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Innovative Interpolating Polynomial Approach to Fractional Integral Inequalities and Real-World Implementations

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Abstract: Our paper explores Hermite–Hadamard inequalities through the application of Abel–Gontscharoff Green's function methodology, which involves interpolating polynomials and Riemann-type generalized fractional integrals. While establishing our main results, we explore new identities. These identities are used to estimate novel findings for functions, such that the second derivative of the functions is monotone, absolutely convex, and concave. A section relating the results of exploration to generalized means and trapezoid formulas is included in the applications. We anticipate that the method presented in this study will inspire further research in this field.

Keywords: Hermite–Hadamard type inequalities; Green's function; interpolating polynomial; fractional integral; generalized means

MSC: 26D10; 26A33; 26D15

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1. Introduction

The theory of inequality has evolved quickly in recent years. It is important to consider how closely related the convexity and inequality theories are to one another. Numerous new definitions, generalizations, and expansions of novel convexity have been provided in recent years, and related advancements in the theory of convexity inequality—particularly integral inequalities theory—have also been acknowledged. The Hermite–Hadamard inequality is one of the most significant causes of this development, among other significant inequalities. This inequality is a well known for convex functions that have been established in various ways and has a number of expansions and generalizations in the literature [1–5]. It defines upper and lower bounds for the integral mean of any convex function defined over a closed and bounded interval encompassing the function's endpoints and midpoint. For a convex function $\intercal$, the Hermite–Hadamard inequality is stated as follows:

$$\intercal\left(\frac{\varsigma_1+\varsigma_2}{2}\right) \leq \frac{1}{\varsigma_1-\varsigma_2}\int_{\varsigma_1}^{\varsigma_2} \intercal(\varrho)d\varrho \leq \frac{\intercal(\varsigma_1)+\intercal(\varsigma_2)}{2}.$$

The above inequality will hold in reverse directions if $\intercal$ is a concave function. This inequality has many fractional extensions like [6–8]. We further refer the reader to [9,10].

Compared to other function classes, convex functions possess a geometric interpretation and find numerous applications in various fields, including mathematics, statistics, optimization theory, finance, decision making, and numerical analysis. They constitute not only fundamental components of inequality theory but also serve as key elements motivating several inequalities. These functions are associated with not just continuity and differentiability, but also inequalities. The exploration of integral inequalities is a captivating pursuit within mathematical analysis. Foundational integral inequalities can significantly contribute to the elucidation of subjective aspects of convexity. Convex functions find utility across multiple domains within mathematical analysis and statistics; however, their role within inequality theory is of paramount significance. In this context, a plethora of classical and analytical inequalities have been established, most notably the Hermite–Hadamard, Ostrowski, Simpson, Fejer, and Hardy-type inequalities [11,12]. The extensive body of literature concerning integral inequalities for convex functions underscores the immense importance of this subject [13–17]. The convex function is defined as follows:

Definition 1 ([17])**.** *A real valued function* $\top : [\varsigma_1, \varsigma_2] \to R$ *is said to be convex (concave) if the inequality*

$$\top(\varphi\varsigma_1 + (1-\varphi)\varsigma_2) \leq (\geq)\ \varphi \top(\varsigma_1) + (1-\varphi) \top(\varsigma_2)$$

holds for all $0 \leq \varphi \leq 1$.

Fractional calculus has seen rapid progress in applied and pure mathematics because of its widespread usage in image processing, physics, and other fields. Experts from various fields have been quick to notice the fractional derivative. Classical derivations cannot be used to model the majority of practical problems. Fractional differential equations deal with the complexities of real-world problems [18–22]. For fractional integral operators, several definitions have been used, including the Hadamard integral, the k-Riemann–Liouville (RL) fractional integral, the Caputo–Fabrizio fractional integral, the RL fractional integral, and the conformable fractional integral [23–25]. One can extend such fractional integral operators by including additional parameters, leading to the fractional inequalities i.e., Minkowski, Hermite–Hadamard, Jensen, Ostrowski, and Grüss [23,26–28]. Future study is encouraged by these generalizations to provide additional innovative concepts using unified fractional operators, and to discover inequalities employing such generalised fractional operators.

We must keep in mind the preliminary equations and notations of a few well-known RL and k-RL fractional integral operators in order to derive certain results and corollaries. The RL fractional integrals, which are defined as follows, are the most traditional form of fractional integrals that have been described in the literature.

Definition 2 ([29])**.** *Let* $\varsigma_1, \varsigma_2 \in R$ *with* $\varsigma_1 < \varsigma_2$ *and* $\top \in L[\varsigma_1, \varsigma_2]$. *Then, the left-sided RL and right-sided RL fractional integrals* $\Im^{\varpi}_{\varsigma_1^+} \top$ *and* $\Im^{\varpi}_{\varsigma_2^-} \top$ *of order* $\varpi > 0$ *on a finite interval* $[\varsigma_1, \varsigma_2]$ *are defined by*

$$\Im^{\varpi}_{\varsigma_1^+} \top(\chi) = \frac{1}{\Gamma(\varpi)} \int_{\varsigma_1}^{\chi} (\chi - \varphi)^{\varpi - 1} \top(\varphi) d\varphi, \quad \chi > \varsigma_1$$

and

$$\Im^{\varpi}_{\varsigma_2^-} \top(\chi) = \frac{1}{\Gamma(\varpi)} \int_{\chi}^{\varsigma_2} (\varphi - \chi)^{\varpi - 1} \top(\varphi) d\varphi, \quad \chi < \varsigma_2,$$

respectively. The conventional Euler's gamma function, denoted by $\Gamma(\varpi)$, *is defined in the following manner:*

$$\Gamma(\varpi) = \int_0^{\infty} \varrho^{\varpi - 1} e^{-\varrho} d\varrho, \quad Re(\varpi) > 0. \tag{1}$$

Definition 3 ([30]). *The definition of κ-fractional integrals with order ϖ and parameters $\kappa > 0$ and $\varsigma \geq 0$ is as follows:*

$$\Im_{\varsigma_1^+}^{\varpi,\kappa} \intercal(\chi) = \frac{1}{\kappa\Gamma_\kappa(\varpi)} \int_{\varsigma_1}^{\chi} (\chi - \varphi)^{\frac{\varpi}{\kappa}-1} \intercal(\varphi) d\varphi, \quad \chi > \varsigma_1$$

and

$$\Im_{\varsigma_2^-}^{\varpi,\kappa} \intercal(\chi) = \frac{1}{\kappa\Gamma_\kappa(\varpi)} \int_{\chi}^{\varsigma_2} (\varphi - \chi)^{\frac{\varpi}{\kappa}-1} \intercal(\varphi) d\varphi, \quad \chi < \varsigma_2.$$

The κ-gamma function denoted as Γ_κ, as defined by Diaz et al. [31], can be expressed as follows:

$$\Gamma_\kappa(\varpi) = \int_0^\infty \varphi^{\varpi-1} e^{\frac{-\varphi^\kappa}{\kappa}} d\varphi.$$

Corresponding to the choice of $\kappa = 1$, the classical RL fractional integral is obtained as given in (1).

Definition 4 ([28]). *A real valued function $\intercal : I \to R$, where I is the range of continuous function $g : [\varsigma_1, \varsigma_2] \to R$. Then, the Jensen's integral inequality,*

$$\intercal\left(\frac{\int_{\varsigma_1}^{\varsigma_2} g(\varrho) d\lambda(\varrho)}{\int_{\varsigma_1}^{\varsigma_2} d\lambda(\varrho)}\right) \leq \frac{\int_{\varsigma_1}^{\varsigma_2} \intercal(g(\varrho)) d\lambda(\varrho)}{\int_{\varsigma_1}^{\varsigma_2} d\lambda(\varrho)},$$

holds if $\intercal$ is continuous, provided that λ is nondecreasing, bounded, and $\lambda(\varsigma_1) \neq \lambda(\varsigma_2)$).

The Abel–Gontscharoff polynomial and theorem for the two-point right focal" problem are referenced in [32]. The Abel–Gontscharoff polynomial for two-point right focal interpolating polynomial can be stated as a special choice for $n = 2$.

$$\intercal(\chi) = \intercal(\varsigma_1) + (\chi - \varsigma_1) \intercal'(\varsigma_2) + \int_{\varsigma_1}^{\varsigma_2} G(\chi, \psi) \intercal''(\psi) d\psi, \tag{2}$$

where $G(\chi, \psi)$ is the Green's function.

The following four functions were introduced by Mehmood et al. [33] based on the Abel–Gontscharoff Green's function.

$$G_1(\chi, \psi) = \begin{cases} \varsigma_1 - \psi, & \text{for } \varsigma_1 \leq \psi \leq \chi; \\ \varsigma_1 - \chi, & \text{for } \chi \leq \psi \leq \varsigma_2; \end{cases} \tag{3}$$

$$G_2(\chi, \psi) = \begin{cases} \chi - \varsigma_2, & \text{for } \varsigma_1 \leq \psi \leq \chi; \\ \psi - \varsigma_2, & \text{for } \chi \leq \psi \leq \varsigma_2; \end{cases} \tag{4}$$

$$G_3(\chi, \psi) = \begin{cases} \chi - \varsigma_1, & \text{for} \varsigma_1 \leq \psi \leq \chi; \\ \psi - \varsigma_1, & \text{for } \chi \leq \psi \leq \varsigma_2; \end{cases} \tag{5}$$

$$G_4(\chi, \psi) = \begin{cases} \varsigma_2 - \psi, & \text{for } \varsigma_1 \leq \psi \leq \chi; \\ \varsigma_2 - \chi, & \text{for } \chi \leq \psi \leq \varsigma_2. \end{cases} \tag{6}$$

Sarikaya et al. [34] derived the subsequent inequality of Hermite–Hadamard type for fractional integrals.

Theorem 1. *Let* $\mathsf{T} : [\varsigma_1, \varsigma_2] \to R$ *be a positive function satisfying* $0 \leq \varsigma_1 < \varsigma_2$ *and* $\mathsf{T} \in L[\varsigma_1, \varsigma_2]$*. If* T *is a convex function over* $[\varsigma_1, \varsigma_2]$*, then the following inequalities hold for fractional integrals.*

$$\mathsf{T}\left(\frac{\varsigma_1 + \varsigma_2}{2}\right) \leq \frac{\Gamma(\varpi + 1)}{2(\varsigma_2 - \varsigma_1)^{\varpi}}\left(\Im^{\varpi}_{\varsigma_1^+} \mathsf{T}(\varsigma_2) + \Im^{\varpi}_{\varsigma_2^-} \mathsf{T}(\varsigma_1)\right) \leq \frac{\mathsf{T}(\varsigma_1) + \mathsf{T}(\varsigma_2)}{2}, \tag{7}$$

where $\varpi > 0$.

Inspired by the research conducted by Khan et al. as reported in [29], we intend to develop Hadamard-type fractional integral inequalities by utilizing appropriate Green's functions. The findings we present in the following section extend the current body of knowledge and serve as a source of inspiration for researchers engaged in the study of mathematical inequalities. Finally, we refer the reader to learn about fractional calculus and its applications from the articles [35–41] and from the books [42,43]. The convex functions and related inequalities can also be studied in [44–53].

2. Generalized Fractional Integral Inequalities via Special Green's Functions

In this section, we derive generalized fractional integral inequalities via Abel–Gontscharoff Green's function given in (3).

Theorem 2. *Let* T *be a function that is twice differentiable and convex on the interval* $[\varsigma_1, \varsigma_2]$*. Then, the following double inequality holds for any positive values of* ϖ *and* κ*:*

$$\mathsf{T}\left(\frac{\kappa\varsigma_1 + \varpi\varsigma_2}{\varpi + \kappa}\right) \leq \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \mathsf{T}(\varsigma_1) \leq \frac{\kappa\, \mathsf{T}(\varsigma_1) + \varpi\, \mathsf{T}(\varsigma_2)}{\varpi + \kappa}. \tag{8}$$

Proof. By substituting $\chi = \frac{\kappa\varsigma_1 + \varpi\varsigma_2}{\varpi + \kappa}$ into the Abel–Gontscharoff polynomial for the two-point right focal interpolating polynomial given by Equation (2), we derive

$$\begin{aligned}\mathsf{T}\left(\frac{\kappa\varsigma_1 + \varpi\varsigma_2}{\varpi + \kappa}\right) &= \mathsf{T}(\varsigma_1) + \left(\frac{\kappa\varsigma_1 + \varpi\varsigma_2}{\varpi + \kappa} - \varsigma_1\right)\mathsf{T}'(\varsigma_2) + \int_{\varsigma_1}^{\varsigma_2} G\left(\frac{\kappa\varsigma_1 + \varpi\varsigma_2}{\varpi + \kappa}, \psi\right)\mathsf{T}''(\psi)\, d\psi \\ &= \mathsf{T}(\varsigma_1) + \frac{\varpi(\varsigma_2 - \varsigma_1)}{\varpi + \kappa}\mathsf{T}'(\varsigma_2) + \int_{\varsigma_1}^{\varsigma_2} G\left(\frac{\kappa\varsigma_1 + \varpi\varsigma_2}{\varpi + \kappa}, \psi\right)\mathsf{T}''(\psi)d\psi. \end{aligned} \tag{9}$$

After multiplying both sides of Equation (2) by $\frac{\varpi(\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1}}{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}}$ and integrating with respect to χ, we obtain

$$\begin{aligned}&\frac{\varpi}{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa} - 1}} \int_{\varsigma_1}^{\varsigma_2} (\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1}\mathsf{T}(\chi)d\chi \\ &= \frac{\varpi}{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa} - 1}}\left(\int_{\varsigma_1}^{\varsigma_2} (\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1}\mathsf{T}(\varsigma_1)d\chi\right. \\ &\quad \left. + \int_{\varsigma_1}^{\varsigma_2} (\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1}(\chi - \varsigma_1)\mathsf{T}'(\varsigma_2)d\chi + \int_{\varsigma_1}^{\varsigma_2}\int_{\varsigma_1}^{\varsigma_2} G(\chi, \psi)(\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1}\mathsf{T}''(\psi)d\psi d\chi\right)\end{aligned}$$

$$= \frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\left(\frac{\kappa \mathsf{T}(\varsigma_1)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}{\varpi} + \frac{\kappa \mathsf{T}'(\varsigma_2)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+1}}{\varpi+\kappa} + \int_{\varsigma_1}^{\varsigma_2}\int_{\varsigma_1}^{\varsigma_2} G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}\mathsf{T}''(\psi)d\psi d\chi\right).$$

By using the relation $\Gamma_\kappa(\varpi+\kappa)=\varpi\Gamma_\kappa(\kappa)$, this can also be written as

$$\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1) = \mathsf{T}(\varsigma_1) + \frac{\varpi(\mathsf{T}'(\varsigma_2)(\varsigma_2-\varsigma_1))}{(\varpi+\kappa)} + \frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2}\int_{\varsigma_1}^{\varsigma_2} G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}\mathsf{T}''(\psi)d\psi d\chi. \tag{10}$$

From (9) and (10), we can write

$$\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right) - \frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1) = \int_{\varsigma_1}^{\varsigma_2}\left(G\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa},\psi\right) - \frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2} G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}d\chi\right)\mathsf{T}''(\psi)d\psi. \tag{11}$$

By employing the formula (3) and implementing certain simplifications, we arrive at the following:

$$\int_{\varsigma_1}^{\varsigma_2} G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}d\chi = \frac{\kappa^2}{\varpi(\varpi+\kappa)}\left((\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1} - \left(\frac{\varpi}{\kappa}+1\right)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\psi-\varsigma_1)\right) \tag{12}$$

and

$$G\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa},\psi\right) = \begin{cases} \varsigma_1-\psi, & \text{for } \varsigma_1\le\psi\le\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}; \\ \frac{\varpi(\varsigma_1-\varsigma_2)}{\varpi+\kappa}, & \text{for } \frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\le\psi\le\varsigma_2. \end{cases} \tag{13}$$

If $\varsigma_1\le\psi\le\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}$, then by utilizing (12) and (13) in (11), we have

$$G\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa},\psi\right) - \frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2} G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}d\chi$$
$$= (\varsigma_1-\psi) - \frac{\kappa\left((\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1} - (\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\psi-\varsigma_1)(\frac{\varpi}{\kappa}+1)\right)}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}$$

$$\begin{aligned} &= \frac{\left((\varsigma_1-\psi)(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}} - \kappa(\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1} + \kappa(\psi-\varsigma_1)(\frac{\varpi}{\kappa}+1)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}\right)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi+\kappa)} \\ &= -\frac{\kappa(\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi+\kappa)} \le 0. \end{aligned} \tag{14}$$

Again, for the choice $\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa} \leq \psi \leq \varsigma_2$, and making use of (12) and (13) in (11), we get

$$\begin{aligned}
& G\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa},\psi\right) - \frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2} G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}d\chi \\
&= \frac{\varpi(\varsigma_1-\varsigma_2)}{\varpi+\kappa} - \frac{\kappa\left((\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1} - (\psi-\varsigma_1)(\frac{\varpi}{\kappa}+1)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}\right)}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}} \\
&= \frac{\varpi(\varsigma_1-\varsigma_2)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}} - \kappa\left((\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1} - (\psi-\varsigma_1)(\frac{\varpi}{\kappa}+1)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}\right)}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}} \\
&= \frac{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi(\psi-\varsigma_2)+\kappa(\psi-\varsigma_1)) - \kappa(\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi+\kappa)}.
\end{aligned} \tag{15}$$

Let

$$h(\psi) = \frac{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi(\psi-\varsigma_2)+\kappa(\psi-\varsigma_1)) - \kappa(\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi+\kappa)}.$$

It is notable that

$$h(\varsigma_2) = 0 \quad \text{and} \quad h'(\psi) = 1 - \frac{(\psi-\varsigma_1)^{\frac{\varpi}{\kappa}}}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}} > 0. \tag{16}$$

Therefore, followed by (15) to (16), we can write

$$G\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa},\psi\right) - \frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2} G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}d\chi \leq 0. \tag{17}$$

Since T is convex and implies $\mathsf{T}''(\psi) \geq 0$, and by making use of (14), (17) in (11), we can write

$$\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right) \leq \frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1). \tag{18}$$

This establishes the left half of the inequality in Equation (8).

Now, we proceed to prove the right half inequality of Equation (8). To accomplish this, we select $\chi = \varsigma_2$ in Equation (2), yielding

$$\mathsf{T}(\varsigma_2) = \mathsf{T}(\varsigma_1) + (\varsigma_2-\varsigma_1)\mathsf{T}'(\varsigma_2) + \int_{\varsigma_1}^{\varsigma_2} G(\varsigma_2,\psi)\mathsf{T}''(\psi)d\psi. \tag{19}$$

Multiplying both sides by $\frac{\varpi}{\kappa}$ and then dividing by $(\frac{\varpi}{\kappa}+1)$, Equation (19) is equivalent to

$$\frac{\kappa\mathsf{T}(\varsigma_1)+\varpi\mathsf{T}(\varsigma_2)}{\varpi+\kappa} = \mathsf{T}(\varsigma_1) + \frac{\varpi(\varsigma_2-\varsigma_1)\mathsf{T}'(\varsigma_2)}{\varpi+\kappa} + \frac{\varpi}{\varpi+\kappa}\int_{\varsigma_1}^{\varsigma_2} G(\varsigma_2,\psi)\mathsf{T}''(\psi)d\psi; \tag{20}$$

by subtracting (10) from (20), we get the inequality

$$\begin{aligned}
& \frac{\kappa\mathsf{T}(\varsigma_1)+\varpi\mathsf{T}(\varsigma_2)}{\varpi+\kappa} - \frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1) \\
&= \int_{\varsigma_1}^{\varsigma_2}\left(\frac{\varpi}{\varpi+\kappa}G(\varsigma_2,\psi) - \frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2} G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}d\chi\right)\mathsf{T}''(\psi)d\psi.
\end{aligned} \tag{21}$$

By including the Green formula $(\varsigma_1 - \psi)$ for $\varsigma_1 \leq \psi \leq \varsigma_2$ and Equation (12), we can express it as follows:

$$\begin{aligned}
&\frac{\varpi G(\varsigma_2, \psi)}{\varpi + \kappa} - \frac{\varpi}{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \int_{\varsigma_1}^{\varsigma_2} G(\chi, \psi)(\chi - \varsigma_1)^{\frac{\varpi}{\kappa}-1} d\chi \\
&= \frac{\varpi(\varsigma_1 - \psi)}{\varpi + \kappa} - \frac{\kappa\Big((\psi - \varsigma_1)^{\frac{\varpi}{\kappa}+1} - (\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\psi - \varsigma_1)(\frac{\varpi}{\kappa} + 1)\Big)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi + \kappa)} \\
&= \frac{\kappa(\psi - \varsigma_1)\Big((\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}} - (\psi - \varsigma_1)^{\frac{\varpi}{\kappa}}\Big)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi + \kappa)} \geq 0.
\end{aligned} \tag{22}$$

By using the convexity of $\intercal$ and Equations (21) and (22), we obtain

$$\frac{\kappa \intercal(\varsigma_1) + \varpi \intercal(\varsigma_2)}{\varpi + \kappa} \geq \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \intercal(\varsigma_1). \tag{23}$$

By combining the inequalities (18) and (23), we obtained the required result. □

Remark 1. *By substituting $\kappa = 1$ into inequality (8), we obtain the following results as presented in ([29], Theorem 2.2)*

$$\intercal\left(\frac{\varsigma_1 + \varpi\varsigma_2}{\varpi + 1}\right) \leq \frac{\Gamma(\varpi + 1)}{(\varsigma_2 - \varsigma_1)^{\varpi}} \Im^{\varpi}_{\varsigma_2^-} \intercal(\varsigma_1) \leq \frac{\intercal(\varsigma_1) + \varpi \intercal(\varsigma_2)}{\varpi + 1}.$$

Theorem 3. *Let $\intercal$ be a function that is twice differentiable on the interval $[\varsigma_1, \varsigma_2]$, and let ϖ and κ be positive. Then, the following inequalities hold.*

(i) *If $|\intercal''|$ is an increasing function, then*

$$\left|\frac{\kappa \intercal(\varsigma_1) + \varpi \intercal(\varsigma_2)}{\varpi + \kappa} - \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \intercal(\varsigma_1)\right| \leq \frac{|\intercal''(\varsigma_2)|\varpi\kappa(\varsigma_2 - \varsigma_1)^2}{2(\kappa + \varpi)(\varpi + 2\kappa)}; \tag{24}$$

(ii) *If $|\intercal''|$ is decreasing function, then*

$$\left|\frac{\kappa \intercal(\varsigma_1) + \varpi \intercal(\varsigma_2)}{\varpi + \kappa} - \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \intercal(\varsigma_1)\right| \leq \frac{|\intercal''(\varsigma_1)|\varpi\kappa(\varsigma_2 - \varsigma_1)^2}{2(\kappa + \varpi)(\varpi + 2\kappa)}; \tag{25}$$

(iii) *If $|\intercal''|$ is a convex function, then*

$$\begin{aligned}
&\left|\frac{\kappa \intercal(\varsigma_1) + \varpi \intercal(\varsigma_2)}{\varpi + \kappa} - \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \intercal(\varsigma_1)\right| \\
&\leq \frac{\max(|\intercal''(\varsigma_1)|, |\intercal''(\varsigma_2)|)\varpi\kappa(\varsigma_2 - \varsigma_1)^2}{2(\kappa + \varpi)(\varpi + \kappa)}.
\end{aligned} \tag{26}$$

Proof. *(i)* From (21) and (22), the increasing monotonicity of $|\intercal''|$ and

$$(\psi - \varsigma_1)(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}} - (\psi - \varsigma_1)^{\frac{\varpi}{\kappa}+1} \geq 0,$$

for $\varsigma_1 \leq \psi \leq \varsigma_2$. Consider

$$
\begin{aligned}
&\left| \frac{\kappa \,\mathsf{T}\,(\varsigma_1) + \varpi \,\mathsf{T}\,(\varsigma_2)}{\varpi + \kappa} - \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \mathsf{T}\,(\varsigma_1) \right| \\
&\leq \frac{\kappa |\mathsf{T}''(\varsigma_2)|}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi + \kappa)} \int\limits_{\varsigma_1}^{\varsigma_2} \Big((\psi - \varsigma_1)(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}} - (\psi - \varsigma_1)^{\frac{\varpi}{\kappa}+1} \Big) d\psi \\
&\leq \frac{\kappa |\mathsf{T}''(\varsigma_2)|}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi + \kappa)} \left(\frac{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varsigma_2 - \varsigma_1)^2}{2} - \frac{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}+2}}{\varpi + 2\kappa} \right) \\
&= \frac{\kappa |\mathsf{T}''(\varsigma_2)| \big(\varpi(\varsigma_2 - \varsigma_1)^2\big)}{2(\varpi + \kappa)(\varpi + 2\kappa)}.
\end{aligned}
$$

This corresponds to inequality (24).

(ii) By employing (21) and (22) and following the same procedure as in case (i), we derive

$$
\begin{aligned}
&\left| \frac{\kappa \,\mathsf{T}\,(\varsigma_1) + \varpi \,\mathsf{T}\,(\varsigma_2)}{\varpi + \kappa} - \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \mathsf{T}\,(\varsigma_1) \right| \\
&\leq \frac{\kappa}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi + \kappa)} \int\limits_{\varsigma_1}^{\varsigma_2} \Big((\psi - \varsigma_1)(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}} - (\psi - \varsigma_1)^{\frac{\varpi}{\kappa}+1} \Big) |\,\mathsf{T}''\,(\psi)| d\psi \\
&\leq \frac{\kappa |\mathsf{T}''(\varsigma_1)|}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi + \kappa)} \left(\frac{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varsigma_2 - \varsigma_1)^2}{2} - \frac{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}+2}}{\varpi + 2\kappa} \right) \\
&= \frac{\kappa |\mathsf{T}''(\varsigma_1)| \big(\varpi(\varsigma_2 - \varsigma_1)^2\big)}{2(\varpi + \kappa)(\varpi + 2\kappa)}.
\end{aligned}
$$

(iii) From (21) and (22),

$$
\begin{aligned}
&\left| \frac{\kappa \,\mathsf{T}\,(\varsigma_1) + \varpi \,\mathsf{T}\,(\varsigma_2)}{\varpi + \kappa} - \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \mathsf{T}\,(\varsigma_1) \right| \\
&\leq \frac{\kappa}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi + \kappa)} \int\limits_{\varsigma_1}^{\varsigma_2} \Big((\psi - \varsigma_1)(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}} - (\psi - \varsigma_1)^{\frac{\varpi}{\kappa}+1} \Big) |\mathsf{T}''(\psi)| d\psi.
\end{aligned}
\tag{27}
$$

By using (27) and the fact that $|\,\mathsf{T}''\,|$ is a convex function on the interval $[\varsigma_1, \varsigma_2]$, we get

$$
\begin{aligned}
&\left| \frac{\kappa \,\mathsf{T}\,(\varsigma_1) + \varpi \,\mathsf{T}\,(\varsigma_2)}{\varpi + \kappa} - \frac{\Gamma_\kappa(\varpi + \kappa)}{(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \Im^{\varpi,\kappa}_{\varsigma_2^-} \mathsf{T}\,(\varsigma_1) \right| \\
&\leq \frac{\kappa \max |\mathsf{T}''(\varsigma_1)|, |\mathsf{T}''(\varsigma_2)|}{(\varpi + \kappa)((\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}})} \int\limits_{\varsigma_1}^{\varsigma_2} \Big((\psi - \varsigma_1)(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}} - (\psi - \varsigma_1)^{\frac{\varpi}{\kappa}+1} \Big) d\psi \\
&= \frac{\max(|\,\mathsf{T}''\,(\varsigma_1)|, |\,\mathsf{T}''\,(\varsigma_2)|)\kappa\varpi(\varsigma_2 - \varsigma_1)^2}{2(\varpi + \kappa)(\varpi + 2\kappa)}.
\end{aligned}
$$

This corresponds to inequality (26). □

Remark 2. *By setting $\kappa = 1$ in inequalities (24)–(26), we derive the following results as presented in ([29], Theorem 2.6)*

$$\left|\frac{\mathsf{T}(\varsigma_1)+\varpi\,\mathsf{T}\,(\varsigma_2)}{\varpi+1}-\frac{\Gamma(\varpi+1)}{(\varsigma_2-\varsigma_1)^{\varpi}}\Im^{\varpi}_{\varsigma_2^-}\,\mathsf{T}\,(\varsigma_1)\right|\le\frac{|\mathsf{T}''(\varsigma_2)|\varpi(\varsigma_2-\varsigma_1)^2}{2(\varpi+1)(\varpi+2)},$$
$$\left|\frac{\mathsf{T}(\varsigma_1)+\varpi\,\mathsf{T}\,(\varsigma_2)}{\varpi+1}-\frac{\Gamma(\varpi+1)}{(\varsigma_2-\varsigma_1)^{\varpi}}\Im^{\varpi}_{\varsigma_2^-}\,\mathsf{T}\,(\varsigma_1)\right|\le\frac{|\mathsf{T}''(\varsigma_1)|\varpi(\varsigma_2-\varsigma_1)^2}{2(\varpi+1)(\varpi+2)}$$

and

$$\left|\frac{\mathsf{T}(\varsigma_1)+\varpi\,\mathsf{T}\,(\varsigma_2)}{\varpi+1}-\frac{\Gamma(\varpi+1)}{(\varsigma_2-\varsigma_1)^{\varpi}}\Im^{\varpi}_{\varsigma_2^-}\,\mathsf{T}\,(\varsigma_1)\right|\le\frac{\max(|\mathsf{T}''(\varsigma_1)|,|\mathsf{T}''(\varsigma_2)|)\varpi(\varsigma_2-\varsigma_1)^2}{2(\varpi+1)(\varpi+2)}.$$

Theorem 4. *Let T be a function that is twice differentiable on the interval $[\varsigma_1,\varsigma_2]$, and let ϖ and κ be positive. The following statements hold.*

(i) *If $|\,\mathsf{T}''\,|$ is an increasing function, then*

$$\begin{aligned}&\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\,\mathsf{T}\,(\varsigma_1)\right|\\&\le\frac{\varpi\kappa^2(\varsigma_2-\varsigma_1)^2}{(\varpi+\kappa)^{\frac{\varpi}{\kappa}+3}(\varpi+2\kappa)}\left(\left|\mathsf{T}''(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa})\right|(\varpi)^{\frac{\varpi}{\kappa}+1}\right.\\&\left.+\left|\mathsf{T}''(\varsigma_2)\right|\left(\frac{(\varpi+\kappa)^{\frac{\varpi}{\kappa}+1}-2(\varpi)^{\frac{\varpi}{\kappa}+1}}{2}\right)\right);\end{aligned}$$

(ii) *If $|\,\mathsf{T}''\,|$ is a decreasing function, then*

$$\begin{aligned}&\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\,\mathsf{T}\,(\varsigma_1)\right|\\&\le\frac{\varpi\kappa^2(\varsigma_2-\varsigma_1)^2}{(\varpi+\kappa)^{\frac{\varpi}{\kappa}+3}(\varpi+2\kappa)}\left(\left|\mathsf{T}''(\varsigma_1)\right|(\varpi)^{\frac{\varpi}{\kappa}+1}\right.\\&\left.+\left|\mathsf{T}''\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)\right|\left(\frac{(\varpi+\kappa)^{\frac{\varpi}{\kappa}+1}-2(\varpi)^{\frac{\varpi}{\kappa}+1}}{2}\right)\right);\end{aligned}$$

(iii) *If $|\,\mathsf{T}''\,|$ is a convex function, then*

$$\begin{aligned}&\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\,\mathsf{T}\,(\varsigma_1)\right|\\&\le\frac{\varpi\kappa^2(\varsigma_2-\varsigma_1)^2}{(\varpi+\kappa)^{\frac{\varpi}{\kappa}+3}(\varpi+2\kappa)}\left(\max\left(\left|\mathsf{T}''(\varsigma_1)\right|,\left|\mathsf{T}''\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)\right|\right)(\varpi)^{\frac{\varpi}{\kappa}+1}\right.\\&\left.+\max\left(\left|\mathsf{T}''\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)\right|,\left|\mathsf{T}''(\varsigma_2)\right|\right)\left(\frac{(\varpi+\kappa)^{\frac{\varpi}{\kappa}+1}-2(\varpi)^{\frac{\varpi}{\kappa}+1}}{2}\right)\right).\end{aligned}\tag{28}$$

Proof. *(i)* By using (11), (14) and (15), we get

$$\begin{aligned}
&\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\\
&=\int\limits_{\varsigma_1}^{\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}}\left(G\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa},\psi\right)-\frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int\limits_{\varsigma_1}^{\varsigma_2}G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}d\chi\right)\mathsf{T}''(\psi)d\psi\\
&\quad+\int\limits_{\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}}^{\varsigma_2}\left(G\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa},\psi\right)-\frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int\limits_{\varsigma_1}^{\varsigma_2}G(\chi,\psi)(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}d\chi\right)\mathsf{T}''(\psi)d\psi\\
&=-\frac{\kappa}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\left(\int\limits_{\varsigma_1}^{\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}}(\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}\mathsf{T}''(\psi)d\psi+\int\limits_{\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}}^{\varsigma_2}\Big((\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}\right.\\
&\quad\left.+\frac{\varpi}{\kappa}(\varsigma_2-\psi)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}-(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\psi-\varsigma_1)\Big)\mathsf{T}''(\psi)d\psi\right).
\end{aligned}\tag{29}$$

The triangle inequality, and the increasing monotonicity of $|\mathsf{T}''(\psi)|$, enabled us to obtain the following from Equation (29):

$$\begin{aligned}
&\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|\\
&\leq\frac{\kappa}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\left(\left|\mathsf{T}''\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)\right|\int\limits_{\varsigma_1}^{\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}}(\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}d\psi\right.\\
&\quad\left.+|\mathsf{T}''(\varsigma_2)|\int\limits_{\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}}^{\varsigma_2}\Big((\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}+\frac{\varpi}{\kappa}(\varsigma_2-\psi)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}-(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\psi-\varsigma_1)\Big)d\psi\right)\\
&=\frac{\varpi\kappa^2(\varsigma_2-\varsigma_1)^2}{(\varpi+\kappa)^{\frac{\varpi}{\kappa}+3}(\varpi+2\kappa)}\left(\left|\mathsf{T}''(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa})\right|(\varpi)^{\frac{\varpi}{\kappa}+1}+|\mathsf{T}''(\varsigma_2)|\left(\frac{(\varpi+\kappa)^{\frac{\varpi}{\kappa}+1}-2(\varpi)^{\frac{\varpi}{\kappa}+1}}{2}\right)\right).
\end{aligned}$$

This completes part *(i)* of the result.

(ii) The proof of this part can be carried out using the same procedure as described above.

(iii) Given that every convex function T defined on an interval $[\varsigma_1,\varsigma_2]$ is bounded above by $\max \mathsf{T}(\varsigma_1),\mathsf{T}(\varsigma_2)$, we have the following inequality:

$$\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|$$

$$\le \frac{\kappa}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Bigg(\max\left(\left|\mathsf{T}''(\varsigma_1)\right|,\left|\mathsf{T}''\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)\right|\right)$$
$$\times\int\limits_{\varsigma_1}^{\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}}(\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}d\psi+\max\left(\left|\mathsf{T}''\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)\right|,\left|\mathsf{T}''(\varsigma_2)\right|\right)$$
$$\times\int\limits_{\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}}^{\varsigma_2}\left((\psi-\varsigma_1)^{\frac{\varpi}{\kappa}+1}+\frac{\varpi}{\kappa}(\varsigma_2-\psi)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}-(\psi-\varsigma_1)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}\right)d\psi\Bigg).$$

By calculation, we get inequality (28). □

Remark 3. *By choosing $\kappa=1$, we get the results presented in ([29], Theorem 2.4).*

Theorem 5. *Suppose that T is a function that is twice differentiable and $|\mathsf{T}''|$ is convex on the interval $[\varsigma_1,\varsigma_2]$. Then, for $\varpi,\kappa>0$, the following inequality holds:*

$$\begin{aligned}&\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|\\&\le\frac{\kappa^3(\varsigma_2-\varsigma_1)^2}{6(\varpi+\kappa)^3(\varpi+3\kappa)}\Bigg(\left|\mathsf{T}''(\varsigma_1)\right|\frac{(7\varpi^2+17\varpi\kappa+12\kappa^2)}{\varpi+2\kappa}\\&\quad+\left|\mathsf{T}''(\varsigma_2)\right|\left(\frac{9\varpi^2}{\kappa}+23\varpi+12\kappa\right)\Bigg).\end{aligned}\tag{30}$$

Proof. By substituting $\psi=\varrho\varsigma_1+(1-\varrho)\varsigma_2$ with $\varrho\in[0,1]$, Equation (29) can be written as

$$\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)$$
$$=\frac{-\kappa}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Bigg(-\int\limits_{1}^{\frac{\kappa}{\varpi+\kappa}}(\varrho\varsigma_1+\varsigma_2-\varrho\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+1}\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)$$
$$\times(\varsigma_2-\varsigma_1)d\varrho-\int\limits_{\frac{\kappa}{\varpi+\kappa}}^{0}\Bigg((\varrho\varsigma_1+\varsigma_2-\varrho\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+1}+\frac{\varpi}{\kappa}\Big(\varsigma_2-\varrho\varsigma_1$$
$$-\varsigma_2+\varrho\varsigma_2\Big)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}-(\varrho\varsigma_1+\varsigma_2-\varrho\varsigma_2-\varsigma_1)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}\Bigg)$$
$$\times\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)(\varsigma_2-\varsigma_1)d\varrho\Bigg)$$
$$=\frac{-\kappa}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Bigg(\int\limits_{\frac{\kappa}{\varpi+\kappa}}^{1}(1-\varrho)^{\frac{\varpi}{\kappa}+1}(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+1}$$
$$\times\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)(\varsigma_2-\varsigma_1)d\varrho+\int\limits_{0}^{\frac{\kappa}{\varpi+\kappa}}\Bigg((1-\varrho)^{\frac{\varpi}{\kappa}+1}(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+1}$$

$$
\begin{aligned}
&+\frac{\varpi}{\kappa}\varrho(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+1}-(1-\varrho)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+1}\Bigg)\\
&\times \mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)(\varsigma_2-\varsigma_1)d\varrho\Bigg)\\
=&-\frac{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+2}}{(\varpi+\kappa)(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\left(\int_{\frac{\kappa}{\varpi+\kappa}}^{1}(1-\varrho)^{\frac{\varpi}{\kappa}+1}\,\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho\right.\\
&+\int_0^{\frac{\kappa}{\varpi+\kappa}}(1-\varrho)^{\frac{\varpi}{\kappa}+1}\,\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho\\
&\left.+\int_0^{\frac{\kappa}{\varpi+\kappa}}\left(\varrho+\frac{\varpi}{\kappa}\varrho-1\right)^{\frac{\varpi}{\kappa}+1}\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho\right)\\
=&\frac{-\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\left(\int_0^1(1-\varrho)^{\frac{\varpi}{\kappa}+1}\,\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho\right.\\
&\left.-\int_0^{\frac{\kappa}{\varpi+\kappa}}\left(1-\varrho-\frac{\varpi}{\kappa}\varrho\right)\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho\right).
\end{aligned}
\tag{31}
$$

Taking absolute value on both sides, using the fact $1-\varrho-\frac{\varpi}{\kappa}\varrho<1-\varrho$ and applying triangular inequality, we get

$$
\begin{aligned}
&\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im_{\varsigma_2^-}^{\varpi,\kappa}\,\mathsf{T}(\varsigma_1)\right|\\
&\leq\frac{\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\left(\int_0^1(1-\varrho)^{\frac{\varpi}{\kappa}+1}|\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)|d\varrho\right.\\
&\left.+\int_0^{\frac{\kappa}{\varpi+\kappa}}\left(1-\varrho-\frac{\varpi}{\kappa}\varrho\right)|\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)|d\varrho\right)\\
&\leq\frac{\kappa(\varsigma_2-\varsigma_1)^2}{(\varpi+\kappa)}\left(\int_0^{\frac{\kappa}{\varpi+\kappa}}\left(1-\varrho-\frac{\varpi}{\kappa}\varrho\right)(\varrho|\mathsf{T}''(\varsigma_1)|+(1-\varrho)|\mathsf{T}''(\varsigma_2)|)d\varrho\right.\\
&\left.+\int_0^1(1-\varrho)^{\frac{\varpi}{\kappa}+1}(\varrho|\mathsf{T}''(\varsigma_1)|+(1-\varrho)|\mathsf{T}''(\varsigma_2)|)d\varrho\right)\\
&\leq\frac{\kappa^3(\varsigma_2-\varsigma_1)^2}{6(\varpi+\kappa)^3(\varpi+3\kappa)}\left(|\mathsf{T}''(\varsigma_1)|\frac{(7(\varpi^2)+17\varpi\kappa+12\kappa^2)}{(\varpi+2\kappa)}\right.\\
&\left.+|\mathsf{T}''(\varsigma_2)|\left(9\left(\frac{\varpi^2}{\kappa}\right)+23\varpi+12\kappa\right)\right),
\end{aligned}
$$

This proved the desired result. □

Remark 4. *By setting $\kappa=1$, we obtain Theorem 2.8 presented in [29].*

Theorem 6. *Suppose* T *is a function that is twice differentiable and* $|\mathsf{T}''|$ *is convex on the interval* $[\varsigma_1, \varsigma_2]$*. Then, we formulate the following inequality:*

$$\begin{aligned}&\left|\frac{\kappa\,\mathsf{T}(\varsigma_1)+\varpi\,\mathsf{T}(\varsigma_2)}{\varpi+\kappa}-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|\\&\leq\frac{\kappa\varpi(\varsigma_2-\varsigma_1)^2}{3(\varpi+\kappa)(\varpi+3\kappa)}\left(\left|\mathsf{T}''(\varsigma_1)\right|\frac{(\varpi+5\kappa)}{2(\varpi+2\kappa)}+\left|\mathsf{T}''(\varsigma_2)\right|\right).\end{aligned}$$

Proof. Let $\varrho \in [0,1]$ and $\psi = \varrho\varsigma_1 + (1-\varrho)\varsigma_2$. Then, from (21) and (22), we obtain

$$\begin{aligned}&\frac{\kappa\,\mathsf{T}(\varsigma_1)+\varpi\,\mathsf{T}(\varsigma_2)}{\varpi+\kappa}-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\\&=\frac{-\kappa}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\varpi+\kappa)}\int_1^0\Bigg((\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}(\varrho\varsigma_1+(1-\varrho)\varsigma_2-\varsigma_1)\\&\quad-(\varrho\varsigma_1+(1-\varrho)\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}+1}\Bigg)\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)(\varsigma_2-\varsigma_1)d\varrho\\&=\frac{\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\int_0^1\left(1-\varrho-(1-\varrho)^{\frac{\varpi}{\kappa}+1}\right)\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho.\end{aligned} \tag{32}$$

By considering the relation of absolute value in an inequality and using the convexity of $|\mathsf{T}''|$, we get

$$\begin{aligned}&\left|\frac{\kappa\,\mathsf{T}(\varsigma_1)+\varpi\,\mathsf{T}(\varsigma_2)}{\varpi+\kappa}-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|\\&\leq\frac{\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\int_0^1\left(1-\varrho-(1-\varrho)^{\frac{\varpi}{\kappa}+1}\right)\left|\mathsf{T}''(\varrho\varsigma_1+(1-\varrho)\varsigma_2)\right|d\varrho\\&\leq\frac{\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\int_0^1\left(1-\varrho-(1-\varrho)^{\frac{\varpi}{\kappa}+1}\right)\left(\varrho|\mathsf{T}''(\varsigma_1)|+(1-\varrho)\left|\mathsf{T}''(\varsigma_2)\right|\right)d\varrho\\&=\frac{\varpi\kappa(\varsigma_2-\varsigma_1)^2}{3(\varpi+\kappa)(\varpi+3\kappa)}\left(\left|\mathsf{T}''(\varsigma_1)\right|\frac{(\varpi+5\kappa)}{2(\varpi+2\kappa)}+\left|\mathsf{T}''(\varsigma_2)\right|\right).\end{aligned}$$

This completes the proof. □

Remark 5. *Corresponding to the choice* $\kappa = 1$ *in Theorem 6, we obtain the result explored by Muhammad Adil Khan et al. in ([29], Theorem 2.10).*

Theorem 7. *Suppose that* T *is a function that is twice differentiable and that* $|\mathsf{T}''|$ *is concave on the interval* $[\varsigma_1, \varsigma_2]$*. Then, the following inequality,*

$$\begin{aligned}&\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|\\&\leq\frac{\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\left(\frac{\kappa}{\varpi+2\kappa}\left|\mathsf{T}''\left(\frac{\kappa(\varsigma_1+2\varsigma_2+\frac{\varpi}{\kappa}\varsigma_2)}{\varpi+3\kappa}\right)\right|\right.\\&\quad\left.+\frac{\kappa}{2(\varpi+\kappa)}\left|\mathsf{T}''\left(\frac{\kappa(\varsigma_1+2\varsigma_2+3\frac{\varpi}{\kappa}\varsigma_2)}{3(\varpi+\kappa)}\right)\right|\right),\end{aligned}$$

holds.

Proof. It follows from (31) and using Jensen's integral inequality that

$$
\begin{aligned}
&\left|\mathsf{T}\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2}{\varpi+\kappa}\right)-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|\\
&\le\frac{\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\left(\int_0^1(1-\varrho)^{\frac{\varpi}{\kappa}+1}d\varrho\left|\mathsf{T}''\left(\frac{\int_0^1(1-\varrho)^{\frac{\varpi}{\kappa}+1}(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho}{\int_0^1(1-\varrho)^{\frac{\varpi}{\kappa}+1}d\varrho}\right)\right|\right.\\
&\quad\left.+\int_0^{\frac{\kappa}{\varpi+\kappa}}\left(1-\varrho-\frac{\varpi}{\kappa}\varrho\right)d\varrho\left|\mathsf{T}''\left(\frac{\int_0^1(1-\varrho-\frac{\varpi}{\kappa}\varrho)(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho}{\int_0^1(1-\varrho-\frac{\varpi}{\kappa}\varrho)d\varrho}\right)\right|\right.\\
&\le\frac{\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\left(\frac{\kappa}{\varpi+2\kappa}\left|\mathsf{T}''\left(\frac{\kappa^2(\varsigma_1+2\varsigma_2+\frac{\varpi}{\kappa}\varsigma_2)(\varpi+2\kappa)}{\kappa(\varpi+2\kappa)(\varpi+3\kappa)}\right)\right|\right.\\
&\quad\left.+\frac{\kappa}{2(\varpi+\kappa)}\left|\mathsf{T}''\left(\frac{2\kappa^2(\varsigma_1+2\varsigma_2+3\frac{\varpi}{\kappa}\varsigma_2)(\varpi+\kappa)}{6\kappa(\varpi+\kappa)^2}\right)\right|\right)\\
&=\frac{\kappa(\varsigma_2-\varsigma_1)^2}{\varpi+\kappa}\left(\frac{\kappa}{2(\varpi+\kappa)}\left|\mathsf{T}''\left(\frac{\kappa\varsigma_1+3\varpi\varsigma_2+2\kappa\varsigma_2}{3(\varpi+\kappa)}\right)\right|\right.\\
&\quad\left.+\frac{\kappa}{\varpi+2\kappa}\left|\mathsf{T}''\left(\frac{\kappa\varsigma_1+\varpi\varsigma_2+2\kappa\varsigma_2}{\varpi+3\kappa}\right)\right|\right).
\end{aligned}
$$

Hence, the required result is proved. □

Remark 6. *By setting $\kappa = 1$ in Theorem 7, we arrive at the result presented in ([29], Theorem 2.12).*

Theorem 8. *Suppose that T is a function that is twice differentiable and that $|\mathsf{T}''|$ is a concave function on the interval $[\varsigma_1, \varsigma_2]$. Then, for any positive values of ϖ and κ, the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\kappa\,\mathsf{T}(\varsigma_1)+\varpi\,\mathsf{T}(\varsigma_2)}{\varpi+\kappa}-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|\\
&\le\frac{\kappa\varpi(\varsigma_2-\varsigma_1)^2}{2(\varpi+\kappa)(\varpi+2\kappa)}\left|\mathsf{T}''\left(\frac{\varsigma_1(\varpi+5\kappa)+2\varsigma_2(\varpi+2\kappa)}{3(\varpi+3\kappa)}\right)\right|.
\end{aligned}\tag{33}
$$

Proof. By using Jensen's inequality on (32), we get

$$
\begin{aligned}
&\left|\frac{\kappa\,\mathsf{T}(\varsigma_1)+\varpi\,\mathsf{T}(\varsigma_2)}{\varpi+\kappa}-\frac{\Gamma_\kappa(\varpi+\kappa)}{(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\Im^{\varpi,\kappa}_{\varsigma_2^-}\mathsf{T}(\varsigma_1)\right|\\
&\le\frac{\kappa(\varsigma_2-\varsigma_1)^2}{(\varpi+\kappa)}\int_0^1\left(-\varrho+1-(1-\varrho)^{\frac{\varpi}{\kappa}+1}\right)d\varrho\\
&\quad\times\left|\mathsf{T}''\left(\frac{\int_0^1\left(1-\varrho-(1-\varrho)^{\frac{\varpi}{\kappa}+1}\right)(\varrho\varsigma_1+(1-\varrho)\varsigma_2)d\varrho}{\int_0^1\left(-\varrho+1-(1-\varrho)^{\frac{\varpi}{\kappa}+1}\right)d\varrho}\right)\right|\\
&=\frac{\kappa\varpi(\varsigma_2-\varsigma_1)^2}{2(\varpi+\kappa)(\varpi+2\kappa)}\left|\mathsf{T}''\left(\frac{(\varpi+5\kappa)\varsigma_1+2\varsigma_2(\varpi+2\kappa)}{3(\varpi+3\kappa)}\right)\right|.
\end{aligned}
$$

□

Remark 7. *The choice of $\kappa = 1$ in Theorem 8 gives the result presented in ([29], Theorem 2.14)*

Remark 8. *If an alternative approach is taken using Green's function G_2, as defined in Equation (4), analogous results to those expounded in this article can be obtained. Nevertheless, the utilization of the method proposed in this article in conjunction with Green's functions G_3 (described by Equation (5)) or G_4 (described by Equation (6)) exclusively reproduces previously established findings. This strategic choice substantiates the exclusive focus on Green's function G_1 within the discourse of this article. As an intellectual exercise, interested scholars are encouraged to explore the remaining three Green functions to elicit their respective outcomes.*

3. Some Relations of Means with Trapezoid Formulae

In this section, we will present some propositions that highlight both the practical use and immense significance of our outcomes. At the same time, we will verify the validity of the conclusions by using special means of real numbers ς_1 and ς_2, where $\varsigma_1 \neq \varsigma_2$. These propositions present the estimates of differences of generalized means that are concluded from our main results. First, we need to recall the following relations. For $\varsigma_1, \varsigma_2 > 0$, one of the important average values is

(i) The Arithmetic mean:

$$A = A(\varsigma_1, \varsigma_2) = \frac{\varsigma_1 + \varsigma_2}{2}; \tag{34}$$

(ii) The Harmonic mean:

$$H = H(\varsigma_1, \varsigma_2) = \frac{2}{\frac{1}{\varsigma_1} + \frac{1}{\varsigma_2}}; \tag{35}$$

(iii) The logarithmic mean:

$$L(\varsigma_1, \varsigma_2) = \frac{\varsigma_2 - \varsigma_1}{\ln \varsigma_2 - \ln \varsigma_1}, \quad \varsigma_1 \neq \varsigma_2; \tag{36}$$

(iv) The generalized logarithmic mean presented in [54] is defined as follows:

$$L_m^m(\varsigma_1, \varsigma_2) = \frac{\varsigma_2^{m+1} - \varsigma_1^{m+1}}{(m+1)(\varsigma_2 - \varsigma_1)}, \quad m \in \mathbb{Z} \setminus \{-1, 0\}, \varsigma_1 \neq \varsigma_2. \tag{37}$$

Proposition 1. *Let $\varsigma_1, \varsigma_2 \in \mathbb{R}$, $\varsigma_1 < \varsigma_2$, then we can derive the following inequalities:*

$$|A(e^{\varsigma_1}, e^{\varsigma_2}) - L(e^{\varsigma_1}, e^{\varsigma_2})| \leq \frac{e^{\varsigma_2}(\varsigma_2 - \varsigma_1)^2}{12},$$
$$|A(e^{\varsigma_1}, e^{\varsigma_2}) - L(e^{\varsigma_1}, e^{\varsigma_2})| \leq \frac{e^{\varsigma_1}(\varsigma_2 - \varsigma_1)^2}{12},$$
$$|A(e^{\varsigma_1}, e^{\varsigma_2}) - L(e^{\varsigma_1}, e^{\varsigma_2})| \leq \frac{\max(e^{\varsigma_1}, e^{\varsigma_2})(\varsigma_2 - \varsigma_1)^2}{12}$$

and

$$|A(e^{\varsigma_1}, e^{\varsigma_2}) - L(e^{\varsigma_1}, e^{\varsigma_2})| \leq \frac{(\varsigma_2 - \varsigma_1)^2(e^{\varsigma_1} + e^{\varsigma_2})}{24}.$$

Proof. By utilizing Theorem 3 and performing some simplifications, we can express them:

$$\left| \frac{\kappa \top(\varsigma_1) + \varpi \top(\varsigma_2)}{\varpi + \kappa} - \frac{\varpi}{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \int_{\varsigma_1}^{\varsigma_2} (\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1} \top(\chi) d\chi \right| \leq \frac{\varpi\kappa |\top''(\varsigma_2)| (\varsigma_2 - \varsigma_1)^2}{2(\varpi + \kappa)(\varpi + 2\kappa)}.$$

By substituting $\kappa = \varpi$, $\mathsf{T}(\varsigma) = e^{\varsigma}$ and using simple calculation, we obtain

$$\left|\frac{e^{\varsigma_1}+e^{\varsigma_2}}{2}-\frac{e^{\varsigma_2}-e^{\varsigma_1}}{(\varsigma_2-\varsigma_1)}\right| \le \frac{e^{\varsigma_2}(\varsigma_2-\varsigma_1)^2}{12}.$$

Now, making use of (34) and (37), we arrive at the result

$$|A(e^{\varsigma_1},e^{\varsigma_2})-L(e^{\varsigma_1},e^{\varsigma_2})| \le \frac{|e^{\varsigma_2}|(\varsigma_2-\varsigma_1)^2}{12}.$$

The remaining inequalities can be obtained by applying the same procedure as in parts (ii) and (iii) of Theorem 3.

Theorem 6 can also be proved by using same function as in Theorem 3, after making some simplifications to Theorem 6

$$\begin{aligned}&\left|\frac{\kappa\,\mathsf{T}\,(\varsigma_1)+\varpi\,\mathsf{T}\,(\varsigma_2)}{\varpi+\kappa}-\frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2}(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}\,\mathsf{T}\,(\chi)d\chi\right|\\&\le \frac{\kappa\varpi(\varsigma_2-\varsigma_1)^2}{3(\varpi+\kappa)(\varpi+3\kappa)}\left(\left|\mathsf{T}''(\varsigma_1)\left(\left|\frac{(\varpi+5\kappa)}{2(\varpi+2\kappa)}+\right|\mathsf{T}''\,(\varsigma_2)\right|\right).\end{aligned}$$

By substituting $\kappa = \varpi$, $\mathsf{T}(\varsigma) = e^{\varsigma}$ and using simple calculation, we obtain

$$\left|\frac{e^{\varsigma_1}+e^{\varsigma_2}}{2}-\frac{e^{\varsigma_2}-e^{\varsigma_1}}{(\varsigma_2-\varsigma_1)}\right| \le \frac{(\varsigma_2-\varsigma_1)^2(e^{\varsigma_1}+e^{\varsigma_2})}{24}.$$

By employing (34) and (37), we obtain the following result

$$|A(e^{\varsigma_1},e^{\varsigma_2})-L(e^{\varsigma_1},e^{\varsigma_2})| \le \frac{(\varsigma_2-\varsigma_1)^2(e^{\varsigma_1}+e^{\varsigma_2})}{24}.$$

□

Proposition 2. *Let $\varsigma_1,\varsigma_2 \in \mathbb{R}^+$ with $\varsigma_1 < \varsigma_2$, then the inequalities hold:*

$$\begin{aligned}\left|H^{-1}(\varsigma_1,\varsigma_2)-L^{-1}(\varsigma_1,\varsigma_2)\right| &\le \frac{(\varsigma_2-\varsigma_1)^2|\varsigma_2^{-3}|}{6},\\ \left|H^{-1}(\varsigma_1,\varsigma_2)-L^{-1}(\varsigma_1,\varsigma_2)\right| &\le \frac{(\varsigma_2-\varsigma_1)^2|\varsigma_1^{-3}|}{6},\\ \left|H^{-1}(\varsigma_1,\varsigma_2)-L^{-1}(\varsigma_1,\varsigma_2)\right| &\le \frac{\max\left(\left|\varsigma_1^{-3}\right|,\left|\varsigma_2^{-3}\right|\right)(\varsigma_2-\varsigma_1)^2}{6}\end{aligned}$$

and

$$\left|H^{-1}(\varsigma_1,\varsigma_2)-L^{-1}(\varsigma_1,\varsigma_2)\right| \le \frac{(\varsigma_2-\varsigma_1)^2\left|\varsigma_1^{-3}+\varsigma_2^{-3}\right|}{12}.$$

Proof. By utilizing Theorem 3 and simplifying the expressions, we derive

$$\left|\frac{\kappa\,\mathsf{T}\,(\varsigma_1)+\varpi\,\mathsf{T}\,(\varsigma_2)}{\varpi+\kappa}-\frac{\varpi}{\kappa(\varsigma_2-\varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2}(\chi-\varsigma_1)^{\frac{\varpi}{\kappa}-1}\,\mathsf{T}\,(\chi)d\chi\right| \le \frac{\varpi\kappa|\mathsf{T}''(\varsigma_2)|(\varsigma_2-\varsigma_1)^2}{2(\varpi+\kappa)(\varpi+2\kappa)}.$$

By substituting $\kappa = \varpi$ and $\top(\varsigma) = \frac{1}{\varsigma}$, where $\varsigma > 0$, we obtain

$$\left|\frac{\varsigma_2 + \varsigma_1}{2\varsigma_1\varsigma_2} - \frac{\ln \varsigma_2 - \ln \varsigma_1}{(\varsigma_2 - \varsigma_1)}\right| \le \frac{\left|\varsigma_2^{-3}\right|(\varsigma_2 - \varsigma_1)^2}{6}$$

Now, by using Equations (35) and (36), we get

$$\left|H^{-1}(\varsigma_1, \varsigma_2) - L^{-1}(\varsigma_1, \varsigma_2)\right| \le \frac{(\varsigma_2 - \varsigma_1)^2\left|\varsigma_2^{-3}\right|}{6}.$$

Similarly, by some simple calculations in Theorem 6,

$$\begin{aligned}&\left|\frac{\kappa \top(\varsigma_1) + \varpi \top(\varsigma_2)}{\varpi + \kappa} - \frac{\varpi}{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2} (\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1} \top(\chi) d\chi\right| \\ &\le \frac{\kappa\varpi(\varsigma_2 - \varsigma_1)^2}{3(\varpi + \kappa)(\varpi + 3\kappa)}\left(\left|\top''(\varsigma_1)\right|\frac{(\varpi + 5\kappa)}{2(\varpi + 2\kappa)} + \left|\top''(\varsigma_2)\right|\right);\end{aligned}$$

using the same polynomial function, we obtain

$$\left|\frac{\varsigma_2 + \varsigma_1}{2\varsigma_1\varsigma_2} - \frac{\ln \varsigma_2 - \ln \varsigma_1}{(\varsigma_2 - \varsigma_1)}\right| \le \frac{(\varsigma_2 - \varsigma_1)^2\left|\varsigma_1^{-3} + \varsigma_2^{-3}\right|}{12},$$

which is the required result. □

Proposition 3. *Let $\varsigma_1, \varsigma_2 \in R^+$ with $\varsigma_1 < \varsigma_2$, then the inequalities*

$$\left|A(\varsigma_1^m, \varsigma_2^m) - L_m^m(\varsigma_1, \varsigma_2)\right| \le \frac{(\varsigma_2 - \varsigma_1)^2|m(m-1)|\varsigma_2^{m-2}}{12},$$

$$\left|A(\varsigma_1^m, \varsigma_2^m) - L_m^m(\varsigma_1, \varsigma_2)\right| \le \frac{(\varsigma_2 - \varsigma_1)^2|m(m-1)|\varsigma_1^{m-2}}{12},$$

$$\left|A(\varsigma_1^m, \varsigma_2^m) - L_m^m(\varsigma_1, \varsigma_2)\right| \le \frac{\max\left(|m(m-1)|\varsigma_1^{m-2}, |m(m-1)|\varsigma_2^{m-2}\right)(\varsigma_2 - \varsigma_1)^2}{12}$$

and

$$\left|A(\varsigma_1^m, \varsigma_2^m) - L_m^m(\varsigma_1, \varsigma_2)\right| \le \frac{(\varsigma_2 - \varsigma_1)^2}{24}\left(|m(m-1)|\left(\varsigma_1^{m-2} + \varsigma_2^{m-2}\right)\right)$$

are true for $m \in \mathbb{Z}$ with $|m(m-1)| \ge 2$.

Proof. Using Theorem 3 and making some simplification, we get

$$\left|\frac{\kappa \top(\varsigma_1) + \varpi \top(\varsigma_2)}{\varpi + \kappa} - \frac{\varpi}{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}}\int_{\varsigma_1}^{\varsigma_2} (\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1} \top(\chi) d\chi\right| \le \frac{\varpi\kappa\left|\top''(\varsigma_2)\right|(\varsigma_2 - \varsigma_1)^2}{2(\varpi + \kappa)(\varpi + 2\kappa)}.$$

By substituting $\kappa = \varpi$ and $\top(\varsigma) = \varsigma^m$, where $\varsigma > 0$ and $|m(m-1)| \ge 2$, we obtain

$$\left|\frac{\varsigma_1^m + \varsigma_2^m}{2} - \frac{\varsigma_2^{m+1} - \varsigma_1^{m+1}}{(m+1)(\varsigma_2 - \varsigma_1)}\right| \le \frac{\left|m(m-1)\varsigma_2^{m-2}\right|(\varsigma_2 - \varsigma_1)^2}{12}.$$

Now, by using Equations (34) and (37), we get

$$|A(\varsigma_1^m, \varsigma_2^m) - L_m^m(\varsigma_1, \varsigma_2)| \leq \frac{(\varsigma_2 - \varsigma_1)^2 |m(m-1)| \varsigma_2^{m-2}}{12}.$$

Similarly, by some simple calculations in Theorem 6,

$$\left| \frac{\kappa\, \mathsf{T}(\varsigma_1) + \varpi\, \mathsf{T}(\varsigma_2)}{\varpi + \kappa} - \frac{\varpi}{\kappa(\varsigma_2 - \varsigma_1)^{\frac{\varpi}{\kappa}}} \int_{\varsigma_1}^{\varsigma_2} (\chi - \varsigma_1)^{\frac{\varpi}{\kappa} - 1}\, \mathsf{T}(\chi) d\chi \right|$$
$$\leq \frac{\kappa\varpi(\varsigma_2 - \varsigma_1)^2}{3(\varpi + \kappa)(\varpi + 3\kappa)} \left(|\mathsf{T}''(\varsigma_1)| \frac{(\varpi + 5\kappa)}{2(\varpi + 2\kappa)} + |\mathsf{T}''(\varsigma_2)| \right),$$

using the same polynomial function, we obtain

$$\left| \frac{\varsigma_1^m + \varsigma_2^m}{2} - \frac{\varsigma_2^{m+1} - \varsigma_1^{m+1}}{(m+1)(\varsigma_2 - \varsigma_1)} \right| \leq \frac{(\varsigma_2 - \varsigma_1)^2}{24} \left(|m(m-1)| \left(\varsigma_1^{m-2} + \varsigma_2^{m-2} \right) \right).$$

□

4. Concluding Remarks

In mathematics, inequalities play a crucial role as they are widely employed in diverse fields of study. They enable us to compare and contrast the relative magnitudes of distinct mathematical expressions, thereby facilitating a deeper comprehension of their interrelationships. Inequalities are not only critical for theoretical purposes but also have significant practical applications in optimization problems and statistical data analysis. The comprehension of inequalities is a fundamental aspect of mathematical literacy as it enables individuals to evaluate and interpret quantitative information and make well-informed decisions in various aspects of their lives. The inequality theory makes use of well-known results such as the Hölder's and Jensen's inequalities to derive compelling consequences. In the work presented, we used some special Green's functions and utilized the Jensen's inequality to establish the fractional Hermite–Hadamard type inequalities. We have adopted an innovative approach to obtain the novel findings. Specifically, we have utilized quadrature formulae to estimate differences between particular average values. The third section of our research deals with practical applications to real-world problems. In this section, we estimate the errors of generalized means differences that are very important in real life problems. Going forward, our objective is to explore additional inequalities by integrating Green's functions $G2$, $G3$, and $G4$. We hope that our research will encourage other scholars who are working on fractional integral inequalities using different types of convex functions.

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