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Article

Coefficient Bounds and Fekete–Szegö Inequalities for a Two Families of Bi-Univalent Functions Related to Gegenbauer Polynomials

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Abstract: The purpose of this article is to introduce and study certain families of normalized certain functions with symmetric points connected to Gegenbauer polynomials. Moreover, we determine the upper bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and resolve the Fekete–Szegöproblem for these functions. In addition, we establish links to a few of the earlier discovered outcomes.

Keywords: coefficient estimates; Gegenbauer polynomial; bi-univalent function; Fekete–Szegö problem

MSC: 30C45; 30C50

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1. Introduction

In [1], Legendre studied orthogonal polynomials comprehensively. These polynomials are recognized to be crucial in solving approximation theory problems. They occur in the theory of differential and integral equations as well as in mathematical statistics, automatic control and axially symmetric potential theory and are known from [2,3] and other articles. In a practical sense, the Gegenbauer polynomials are special cases of orthogonal polynomials. They are representatively related with typically real functions, T_R as discovered in [4]. Because of the relation $T_R = \overline{co}S_R$ and its function of estimating coefficient bounds, real functions typically play a significant role in geometric function theory. Here, S_R stands for the family of univalent functions in the unit disk with real coefficients, and $\overline{co}S_R$ stands for the closed convex hull of S_R.

On this subject in geometric function theory, the so-called Fekete–Szegö-type inequalities (or problems) study the estimate for $\left|a_3 - \mu a_2^2\right|$ for holomorphic univalent functions (cf. [5]).

Assume that $\mathcal{A}$ refers to the collection of functions f, which are holomorphic in the open unit disk $U = \{\eta \in \mathbb{C} : |\eta| < 1\}$ that have the type

$$f(\eta) = \eta + \sum_{n=2}^{\infty} a_n \eta^n \,. \tag{1}$$

Further, the subfamily of $\mathcal{A}$ in U is symbolized by S.

From [6], we have

$$f^{-1}(f(\eta)) = \eta, \ (\eta \in U)$$

and

$$f\left(f^{-1}(\xi)\right) = \xi, \ \left(|\xi| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(\xi) = f^{-1}(\xi) = \xi - a_2\xi^2 + \left(2a_2^2 - a_3\right)\xi^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)\xi^4 + \ldots \tag{2}$$

and f^{-1} is an inverse of $f \in S$.

A function $f \in \mathcal{A}$ is named bi-univalent in U if both f and its inverse f^{-1} are univalent in U. Suppose Σ indicates the collection of bi-univalent functions in U given by (1). From Srivastava et al.'s [7] pioneering work on the topic, a large number of works connected with this topic have been discussed (see, for example, [4,8–47]). The family Σ is not empty, as can be shown. Several examples of functions from the collection Σ are

$$\frac{\eta}{1-\eta}, \quad \frac{1}{2}\log\left(\frac{1+\eta}{1-\eta}\right) \text{ and } -\log(1-\eta)$$

and their inverses are

$$\frac{\xi}{1+\xi}, \quad \frac{e^{2\xi}-1}{e^{2\xi}+1} \text{ and } \frac{e^{\xi}-1}{e^{\xi}},$$

respectively. As other examples without Σ, we have

$$\eta - \frac{\eta^2}{2} \text{ and } \frac{\eta}{1-\eta^2}.$$

Until now, the study of the estimates problem for functions $f \in \Sigma$ associated with the following Taylor–Maclaurin coefficients $|a_n|$, $(n = 3, 4, \ldots)$ is still open.

In [48], the authors investigated the family S_{sc}^* of starlike functions under the following condition:

$$\operatorname{Re}\left\{\frac{2\eta f'(\eta)}{\left(f(\eta) - \overline{f(-\overline{\eta})}\right)'}\right\} > 0, \quad \eta \in U.$$

The family C_{sc} represents all convex functions if

$$\operatorname{Re}\left\{\frac{(2\eta f'(\eta))'}{\left(f(\eta) - \overline{f(-\overline{\eta})}\right)'}\right\} > 0, \quad \eta \in U.$$

Let f and g be holomorphic functions in U. We say that the function f is named subordinate to g if there exists a Schwarz function ξ holomorphic in U with $\xi(0) = 0$ and $|\xi(\eta)| < 1$ $(\eta \in U)$ such that $f(\eta) = g(\xi(\eta))$. This subordination is indicated by $f \prec g$ or $f(\eta) \prec g(\eta)$ $(\eta \in U)$. It is well known that (see [49]) if g is a univalent function in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The Gegenbauer polynomial $H_\delta(\eta, \tau)$ is defined by [50,51] (also, see [52])

$$H_\delta(\eta, \tau) = \frac{1}{(1 - 2\tau\eta + \eta^2)^\delta} \ \tau \in [-1, 1], \ \ \eta \in U,$$

where δ is a nonzero real constant. For fixed τ, the analytic function H_δ, depending on η, has the following expansion involving Gegenbauer polynomials $\mathcal{G}_n^\delta(\tau)$ as coefficients:

$$H_\delta(\eta, \tau) = \frac{1}{(1 - 2\tau\eta + \eta^2)^\delta} = \sum_{n=0}^{\infty} \mathcal{G}_n^\delta(\tau)\eta^n,$$

When $\delta = 0$, then we obtain the generating function of the Gegenbauer polynomial in the form

$$H_0(\eta, \tau) = 1 - \log\left(1 - 2\tau\eta + \eta^2\right) = \sum_{n=0}^{\infty} \mathcal{G}_n^0(\tau)\eta^n.$$

When δ is more than $-\frac{1}{2}$ [3,53], the recurrence relations of Gegenbauer polynomials are given as

$$\mathcal{G}_n^\delta(\tau) = \frac{1}{2}[2\tau(n+\delta-1)(\tau) - (n+2\delta-2)(\tau)]\ \mathcal{G}_{n-1}^\delta,$$

with the initial values

$$\mathcal{G}_2^\delta(\tau) = 2\delta(\delta+1)\tau^2 - \delta \quad \mathcal{G}_1^\delta(\tau) = 2\delta t \quad \text{and} \quad \mathcal{G}_0^\delta(\tau) = 1. \tag{3}$$

Remark 1. *Some special cases of $\mathcal{G}_n^\delta(\tau)$ are the following:*
(1) Taking $\delta = 1$, we obtain the Chebyshev polynomials.
(2) When $\delta = \frac{1}{2}$, we obtain the Legendre polynomials.

2. Main Results

To start, we define the families $\mathcal{M}_\Sigma^{\mathrm{sc}}(\alpha, \tau, \delta)$ and $\mathcal{N}_\Sigma^{\mathrm{sc}}(\alpha, \tau, \delta)$ as follows.

Definition 1. *For $0 \le \alpha \le 1$, $\tau \in \left(\frac{1}{2}, 1\right]$ and $\delta \neq 0$, a function $f \in \Sigma$ is said to be in the family $\mathcal{M}_\Sigma^{sc}(\alpha, \tau, \delta)$ if it fulfills the following subordinations:*

$$\frac{2\eta f'(\eta)}{f(\eta) - \overline{f(-\overline{\eta})}} + \frac{2\left(\eta f'(\eta)\right)'}{\left(f(\eta) - \overline{f(-\overline{\eta})}\right)'} - \frac{2\alpha\eta^2 f''(\eta) + 2\eta f'(\eta)}{\alpha\eta\left(f(\eta) - \overline{f(-\overline{\eta})}\right)' + (1-\alpha)\left(f(\eta) - \overline{f(-\overline{\eta})}\right)}$$

$$\prec H_\delta(\eta, \tau) = \frac{1}{(1 - 2\tau\eta + \eta^2)^\delta}$$

and

$$\frac{2\xi g'(\xi)}{g(\xi) - \overline{g(-\overline{\xi})}} + \frac{2\left(\xi g'(\xi)\right)'}{\left(g(\xi) - \overline{g(-\overline{\xi})}\right)'} - \frac{2\alpha\xi^2 g''(\xi) + 2\xi g'(\xi)}{\alpha\xi\left(g(\xi) - \overline{g(-\overline{\xi})}\right)' + (1-\alpha)\left(g(\xi) - \overline{g(-\overline{\xi})}\right)}$$

$$\prec H_\delta(\xi, \tau) = \frac{1}{(1 - 2\tau\xi + \xi^2)^\delta},$$

where the function $g = f^{-1}$ is defined by (2).

Taking $\delta = 1$ in Definition 1, the family $\mathcal{M}_\Sigma^{\mathrm{sc}}(\alpha, t, \delta)$ reduces to the result of Wanas and Majeed (see [54]).

Definition 2. *For $0 \le \alpha \le 1$, $\tau \in \left(\frac{1}{2}, 1\right]$ and $\delta \neq 0$, a function $f \in \Sigma$ is said to be in the family $\mathcal{N}_\Sigma^{sc}(\alpha, \tau, \delta)$ if it fulfills the following subordinations:*

$$\frac{2\left[\alpha\eta^3 f'''(\eta) + (\alpha+1)\eta^2 f''(\eta) + \eta f'(\eta)\right]}{\alpha\left(\eta^2\left(f(\eta) - \overline{f(-\overline{\eta})}\right)'' + \left(f(\eta) - \overline{f(-\overline{\eta})}\right)\right) + (1-\alpha)\eta\left(f(\eta) - \overline{f(-\overline{\eta})}\right)'}$$

$$\prec H_\delta(\eta, \tau) = \frac{1}{(1 - 2\tau\eta + \eta^2)^\delta}$$

and

$$\frac{2\left[\alpha\xi^3 g'''(\xi)+(\alpha+1)\xi^2 g''(\xi)+\xi g'(\xi)\right]}{\alpha\left(\xi^2\left(g(\xi)-\overline{g(-\overline{\xi})}\right)''+\left(g(\xi)-\overline{g(-\overline{\xi})}\right)\right)+(1-\alpha)\xi\left(g(\xi)-\overline{g(-\overline{\xi})}\right)'}$$

$$\prec H_\delta(\xi,\tau)=\frac{1}{(1-2\tau\xi+\xi^2)^\delta},$$

where the function $g=f^{-1}$ is given by (2).

As special case, $\delta=1$ in Definition 2 gives the family $S^c_\Sigma(\alpha,\tau)$ (cf. [55]).

Theorem 1. *For $t\in(\frac{1}{2},1]$, $0\le\alpha\le1$ and $\delta\neq0$, let $f\in\mathcal{A}$ be in the family $\mathcal{M}^{sc}_\Sigma(\alpha,\tau,\delta)$. Then,*

$$|a_2|\le\frac{\tau\ \sqrt{2|\delta|\tau}\ |\delta|}{\sqrt{\left|\delta(2-\alpha)^2-2\left[\delta(\delta+1)(2-\alpha)^2-\delta^2(3-2\alpha)\right]\tau^2\right|}}$$

and

$$|a_3|\le\frac{\delta^2\tau^2}{(2-\alpha)^2}+\frac{|\delta|\tau}{3-2\alpha}.$$

Proof. Let $f\in\mathcal{M}^{sc}_\Sigma(\alpha,\tau,\delta)$. Then, $u,v:U\longrightarrow U$ can be given by

$$u(\eta)=u_1\eta+u_2\eta^2+u_3\eta^3+\ldots\qquad(\eta\in U)\tag{4}$$

and

$$v(\xi)=v_1\xi+v_2\xi^2+v_3\xi^3+\ldots\qquad(\xi\in U),\tag{5}$$

with $u(0)=v(0)=0$, $|u(\eta)|<1$, $|v(\xi)|<1$, $\eta,\xi\in U$ such that

$$\begin{aligned}&\frac{2\eta f'(\eta)}{f(\eta)-\overline{f(-\overline{\eta})}}+\frac{2\left(\eta f'(\eta)\right)'}{\left(f(\eta)-\overline{f(-\overline{\eta})}\right)'}-\frac{2\alpha\eta^2 f''(\eta)+2\eta f'(\eta)}{\alpha\eta\left(f(z)-\overline{f(-\overline{\eta})}\right)'+(1-\alpha)\left(f(\eta)-\overline{f(-\overline{\eta})}\right)}\\&=1+\mathcal{G}^\delta_1(\tau)u(\eta)+\mathcal{G}^\delta_2(\tau)u^2(\eta)+\ldots\end{aligned}\tag{6}$$

and

$$\begin{aligned}&\frac{2\xi g'(\xi)}{g(\xi)-\overline{g(-\overline{\xi})}}+\frac{2\left(\xi g'(\xi)\right)'}{\left(g(\xi)-\overline{g(-\overline{\xi})}\right)'}-\frac{2\alpha\xi^2 g''(\xi)+2\xi g'(\xi)}{\alpha\xi\left(g(\xi)-\overline{g(-\overline{\xi})}\right)'+(1-\alpha)\left(g(\xi)-\overline{g(-\overline{\xi})}\right)}\\&=1+\mathcal{G}^\delta_1(\tau)v(\xi)+\mathcal{G}^\delta_2(t)v^2(\xi)+\ldots\ .\end{aligned}\tag{7}$$

Combining (4)–(7), we deduce that

$$\begin{aligned}&\frac{2\eta f'(\eta)}{f(\eta)-\overline{f(-\overline{\eta})}}+\frac{2\left(\eta f'(\eta)\right)'}{\left(f(\eta)-\overline{f(-\overline{\eta})}\right)'}-\frac{2\alpha\eta^2 f''(\eta)+2\eta f'(\eta)}{\alpha\eta\left(f(\eta)-\overline{f(-\overline{\eta})}\right)'+(1-\alpha)\left(f(\eta)-\overline{f(-\overline{\eta})}\right)}\\&=1+\mathcal{G}^\delta_1(\tau)u_1\eta+[\mathcal{G}^\delta_1(\tau)u_2+\mathcal{G}^\delta_2(\tau)u_1^2]\eta^2+\ldots\end{aligned}\tag{8}$$

and

$$\begin{aligned}&\frac{2\xi g'(\xi)}{g(\xi)-\overline{g(-\overline{w})}}+\frac{2\left(wg'(\xi)\right)'}{\left(g(\xi)-\overline{g(-\overline{\xi})}\right)'}-\frac{2\alpha\xi^2 g''(\xi)+2\xi g'(\xi)}{\alpha\xi\left(g(\xi)-\overline{g(-\overline{\xi})}\right)'+(1-\alpha)\left(g(\xi)-\overline{g(-\overline{\xi})}\right)}\\&=1+\mathcal{G}^\delta_1(\tau)v_1\xi+[\mathcal{G}^\delta_1(\tau)v_2+\mathcal{G}^\delta_2(\tau)v_1^2]\xi^2+\ldots\ .\end{aligned}\tag{9}$$

It is known that if $|u(z)| < 1$ and $|v(\xi)| < 1$, $\eta, \xi \in U$, then

$$|u_i| \le 1 \quad \text{and} \quad |v_i| \le 1 \quad \text{for each } i \in \mathbb{N}. \tag{10}$$

Comparing the corresponding coefficients in (8) and (9), we deduce that

$$2(2-\alpha)a_2 = \mathcal{G}_1^{\delta}(\tau)u_1, \tag{11}$$

$$2(3-2\alpha)a_3 = \mathcal{G}_1^{\delta}(\tau)u_2 + \mathcal{G}_2^{\delta}(t)u_1^2, \tag{12}$$

$$-2(2-\alpha)a_2 = \mathcal{G}_1^{\delta}(\tau)v_1 \tag{13}$$

and

$$2(3-2\alpha)(2a_2^2 - a_3) = \mathcal{G}_1^{\delta}(\tau)v_2 + \mathcal{G}_2^{\delta}(t)v_1^2. \tag{14}$$

From (11) and (13), we have

$$u_1 = -v_1 \tag{15}$$

and

$$8(2-\alpha)^2 a_2^2 = \left(\mathcal{G}_1^{\delta}(\tau)\right)^2 \left(u_1^2 + v_1^2\right). \tag{16}$$

Adding (12) to (14) yields

$$4(3-2\alpha)a_2^2 = \mathcal{G}_1^{\delta}(\tau)(u_2+v_2) + \mathcal{G}_2^{\delta}(\tau)\left(u_1^2+v_1^2\right). \tag{17}$$

Substituting the value of $u_1^2 + v_1^2$ from (16) in (17), we have

$$\left[4(3-2\alpha) - \frac{8\mathcal{G}_2^{\delta}(\tau)}{\left(\mathcal{G}_1^{\delta}(\tau)\right)^2}(2-\alpha)^2\right]a_2^2 = \mathcal{G}_1^{\delta}(\tau)(u_2+v_2). \tag{18}$$

Through further calculations using (3), (10) and (18), we conclude that

$$|a_2| \le \frac{|\delta|\tau\sqrt{2|\delta|\tau}}{\sqrt{\left|\delta(2-\alpha)^2 - 2\left[\delta(\delta+1)(2-\alpha)^2 - \delta^2(3-2\alpha)\right]\tau^2\right|}}.$$

Subtracting (14) by (12), we observe that

$$4(3-2\alpha)\left(a_3 - a_2^2\right) = \mathcal{G}_1^{\delta}(\tau)(u_2-v_2) + \mathcal{G}_2^{\delta}(\tau)\left(u_1^2 - v_1^2\right). \tag{19}$$

In view of (15) and (16), we obtain from (19) that

$$a_3 = \frac{\left(\mathcal{G}_1^{\delta}(\tau)\right)^2}{8(2-\alpha)^2}\left(u_1^2+v_1^2\right) + \frac{\mathcal{G}_1^{\delta}(\tau)}{4(3-2\alpha)}(u_2-v_2).$$

Thus, applying (3), we obtain

$$|a_3| \le \frac{\delta^2\tau^2}{(2-\alpha)^2} + \frac{|\delta|\tau}{3-2\alpha}.$$

□

When $\delta = 1$, we have the result in [54].

Corollary 1 ([54])**.** *For $0 \le \alpha \le 1$ and $\tau \in \left(\frac{1}{2}, 1\right]$, let $f \in \mathcal{A}$ be in the family $\mathcal{F}^{sc}_{\Sigma}(\alpha, \tau)$. Then,*

$$|a_2| \le \frac{\tau\sqrt{2\tau}}{\sqrt{\left|(2-\alpha)^2 - 2(2\alpha^2 - 6\alpha + 5)\tau^2\right|}}$$

and

$$|a_3| \le \frac{\tau^2}{(2-\alpha)^2} + \frac{\tau}{3-2\alpha}\,.$$

Theorem 2. *For $\tau \in \left(\frac{1}{2}, 1\right]$, $0 \le \alpha \le 1$ and $\delta \ne 0$, let $f \in \mathcal{A}$ be in the family $\mathcal{N}^{sc}_{\Sigma}(\alpha, \tau, \delta)$. Then,*

$$|a_2| \le \frac{|\delta|\tau\sqrt{2|\delta|\tau}}{\sqrt{\left|\delta(\alpha+2)^2 - 2\left[\delta(\delta+1)(\alpha+2)^2 - \delta^2(4\alpha+3)\right]\tau^2\right|}}$$

and

$$|a_3| \le \frac{\delta^2\tau^2}{(\alpha+2)^2} + \frac{|\delta|\tau}{4\alpha+3}\,.$$

Proof. Let $f \in \mathcal{N}^{sc}_{\Sigma}(\alpha, \tau, \delta)$. Then $u, v : U \longrightarrow U$

$$\begin{aligned}&\frac{2\left[\alpha\eta^3 f'''(\eta)+(\alpha+1)\eta^2 f''(\eta)+\eta f'(\eta)\right]}{\alpha\left(\eta^2\left(f(\eta)-\overline{f(-\overline{\eta})}\right)''+\left(f(\eta)-\overline{f(-\overline{\eta})}\right)\right)+(1-\alpha)\eta\left(f(\eta)-\overline{f(-\overline{\eta})}\right)'}\\&= 1+\mathcal{G}^{\delta}_1(\tau)u(\eta)+\mathcal{G}^{\delta}_2(\tau)u^2(\eta)+\ldots\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\frac{2\left[\alpha\xi^3 g'''(\xi)+(\alpha+1)\xi^2 g''(\xi)+\xi g'(\xi)\right]}{\alpha\left(\xi^2\left(g(\xi)-\overline{g(-\overline{\xi})}\right)''+\left(g(\xi)-\overline{g(-\overline{\xi})}\right)\right)+(1-\alpha)\xi\left(g(\xi)-\overline{g(-\overline{\xi})}\right)'}\\&= 1+\mathcal{G}^{\delta}_1(\tau)v(\xi)+\mathcal{G}^{\delta}_2(\tau)v^2(\xi)+\ldots\ ,\end{aligned} \tag{21}$$

where u and v are defined as in (4) and (5), respectively. Combining (20) and (21) yields

$$\begin{aligned}&\frac{2\left[\alpha\eta^3 f'''(\eta)+(\alpha+1)\eta^2 f''(\eta)+\eta f'(\eta)\right]}{\alpha\left(\eta^2\left(f(\eta)-\overline{f(-\overline{\eta})}\right)''+\left(f(\eta)-\overline{f(-\overline{\eta})}\right)\right)+(1-\alpha)\eta\left(f(z\eta)-\overline{f(-\overline{\eta})}\right)'}\\&= 1+\mathcal{G}^{\delta}_1(\tau)u_1 z+\left[\mathcal{G}^{\delta}_1(\tau)u_2+\mathcal{G}^{\delta}_2(\tau)u_1^2\right]\eta^2+\ldots\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\frac{2\left[\alpha\xi^3 g'''(\xi)+(\alpha+1)\xi^2 g''(\xi)+\xi g'(\xi)\right]}{\alpha\left(\xi^2\left(g(\xi)-\overline{g(-\overline{\xi})}\right)''+\left(g(\xi)-\overline{g(-\overline{\xi})}\right)\right)+(1-\alpha)\xi\left(g(\xi)-\overline{g(-\overline{\xi})}\right)'}\\&= 1+\mathcal{G}^{\delta}_1(\tau)v_1\xi+\left[\mathcal{G}^{\delta}_1(\tau)v_2+\mathcal{G}^{\delta}_2(\tau)v_1^2\right]\xi^2+\ldots\ .\end{aligned} \tag{23}$$

Comparing the corresponding coefficients in (22) and (23), we observe that

$$2(\alpha+2)a_2 = \mathcal{G}^{\delta}_1(\tau)u_1, \tag{24}$$

$$2(4\alpha+3)a_3 = \mathcal{G}^{\delta}_1(\tau)u_2 + \mathcal{G}^{\delta}_2(t)u_1^2, \tag{25}$$

$$-2(\alpha+2)a_2 = \mathcal{G}^{\delta}_1(\tau)v_1 \tag{26}$$

and

$$2(4\alpha+3)(2a_2^2-a_3)=\mathcal{G}_1^{\delta}(\tau)v_2+\mathcal{G}_2^{\delta}(\tau)v_1^2. \tag{27}$$

In view of (24) and (26), we have

$$u_1=-v_1 \tag{28}$$

and

$$8(\alpha+2)^2a_2^2=\left(\mathcal{G}_1^{\delta}(\tau)\right)^2\left(u_1^2+v_1^2\right). \tag{29}$$

If we add (25) to (27), we conclude that

$$4(4\alpha+3)a_2^2=\mathcal{G}_1^{\delta}(\tau)(u_2+v_2)+\mathcal{G}_2^{\delta}(\tau)\left(u_1^2+v_1^2\right). \tag{30}$$

Substituting the value of $u_1^2+v_1^2$ from (29) into (30), we arrive at

$$\left[4(4\alpha+3)-\frac{8\mathcal{G}_2^{\delta}(\tau)}{\left(\mathcal{G}_1^{\delta}(\tau)\right)^2}(\alpha+2)^2\right]a_2^2=\mathcal{G}_1^{\delta}(\tau)(u_2+v_2),$$

or, equivalently,

$$a_2^2=\frac{\left(\mathcal{G}_1^{\delta}(\tau)\right)^3(u_2+v_2)}{4(4\alpha+3)\left(\mathcal{G}_1^{\delta}(\tau)\right)^2-8(\alpha+2)^2\mathcal{G}_2^{\delta}(\tau)}, \tag{31}$$

Through further calculations using (3), (10) and (31), we find that

$$|a_2|\le\frac{\tau\sqrt{2|\delta|\tau}\,|\delta|}{\sqrt{\left|\delta(\alpha+2)^2-2\left[\delta(\delta+1)(\alpha+2)^2-\delta^2(4\alpha+3)\right]\tau^2\right|}}.$$

Next, if we subtract (27) from (25), we deduce that

$$4(4\alpha+3)\left(a_3-a_2^2\right)=\mathcal{G}_1^{\delta}(\tau)(u_2-v_2)+\mathcal{G}_2^{\delta}(\tau)\left(u_1^2-v_1^2\right). \tag{32}$$

In light of (28) and (29), we obtain from (32) that

$$a_3=\frac{\left(\mathcal{G}_1^{\delta}(\tau)\right)^2}{8(\alpha+2)^2}\left(u_1^2+v_1^2\right)+\frac{\mathcal{G}_1^{\delta}(\tau)}{4(4\alpha+3)}(u_2-v_2).$$

Thus, applying (3), we obtain

$$|a_3|\le\frac{\delta^2\tau^2}{(\alpha+2)^2}+\frac{|\delta|\tau}{4\alpha+3}.$$

□

Taking $\delta=1$ in Theorem 2, we reduce to the results by Wanas and Yalçin [55] as follows.

Corollary 2 ([55])**.** *For $0\le\alpha\le1$ and $t\in\left(\frac{1}{2},1\right]$, let $f\in\mathcal{A}$ be in the family $S_{\Sigma}^{c}(\alpha,\tau)$. Then,*

$$|a_2|\le\frac{\tau\sqrt{2t}}{\sqrt{\left|(\alpha+2)^2-2(2\alpha^2+4\alpha+5)\tau^2\right|}}$$

and

$$|a_3| \leq \frac{\tau^2}{(\alpha+2)^2} + \frac{\tau}{4\alpha+3}.$$

Following, we give the "Fekete-Szegö problem" for the families $\mathcal{M}_{\Sigma}^{sc}(\alpha,\tau,\delta)$ and $\mathcal{N}_{\Sigma}^{sc}(\alpha,t,\delta)$.

Theorem 3. *For $0 \leq \alpha \leq 1$, $\tau \in \left(\frac{1}{2},1\right]$, $\mu \in \mathbb{R}$ and δ is a nonzero real constant. Let $f \in \mathcal{A}$ be in the family $\mathcal{M}_{\Sigma}^{sc}(\alpha,\tau,\delta)$. Then,*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\tau|\delta|}{3-2\alpha}; \\ \textit{for } \ |\mu-1| \leq \frac{1}{2(3-2\alpha)}\left|\frac{(2-\alpha)^2}{\delta\tau^2} - 2\left[\frac{(\delta+1)}{\delta}(2-\alpha)^2-(3-2\alpha)\right]\right|, \\ \\ \frac{2\tau^3\left|\delta^3\right||\mu-1|}{\left|\delta(2-\alpha)^2-2\left[\delta(\delta+1)(2-\alpha)^2-\delta^2(3-2\alpha)\right]t^2\right|}; \\ \textit{for } \ |\mu-1| \geq \frac{1}{2(3-2\alpha)}\left|\frac{(2-\alpha)^2}{\delta\tau^2} - 2\left[\frac{(\delta+1)}{\delta}(2-\alpha)^2-(3-2\alpha)\right]\right|. \end{cases}$$

Proof. From (18) and (19), we obtain

$$\begin{aligned} a_3-\mu a_2^2 &= (1-\mu)\frac{\left(\mathcal{G}_1^{\delta}(\tau)\right)^3(u_2+v_2)}{4(3-2\alpha)\left(\mathcal{G}_1^{\delta}(\tau)\right)^2-8(2-\alpha)^2\mathcal{G}_2^{\delta}(\tau)} + \frac{\mathcal{G}_1^{\delta}(\tau)(u_2-v_2)}{4(3-2\alpha)} \\ &= \mathcal{G}_1^{\delta}(\tau)\left[\left(\psi(\mu)+\frac{1}{4(3-2\alpha)}\right)u_2+\left(\psi(\mu)-\frac{1}{4(3-2\alpha)}\right)v_2\right], \end{aligned}$$

where

$$\psi(\mu) = \frac{\left(\mathcal{G}_1^{\delta}(\tau)\right)^2(1-\mu)}{4\left[(3-2\alpha)\left(\mathcal{G}_1^{\delta}(t)\right)^2-2(2-\alpha)^2\mathcal{G}_2^{\delta}(\tau)\right]}.$$

According to (3), we deduce that

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\tau|\delta|}{3-2\alpha}, & 0 \leq |\psi(\mu)| \leq \frac{1}{4(3-2\alpha)} \\ 4\tau|\delta||\psi(\mu)|, & |\psi(\mu)| \geq \frac{1}{4(3-2\alpha)} \end{cases}.$$

After some computations, we obtain

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\tau|\delta|}{3-2\alpha}; \\ \text{for } \ |\mu-1| \leq \frac{1}{2(3-2\alpha)}\left|\frac{(2-\alpha)^2}{\delta t^2} - 2\left[\frac{(\delta+1)}{\delta}(2-\alpha)^2-(3-2\alpha)\right]\right|, \\ \\ \frac{2\tau^3\left|\delta^3\right||\mu-1|}{\left|\delta(2-\alpha)^2-2\left[\delta(\delta+1)(2-\alpha)^2-\delta^2(3-2\alpha)\right]t^2\right|}; \\ \text{for } \ |\mu-1| \geq \frac{1}{2(3-2\alpha)}\left|\frac{(2-\alpha)^2}{\delta\tau^2} - 2\left[\frac{(\delta+1)}{\delta}(2-\alpha)^2-(3-2\alpha)\right]\right|. \end{cases}$$

□

For $\delta = 1$, Theorem 3 yields the family $\mathcal{F}_{\Sigma}^{sc}(\alpha,\tau)$, defined by Wanas and Majeed [54].

Corollary 3 ([54])**.** *For* $0 \le \alpha \le 1$, $\tau \in \left(\frac{1}{2}, 1\right]$ *and* $\mu \in \mathbb{R}$, *let* $f \in \mathcal{A}$ *be in the family* $\mathcal{F}_{\Sigma}^{sc}(\alpha, \tau)$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{\tau}{3-2\alpha}; \\ \text{for } \ |\mu - 1| \le \frac{1}{2(3-2\alpha)}\left|\frac{(2-\alpha)^2}{\tau^2} - 2\left(2\alpha^2 - 6\alpha + 5\right)\right| \\ \\ \frac{2\tau^3|\mu-1|}{\left|2(3-2\alpha)\tau^2 - (2-\alpha)^2(4\tau^2-1)\right|}; \\ \text{for } \ |\mu - 1| \ge \frac{1}{2(3-2\alpha)}\left|\frac{(2-\alpha)^2}{\tau^2} - 2\left(2\alpha^2 - 6\alpha + 5\right)\right| \end{cases}.$$

For $\mu = 1$, Theorem 3 gives the next corollary:

Corollary 4. *For* $0 \le \alpha \le 1$, $\tau \in \left(\frac{1}{2}, 1\right]$ *and* $\delta \ne 0$, *let* $f \in \mathcal{A}$ *be in the family* $\mathcal{M}_{\Sigma}^{sc}(\alpha, \tau, \delta)$. *Then,*

$$\left|a_3 - a_2^2\right| \le \frac{\tau|\delta|}{3 - 2\alpha}.$$

Theorem 4. *For* $0 \le \alpha \le 1$, $\tau \in \left(\frac{1}{2}, 1\right]$, $\mu \in \mathbb{R}$ *and* $\delta \ne 0$, *let* $f \in \mathcal{A}$ *be in the family* $\mathcal{N}_{\Sigma}^{sc}(\alpha, \tau, \delta)$. *Then,*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{\tau|\delta|}{4\alpha+3}; \\ \text{for } \ |\mu - 1| \le \frac{1}{2(4\alpha+3)}\left|\frac{(\alpha+2)^2}{\delta\tau^2} - 2\left[\frac{(\delta+1)}{\delta}(\alpha+2)^2 - (4\alpha+3)\right]\right| \\ \\ \frac{2t^3|\delta^3||\mu-1|}{\left|\delta(\alpha+2)^2 - 2\left[\delta(\delta+1)(\alpha+2)^2 - \delta^2(4\alpha+3)\right]\tau^2\right|} \\ \text{for } \ |\mu - 1| \ge \frac{1}{2(4\alpha+3)}\left|\frac{(\alpha+2)^2}{\delta\tau^2} - 2\left[\frac{(\delta+1)}{\delta}(\alpha+2)^2 - (4\alpha+3)\right]\right| \end{cases}.$$

Proof. From (31) and (32), we find that

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{\left(\mathcal{G}_1^{\delta}(\tau)\right)^3(u_2 + v_2)}{4(4\alpha+3)\left(\mathcal{G}_1^{\delta}(\tau)\right)^2 - 8(\alpha+2)^2\mathcal{G}_2^{\delta}(\tau)} + \frac{\mathcal{G}_1^{\delta}(\tau)(u_2 - v_2)}{4(4\alpha+3)} \\ &= \mathcal{G}_1^{\delta}(t)\left[\left(\varphi(\mu) + \frac{1}{4(4\alpha+3)}\right)u_2 + \left(\varphi(\mu) - \frac{1}{4(4\alpha+3)}\right)v_2\right], \end{aligned}$$

where

$$\varphi(\mu) = \frac{\left(\mathcal{G}_1^{\delta}(\tau)\right)^2(1-\mu)}{4\left[(4\alpha+3)\left(\mathcal{G}_1^{\delta}(\tau)\right)^2 - 2(\alpha+2)^2\mathcal{G}_2^{\delta}(\tau)\right]}.$$

According to (3), we deduce that

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{\tau|\delta|}{4\alpha+3}, & 0 \le |\varphi(\mu)| \le \frac{1}{4(4\alpha+3)} \\ \\ 4\tau|\delta||\varphi(\mu)|, & |\varphi(\mu)| \ge \frac{1}{4(4\alpha+3)} \end{cases}.$$

After some computations, we obtain

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\tau|\delta|}{4\alpha+3}; \\ \text{for } |\mu-1| \leq \frac{1}{2(4\alpha+3)}\left|\frac{(\alpha+2)^2}{\delta\tau^2}-2\left[\frac{(\delta+1)}{\delta}(\alpha+2)^2-(4\alpha+3)\right]\right| \\ \\ \frac{2\tau^3|\delta^3||\mu-1|}{\left|\delta(\alpha+2)^2-2\left[\delta(\delta+1)(\alpha+2)^2-\delta^2(4\alpha+3)\right]\tau^2\right|} \\ \text{for } |\mu-1| \geq \frac{1}{2(4\alpha+3)}\left|\frac{(\alpha+2)^2}{\delta\tau^2}-2\left[\frac{(\delta+1)}{\delta}(\alpha+2)^2-(4\alpha+3)\right]\right| \end{cases}.$$

□

When $\delta=1$ in Theorem 4, we obtain the results in [55] for the family $S_\Sigma^c(\alpha,t)$.

Corollary 5 ([55])**.** *For $0\leq\alpha\leq 1$, $\tau\in\left(\frac{1}{2},1\right]$ and $\mu\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the family $S_\Sigma^c(\alpha,\tau)$. Then,*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\tau}{4\alpha+3}; \\ \textit{for } |\mu-1| \leq \frac{1}{2(4\alpha+3)}\left|\frac{(\alpha+2)^2}{\tau^2}-2\left(2\alpha^2+4\alpha+5\right)\right| \\ \\ \frac{2\tau^3|\mu-1|}{\left|2(4\alpha+3)\tau^2-(\alpha+2)^2(4\tau^2-1)\right|}; \\ \textit{for } |\mu-1| \geq \frac{1}{2(4\alpha+3)}\left|\frac{(\alpha+2)^2}{\tau^2}-2\left(2\alpha^2+4\alpha+5\right)\right| \end{cases}.$$

Theorem 4 turns into the following corollary at $\mu=1$:

Corollary 6. *For $0\leq\alpha\leq 1$, $\tau\in\left(\frac{1}{2},1\right]$ and $\delta\neq 0$, let $f\in\mathcal{A}$ be in the family $\mathcal{N}_\Sigma^{sc}(\alpha,\tau,\delta)$. Then,*

$$\left|a_3-a_2^2\right| \leq \frac{\tau|\delta|}{4\alpha+3}.$$

3. Conclusions

The primary goal was to define new families $\mathcal{M}_\Sigma^{sc}(\alpha,\tau,\delta)$ and $\mathcal{N}_\Sigma^{sc}(\alpha,\tau,\delta)$ of holomorphic and bi-univalent functions with respect to symmetric points, which are defined by Gegenbauer polynomials. We investigated the Taylor–Maclaurin coefficient inequalities of functions in these new families and viewed the famous Fekete–Szegö inequalities. Further, by specifying the parameters, the consequences of these families have been mentioned.

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