axioms

MDPI

Article

On Hybrid Hyper k-Pell, k-Pell–Lucas, and Modified k-Pell Numbers

Elen Viviani Pereira Spreafico [1,†], Paula Catarino [2,*,†] and Paulo Vasco [2,†]

[1] Institute of Mathematics, Federal University of Mato Grosso do Sul, Campo Grande 79074-460, Brazil; elen.spreafico@ufms.br
[2] Department of Mathematics, University of Trás-os-Montes e Alto Douro, 5001-801 Vila Real, Portugal; pcatarin@utad.pt, pvasco@utad.pt
* Correspondence: pcatarin@utad.pt
† These authors contributed equally to this work.

Abstract: Many different number systems have been the topic of research. One of the recently studied number systems is that of hybrid numbers, which are generalizations of other number systems. In this work, we introduce and study the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell numbers. In order to study these new sequences, we established new properties, generating functions, and the Binet formula of the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell sequences. Thus, we present some algebraic properties, recurrence relations, generating functions, the Binet formulas, and some identities for the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell numbers.

Keywords: hybrid hyper k-Pell; hybrid hyper k-Pell–Lucas; hybrid hyper Modified k-Pell numbers; generating functions; the Binet formula; identities

MSC: 11B37; 11B39; 11R52

Citation: Spreafico, E. V. P.; Catarino, P.; Vasco, P. On Hybrid Hyper k-Pell, k-Pell–Lucas, and Modified k-Pell Numbers. *Axioms* **2023**, *12*, 1047. https://doi.org/10.3390/axioms12111047

Academic Editors: Mircea Merca and Hari Mohan Srivastava

Received: 18 September 2023
Revised: 30 October 2023
Accepted: 9 November 2023
Published: 11 November 2023

1. Preliminaries and Background

For some years now, many numerical sequences have attracted the attention and curiosity of various researchers. The best known is the famous Fibonacci sequence, $\{F_n\}_{n\geq 0}$, defined by the recurrence $F_n = F_{n-1} + F_{n-2}$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Many generalizations of Fibonacci numbers have been considered in recent years. One of them is the Lucas numbers, denoted by $\{L_n\}_{n\geq 0}$, defined by the same recurrence relation of Fibonacci, but with different initial conditions given by $L_0 = 2$ and $L_1 = 1$.

Another important sequence, closely related to the Fibonacci sequence, is the Pell sequence. The Pell sequence, denoted by $\{P_n\}_{n\geq 0}$, is a sequence of integers that was first studied by mathematician John Pell in the 17th Century and is defined by the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$, with initial conditions $P_0 = 0$ and $P_1 = 1$ [1,2]. These numbers have many interesting properties and have been studied in various fields such as number theory and combinatorics. For example, Pell numbers can be used to solve certain combinatorial enumeration problems and to find square triangular numbers. For those interested in learning more about number sequences, you can consult the encyclopedia [3]; for applications of these numbers and their relatives, you can consult the books [4,5].

Many variations of the Pell and Fibonacci sequences have been studied, such as Pell–Lucas sequences, denoted by $\{Q_n\}_{n\geq 0}$, which are characterized by the same linear recurrence of Pell numbers, but with different initial conditions, namely $Q_n = 2Q_{n-1} + Q_{n-2}$, with initial conditions $Q_0 = 2$ and $Q_1 = 2$.

In general, research on numerical sequences advances by considering generalizations such as arbitrary coefficients, arbitrary initial conditions, or the extension of an index to integers. For each generalization considered, it is interesting to find closed formulas of the

Binet type, or the generating function, in order to seek better mechanisms for computing it without using recurrence. It is also interesting to look for identities that involve them and study their growth and convergence.

Still on Fibonacci and Pell numbers, and their generalizations, we would like to highlight some articles to better contextualize the sequence that we will define and work on here. The k-Fibonacci numbers were studied in [6] and defined by

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2},$$

with $F_{k,0} = 0$ and $F_{k,1} = 1$. Similarly, a generalization for Pell numbers, k-Pell numbers, was introduced by the second author in [7], denoted by $\{P_{k,n}\}_{n\geq 0}$ and defined by

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \tag{1}$$

with $P_{k,0} = 0$ and $P_{k,1} = 1$. Other important generalizations are the k-Pell–Lucas, $\{Q_{k,n}\}_{n\geq 0}$ and Modified k-Pell, $\{q_{k,n}\}_{n\geq 0}$, defined, respectively, by the recurrences relations

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \tag{2}$$

$$q_{k,n} = 2q_{k,n-1} + kq_{k,n-2}, \tag{3}$$

with $Q_{k,0} = Q_{k,1} = 2$ and $q_{k,0} = q_{k,1} = 1$ as the respective initial conditions.

These sequences were investigated, and several results, such as the Binet formula, generating functions, Cassini's identities, as well as the matrix approach were established (see more about these sequences in [7–12] and the references therein).

Now, consider the results given by Dil and Mező in [13]. The authors introduced a symmetric algorithm obtained by the recurrence relation $a_n^k = a_{n-1}^k + a_n^{k-1}$, and studied a generalization of the Fibonacci and Lucas numbers, the hyper Fibonacci numbers and hyper Lucas numbers, as an application of the results obtained with the symmetric algorithm. The sequence of hyper Fibonacci numberswas defined as

$$F_n^{(r)} = \sum_{i=0}^{n} F_i^{(r-1)},$$

for non-negative integers n and r, with initial conditions $F_n^{(0)} = F_n$, $F_0^{(r)} = 0$, and $F_1^{(r)} = 1$.

Motivated by the hyper Fibonacci numbers, in [14] was introduced the "hyper" approach for the k-Pell, k-Pell–Lucas, and Modified k-Pell sequences. The authors established some properties and discussed the concavity, convexity, log-concavity, and log-convexity properties for these sequences.

On the other hand, there is a natural extension that we can make to numerical sequences, considering the numerical set in which its elements are inserted. Different number systems have been studied, and the relationship with numerical sequences has been explored. Horadan in [15] introduced the first extension of this type by defining the Fibonacci numbers in the complex and quaternions number systems. Horadam introduced the concept of complex Fibonacci numbers as the Gaussian Fibonacci sequence $\{GF_n\}_n \geq 0$ defined by

$$GF_n = GF_{n-1} + GF_{n-2},$$

with initial conditions $GF_0 = i$ and $GF_1 = 1$, where i is the imaginary unit ($i^2 = -1$) and F_n is the n-th Fibonacci number (for instance, see more in [16]). It was shown that $GF_n = F_n + F_{n+1}i$, and motivated by this equality, the Fibonacci quaternions were introduced, defined by the recurrence relation

$$Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+1}k,$$

where i, j, k are units with the properties $i^2 = j^2 = k^2 = ijk = -1$ and F_n is the n-th Fibonacci number.

Complex, hyperbolic, and dual-numbers are two-dimensional systems that have been studied extensively over the last century for their potential applications in various fields. In the realm of Physics, these numbers are used to represent space–time through the concept of a hyper complex ring. This suggests that space–time can be seen as a structure created by the algebra of hybrid numbers. In other words, the study of hybrid numbers can help us understand and explain all types of space–times (for instance, see [17]).

The hybrid number system was introduced by Özdemir in [18]. A hybrid number can be viewed as a generalization of the complex, hyperbolic, and dual-number. The set of hybrid numbers, denoted by $\mathbb{K}$, is defined as

$$\mathbb{K} = \{a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}.$$

The addition of hybrid numbers is performed componentwise, and this operation is commutative and associative, while the multiplication is not commutative, but has the property of associativity. For more details related to this number system, see the work developed by Özdemir in [18], where the author examined this new ring of numbers, which is non-commutative and has the unit element. The sequences of hybrid numbers are studied from several perspectives using, for example, the analytic and matrix approach (see [19,20] and the references therein). It is important to highlight hybrid Fibonacci numbers here. This sequence was introduced by Szynal-Liana and Wloch in [19]. The authors derived some properties using classical Fibonacci identities. The "hybrid" version of the Pell, Pell–Lucas, and Jacobsthal numbers was considered by the same authors (see, for instance [21–23]).

Consider the hyper k-Pell, the hyper k-Pell–Lucas, and the hyper Modified k-Pell numbers that were introduced by Catarino, Alves, and Campos in [14]. Some properties and identities satisfied by these sequences are present in [14]. In this paper, it is our intention to introduce new sequences that are the "hybrid version" of them.

In order to study these new sequences, we need to study other properties, generating functions, and Binet's formula of the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell sequences, which will be stated in the next section. The last section is dedicated to the new sequences, which were the subject of a study motivated by the work of Yasemin in [24] with the hybrid hyper Fibonacci and hybrid hyper Lucas numbers.

2. "Hyper" Version of the k-Pell, k-Pell–Lucas, and Modified k-Pell Numbers

As we have mentioned before, in this section, we recall the hyper k-Pell, k-Pell–Lucas, and Modified k-Pell sequences introduced in [14] and present some properties that will be necessary for the study of the new sequences introduced in the next section of this paper.

For non-negative integers r, n, and k, the n-th term of the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell sequences is defined, respectively, as follows:

$$P_{k,n}^{(r)} = \sum_{i=0}^{n} P_{k,i}^{(r-1)}, \ \ P_{k,n}^{(0)} = P_{k,n}, \ \ P_{k,0}^{(r)} = 0, \ \ P_{k,1}^{(r)} = 1, \tag{4}$$

$$Q_{k,n}^{(r)} = \sum_{i=0}^{n} Q_{k,i}^{(r-1)}, \ Q_{k,n}^{(0)} = Q_{k,n}, \ Q_{k,0}^{(r)} = 2, \ Q_{k,1}^{(r)} = 2(r+1) \tag{5}$$

and

$$q_{k,n}^{(r)} = \sum_{i=0}^{n} q_{k,i}^{(r-1)}, \ q_{k,n}^{(0)} = q_{k,n}, \ q_{k,0}^{(r)} = 1, \ q_{k,1}^{(r)} = r+1. \tag{6}$$

It is clear that taking $r = 0$ in (4)–(6), the k-Pell given by (1), k-Pell–Lucas given by (2), and Modified k-Pell numbers given by (3) are obtained, respectively.

2.1. Some Properties of These Sequences

This subsection is dedicated to introducing some properties that are satisfied by the sequences $\{P_{k,n}^{(r)}\}_{n\geq 0}$, $\{Q_{k,n}^{(r)}\}_{n\geq 0}$ and $\{q_{k,n}^{(r)}\}_{n\geq 0}$, which will be necessary for the study of the new numerical sequences that will be introduced in the next section.

Given the recurrence relation (1) of k-Pell numbers, we immediately derive the recurrence relation for $\{P_{k,n}^{(0)}\}_{n\geq 0}$ that coincides with the recurrence relation of the sequence of k-Pell numbers. Also, we can obtain the recurrence relation for $\{P_{k,n}^{(1)}\}_{n\geq 0}$. In fact, taking into account the recurrence (1) and the initial conditions of the k-Pell sequence, we obtain:

$$\begin{aligned} P_{k,n}^{(1)} &= \sum_{i=0}^{n} P_{k,i}^{(0)} = \sum_{i=0}^{n} P_{k,i} = P_{k,0} + P_{k,1} + \sum_{i=2}^{n} P_{k,i} = 1 + \sum_{i=2}^{n} P_{k,i} \\ &= 1 + \sum_{i=2}^{n} (2P_{k,i-1} + kP_{k,i-2}) = 1 + 2\sum_{i=2}^{n} P_{k,i-1} + k\sum_{i=2}^{n} P_{k,i-2} \\ &= 1 + 2\sum_{i=0}^{n-1} P_{k,i} + k\sum_{i=0}^{n-2} P_{k,i} = 2P_{k,n-1}^{(1)} + kP_{k,n-2}^{(1)} + 1. \end{aligned}$$

In a similar process, using the previous recurrence relation and the initial conditions, the recurrence relation for $\{P_{k,n}^{(2)}\}_{n\geq 0}$ is given by $P_{k,n}^{(2)} = 2P_{k,n-1}^{(2)} + kP_{k,n-2}^{(2)} + n$.

Under the previous discussion, we have the next result.

Proposition 1. *For $n \geq 2$ and a non-negative integer k, the following recurrence relations hold:*

$$P_{k,n}^{(0)} = 2P_{k,n-1}^{(0)} + kP_{k,n-2}^{(0)}, \tag{7}$$

$$P_{k,n}^{(1)} = 2P_{k,n-1}^{(1)} + kP_{k,n-2}^{(1)} + 1, \tag{8}$$

$$P_{k,n}^{(2)} = 2P_{k,n-1}^{(2)} + kP_{k,n-2}^{(2)} + n. \tag{9}$$

For the statement of the general case, consider the following lemma given in [25].

Lemma 1 (Lemma 2.1 [25])**.** *Consider the arithmetic progression $\{a_n\}_{n\geq 0} = \{a_n^{(0)}\}_{n\geq 0}$ defined by $a_n = 1$, for all non-negative integers n. Consider the arithmetic progression of order $r \geq 1$, $\{a_n^{(r)}\}_{n\geq r}$, defined by the partial sums of the arithmetic progression of order $r-1$, $\{a_n^{(r-1)}\}_{n\geq r-1}$. Then, the sequence $\{a_n^{(r)}\}_{n\geq r}$, is given by the polynomial of degree r :*

$$a_n^{(r)} = \sum_{k=1}^{n} a_k^{(r-1)} = \binom{n}{r} = \frac{(n)(n-1)\cdots(n-r+1)}{r!},$$

for $n \geq r$ and $r \geq 1$.

For the general case, we have:

Proposition 2. *For a non-negative integer k and positive integers $n \geq 2$ and $r \geq 1$, the hyper k-Pell sequence satisfies the recurrence relation:*

$$P_{k,n}^{(r)} = 2P_{k,n-1}^{(r)} + kP_{k,n-2}^{(r)} + \sum_{j=1}^{r} \binom{n+r-2-j}{r-j}. \tag{10}$$

Proof. We used induction on r. For $r = 1$ and $r = 2$, the result derives from the previous proposition. Now, suppose that this statement is valid for all positive integers less than or equal to r, and we will show that this holds for $r + 1$. In fact,

$$\begin{aligned} P_{k,n}^{(r+1)} &= \sum_{i=0}^{n} P_{k,i}^{(r)} = P_{k,0}^{(r)} + P_{k,1}^{(r)} + \sum_{i=2}^{n} P_{k,i}^{(r)} = 1 + \sum_{i=2}^{n} P_{k,i}^{(r)} \\ &= 1 + \sum_{i=2}^{n}\left(2P_{k,i-1}^{(r)} + kP_{k,i-2}^{(r)} + \sum_{j=1}^{r}\binom{n+r-2-j}{r-j}\right) \\ &= 2P_{k,n-1}^{(r+1)} + kP_{k,n-2}^{(r+1)} + \left(1 + \sum_{j=1}^{r}\sum_{i=2}^{n}\binom{i+r-2-j}{r-j}\right) \\ &= 2P_{k,n-1}^{(r+1)} + kP_{k,n-2}^{(r+1)} + \sum_{j=1}^{r+1}\binom{n+r-1-j}{r+1-j} \end{aligned}$$

and by Lemma 1, the result follows. □

Using similar reasoning to that used for the recurrence relations in the cases of $r = 0, 1$ in $\{P_{k,n}^{(r)}\}_{n\geq 0}$, for the sequences defined in (5) and (6), we have the following result, whose proof we omit:

Proposition 3. *For $n \geq 2$ and a non-negative integer k, the following recurrence relations hold:*

$$Q_{k,n}^{(0)} = 2Q_{k,n-1}^{(0)} + kQ_{k,n-2}^{(0)},\ Q_{k,n}^{(1)} = 2Q_{k,n-1}^{(1)} + kQ_{k,n-2}^{(1)} \tag{11}$$

$$q_{k,n}^{(0)} = 2q_{k,n-1}^{(0)} + kq_{k,n-2}^{(0)},\ q_{k,n}^{(1)} = 2q_{k,n-1}^{(1)} + kq_{k,n-2}^{(1)} \tag{12}$$

with $Q_{k,0} = Q_{k,1} = 2$, $Q_{k,0}^{(1)} = 2$, $Q_{k,n}^{(1)} = 4$ and $q_{k,0} = q_{k,1} = 1$, $q_{k,0}^{(1)} = 1$, $q_{k,1}^{(1)} = 2$ as the respective initial conditions.

The third item of Proposition 3 in [14] established the following relation: $q_{k,n}^{(r)} = P_{k,n+1}^{(r-1)}$. Then, by Proposition 2, it is verified that

$$\begin{aligned} q_{k,n}^{(r)} &= 2P_{k,n}^{(r-1)} + kP_{k,n-1}^{(r-1)} + \sum_{j=1}^{r-1}\binom{n+r-2-j}{r-1-j} \\ &= 2q_{k,n-1}^{(r)} + kq_{k,n-2}^{(r)} + \sum_{j=1}^{r-1}\binom{n+r-2-j}{r-1-j}. \end{aligned}$$

Similarly, the second item of Proposition 3 in [14] established the following relation: $Q_{k,n}^{(r)} = 2P_{k,n+1}^{(r-1)}$. Then, by Proposition 2, it is verified that

$$\begin{aligned} Q_{k,n}^{(r)} &= 4P_{k,n}^{(r-1)} + 2kP_{k,n-1}^{(r-1)} + 2\sum_{j=1}^{r-1}\binom{n+r-2-j}{r-1-j} \\ &= 2Q_{k,n-1}^{(r)} + kQ_{k,n-2}^{(r)} + 2\sum_{j=1}^{r-1}\binom{n+r-2-j}{r-1-j}. \end{aligned}$$

Under the previous discussion, we can establish the following recurrence relations.

Proposition 4. *For integer numbers $k \geq 0$, $n \geq 2$, and $r \geq 1$, the hyper k-Pell–Lucas and the hyper Modified k-Pell sequences satisfy the recurrence relations:*

$$Q_{k,n}^{(r)} = 2Q_{k,n-1}^{(r)} + kQ_{k,n-2}^{(r)} + 2\sum_{j=1}^{r-1}\binom{n+r-2-j}{r-1-j}. \tag{13}$$

$$q_{k,n}^{(r)}=2q_{k,n-1}^{(r)}+kq_{k,n-2}^{(r)}+\sum_{j=1}^{r-1}\binom{n+r-2-j}{r-1-j}, \tag{14}$$

with $Q_{k,0}^{(r)}=2,\ Q_{k,1}^{(r)}=2(r+1),\ q_{k,0}^{(r)}=1,$ *and* $q_{k,1}^{(r)}=r+1.$

Propositions 1–4 give us the recurrence relations for the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell sequences. The recurrence relations allow us to study the generating functions and the Binet formula, by providing the explicit formulas for the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell sequences.

2.2. The Generating Function and Binet's Formula

Next, we shall give the generating functions for the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell sequences. We shall write such a sequence as a power series where each term of the sequence corresponds to the coefficients of the series. Considering these sequences, the generating functions $f_P(t)$, $f_Q(t)$, and $f_q(t)$ are defined, respectively, by $f_P(t)=\sum_{n=0}^{\infty}P_{k,n}^{(r)}t^n$, $f_Q(t)=\sum_{n=0}^{\infty}Q_{k,n}^{(r)}t^n$, and $f_q(t)=\sum_{n=0}^{\infty}q_{k,n}^{(r)}t^n$.

Theorem 1. *For non-negative integers k,n, and r, the generating functions for the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell sequences are, respectively,*

$$f_P(t)=\frac{t}{(1-2t-kt^2)}\left(1+\sum_{j=1}^{r}\frac{t}{(1-t)^{r-j+1}}\right), \tag{15}$$

$$f_Q(t)=\frac{1}{(1-2t-kt^2)}\left(2+\sum_{j=1}^{r-1}\frac{2t}{(1-t)^{r-j}}\right), \tag{16}$$

$$f_q(t)=\frac{1}{(1-2t-kt^2)}\left(1+\sum_{j=1}^{r-1}\frac{t}{(1-t)^{r-j}}\right). \tag{17}$$

Proof. We have

$$\begin{aligned}
f_P(t)&=\sum_{n=0}^{\infty}P_{k,n}^{(r)}t^n=P_{k,0}^{(r)}+P_{k,1}^{(r)}t+\sum_{n=2}^{\infty}P_{k,n}^{(r)}t^n\\
&=t+\sum_{n=2}^{\infty}\left(2P_{k,n-1}^{(r)}+kP_{k,n-2}^{(r)}+\sum_{j=1}^{r}\binom{n+r-2-j}{r-j}\right)t^n\\
&=t+2\sum_{n=2}^{\infty}P_{k,n-1}^{(r)}t^n+k\sum_{n=2}^{\infty}P_{k,n-2}^{(r)}t^n+\sum_{n=2}^{\infty}\sum_{j=1}^{r}\binom{n+r-2-j}{r-j}t^n\\
&=t+2t\left(\sum_{n=0}^{\infty}P_{k,n}^{(r)}t^n\right)+kt^2\left(\sum_{n=0}^{\infty}P_{k,n}^{(r)}t^n\right)+t^2\sum_{j=1}^{r}\sum_{n=0}^{\infty}\binom{n+r-j}{r-j}t^n.
\end{aligned}$$

Hence,

$$\left(1-2t-kt^2\right)\sum_{n=0}^{\infty}P_{k,n}^{(r)}t^n=t+t^2\sum_{j=1}^{r}\sum_{n=0}^{\infty}\binom{n+r-j}{r-j}t^n.$$

Since $t^2\sum_{j=1}^{r}\sum_{n=0}^{\infty}\binom{n+r-j}{r-j}t^n=t^2\sum_{j=1}^{r}\frac{1}{(1-t)^{r-j+1}}$, then, with some more calculations, we obtain the desired result. The proof of the identity (16) can be performed in a similar way to what we have just shown, taking into account the respective initial conditions. As for the last generating function, all we have to do is look at Item 1 of Proposition 3 in [14], and the result follows immediately. □

In what follows, we will present the Binet formula for these numerical sequences. Thus, we have:

Theorem 2. *For non-negative integers k, n, and r, the following identities hold*

$$P_{k,n}^{(0)} = \frac{1}{2\sqrt{1+k}}(\alpha_1^n - \alpha_2^n), \tag{18}$$

$$P_{k,n}^{(r)} = C_1\alpha_1^n + C_2\alpha_2^n + \sum_{j=0}^{r-1} A_j n^j, \tag{19}$$

where $\alpha_1 = 1 + \sqrt{1+k}$, $\alpha_2 = 1 - \sqrt{1+k}$, and A_j are constants for each $0 \leq j \leq r$, obtained by solving the equation:

$$\sum_{j=0}^{r} A_j n^j = \sum_{j=0}^{r} A_j (n-1)^j + \sum_{j=0}^{r} A_j (n-2)^j + \sum_{j=1}^{r} \binom{n+r-2-j}{r-j},$$

for each fixed r, and C_1, C_2 are obtained by solving the Vandermonde system with initial conditions $P_{k,0}^{(r)} + A_0$ and $P_{k,1}^{(r)} + \sum_{j=0}^{r} A_j$.

Proof. In order to prove (19), consider Expression (10), whose characteristic equation is nonhomogeneous. The respective solutions are the sum of the solutions of the homogeneous part of (10) and a particular solution. Consider the homogeneous part of (10) and the associated characteristic equation $P(x) = x^2 - 2x - k = 0$. The roots of $P(x)$ are simply given by $\alpha_1 = 1 + \sqrt{1+k}$ and $\alpha_2 = 1 - \sqrt{1+k}$. Then, the solutions of the homogeneous part of (10) is under the form $C_1\alpha_1^n + C_2\alpha_2^n$, where $C_1 = \frac{1}{2\sqrt{1+k}}$ and $C_2 = -\frac{1}{2\sqrt{1+k}}$. By fixing r and considering $C_p(n) = \sum_{j=1}^{r} \binom{n+r-2-j}{r-j}$, a particular solution is given by $\sum_{j=0}^{r} A_j n^j$, where A_j are constants for each $0 \leq j \leq r$, obtained by solving the equation

$$\sum_{j=0}^{r} A_j n^j = 2\sum_{j=0}^{r} A_j (n-1)^j + k\sum_{j=0}^{r} A_j (n-2)^j + C_p(n).$$

Given the particular solution, by replacing $P_{k,n}^{(r)} = y_{k,n}^{(r)} + \sum_{j=0}^{r} A_j n^j$, we obtain the associated homogeneous recurrence relation $y_{k,n}^{(r)} = 2y_{k,n-1}^{(r)} + ky_{k,n-2}^{(r)}$, the closed formula of which is given by $C_1\alpha_1^n + C_2\alpha_2^n$, where $\alpha_1 = 1 + \sqrt{1+k}$, $\alpha_2 = 1 - \sqrt{1+k}$, and C_1, C_2 are obtained by solving the Vandermonde system with the initial conditions $y_{k,0}^{(r)} = P_{k,0}^{(r)} + A_0$ and $y_{k,1}^{(r)} = P_{k,1}^{(r)} + \sum_{j=0}^{r} A_j$. □

For $r = 1$, we obtain the function $C_p(n) = 1$, then the particular solution is given by constant $A = \frac{-1}{k+1}$. Therefore, the next result is verified.

Corollary 1. *For non-negative integers n and k, the n-th hyper k-Pell number of order one is given as*

$$P_{k,n}^{(1)} = \frac{1+\sqrt{1+k}+k}{2\sqrt{1+k}(1+k)}\alpha_1^n + \frac{-1+\sqrt{1+k}-k}{2\sqrt{1+k}(1+k)}\alpha_2^n - \frac{1}{k+1}, \tag{20}$$

where $\alpha_1 = 1 + \sqrt{1+k}$ and $\alpha_2 = 1 - \sqrt{1+k}$.

Similarly, the results of Proposition 4 give us the Binet formula for the hyper k-Pell–Lucas and hyper Modified k-Pell numbers.

Theorem 3. *For non-negative integers k, n, and r, the following identities hold:*

$$Q_{k,n}^{(0)} = \alpha_1^n + \alpha_2^n, \tag{21}$$

$$q_{k,n}^{(0)} = \frac{1}{2}(\alpha_1^n + \alpha_2^n), \tag{22}$$

$$Q_{k,n}^{(1)} = \frac{1}{\sqrt{1+k}}\left(\alpha_1^{n+1} - \alpha_2^{n+1}\right), \tag{23}$$

$$q_{k,n}^{(1)} = \frac{1}{2\sqrt{1+k}}\left(\alpha_1^{n+1} - \alpha_2^{n+1}\right), \tag{24}$$

$$Q_{k,n}^{(r)} = C_1\alpha_1^n + C_2\alpha_2^n + \sum_{j=0}^{r-1} A_j n^j, \tag{25}$$

$$q_{k,n}^{(r)} = C_1\alpha_1^n + C_2\alpha_2^n + \sum_{j=0}^{r-1} A_j n^j, \tag{26}$$

where $\alpha_1 = 1 + \sqrt{1+k}$, $\alpha_2 = 1 - \sqrt{1+k}$, $\sum_{j=0}^{r-1} A_j n^j$ *is the particular solution and* C_1, C_2 *are constants obtained by solving the Vandermonde system with initial conditions* $Q_{k,0}^{(r)} + A_0$ *and* $Q_{k,1}^{(r)} + \sum_{j=0}^{r} A_j$ *and* $q_{k,0}^{(r)} + A_0$ *and* $q_{k,1}^{(r)} + \sum_{j=0}^{r} A_j$*, respectively.*

3. "Hybrid Hyper" Version of the *k*-Pell, *k*-Pell–Lucas, and Modified *k*-Pell Numbers

In this section, motivated by the conception of "hybrid" and "hyper" for each sequence of k-Pell, k-Pell–Lucas, and Modified k-Pell numbers, we present the following definition:

Definition 1. *For non-negative integers* r, n, *and* k, *the* n*-th hybrid hyper* k*-Pell,* k*-Pell–Lucas, and Modified* k*-Pell numbers are defined, respectively, as follows:*

$$HP_{k,n}^{(r)} = P_{k,n}^{(r)} + iP_{k,n+1}^{(r)} + \varepsilon P_{k,n+2}^{(r)} + hP_{k,n+3}^{(r)}, \tag{27}$$

where $P_{k,n}^{(r)}$ *is the* n*-th hyper* k*-Pell number given by Expression (4):*

$$HQ_{k,n}^{(r)} = Q_{k,n}^{(r)} + iQ_{k,n+1}^{(r)} + \varepsilon Q_{k,n+2}^{(r)} + hQ_{k,n+3}^{(r)}, \tag{28}$$

where $Q_{k,n}^{(r)}$ *is the* n*-th hyper* k*-Pell–Lucas number given by Expression (5), and*

$$Hq_{k,n}^{(r)} = q_{k,n}^{(r)} + iq_{k,n+1}^{(r)} + \varepsilon q_{k,n+2}^{(r)} + hq_{k,n+3}^{(r)}, \tag{29}$$

where $q_{k,n}^{(r)}$ *is the* n*-th hyper Modified* k*-Pell number given by Expression (6).*

Observe that, for $r = 0$, we have that $HP_{k,n}^{(0)} = HP_{k,n}$ is the n-th hybrid k-Pell number, $HQ_{k,n}^{(0)} = HQ_{k,n}$ is the n-th hybrid k-Pell–Lucas number, and $Hq_{k,n}^{(0)} = Hq_{k,n}$ is the n-th hybrid Modified k-Pell number.

3.1. Some Properties of These Sequences

In this subsection, we will study some properties of the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell numbers. Several identities of the hyper k-Pell, hyper k-Pell–Lucas, and hyper Modified k-Pell numbers were established in [14] and can be extended to the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell numbers. First, consider the result of Proposition 1 in [14]:

$$P_{k,n}^{(r)} = P_{k,n}^{(r-1)} + P_{k,n-1}^{(r)}. \tag{30}$$

Then, by replacing Expression (30) in Expression (27), we obtain

$$\begin{aligned} HP_{k,n}^{(r)} &= (P_{k,n}^{(r-1)} + P_{k,n-1}^{(r)}) + i\left(P_{k,n+1}^{(r-1)} + P_{k,n}^{(r)}\right) + \varepsilon\left(P_{k,n+2}^{(r-1)} + P_{k,n+1}^{(r)}\right) + h\left(P_{k,n+3}^{(r-1)} + P_{k,n+2}^{(r)}\right) \\ &= HP_{k,n}^{(r-1)} + HP_{k,n-1}^{(r)}. \end{aligned}$$

Similarly, the result can be provided for the hybrid hyper k-Pell–Lucas and the hybrid hyper Modified k-Pell sequences. The following proposition gives us these formulas.

Proposition 5. *For non-negative integers k, $n \geq 1$, and $r \geq 1$, the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell sequences satisfy the recurrence relations:*

$$HP_{k,n}^{(r)} = HP_{k,n}^{(r-1)} + HP_{k,n-1}^{(r)},$$

$$HQ_{k,n}^{(r)} = HQ_{k,n}^{(r-1)} + HQ_{k,n-1}^{(r)},$$

$$Hq_{k,n}^{(r)} = Hq_{k,n}^{(r-1)} + Hq_{k,n-1}^{(r)}.$$

As a consequence, for $r = 1$, we obtain the following corollary.

Corollary 2. *For $n \geq 1$, the identities below hold:*

$$HP_{k,n}^{(1)} = HP_{k,n} + HP_{k,n-1}^{(1)},$$

$$HQ_{k,n}^{(1)} = HQ_{k,n} + HQ_{k,n-1}^{(1)},$$

$$Hq_{k,n}^{(1)} = Hq_{k,n} + Hq_{k,n-1}^{(1)},$$

where $HP_{k,n}$ is the n-th hybrid k-Pell number, $HQ_{k,n}$ is the n-th hybrid k-Pell–Lucas number, and $Hq_{k,n}$ is the n-th hybrid Modified k-Pell number.

Using Definition 1, Corollary 2, and the identities of Proposition 2 in [14], we can describe a different expression for the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell sequences, for $r = 1$, whose proofs will be omitted.

Proposition 6. *For non-negative integers k, $n \geq 1$, and $r = 1$, the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell sequences satisfy the identities below:*

$$HP_{k,n}^{(1)} = HP_{k,n} + \frac{1}{k+1}(Hq_{k,n} - (1 + i + \varepsilon + h)),$$

$$HQ_{k,n}^{(1)} = \frac{1}{k+1}((2k+1)HQ_{k,n} + HQ_{k,n+1}),$$

$$Hq_{k,n}^{(1)} = Hq_{k,n} + \frac{1}{2}\left(HQ_{k,n}^{(1)} + HQ_{k,n}\right),$$

where $HP_{k,n}$ is the n-th hybrid k-Pell number, $HQ_{k,n}$ is the n-th hybrid k-Pell–Lucas number, and $Hq_{k,n}$ is the n-th hybrid Modified k-Pell number.

Using the results of Section 2.1, we will provide a recursive relation for the hybrid hyper k-Pell, k-Pell–Lucas, and Modified k-Pell numbers. By replacing Expression (10) in Expression (27), we obtain

$$\begin{aligned}HP_{k,n}^{(r)} &= (2P_{k,n-1}^{(r)} + kP_{k,n-2}^{(r)} + \sum_{j=1}^{r} \binom{n+r-2-j}{r-j}) \\ &+ i\left(2P_{k,n}^{(r)} + kP_{k,n-1}^{(r)} + \sum_{j=1}^{r} \binom{n+r-1-j}{r-j}\right) \\ &+ \varepsilon\left(2P_{k,n+1}^{(r)} + kP_{k,n}^{(r)} + \sum_{j=1}^{r} \binom{n+r-j}{r-j}\right) \\ &+ h\left(2P_{k,n+2}^{(r)} + kP_{k,n+1}^{(r)} + \sum_{j=1}^{r} \binom{n+r+1-j}{r-j}\right) \\ &= 2HP_{k,n-1}^{(r)} + kHP_{k,n-2}^{(r)} + Z_{k,n}^{(r)},\end{aligned}$$

where $Z_{k,n}^{(r)} = \sum_{j=1}^{r} \binom{n+r-2-j}{r-j} + i\binom{n+r-1-j}{r-j} + \varepsilon\binom{n+r-j}{r-j} + h\binom{n+r+1-j}{r-j}$. Observe that the initial values can be determined by replacing $n = 0$ and $n = 1$ in Expression (27), given by $HP_{k,0}^{(r)} = i + \varepsilon P_{k,2}^{(r)} + hP_{k,3}^{(r)}$ and $HP_{k,1}^{(r)} = 1 + iP_{k,2}^{(r)} + \varepsilon P_{k,3}^{(r)} + hP_{k,4}^{(r)}$.

Similarly, by replacing Expression (13) in Expression (28) and Expression (14) in Expression (29), we obtain the following result.

Proposition 7. *For non-negative integers k, $n \geq 2$, and $r \geq 1$, the hybrid hyper k-Pell, the hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell sequences satisfy the recurrence relations:*

$$HP_{k,n}^{(r)} = 2HP_{k,n-1}^{(r)} + kHP_{k,n-2}^{(r)} + Z_{k,n}^{(r)}, \tag{31}$$

with initial conditions $HP_{k,0}^{(r)} = i + \varepsilon P_{k,2}^{(r)} + hP_{k,3}^{(r)}$ and $HP_{k,1}^{(r)} = 1 + iP_{k,2}^{(r)} + \varepsilon P_{k,3}^{(r)} + hP_{k,4}^{(r)}$,

$$HQ_{k,n}^{(r)} = 2HQ_{k,n-1}^{(r)} + kHQ_{k,n-2}^{(r)} + 2T_{k,n}^{(r)}, \tag{32}$$

with initial conditions $HQ_{k,0}^{(r)} = 2 + 2i + \varepsilon Q_{k,2}^{(r)} + hQ_{k,3}^{(r)}$ and $HQ_{k,1}^{(r)} = 2 + iQ_{k,2}^{(r)} + \varepsilon Q_{k,3}^{(r)} + hQ_{k,4}^{(r)}$, and

$$Hq_{k,n}^{(r)} = 2Hq_{k,n-1}^{(r)} + kHq_{k,n-2}^{(r)} + T_{k,n}^{(r)}, \tag{33}$$

with initial conditions $Hq_{k,0}^{(r)} = 1 + i + \varepsilon q_{k,2}^{(r)} + hq_{k,3}^{(r)}$ and $Hq_{k,1}^{(r)} = 1 + iq_{k,2}^{(r)} + \varepsilon q_{k,3}^{(r)} + hq_{k,4}^{(r)}$, where $Z_{k,n}^{(r)} = \sum_{j=1}^{r} \binom{n+r-2-j}{r-j} + i\binom{n+r-1-j}{r-j} + \varepsilon\binom{n+r-j}{r-j} + h\binom{n+r+1-j}{r-j}$ and $T_{k,n}^{(r)} = \sum_{j=1}^{r-1} \binom{n+r-2-j}{r-1-j} + i\binom{n+r-1-j}{r-1-j} + \varepsilon\binom{n+r-j}{r-1-j} + h\binom{n+r+1-j}{r-1-j}$.

Proposition 7 shows us that the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell sequences can be see as a nonhomogeneous linear recurrence relation of order 2. These results allow us to provide the generating function and the Binet formula for these sequences.

3.2. The Generating Function and Binet's Formula

Motivated by the results in Section 2.2, this subsection is devoted to establishing the generating function and the Binet formula for the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell sequences.

Consider that the generating functions $f_{HP}(t)$, $f_{HQ}(t)$, and $f_{Hq}(t)$ are defined, respectively, by $f_{HP}(t) = \sum_{n=0}^{\infty} HP_{k,n}^{(r)} t^n$, $f_{HQ}(t) = \sum_{n=0}^{\infty} HQ_{k,n}^{(r)} t^n$, and $f_{Hq}(t) = \sum_{n=0}^{\infty} Hq_{k,n}^{(r)} t^n$. We have

$$\begin{aligned}
f_{HP}(t) &= \sum_{n=0}^{\infty} HP_{k,n}^{(r)} t^n = HP_{k,0}^{(r)} + HP_{k,1}^{(r)} t + \sum_{n=2}^{\infty} HP_{k,n}^{(r)} t^n \\
&= (i + \varepsilon P_{k,2}^{(r)} + hP_{k,3}^{(r)}) + (1 + iP_{k,2}^{(r)} + \varepsilon P_{k,3}^{(r)} + hP_{k,4}^{(r)})t \\
&+ \sum_{n=2}^{\infty} \left(2HP_{k,n-1}^{(r)} + kHP_{k,n-2}^{(r)} + Z_{k,n}^{(r)}\right) t^n \\
&= (i + \varepsilon P_{k,2}^{(r)} + hP_{k,3}^{(r)}) + (1 + iP_{k,2}^{(r)} + \varepsilon P_{k,3}^{(r)} + hP_{k,4}^{(r)})t - 2t(i + \varepsilon P_{k,2}^{(r)} + hP_{k,3}^{(r)}) \\
&+ 2t \sum_{n=0}^{\infty} HP_{k,n}^{(r)} t^n + kt^2 \sum_{n=0}^{\infty} HP_{k,n}^{(r)} t^n + \sum_{n=2}^{\infty} Z_{k,n}^{(r)} t^n.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left(1 - 2t - kt^2\right) \sum_{n=0}^{\infty} HP_{k,n}^{(r)} t^n &= i + \varepsilon P_{k,2}^{(r)} + hP_{k,3}^{(r)} \\
&+ t\left(1 + i(P_{k,2}^{(r)} - 2) + \varepsilon(P_{k,3}^{(r)} - 2P_{k,2}^{(r)}) + h(P_{k,4}^{(r)}) - 2P_{k,3}^{(r)}\right) \\
&+ \sum_{n=2}^{\infty} Z_{k,n}^{(r)} t^n.
\end{aligned}$$

Observe that, since

$$t^2 \sum_{j=1}^{r} \sum_{n=0}^{\infty} \binom{n+r-j}{r-j} t^n = t^2 \sum_{j=1}^{r} \frac{1}{(1-t)^{r-j+1}},$$

$$t \sum_{j=1}^{r} \left(\sum_{n=0}^{\infty} \binom{n+r-j}{r-j} t^n - 1 \right) = t \sum_{j=1}^{r} \frac{1}{(1-t)^{r-j+1}} - rt,$$

then

$$\begin{aligned}
\sum_{n=2}^{\infty} Z_{k,n}^{(r)} t^n \quad &= \sum_{j=1}^{r} \sum_{n=2}^{\infty} \tbinom{n+r-2-j}{r-j} t^n + i \sum_{j=1}^{r} \sum_{n=2}^{\infty} \tbinom{n+r-1-j}{r-j} t^n \\
&+\varepsilon \sum_{j=1}^{r} \sum_{n=2}^{\infty} \tbinom{n+r-j}{r-j} t^n + h \sum_{j=1}^{r} \sum_{n=2}^{\infty} \tbinom{n+r+1-j}{r-j} t^n \\
&= t^2 \sum_{j=1}^{r} \tfrac{1}{(1-t)^{r-j+1}} + it \sum_{j=1}^{r} \tfrac{1}{(1-t)^{r-j+1}} - irt \\
&+\varepsilon \sum_{j=1}^{r} \tfrac{1}{(1-t)^{r-j+1}} - 1 - t(r-j) \\
&+ht \sum_{j=1}^{r} \tfrac{1}{(1-t)^{r-j+1}} - 1 - t^2(r-j) - t^3 \frac{(r-j+2)(r-j+1)}{2}.
\end{aligned} \tag{34}$$

Taking into account the respective initial conditions, we can provide the generating function for the hybrid hyper k-Pell–Lucas and hybrid hyper Modified k-Pell sequences. We have

$$\begin{aligned}
f_{HQ}(t) &= \sum_{n=0}^{\infty} HQ_{k,n}^{(r)} t^n = HQ_{k,0}^{(r)} + HQ_{k,1}^{(r)} t + \sum_{n=2}^{\infty} HQ_{k,n}^{(r)} t^n \\
&= (2+2i+\varepsilon Q_{k,2}^{(r)} + hQ_{k,3}^{(r)}) + (2+iQ_{k,2}^{(r)} + \varepsilon Q_{k,3}^{(r)} + hQ_{k,4}^{(r)})t \\
&+ \sum_{n=2}^{\infty}\Big(2HQ_{k,n-1}^{(r)} + kHQ_{k,n-2}^{(r)} + 2Tk,n^{(r)}\Big)t^n \\
&= (2+2i+\varepsilon Q_{k,2}^{(r)} + hQ_{k,3}^{(r)}) + (2+iQ_{k,2}^{(r)} + \varepsilon Q_{k,3}^{(r)} + hQ_{k,4}^{(r)})t - 2t(2+2i+\varepsilon Q_{k,2}^{(r)} + hQ_{k,3}^{(r)}) \\
&+ 2t\sum_{n=0}^{\infty} HQ_{k,n}^{(r)} t^n + kt^2 \sum_{n=0}^{\infty} HQ_{k,n}^{(r)} t^n + 2\sum_{n=2}^{\infty} T_{k,n}^{(r)} t^n.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Big(1-2t-kt^2\Big)\sum_{n=0}^{\infty} HQ_{k,n}^{(r)} t^n &= (2+2i+\varepsilon Q_{k,2}^{(r)} + hQ_{k,3}^{(r)}) \\
&+ t\Big(-2 + i(Q_{k,2}^{(r)} - 4) + \varepsilon(Q_{k,3}^{(r)} - 2Q_{k,2}^{(r)}) + h(Q_{k,4}^{(r)}) - 2Q_{k,3}^{(r)}\Big) \\
&+ 2\sum_{n=2}^{\infty} T_{k,n}^{(r)} t^n,
\end{aligned}$$

where

$$\begin{aligned}
\sum_{n=2}^{\infty} T_{k,n}^{(r)} t^n \quad &= \sum_{j=1}^{r-1}\sum_{n=2}^{\infty}\tbinom{n+r-2-j}{r-j-1}t^n + i\sum_{j=1}^{r-1}\sum_{n=2}^{\infty}\tbinom{n+r-1-j}{r-j-1}t^n \\
&+\varepsilon\sum_{j=1}^{r-1}\sum_{n=2}^{\infty}\tbinom{n+r-j}{r-j-1}t^n + h\sum_{j=1}^{r-1}\sum_{n=2}^{\infty}\tbinom{n+r+1-j}{r-j-1}t^n \\
&= -(r-1)t + t\sum_{j=1}^{r-1}\tfrac{1}{(1-t)^{r-j}} + i\sum_{j=1}^{r-1}\tfrac{1}{(1-t)^{r-j+1}} - 1 - t(r-j) \\
&+\frac{\varepsilon}{t}\sum_{j=1}^{r-1}\tfrac{1}{(1-t)^{r-j}} - 1 - t(r-j) - \frac{(r-j+1)(r-j)}{2}t^2 \\
&+\frac{h}{t^2}\sum_{j=1}^{r-1}\tfrac{1}{(1-t)^{r-j}} - 1 - t(r-j) - \tbinom{r-j+3}{r-j-1}t^2 - \tbinom{r-j+4}{r-j-1}t^3.
\end{aligned} \quad (35)$$

Under the previous discussion, the next result is established.

Theorem 4. *For non-negative integers r, n, and k, the generating functions for the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell sequences are, respectively,*

$$\begin{aligned}
\sum_{n=0}^{\infty} HP_{k,n}^{(r)} t^n \quad &= \frac{1}{(1-2t-kt^2)}(i+\varepsilon P_{k,2}^{(r)} + hP_{k,3}^{(r)}) \\
&+\frac{t}{(1-2t-kt^2)}\Big(1 + i(P_{k,2}^{(r)} - 2) + \varepsilon(P_{k,3}^{(r)} - 2P_{k,2}^{(r)}) + h(P_{k,4}^{(r)} - 2P_{k,3}^{(r)})\Big) \\
&+\frac{1}{(1-2t-kt^2)}\sum_{n=2}^{\infty} Z_{k,n}^{(r)} t^n,
\end{aligned} \quad (36)$$

$$\begin{aligned}
\sum_{n=0}^{\infty} HQ_{k,n}^{(r)} t^n \quad &= \frac{1}{(1-2t-kt^2)}(2+2i+\varepsilon Q_{k,2}^{(r)} + hQ_{k,3}^{(r)}) \\
&+\frac{t}{(1-2t-kt^2)}\Big(2 + i(Q_{k,2}^{(r)} - 4) + \varepsilon(Q_{k,3}^{(r)} - 2Q_{k,2}^{(r)}) + h(Q_{k,4}^{(r)} - 2Q_{k,3}^{(r)})\Big) \\
&+\frac{2}{(1-2t-kt^2)}\sum_{n=2}^{\infty} T_{k,n}^{(r)} t^n,
\end{aligned} \quad (37)$$

$$\begin{aligned}\sum_{n=0}^{\infty} Hq_{k,n}^{(r)} t^n &= \frac{1}{(1-2t-kt^2)}(1+i+\varepsilon q_{k,2}^{(r)} + hq_{k,3}^{(r)}) \\ &+ \frac{t}{(1-2t-kt^2)}\Big(1+i(q_{k,2}^{(r)}-2)+\varepsilon(q_{k,3}^{(r)}-2q_{k,2}^{(r)})+h(q_{k,4}^{(r)}-2q_{k,3}^{(r)})\Big) \\ &+ \frac{1}{(1-2t-kt^2)}\sum_{n=2}^{\infty} T_{k,n}^{(r)} t^n,\end{aligned} \tag{38}$$

where $\sum_{n=2}^{\infty} Z_{k,n}^{(r)} t^n$ *is given by Expression* (34) *and* $\sum_{n=2}^{\infty} T_{k,n}^{(r)} t^n$ *is given by Expression* (35).

Proposition 7 shows that the hybrid hyper k-Pell, hybrid hyper k-Pell–Lucas, and hybrid hyper Modified k-Pell sequences are defined by nonhomogeneous linear recurrence relations. Then, by fixing r and by considering a particular solution given by $\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)n^j$, we obtain the associated homogeneous recurrence relation $y_{k,n}^{(r)} = 2y_{k,n-1}^{(r)} + ky_{k,n-2}^{(r)}$, the closed formula of which is given by $C_1\alpha_1^n + C_2\alpha_2^n$, where $\alpha_1 = 1+\sqrt{1+k}$, $\alpha_2 = 1-\sqrt{1+k}$ and C_1, C_2 are obtained by solving the Vandermonde system with initial conditions $y_{k,0}^{(r)}$ and $y_{k,1}^{(r)}$.

Theorem 5. *For non-negative integers k, n, and r, the following identities hold:*

$$HP_{k,n}^{(0)} = C_1\alpha_1^n + C_2\alpha_2^n, \tag{39}$$

where $C_1 = -\frac{1}{2\sqrt{1+k}}(\alpha_2 HP_{k,0}^{(0)} - HP_{k,1}^{(0)})$ *and* $C_2 = -\frac{1}{2\sqrt{1+k}}(-\alpha_1 HP_{k,0}^{(0)} + HP_{k,1}^{(0)})$,

$$HP_{k,n}^{(r)} = C_1\alpha_1^n + C_2\alpha_2^n + \sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)n^j, \tag{40}$$

where $\alpha_1 = 1+\sqrt{1+k}$, $\alpha_2 = 1-\sqrt{1+k}$, A_j *are constants for each* $0 \le j \le r$, *obtained by solving the equation:*

$$\begin{aligned}\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)n^j &= 2\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)(n-1)^j \\ &\quad + k\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)(n-2)^j + Z_{k,n}^{(r)},\end{aligned}$$

for each fixed r, and C_1, C_2 are obtained by solving the Vandermonde system with initial conditions $HP_{k,0}^{(r)} + A_0$ *and* $HP_{k,1}^{(r)} + \sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)$.

Theorem 6. *For non-negative integers k, n, and r, the following identities hold:*

$$HQ_{k,n}^{(0)} = C_1\alpha_1^n + C_2\alpha_2^n, \tag{41}$$

where $C_1 = -\frac{1}{2\sqrt{1+k}}(\alpha_2 HQ_{k,0}^{(0)} - HQ_{k,1}^{(0)})$ *and* $C_2 = -\frac{1}{2\sqrt{1+k}}(-\alpha_1 HQ_{k,0}^{(0)} + HQ_{k,1}^{(0)})$,

$$HQ_{k,n}^{(1)} = C_1\alpha_1^n + C_2\alpha_2^n, \tag{42}$$

where $C_1 = -\frac{1}{2\sqrt{1+k}}(\alpha_2 HQ_{k,0}^{(1)} - HQ_{k,1}^{(1)})$ *and* $C_2 = -\frac{1}{2\sqrt{1+k}}(-\alpha_1 HQ_{k,0}^{(1)} + HQ_{k,1}^{(1)})$,

$$HQ_{k,n}^{(r)} = C_1\alpha_1^n + C_2\alpha_2^n + \sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)n^j, \tag{43}$$

where $\alpha_1 = 1 + \sqrt{1+k}$, $\alpha_2 = 1 - \sqrt{1+k}$, A_j *are constants for each* $0 \leq j \leq r$, *obtained by solving the equation:*

$$\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)n^j = 2\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)(n-1)^j + k\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)(n-2)^j + 2T_{k,n}^{(r)},$$

for each fixed r, *and* C_1, C_2 *are obtained by solving the Vandermonde system with initial conditions* $HQ_{k,0}^{(r)} + A_0$ *and* $HQ_{k,1}^{(r)} + \sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)$.

Theorem 7. *For non-negative integers* k, n, *and* r, *the following identities hold:*

$$Hq_{k,n}^{(0)} = C_1\alpha_1^n + C_2\alpha_2^n, \tag{44}$$

where $C_1 = -\frac{1}{2\sqrt{1+k}}(\alpha_2 Hq_{k,0}^{(0)} - Hq_{k,1}^{(0)})$ *and* $C_2 = -\frac{1}{2\sqrt{1+k}}(-\alpha_1 Hq_{k,0}^{(0)} + Hq_{k,1}^{(0)})$,

$$Hq_{k,n}^{(1)} = C_1\alpha_1^n + C_2\alpha_2^n, \tag{45}$$

where $C_1 = -\frac{1}{2\sqrt{1+k}}(\alpha_2 Hq_{k,0}^{(1)} - Hq_{k,1}^{(1)})$ *and* $C_2 = -\frac{1}{2\sqrt{1+k}}(-\alpha_1 Hq_{k,0}^{(1)} + Hq_{k,1}^{(1)})$,

$$Hq_{k,n}^{(r)} = C_1\alpha_1^n + C_2\alpha_2^n + \sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)n^j, \tag{46}$$

where $\alpha_1 = 1 + \sqrt{1+k}$, $\alpha_2 = 1 - \sqrt{1+k}$, A_j *are constants for each* $0 \leq j \leq r$, *obtained by solving the equation*

$$\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)n^j = 2\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)(n-1)^j + k\sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)(n-2)^j + T_{k,n}^{(r)},$$

for each fixed r, *and* C_1, C_2 *are obtained by solving the Vandermonde system with initial conditions* $Hq_{k,0}^{(r)} + A_0$ *and* $Hq_{k,1}^{(r)} + \sum_{j=0}^{r}(A_j + iB_j + \varepsilon C_j + hD_j)$.

4. Conclusions

In this paper, we established some properties and identities involving the hyper *k*-Pell, hyper *k*-Pell–Lucas, and hyper Modified *k*-Pell numbers, as recurrence relations, generating functions, and the Binet formula. In addition, we presented the hybrid hyper *k*-Pell, hybrid hyper *k*-Pell–Lucas, and hybrid hyper Modified *k*-Pell numbers, which consist of a new generalization of the hyper *k*-Pell, hyper *k*-Pell–Lucas, and hyper Modified *k*-Pell numbers. Moreover, the algebraic properties of these sequences were studied, and also, the generating function, Binet formula, and several identities were provided.

It seems to us that all the results given here are new in the literature, and these new sequences of numbers are a subject that can still be studied in several aspects.

Author Contributions: E.V.P.S., P.C. and P.V. contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding

Data Availability Statement: Data are contained within the article.

Acknowledgments: The first author is CNPq scholarship holder Process 2007770/2022-5 and expresses sincere thanks to the The Brazilian National Council for Scientific and Technological Development—CNPq—Brazil and Federal University of Mato Grosso do Sul—UFMS/MEC—Brazil for their valuable support. The last two authors are members of the Research Centre CMAT-UTAD (Polo of Research Centre CMAT—Centre of Mathematics of University of Minho) and thank the Portuguese Funds through Fundação para a Ciência e a Tecnologia—FCT—within Projects UIDB/00013/2020 and UIDP/00013/2020.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dasdemir, A. On the Pell, Pell–Lucas and Modified Pell Numbers by Matrix Method. *Appl. Math. Sci.* **2011**, *5*, 3173–3181.
2. Horadam, A.F. Pell identities. *Fibonacci Q.* **1971**, *9*, 245–252.
3. OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. 2013. Available online: http://oeis.org (accessed on 8 November 2023).
4. Koshy, T. *Fibonacci and Lucas Number with Applications*; A Wiley-Interscience Publications: Hoboken, NJ, USA, 2001.
5. Koshy, T. *Pell and Pell–Lucas Numbers with Applications*; Springer: Berlin/Heidelberg, Germany, 2014.
6. Falcón, S.; Plaza, Á. The k-Fibonacci sequence and the Pascal 2-triangle. *Chaos Solitons Fractals* **2007**, *33*, 38–49. [CrossRef]
7. Catarino, P. On some identities and generating functions for *k*-Pell numbers. *Int. J. Math. Anal.* **2013**, *7*, 1877–1884. [CrossRef]
8. Catarino, P.; Campos, H.; Vasco, P. A note on *k*-Pell, *k*-Pell–Lucas and Modified *k*-Pell numbers with arithemetic indexes. *Acta Math. Univ. Comen.* **2020**, *89*, 97–107.
9. Catarino, P.; Vasco, P. Modified *k*-Pell- Sequence Some Identities and Ordinary Generating Function. *Appl. Math. Sci.* **2013**, *7*, 6031–6037. [CrossRef]
10. Falcón, S.; Plaza, Á. Iterated Partial Sums of the k-Fibonacci Sequences. *Axioms* **2022**, *11*, 542. [CrossRef]
11. Panwar, A.; Sisodiya, K.; Rathore, G.P.S. On The Products of *k*-Pell Number and *k*-Pell Lucas number. *IOSR J. Math. (IOSR-JM)* **2017**, *13*, 85–87.
12. Rihane, S.E. *k*-Pell Numbers as Product of Two Repdigits. *Mediterr. J. Math.* **2022**, *19*, 61. [CrossRef]
13. Dil, A.; Mező, I. A symmetric algorithm for hyperharmonic and Fibonacci numbers. *Appl. Math. Comput.* **2008**, *206*, 942–951. [CrossRef]
14. Catarino, P.; Alves, F.R.V.; ; Campos, H. On hyper *k*-Pell, hyper *k*-Pell–Lucas and hyper Modified *k*-Pell sequences. *Integers* **2022**, *22*, A60.
15. Horadam, A.F. Complex Fibonacci Numbers and Fibonacci Quaternions. *Am. Math. Mon.* **1963**, *70*, 289–291. [CrossRef]
16. Jordan, J.H. Gaussian Fibonacci and Lucas Numbers. *Fibonacci Q.* **1965**, *3*, 315–318.
17. Morales, G.C. Investigation of generalized hybrid Fibonacci numbers and their properties. *arXiv* **2018**, arXiv:1806.02231.
18. Özdemir, M. Introduction to Hybrid Numbers. *Adv. Appl. Clifford Algebras* **2018**, *28*, 11. [CrossRef]
19. Szynal-Liana, A.; Wloch, I. The Fibonacci hybrid numbers. *Util. Math.* **2019**, *11*, 3–10.
20. Yılmaz, F.; Özkan, M. On the Generalized Gaussian Fibonacci Numbers and Horadam Hybrid Numbers: A Unified Approach. *Axioms* **2022**, *11*, 255. [CrossRef]
21. Szynal-Liana, A. The horadam hybrid numbers. *Discuss. Math. Gen. Algebra Appl.* **2018**, *38*, 91–98. [CrossRef]
22. Szynal-Liana, A.; Wloch, I. On Pell and Pell–Lucas Hybrid Numbers. *Comment. Math.* **2018**, *58*, 11–17. [CrossRef]
23. Szynal-Liana, A.; Wloch, I. On Jacobsthal and Jacobsthal-Lucas Hybrid Numbers. *Ann. Math. Silesianae* **2019**, *33*, 276–283. [CrossRef]
24. Yasemin, A. Hybrid hyper-Fibonacci and hyper-Lucas numbers. *Notes Number Theory Discret. Math.* **2023**, *29*, 154–170.
25. Spreafico, E.V.P.; Catarino, P. A note on hyper-Leonardo numbers. *Notes Number Theory Discret. Math.* 2023, *submitted*.