, for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{TV}$, since the operation $\odot$ in the right-hand side of (80) is associative, inducing a monoid TV.

Therefore, this well-defined multiplication $\odot$ of (80) is associative by (81) on $\mathfrak{TV}$. Thus, the vector space equipped with $\odot$ forms a (pure-algebraic) algebra over $\mathbb{C}$. Moreover, this algebra is unital in the sense that it contains its unity (or, unit vector) $\mathbb{I} = (\{x\}, x)$,

$$\mathbb{I} = \xi_{\mathbb{I}} \in \mathfrak{TV}, \text{ under } \mathbf{NA\ 78},$$

satisfying

$$\left(\sum_{T \in \mathcal{TV}} t_T T \right) \odot \mathbb{I} = \sum_{T \in \mathcal{TV}} t_T (T \odot \mathbb{I}) = \sum_{T \in \mathcal{TV}} t_T T,$$

implying

$$\xi \odot \mathbb{I} = \xi = \mathbb{I} \odot \xi, \ \forall \xi \in \mathfrak{TV}. \tag{82}$$

Additionally, since

$$(G, v) \odot (K, x) = (G \ _v\#_x \ K, \ v\#x),$$

and

$$(K, x) \odot (G, v) = (K \ _x\#_v \ G, \ x\#v),$$

in the tree-monoid TV, implying that

$$T \odot S = S \odot T, \text{ for all } T, S \in \mathcal{TV},$$

one can check that

$$\xi \odot \eta = \eta \odot \xi \text{ for all } \xi, \eta \in \mathfrak{TV}, \tag{83}$$

by (80).

Theorem 17. *Let $\mathfrak{TV}$ be the vector space (78) generated by the tree-monoid TV, equipped with the vector-multiplication $\odot$ of (80). Then, it is a commutative unital algebra over $\mathbb{C}$.*

Proof. The vector space $\mathfrak{TV}$ equipped with the operation $\odot$ forms an algebra over $\mathbb{C}$ by the well-definedness (80) and the associativity (81). Additionally, it is commutative by (83) and unital by the existence of the unity $\mathbb{I} = \xi_{\mathbb{I}} \in \mathfrak{TV}$ by (82). $\square$

The above theorem shows that the tree-monoid TV generates the commutative unital algebra $\mathfrak{TV}$.

Definition 22. *The commutative unital algebra $\mathfrak{TV}$ of (78), equipped with the vector-multiplication $\odot$ of (80), is called the tree-monoidal algebra. From below, to distinguish with the vector-space notation $\mathfrak{TV}$, we denote this tree-monoidal algebra using $\mathscr{TV}$. This notation $\mathscr{TV}$ means a vector space $\mathfrak{TV}$ with the vector-multiplication $\odot$, as an algebra.*

11. Discrete Statistical Models of $\mathscr{TV}$

Let $\mathscr{TV}$ be the tree-monoidal algebra of Definition 22 generated by the tree-monoid TV, equipped with its algebra multiplication $\odot$ of (80).

11.1. A Tree-Index Statistical Model

On the tree-monoidal algebra $\mathscr{TV}$, define a (pure-algebraic) linear functional Γ by a linear morphism:

$$\Gamma\left(\sum_{(G,v)\in TV} t_{(G,v)}(G,v)\right) \overset{\text{def}}{=} \sum_{(G,v)\in TV} \frac{t_{(G,v)}}{\Gamma(G)}, \tag{84}$$

where Γ on the right-hand side of (84) is the graph-tree index of Definition 25. Note that, by the very definition (84), the morphism Γ is a well-defined linear functional, which is bounded in the sense that:

$$|\Gamma(Y)| < \infty, \ \forall Y \in \mathscr{TV},$$

where $|.|$ is the modulus on $\mathbb{C}$. Moreover, by the commutativity of the tree-monoidal algebra $\mathscr{TV}$. Note that if $O \in \mathscr{TV}$ is the zero element (which is the zero vector of $\mathfrak{TV}$), then it is understood to be:

$$O = 0 \cdot \mathbb{U},$$

and hence,

$$\Gamma(O) = \frac{0}{\Gamma(\mathbb{U})} = \frac{0}{1} = 0.$$

Definition 23. *We call the linear functional Γ of (84), the (graph-tree-)indexing trace on $\mathscr{TV}$.*

Recall that if $K \in \mathcal{G}$ with its loop-part $W \subseteq K$, then

$$\Gamma(K) = [K : W] = |\mathbb{K}_{\mathbb{W}}|,$$

where $\mathbb{K}_{\mathbb{W}}$ is the graph groupoid of the quotient graph K_W. Thus, one has:

$$G \in \mathcal{T} \text{ is a tree,}$$

if and only if

$$\Gamma(G) = [G : \phi] = |\mathbb{G}|, \tag{85}$$

by our axiomatization $G_\phi = G$, where ϕ is the empty part (see Definition 8).

Proposition 6. *The indexing trace Γ of (84) satisfies that:*

$$\Gamma\left(\sum_{(G,v)\in \mathcal{TV}} t_{(G,v)}(G,v)\right) = \sum_{(G,v)} \frac{t_{(G,v)}}{|\mathbb{G}|},$$

and, as a special case,

$$\Gamma(\xi_G) = \frac{|V(G)|}{|\mathbb{G}|}, \tag{86}$$

for all $G \in \mathcal{T}$, where $\mathbb{G}$ are the graph groupoids of $G \in \mathcal{T}$.

Proof. By the definition (84) of the indexing trace Γ and by the formula (85), one obtains the first formula of (86). Thus, one has:

$$\Gamma(\xi_G) = \Gamma\left(\sum_{v\in V(G)} (G,v)\right) = \sum_{v\in V(G)} \frac{1}{|\mathbb{G}|} = \frac{|V(G)|}{|\mathbb{G}|},$$

by (85), for all $G \in \mathcal{T}$. Thus, the second formula of (86) holds as a special case of the first. $\square$

It is easy to check that

$$\Gamma((G,v)) = \Gamma\left(\xi_{(G,v)}\right) = \frac{1}{\Gamma(G)} = \frac{1}{|\mathbb{G}|},$$

for all generating elements $(G,v) \in \mathscr{TV}$.

In (86), note that, since we are handling linear combinations (as finite sums), and since the graph groupoids of our trees contains finitely many elements, we always have:

$$0 \le |\Gamma(\xi)| < \infty, \; \forall \xi \in \mathscr{TV}.$$

Therefore, the resulted quantities of (86) are bounded in $\mathbb{C}$. Thus, one can obtain the following statistical structure (e.g., [18]).

Definition 24. *Let $\mathscr{TV}$ be the tree-monoidal algebra (78) generated by the tree-monoid TV, and let Γ be the indexing trace (84). Then, the pair $(\mathscr{TV}, \Gamma)$ is called the index-tree-monoidal (measure) space.*

By the very definition, the statistical data determined by the moments of an element of the index-tree-monoidal measure space $(\mathscr{TV}, \Gamma)$ are determined by the tree index Γ. In particular, by (86), one has:

$$\Gamma(\xi_G) = \frac{|V(G)|}{|\mathbb{G}|} = \frac{|V(G)|}{\Gamma(G)},$$

for all $G \in \mathcal{T}$. For instance, if $\mathbb{I} = (\{x\}, x) \in \mathcal{TV}$ in $(\mathscr{TV}, \Gamma)$, then:

$$\Gamma(\mathbb{I}) = \frac{1}{\Gamma(\{x\})} = \frac{1}{1} = 1$$

(e.g., see (18), or (20)).

Now, for an arbitrarily fixed $(G,v) \in \mathcal{TV}(N)$ in TV (implying that $|V(G)| = N$, for $N \in \mathbb{N}$), let:

$$\xi_{(G,v)} \overset{\text{denote}}{\underset{\textbf{NA 78}}{=\joinrel=}} (G,v) \in \mathscr{TV}.$$

Observe that the powers $\left\{(G,v)^n\right\}_{n=1}^{\infty} \subset \mathscr{TV}$ of $(G,v) \in \mathscr{TV}$ satisfy that:

$$(G,v)^n = (G^n(v), v) \overset{\text{denote}}{=\joinrel=} G^n(v)\left(= \xi_{G^n(v)}\right) \in \mathscr{TV},$$

where $G^n(v)$ are in the sense of (53), for all $n \in \mathbb{N}$. Indeed,

$$(G,v)^n = \underbrace{(G,v) \odot \cdots \odot (G,v)}_{n\text{-times}} = ((G,v) \odot (G,v)) \odot \cdots \odot (G,v)$$

$$= G^2(v) \odot (G,v) \odot \cdots \odot (G,v) = \dots$$

$$= G^{n-1}(v) \odot (G,v) = (G^n(v), v) \overset{\text{denote}}{=\joinrel=} G^n(v), \tag{87}$$

in $\mathscr{TV}$.

Theorem 18. *Let $(G,v) \overset{\text{denote}}{=\joinrel=} \xi_{(G,v)} \in (\mathscr{TV}, \Gamma)$, for $(G,v) \in TV$. Then*

$$\Gamma\left((G,v)^n\right) = \frac{1}{|\mathbb{G}^n|}, \; \forall n \in \mathbb{N},$$
$$\text{and } \lim_{n \to \infty} \Gamma\left((G,v)^n\right) = 0, \tag{88}$$

where $\mathbb{G}^n$ are the graph groupoids of $G^n(v)$, for all $n \in \mathbb{N}$, and where the limit in (88) is taken from the usual topology on $\mathbb{C}$.

Proof. By (87), one has:
$$(G, v)^n = G^n(v) \text{ in } \mathscr{TV},$$

Hence, if $\mathbb{G}^n$ are the graph groupoids of $G^n(v)$, then:
$$\Gamma\left((G, v)^n\right) = \frac{1}{\Gamma(G^n(v))} = \frac{1}{|\mathbb{G}^n|}, \tag{89}$$

by (86) for all $n \in \mathbb{N}$. Thus, the statistical data of the powers $\left\{(G, v)^n\right\}_{n \in \mathbb{N}}$ of (88) are obtained.

Note now that if $(G, v) \in \mathcal{TV}(N)$ in TV, then

$$G^2(v) \in \mathcal{TV}(2N - 1), \ G^3(v) \in \mathcal{TV}(3N - 2),$$

inductively,
$$G^n(v) \in \mathcal{TV}(nN - (n - 1)) = \mathcal{TV}(n(N - 1) + 1), \tag{90}$$

for all $n \in \mathbb{N}$. Thus, if $\mathbb{G}^n$ are the graph groupoids of $G^n(v) \in \mathcal{TV}$, then the cardinalities of them are strictly increasing in $\mathbb{N}$,

$$0 < \left|\mathbb{G}^1\right| < \left|\mathbb{G}^2\right| < \left|\mathbb{G}^3\right| < \cdots,$$

by (90). Hence:
$$\frac{1}{|\mathbb{G}^1|} > \frac{1}{|\mathbb{G}^2|} > \frac{1}{|\mathbb{G}^3|} > \cdots.$$

Thus,
$$\Gamma\left((G, v)^n\right) = \Gamma(G^n(v)) = \frac{1}{|\mathbb{G}^n|} \to 0,$$

as $n \to \infty$, in $\mathbb{C}$. Therefore, the asymptotic data of (88) hold. $\square$

The above statistical data (88) show that each generating element $(G, v) \in TV$ induces the algebra-element $(G, v) = \xi_{(G,v)} \in \mathscr{TV}$, whose moments approach 0.

Corollary 10. *Let* $X = \sum\limits_{k=1}^{N} t_k(G_k, v_k) \in (\mathscr{TV}, \Gamma)$, *with* $|t_k| < 1$ *for all* $k = 1, \ldots, N$, *for* $N \in \mathbb{N}$. *Then*

$$\lim_{n \to \infty} |\Gamma(X^n)| = 0 \text{ in } \mathbb{C}.$$

Proof. Under the assumption for the $\mathbb{C}$-coefficients that

$$|t_k| < 1, \text{ for all } k = 1, \ldots, N,$$

one can obtain the above asymptotic statistical data according to (90). $\square$

11.2. A Vertex-Cardinality Model

The index-tree-monoidal space $(\mathscr{TV}, \Gamma)$ is well-determined in Definition 13, and the corresponding discrete-measure-theoretic data on $(\mathscr{TV}, \Gamma)$ are considered in Section 11.1 as a discrete statistical model. However, in general, it is somewhat hard to "actually" compute the cardinality of graph groupoids of trees of the tree-family $\mathcal{T}$. In particular, if the size of finite trees of $\mathcal{T}$ is bigger and bigger, or the combinatorial structure of the trees is more and more complicated, computing the cardinalities of the graph groupoids of such trees, which is equivalent to finding their tree-indices, is not easy, even though we know that they are finitely determined in $\mathbb{N}$. Thus, in this section, we introduce another statistical model on our tree-monoidal algebra $\mathscr{TV}$, providing rough upper bounds.

Frankly speaking, in the model of this section, we will ignore some interesting combinatorial data of the trees of $\mathcal{T}$ and corresponding algebraic information of the graph groupoids of the trees. However, this model is interesting as a discrete statistical model (induced from our graphs of $\mathcal{G}$) independently, and it is much easier to handle computationally.

On the tree-monoidal algebra $\mathscr{TV}$, we define a linear functional,

$$v : \mathscr{TV} \to \mathbb{C},$$

by

$$v\left(\sum_{(G,v)\in TV} t_{(G,v)}(G,v) \right) \stackrel{\text{def}}{=} \sum_{(G,v)\in TV} \frac{t_{(G,v)}}{|V(G)|}. \tag{91}$$

Then, it is indeed a well-defined bounded linear functional on the algebra $\mathscr{TV}$.

By the very definition (91) of the linear functional v on $\mathscr{TV}$, one can realize that the combinatorial data from the trees of TV, determined by the admissibility of their graph groupoids, do not affect the quantitative data, while the classification of the generating family

$$TV = \bigsqcup_{n\in\mathbb{N}} TV(n),$$

determines the linear-functional values. For instance, if $(G,v) \in TV(n)$ in TV, for all $v \in V(G)$, then:

$$v\left(\xi_{(G,v)} \right) \stackrel[\mathbf{NA\ 78}]{\text{denote}}{=} v((G,v)) = \frac{1}{|V(G)|} = \frac{1}{n}, \tag{92}$$

Hence, if $\xi_G = \sum_{x\in V(G)} (G,x) \in \mathscr{TV}$, then, by (92), one has:

$$v(\xi_G) = \frac{n}{n} = 1, \text{ for all } G \in \mathcal{T}. \tag{93}$$

Proposition 7. *Let* $X = \sum_{k=1}^{N} t_k(G_k, x_k) \in \mathscr{TV}$, *with* $(G_k, x_k) \in TV(n_k)$, *for all* $k = 1, \ldots, N$, *for* $n_1, \ldots, n_N, N \in \mathbb{N}$, *and let*

$$\xi_G = \sum_{x\in V(G)} (G,x) \in \mathscr{TV}, \text{ for } G \in \mathcal{T},$$

under **NA 78**. *Then*

$$v(X) = \sum_{k=1}^{N} \frac{t_k}{n_k}, \quad \text{and} \quad v(\xi_G) = 1 \tag{94}$$

Proof. Under this hypothesis, the second formula of (94) is shown by (93). The first formula of (94) is proven by the straightforward computation by (87). Indeed, if $X \in \mathscr{TV}$ is given as above, then:

$$v(X) = \sum_{k=1}^{N} \frac{t_k}{|V(G_k)|} = \sum_{k=1}^{N} \frac{t_k}{n_k},$$

since $(G_k, x_k) \in TV(n_k)$, for all $k = 1, \ldots, N$, and hence,

$$|V(G_k)| = n_k, \ \forall k = 1, \ldots, N.$$

□

Now, let $(G,v) \in TV(N)$ in TV, inducing the algebra-element $(G,v) \stackrel{\text{denote}}{=} \xi_{(G,v)}$ in $\mathscr{TV}$ under **NA 78**, for $N \in \mathbb{N}$. As we considered in Section 11.1:

$$(G,v)^n = G^n(v) \stackrel{\text{denote}}{=} (G^n(v), v) \in \mathscr{TV},$$

(under **NA 78**), and

$$G^n(v) \in TV(n(N-1)+1),$$

in TV, implying that

$$|V(G^n(v))| = n(N-1)+1, \tag{95}$$

for all $n \in \mathbb{N}$.

Theorem 19. *Let* $(G, v) \in \mathcal{TV}(N)$ *in TV inducing an algebra-element* $(G, v) \in \mathscr{TV}$ *under NA 78, for* $N \in \mathbb{N}$*. Then:*

$$v\big((G, v)^n\big) = \frac{1}{n(N-1)+1}, \quad \forall n \in \mathbb{N},$$

and hence,

$$\lim_{n \to \infty} v\big((G, v)^n\big) = 0, \tag{96}$$

where the limit is taken from the usual topology for $\mathbb{C}$*.*

Proof. Recall that

$$(G, v)^n = \underbrace{(G, v) \odot \cdots \odot (G, v)}_{n\text{-times}} = G^n(v),$$

in $\mathscr{TV}$. Hence:

$$v\big((G, v)^n\big) = v(G^n(v)) = \frac{1}{|V(G^n(v))|},$$

by (94), for all $n \in \mathbb{N}$. By (95),

$$|V(G^n(v))| = n(N-1)+1,$$

for all $n \in \mathbb{N}$. Thus, the statistical data of (96) hold.

If $n \to \infty$, then $n(N-1)+1 \to \infty$. Hence, the asymptotic data of (96) hold as well. $\quad\square$

By (88) and (96), one can obtain the following result.

Corollary 11. *Let* $(G, v) \in \mathscr{TV}$ *be in the sense of Theorem 19, with* $(G, v) \in \mathcal{TV}(N)$*, for* $N \in \mathbb{N} \setminus \{1\}$*. Then*

$$\Gamma\big((G, v)^n\big) < v\big((G, v)^n\big),$$

however,

$$v(\mathbb{I}^n) = v(\mathbb{I}) = \Gamma(\mathbb{I}) = \Gamma(\mathbb{I}^n), \tag{97}$$

for all $n \in \mathbb{N}$*, where* $\mathbb{I} = (\{x\}, x) \in \mathscr{TV}$ *is the unity.*

Proof. If $N \in \mathbb{N} \setminus \{1\}$, then:

$$(G, v) \neq \mathbb{I} \text{ in } TV, \text{ and hence, in } \mathscr{TV},$$

under **NA 78**. Thus, if $G \in \mathcal{T}$ is a tree, which is not the vertex graph $\{x\} \in \mathcal{T}$, then

$$|V(G)| < |\mathbb{G}|,$$

because $\mathbb{G}$ contains the empty word ϕ other than vertices. Hence:

$$|V(G)| + |\{\phi\}| = |V(G)| + 1 \leq |\mathbb{G}|.$$

Therefore, by (88) and (94), we have:

$$v\big((G, v)^n\big) = \frac{1}{|V(G^n(v))|} > \frac{1}{|\mathbb{G}^n|} = \Gamma\big((G, v)^n\big),$$

where $\mathbb{G}^n$ are the graph groupoids of $G^n(v) \in \mathcal{T}$, for all $n \in \mathbb{N}$. Therefore, the strict inequality of (97) holds.

Now, let $\mathbb{I} = (\{x\}, x) \in \mathcal{TV}(1)$ is the identity of the tree-monoid TV, and $\mathbb{I} \overset{\text{denote}}{=} \xi_{\mathbb{I}} \in \mathscr{TV}$, the unity under **NA 78**. Then, the powers of $\mathbb{I}$ satisfy

$$\mathbb{I}^n = \mathbb{I}^{\odot n} = \mathbb{I}, \text{ in } \mathscr{TV}.$$

Therefore, one has:

$$\Gamma(\mathbb{I}^n) = \Gamma(\mathbb{I}) = \frac{1}{\Gamma(\mathbb{I})} = \frac{1}{|\mathbb{X}|} = \frac{1}{1} = 1,$$

and

$$v(\mathbb{I}^n) = v(\mathbb{I}) = \frac{1}{|V(\mathbb{I})|} = \frac{1}{1} = 1,$$

implying the equalities of (97) for all $n \in \mathbb{N}$. $\quad\square$

The following result is a direct consequence of Corollary 11.

Corollary 12. *Let* $(G, v) \in \mathscr{TV}$ *be in the sense of Theorem 19. Then:*

$$\Gamma\big((G, v)^n\big) \leq v\big((G, v)^n\big),$$

for all $n \in \mathbb{N}$.

Proof. This is shown by (97). $\quad\square$

The above result illustrates that our vertex-cardinality statistical model $(\mathscr{TV}, v)$ on the tree-monoidal algebra $\mathscr{TV}$ provides rough upper bounds for the index-tree-monoidal space $(\mathscr{TV}, \Gamma)$.

Definition 25. *The pair* $(\mathscr{TV}, v)$ *is called the vertex-tree-monoidal (measure) space.*

Now, let W be an element of the vertex-tree-monoidal space:

$$W = \sum_{k=1}^{N} t_k(G_k, v_k) \in (\mathscr{TV}, v), \text{ with } t_k \in \mathbb{C}, \tag{98}$$
$$\text{with } (G_k, v_k) \in \mathcal{TV}(N_k) \subset TV, \text{ for } k = 1, \dots, N,$$

for $N_1, \dots, N_N, N \in \mathbb{N}$.

Observe that if $W \in (\mathscr{TV}, v)$ is in the sense of (98), then:

$$W^n = \sum_{(i_1, \dots, i_n) \in \{1, \dots, N\}^n} \left(\prod_{l=1}^{n} t_{i_l}\right) \left(\overset{n}{\underset{l=1}{\odot}} (G_{i_1}, v_{i_l})\right), \tag{99}$$
$$\text{where } \overset{n}{\underset{l=1}{\odot}} (G_{i_l}, v_{i_l}) = (G_{i_1}, v_{i_1}) \odot (G_{i_2}, v_{i_v}) \odot \cdots \odot (G_{i_n}, v_{i_n}),$$

for all $(i_1, \dots, i_n) \in \{1, \dots, N\}^n$, for all $n \in \mathbb{N}$. If we denote the summands of W^n by:

$$\big\{ t_{i_1, \dots, i_n} \big(K_{i_1, \dots, i_n}, v_{i_1, \dots, i_n} \big) : (i_1, \dots, i_n) \in \{1, \dots, N\}^n \big\},$$

then

$$W^n = \sum_{(i_1, \dots, i_n) \in \{1, \dots, N\}^n} t_{i_1, \dots, i_n} \big(K_{i_1, \dots, i_n}, v_{i_1, \dots, i_n} \big), \tag{100}$$
$$\text{where } t_{i_1, \dots, i_n} = \prod_{l=1}^{n} t_{i_l} \text{ in } \mathbb{C},$$

and

$$K_{i_1, \dots, i_n} = G_{i_1 v_{i_1}} \#_{v_{i_2}} G_{i_2 v_{i_2}} \#_{v_{i_3}} \cdots v_{i_{n-1}} \#_{v_{i_n}} G_{i_n},$$

are the iterated glued graphs, and

$$v_{i_1,\ldots,i_n} = v_{i_1} \# v_{i_2} \# \ldots \# v_{i_n},$$

are the corresponding iterated collapsed vertices, for all $(i_1,\ldots,i_n) \in \{1,\ldots,N\}^n$, for $n \in \mathbb{N}$, by (99).

Since if $(G,v) \in \mathcal{TV}(k_1)$ and $(K,x) \in \mathcal{TV}(k_2)$ in $\mathcal{TV}$, then

$$(G,v) \circledcirc (K,x) = (G\,{}_v\#_x\,K,\, v\#x) \in \mathcal{TV}(k_1 + k_2 - 1),$$

in TV, each factor $\left(K_{i_1,\ldots,i_n}, v_{i_1,\ldots,i_n}\right) \in \mathscr{TV}$ of the summand of W^n, in the sense of (100), satisfies:

$$
\begin{aligned}
\left(K_{i_1,\ldots,i_n}, v_{i_1,\ldots,i_n}\right) &\in \mathcal{TV}\left(n_{i_1} + \cdots + n_{i_n} - (n-1)\right), \\
&\Longleftrightarrow \\
\left(K_{i_1,\ldots,i_n}, v_{i_1,\ldots,i_n}\right) &\in \mathcal{TV}\left(1 - n + \sum_{l=1}^{n} N_{i_l}\right),
\end{aligned}
\tag{101}
$$

by (98) and (99).

Theorem 20. *Let $W \in (\mathscr{TV}, v)$ be in the sense of (98). Then:*

$$
v(W^n) = \sum_{(i_1,\ldots,i_n) \in \{1,\ldots,N\}^n} \frac{\left(\prod_{l=1}^{n} t_{i_l}\right)}{\left(1 - n + \sum_{l=1}^{n} N_{i_l}\right)},
\tag{102}
$$

for all $n \in \mathbb{N}$.

Proof. If $W \in (\mathscr{TV}, v)$ is an element (98), then:

$$
W^n = \sum_{(i_1,\ldots,i_n) \in \{1,\ldots,N\}^n} t_{i_1,\ldots,i_n} \left(K_{i_1,\ldots,i_n}, v_{i_1,\ldots,i_n}\right),
$$

by (99), where $t_{i_1,\ldots,i_n} \in \mathbb{C}$ and $\left(K_{i_1,\ldots,i_n}, v_{i_1,\ldots,i_n}\right) \in \mathscr{TV}$ are in the sense of (100), for all $n \in \mathbb{N}$.

Thus, one has:

$$
v(W^n) = \sum_{(i_1,\ldots,i_n) \in \{1,\ldots,N\}^n} \frac{t_{i_1,\ldots,i_n}}{\left|V\left(K_{i_1,\ldots,i_n}\right)\right|},
$$

by (91). Hence, the formula (102) holds because:

$$
\left|V\left(K_{i_1,\ldots,i_n}\right)\right| = 1 - n + \sum_{l=1}^{n} N_{i_l},
$$

by (101). $\square$

The above moment computation (102) provides the following generalized estimation of (97).

Corollary 13. *Let $Y = \sum_{k=1}^{N} (G_k, v_k)$ be an element of the tree-monoidal algebra $\mathscr{TV}$. If we understand $Y \in \mathscr{TV}$ as an element of the index-tree-monoidal space $(\mathscr{TV}, \Gamma)$, then:*

$$
\Gamma(Y^n) \leq \sum_{(i_1,\ldots,i_n) \in \{1,\ldots,N\}^n} \frac{1}{\left(1 - n + \sum_{l=1}^{n} N_{i_l}\right)},
\tag{103}
$$

for all $n \in \mathbb{N}$.

Proof. If $Y \in \mathscr{T}\mathscr{V}$ is as above, then, similar to (99),

$$Y^n = \sum_{(i_1,\ldots,i_n) \in \{1,\ldots,N\}^n} \left(K_{i_1,\ldots,i_n}, v_{i_1,\ldots,i_n}\right),$$

where the summands are in terms of (100), for all $n \in \mathbb{N}$.

By (97), one has:

$$0 < \Gamma\left(\left(K_{i_1,\ldots,i_n}, v_{i_1,\ldots,i_n}\right)\right) \leq v\left(\left(K_{i_1,\ldots,i_n}, v_{i_1,\ldots,i_n}\right)\right),$$

for all $(i_1,\ldots,i_n) \in \{1,\ldots,N\}^n$. It implies that

$$\Gamma(Y^n) \leq v(Y^n), \text{ for all } n \in \mathbb{N}.$$

Therefore, the inequality (103) holds by (102). $\square$

12. Conclusions and Discussion

In this section, we explain the main ideas of this paper, summarize our main results, and discuss the connections among them.

Let G be a connected finite-directed graph with its graph groupoid $\mathbb{G}$. Then, the graph groupoid algebra M_G is well-defined as a C^*-algebra generated by $\mathbb{G}$, and the trace τ on M_G is naturally defined. Thus, the graph C^*-probability space (M_G, τ) is established. On (M_G, τ), the semicircularity is characterized by the loop-ness on $\mathbb{G}$. If G_u is the undirected graph of G, then the semicircularity on (M_G, τ) is characterized by the loop-ness on $\mathbb{G}$, which is characterized by the condition: G_u is not a tree—i.e., $w \in \mathbb{G}$ is a loop if and only if the w-radial operator L_w is semicircular in (M_G, τ), if and only if G_u is not a tree. Equivalently, the "non-semicircularity" on (M_G, τ) is characterized by the "non-loop-ness" of $\mathbb{G}$ and, equivalently, the "tree-ness" of G_u (or, the "directed-tree-ness" of G).

From a given connected finite-directed graph $G \in \mathcal{G}$ with more than one vertex (unique for the graph isomorphisms), one can take the loop-part $W \subseteq G$ and construct the quotient graph G_W. Then, this quotient graph G_W is a (directed) tree, called the G-tree. This shows that the G-tree G_W implies the "non-semicircularity" inside (M_G, τ), implying that (M_{G_W}, τ) does not contain any semicircular elements. Such a non-semicircularity on (M_G, τ) is quantized by the (graph-)tree index Γ:

$$\Gamma(G) \stackrel{\text{def}}{=} |\mathbb{G}_{\mathbb{W}}| = [G : W].$$

Based on the tree-indexing on $\mathcal{G}$, one can classify the family $\mathcal{G}$ in terms of the tree-family $\mathcal{T}$ (up to graph isomorphisms), implying the non-semicircularity induced by $\mathcal{G}$.

If $TV = \{(G, v) : G \in \mathcal{T}, v \in V(G)\}$, then the corresponding commutative monoid $(TV, \odot)$ is well-defined under the vertex-gluing on $\mathcal{G}$ (and, hence, on $\mathcal{T}$), under the operation $(\odot)$, which is the vertex-fixed gluing process. This monoidal structure $(TV, \odot)$ induces an operad,

$$\mathcal{T}\mathcal{V} = \{\mathcal{T}\mathcal{V}(n) : n \in \mathbb{N}\},$$

which is a monoidal category satisfying

$$\mathcal{T}\mathcal{V}(n_1) \odot \mathcal{T}\mathcal{V}(n_2) = \mathcal{T}\mathcal{V}(n_1 + n_2 - 1),$$

for all $n_1, n_2 \in \mathbb{N}$. Independently, this monoid $(TV, \odot)$ induces a pure-algebraic algebra $\mathscr{T}\mathscr{V}$ over $\mathbb{C}$, and it has certain statistics depending on the tree-index Γ, and that on the vertex-cardinality. In particular, the vertex-cardinality model provides rough upper bounds for the statistical data determined by Γ.

From our main results, one may/can consider further operad-dependent structures, such as the operad algebra generated by our operad $\mathcal{T}\mathcal{V}$, and keep considering how the tree-ness (which classifies the non-semicircularity) affects not only the analysis but also

the topology, as well as the physics. Additinoally, one may/can consider direct, canonical, and interesting connections between statistical data on $\mathcal{TV}$ and the non-semicircularity on graph C^*-probability spaces.

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