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On Boundary Controllability for the Higher-Order Nonlinear Schrödinger Equation

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Abstract: A control problem with final overdetermination is considered for the higher-order nonlinear Schrödinger equation on a bounded interval. The boundary condition on the space derivative is chosen as the control. Results on the global existence of solutions under small input data are established.

Keywords: boundary controllability; terminal overdetermination; higher-order nonlinear Schrödinger equation; initial-boundary value problem

MSC: 93B05; 35Q55; 35Q53

1. Introduction

In this paper, the higher-order nonlinear Schrödinger equation (HNLS)

$$iu_t + au_{xx} + ibu_x + iu_{xxx} + \lambda|u|^{p_0}u + i\beta(|u|^{p_1}u)_x + i\gamma(|u|^{p_1})_xu = f(t, x), \quad (1)$$

posed on an interval $I = (0, R)$ is considered. Here, a, b, λ, β and γ are real constants; $p_0, p_1 \geq 1$, $u = u(t, x)$ and f are complex-valued functions (as well as all other functions below, unless otherwise stated); and $|u|$ is the modulus of the complex number u .

For an arbitrary $T > 0$ in a rectangle $Q_T = (0, T) \times I$, consider an initial-boundary value problem for Equation (1) with an initial condition

$$u(0, x) = u_0(x), \quad x \in [0, R], \quad (2)$$

and boundary conditions

$$u(t, 0) = \mu(t), \quad u(t, R) = \nu(t), \quad u_x(t, R) = h(t), \quad t \in [0, T], \quad (3)$$

where the function h is unknown and must be chosen such that the corresponding solution to problem (1)–(3) satisfies the condition of terminal overdetermination

$$u(T, x) = u_T(x), \quad x \in [0, R], \quad (4)$$

for a given function u_T . The problem of such a type is usually called the boundary-controllability problem.

Equation (1) is a generalized combination of the nonlinear Schrödinger equation (NLS)

$$iu_t + au_{xx} + \lambda|u|^p u = 0$$

and the Korteweg–de Vries equation (KdV)

$$u_t + bu_x + u_{xxx} + uu_x = 0.$$

It has various physical applications; in particular, it models the propagation of femtosecond optical pulses in a monomode optical fiber, accounting for additional effects such



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as third-order dispersion, self-steeping of the pulse and self-frequency shifts (see [1–4] and the references therein).

The first result on the boundary controllability for the KdV equation on a boundary interval appeared in the pioneer paper by L. Rosier [5]. In the case $b = 1$, initial condition (2) and boundary conditions (3) for $\mu = \nu \equiv 0$, it was proved that under a small $u_0, u_T \in L_2(I)$, there existed a solution under the restriction on the length of the interval

$$R \neq 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}, \quad \forall k, l \in \mathbb{N}.$$

In paper [6], this result on the local exact controllability was extended to the truncated HNLS equation with cubic nonlinearity

$$iu_t + au_{xx} + ibu_x + iu_{xxx} + |u|^2u = 0$$

again under homogeneous boundary conditions (3), under restriction on the length of the interval

$$R \neq 2\pi\sqrt{\frac{k^2 + kl + l^2}{3b + a^2}}, \quad \forall k, l \in \mathbb{N}, \tag{5}$$

and under the conditions $b > 0, |a| < 3$ (in fact, in the equation considered in [6], there was a positive coefficient before the third derivative, but it can be easily eliminated by the scaling with respect to t , which is possible since the time interval is arbitrary). The argument repeated the one from [5].

In the present paper, the same result is established for the general HNLS Equation (1) under nonhomogeneous boundary conditions (3) and without any conditions on the coefficients a and b .

In [5], it was shown that if $R = 2\pi\sqrt{(k^2 + kl + l^2)/3}$ for certain natural numbers k and l , the corresponding problem for the linearized KdV equation

$$u_t + u_x + u_{xxx} = 0$$

was not exactly controllable; that is, for any $T > 0$, there existed a finite-dimensional subspace of $L_2(I)$ such that for any function u_T from this subspace, any $h \in L_2(0, T)$ and $u_0 \equiv 0, \mu = \nu \equiv 0$ for the corresponding solution to the initial-boundary value problem satisfied $u|_{t=T} \neq u_T$.

Surprisingly, for the KdV equation itself, the situation is not so. In [7], it was shown that if $R = 2\pi k$ for a certain natural k , the problem was locally exactly controllable in the same sense as in [5] (similar results for other values of R were obtained in [8]). Moreover, in [9], the result of the local exact controllability was established for any R but only for large values of T ($T > T_R$). However, in the recent paper [10], it was shown that the property of the local exact controllability was wrong for all $T > 0$ and R . For the HNLS model, the results of such a type are an open problem.

Results on controllability for the KdV equation on a bounded interval with other controls can be found in [11–14]. For the HNLS model, the results of such a type are an open problem also.

Results on controllability for other models of the KdV type were obtained; for example, in [15] (systems), [16] (higher-order equations) and [17] (a multidimensional case).

Note that in the recent paper [18], the inverse initial-boundary value problem (1)–(3) was considered with an integral overdetermination

$$\int_0^R u(t, x)\omega(x) dx = \varphi(t), \quad t \in [0, T],$$

for the given functions ω and φ . Either the boundary function h or the function F on the right-hand side $f(t, x) = F(t)g(t, x)$ for the given function g were chosen as controls.

The results on well posedness under either small input data or small time intervals were established.

In [19], a direct initial-boundary value problem on a bounded interval with homogeneous boundary conditions (3) for Equation (1) in the case $p_0 = p_1 = p$ was studied. For $p \in [1, 2]$ and the initial function $u_0 \in H^s(I)$, $0 \leq s \leq 3$, the results on the global existence and uniqueness of mild solutions were obtained. For $u_0 \in L_2(I)$ the result on the global existence was extended either to $p \in (2, 3)$ or $p \in (2, 4)$, $\gamma = 0$. Nonhomogeneous boundary conditions were considered in [20] in the real case and nonlinearity uu_x . Note also that in [18], there is a brief survey of other results concerning the direct initial-boundary value problems for Equation (1).

Solutions to the considered problems are constructed in a special functional space

$$X(Q_T) = C([0, T]; L_2(I)) \cap L_2(0, T; H^1(I)),$$

endowed with the norm

$$\|u\|_{X(Q_T)} = \sup_{t \in (0, T)} \|u(t, \cdot)\|_{L_2(I)} + \|u_x\|_{L_2(Q_T)}.$$

For $r > 0$, denote by $\bar{X}_r(Q_T)$ the closed ball $\{u \in X(Q_T) : \|u\|_{X(Q_T)} \leq r\}$.

Next, we introduce the notion of a weak solution to problem (1)–(3).

Definition 1. Let $u_0 \in L_2(I)$, $\mu, \nu, h \in L_2(0, T)$, $f \in L_1(Q_T)$. A function $u \in X(Q_T)$ is called a weak solution to problem (1)–(3) if $u(t, 0) \equiv \mu(t)$, $u(t, R) \equiv \nu(t)$ and for all test functions $\phi(t, x)$, such that $\phi \in C^1([0, T]; L_2(I)) \cap C([0, T]; (H^3 \cap H_0^1)(I))$, $\phi|_{t=T} \equiv 0$, $\phi_x|_{x=0} \equiv 0$, the functions $|u|^{p_0}u, |u|^{p_1}u, |u|^{p_1}u_x \in L_1(Q_T)$, and the following integral identity is verified:

$$\iint_{Q_T} \left[iu\phi_t + au_x\phi_x + ibu\phi_x - iu_x\phi_{xx} - \lambda|u|^{p_0}u\phi + i\beta|u|^{p_1}u\phi_x + i\gamma|u|^{p_1}(u\phi)_x + f\phi \right] dxdt + i \int_0^R u_0\phi|_{t=0} dx + i \int_0^T h\phi_x|_{x=R} dt = 0. \quad (6)$$

Remark 1. Note that $\phi, \phi_x \in C(\bar{Q}_T)$, $\phi_x \in C(\bar{I}; L_2(0, T))$; therefore, the integrals in (6) exist.

To describe the properties of the boundary data μ and ν , introduce the fractional-order Sobolev spaces. Let $\hat{f}(\xi) \equiv \mathcal{F}[f](\xi)$ and $\mathcal{F}^{-1}[f](\xi)$ be the direct and inverse Fourier transforms of a function f , respectively. In particular, for $f \in \mathcal{S}(\mathbb{R})$

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx, \quad \mathcal{F}^{-1}[f](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} f(\xi) d\xi.$$

For $s \in \mathbb{R}$, define the fractional-order Sobolev space

$$H^s(\mathbb{R}) = \{f : \mathcal{F}^{-1}[(1 + |\xi|^s)\hat{f}(\xi)] \in L_2(\mathbb{R})\}$$

and for certain $T > 0$, let $H^s(0, T)$ be a space of restrictions on $(0, T)$ of functions from $H^s(\mathbb{R})$.

Now we can pass to the main result of the paper.

Theorem 1. Let $p_0 \in [1, 4]$, $p_1 \in [1, 2]$, $u_0, u_T \in L_2(I)$, $\mu, \nu \in H^{1/3}(0, T)$, $f \in L_1(0, T; L_2(I))$. Assume also that condition (5) is satisfied if $3b + a^2 > 0$. Denote

$$c_0 = \|u_0\|_{L_2(I)} + \|u_T\|_{L_2(I)} + \|\mu\|_{H^{1/3}(0, T)} + \|\nu\|_{H^{1/3}(0, T)} + \|f\|_{L_1(0, T; L_2(I))}. \quad (7)$$

Then, there exists $\delta > 0$ such that under the assumption $c_0 \leq \delta$, there exists a function $h \in L_2(0, T)$ and the corresponding unique solution to problem (1)–(3) $u \in X(Q_T)$ verifying condition (4).

Remark 2. The smoothness assumption $\mu, \nu \in H^{1/3}(0, T)$ on the boundary data is natural, since if one considers the initial value problem

$$v_t + v_{xxx} = 0, \quad v|_{t=0} = v_0(x) \in L_2(\mathbb{R}),$$

then, by [21], its solution $v \in C(\mathbb{R}; L_2(\mathbb{R}))$ (which can be constructed via the Fourier transform) satisfies the following relations for any $x \in \mathbb{R}$:

$$\|D_t^{1/3}v(\cdot, x)\|_{L_2(\mathbb{R})} = \|v_x(\cdot, x)\|_{L_2(\mathbb{R})} = c\|v_0\|_{L_2(\mathbb{R})}$$

(here, the symbol $D^s f = \mathcal{F}^{-1}[|\xi|^s \widehat{f}(\xi)]$ denotes the Riesz potential).

Further, we use the following simple interpolating inequality: there exists a constant $c = c(R, q)$ such that for any $\varphi \in H^1(I)$,

$$\|\varphi\|_{L_\infty(I)} \leq c\|\varphi'\|_{L_2(I)}^{1/2}\|\varphi\|_{L_2(I)}^{1/2} + c\|\varphi\|_{L_2(I)}, \tag{8}$$

where the second term on the right-hand side is absent if $\varphi \in H_0^1(I)$.

This paper is organized as follows. In Section 2, results on the corresponding linear problem are presented, and Section 3 contains the proof of the nonlinear results.

2. Auxiliary Linear Problem

Besides the nonlinear problem, consider its linear analog and start with the following one with homogeneous boundary conditions:

$$iu_t + au_{xx} + ibu_x + iu_{xxx} = f(t, x), \tag{9}$$

$$u|_{t=0} = u_0(x), \quad u|_{x=0} = u|_{x=R} = u_x|_{x=R} = 0. \tag{10}$$

Define an operator

$$A : D(A) \rightarrow L_2(I), \quad y \mapsto A(y) = -y''' + iay'' - by'$$

with the domain $D(A) = \{y \in H^3(I) : y(0) = y(R) = y'(R) = 0\}$.

Lemma 1. The operator A generates a continuous semigroup of contractions $\{e^{tA}, t \geq 0\}$ in $L_2(I)$.

Proof. This assertion is proved in ([6], Lemma 4.1) but under the restriction $|a| < 3$. However, the slight correction of that proof provides the desired result. In fact, the operator A is obviously closed. Next, for $y \in D(A)$,

$$(Ay, y)_{L_2(I)} = \int_0^R (-y''' + iay'' - by')\bar{y} \, dx.$$

Here,

$$\begin{aligned} - \int_0^R y'''\bar{y} \, dx &= -|y'(0)|^2 + \int_0^R y\bar{y}'''' \, dx, \\ i \int_0^R y''\bar{y} \, dx &= -i \int_0^R |y'|^2 \, dx, \\ - \int_0^R y'\bar{y} \, dx &= \int_0^R y\bar{y}' \, dx, \end{aligned}$$

therefore,

$$\Re(Ay, y)_{L_2(I)} = -\frac{1}{2}|y'(0)|^2 \leq 0$$

and so the operator A is dissipative. Next, the operator $A^*y = y''' - iay'' + by'$ with the domain $D(A^*) = \{y \in H^3(I) : y(0) = y'(0) = y(R) = 0\}$ and similarly for $y \in D(A^*)$:

$$\Re(A^*y, y)_{L_2(I)} = -\frac{1}{2}|y'(R)|^2 \leq 0.$$

Therefore, the operator A^* is also dissipative. The application of the Lumer–Phillips theorem (see [22]) finishes the proof. \square

Remark 3. Note that the weak solution to problem (9) and (10) can be considered in the space $L_1(0, T; L_2(I))$ in the sense of an integral identity

$$\iint_{Q_T} u(i\phi_t - a\phi_{xx} + ib\phi_x + i\phi_{xxx}) dxdt + \iint_{Q_T} f\phi dxdt + i \int_0^R u_0\phi|_{t=0} dx = 0,$$

valid for any test function from Definition 1. Then, the general theory of semigroups (see [22]) provides that for $u_0 \in L_2(I)$, $f \in L_1(0, T; L_2(I))$, there exists a weak solution $u \in C([0, T]; L_2(I))$ to problem (9) and (10),

$$u(t, \cdot) = e^{tA}u_0 + \int_0^t e^{(t-\tau)A}f(\tau, \cdot) d\tau,$$

$$\|u\|_{C([0,T];L_2(I))} \leq \|u_0\|_{L_2(I)} + \|f\|_{L_1(0,T;L_2(I))}, \tag{11}$$

which is unique in $L_1(0, T; L_2(I))$. Moreover, for $u_0 \in D(A)$, $f \in C^1([0, T]; L_2(I))$, this solution is regular; that is, $u \in C^1([0, T]; L_2(I)) \cap C([0, T]; D(A))$.

Lemma 2. Let $u_0 \in L_2(I)$, $f \equiv f_0 - f_{1x}$, where $f_0 \in L_1(0, T; L_2(I))$, $f_1 \in L_2(Q_T)$. Then, there exists a unique weak solution to problem (9) and (10) $u \in X(Q_T)$ and a function $\theta \in L_2(0, T)$, such that for a certain constant $c = c(T)$, nondecreasing with respect to T ,

$$\|u\|_{X(Q_T)} + \|\theta\|_{L_2(0,T)} \leq c(\|u_0\|_{L_2(I)} + \|f_0\|_{L_1(0,T;L_2(I))} + \|f_1\|_{L_2(Q_T)}), \tag{12}$$

and for a.e. $t \in (0, T)$,

$$\begin{aligned} & \frac{d}{dt} \int_0^R |u(t, x)|^2 \rho(x) dx + |\theta(t)|^2 + 3 \int_0^R |u_x|^2 \rho' dx \\ & = b \int_0^R |u|^2 \rho' dx + 2a\Im \int_0^R u_x \bar{u} \rho' dx + 2\Im \int_0^R f_0 \bar{u} \rho dx + 2\Im \int_0^R f_1 (\bar{u} \rho)_x dx, \end{aligned} \tag{13}$$

where one can choose both $\rho(x) \equiv 1$ and $\rho(x) \equiv 1 + x$. Moreover, if $u_0 \in D(A)$ and $f \in C^1([0, T]; L_2(I))$, then $\theta \equiv u_x|_{x=0}$.

Proof. First, consider regular solutions in the case $u_0 \in D(A)$, $f \in C^1([0, T]; L_2(I))$. Then, multiplying equality (9) by $2\bar{u}(t, x)\rho(x)$, extracting the imaginary part and integrating one, we obtain an equality

$$\begin{aligned} & \int_0^R |u(t, x)|^2 \rho(x) dx + \int_0^t |u_x(\tau, 0)|^2 d\tau + 3 \iint_{Q_t} |u_x|^2 \rho' dx d\tau \\ & = \int_0^R |u_0|^2 \rho dx + b \iint_{Q_t} |u|^2 \rho' dx d\tau + 2a\Im \iint_{Q_t} u_x \bar{u} \rho' dx d\tau \\ & \quad + 2\Im \iint_{Q_t} f_0 \bar{u} \rho dx d\tau + 2\Im \iint_{Q_t} f_1 (\bar{u} \rho)_x dx d\tau. \end{aligned} \tag{14}$$

Choose $\rho(x) \equiv 1 + x$; then,

$$\left| 2a \int_0^R u_x \bar{u} dx \right| \leq a^2 \int_0^R |u|^2 dx + \int_0^R |u_x|^2 dx,$$

$$\left| 2 \int_0^R f_1(\bar{u}\rho)_x dx \right| \leq ((1 + R)^2 + 1) \int_0^R |f_1|^2 dx + \int |u_x|^2 dx + \int_0^R |u|^2 dx,$$

and equality (14) provides estimate (12) in the regular case. This estimate gives an opportunity to establish the existence of a weak solution with property (12) in the general case via closure. Moreover, equality (14) is also verified. In particular, this equality implies that the function $\|u(t, \cdot)\rho^{1/2}\|_{L_2(I)}^2$ is absolutely continuous on $[0, T]$, and then (13) follows. \square

Corollary 1. *There exists a linear bounded operator $P : L_2(I) \rightarrow L_2(0, T)$ such that for any $u_0 \in L_2(I)$,*

$$\|Pu_0\|_{L_2(0,T)} \leq \|u_0\|_{L_2(I)}, \tag{15}$$

for the corresponding weak solution $u \in X(Q_T)$ to problem (9) and (10) in the case $f_0 = f_1 \equiv 0$,

$$\|u_0\|_{L_2(I)}^2 \leq \frac{1}{T} \|u\|_{L_2(Q_T)}^2 + \|Pu_0\|_{L_2(0,T)}^2, \tag{16}$$

and $Pu_0 = u_x|_{x=0}$ if $u_0 \in D(A)$.

Proof. In the case $|a| < 3$, this assertion was proved in ([6], Lemma 4.2). Choosing (13) $\rho(x) \equiv 1$, we obtain estimate (15) for $Pu_0 \equiv \theta$. Next, again for $\rho(x) \equiv 1$, multiplying equality (13) by $(T - t)$ and integrating with respect to t , we derive an equality

$$\iint_{Q_T} |u|^2 dxdt - T \int_0^R |u_0|^2 dx + \int_0^T (T - t)|\theta(t)|^2 dt = 0,$$

which implies inequality (16). \square

Three following lemmas are proved in [6] in the case $|a| < 3, b > 0$. The proof in the general case is similar; however, we present it here in a more transparent way. The first auxiliary lemma is concerned with the properties of the operator A .

Lemma 3. *Let the function $y \in D(A), y \not\equiv 0$, be the eigenfunction of the operator A and $y'(0) = 0$. Then, $3b + a^2 > 0$ and $R = 2\pi\sqrt{(k^2 + kl + l^2)/(3b + a^2)}$ for certain natural numbers k and l .*

Proof. Let $\varkappa = y''(0), \sigma = y''(R), Ay = \lambda y$ for certain $\lambda \in \mathbb{C}$.

Extend the function y by zero outside the segment $[0, R]$; note that $y \in H^2(\mathbb{R})$. Then, in $S'(\mathbb{R})$,

$$\lambda y + y''' - iay'' + by' = \varkappa\delta_0 - \sigma\delta_R,$$

where δ_{x_0} denotes the Dirac measure at the point x_0 . Applying the Fourier transform, we derive an equality

$$(\lambda - i\zeta^3 + ia\zeta^2 + ib\zeta)\hat{y}(\zeta) = \varkappa - \sigma e^{-iR\zeta},$$

whence for $p = i\lambda$,

$$\hat{y}(\zeta) = i \frac{\varkappa - \sigma e^{-iR\zeta}}{\zeta^3 - a\zeta^2 - b\zeta + p}.$$

Since the function y has compact support, the function \hat{y} can be extended to the entire function on \mathbb{C} . Note that $(\varkappa, \sigma) \neq (0, 0)$; otherwise, $y \equiv 0$. The roots of the function $\varkappa - \sigma e^{-iR\zeta}$ are simple and have the form $\zeta_0 + 2\pi n/R$ for a certain complex number ζ_0 and integer number n . Then, the roots of the function $\zeta^3 - a\zeta^2 - b\zeta + p$ must also be simple and coincide with the roots of the numerator. As a result, for a certain complex number ζ_0 and natural k, l , the roots of the denominator can be written in such a form:

$$\zeta_0, \quad \zeta_1 = \zeta_0 + k\frac{2\pi}{R}, \quad \zeta_2 = (k + l)\frac{2\pi}{R}.$$

Exploiting the Vieta formulas

$$\zeta_0 + \zeta_1 + \zeta_2 = a, \quad \zeta_0\zeta_1 + \zeta_0\zeta_2 + \zeta_1\zeta_2 = -b,$$

we express ζ_0 from the first one, substitute it into the second one and derive an equality:

$$a^2 + 3b = (k^2 + kl + l^2) \frac{4\pi^2}{R^2}.$$

□

Remark 4. It can be shown that the restriction on the size of the interval is also sufficient for the existence of such eigenfunctions, but this is not used further.

Lemma 4. For $T > 0$, let \mathcal{M}_T denote the space of initial functions $u_0 \in L_2(I)$ such that $Pu_0 = 0$ in $L_2(0, T)$. Then, $\mathcal{M}_T = \{0\}$ for all $T > 0$ if $3b + a^2 \leq 0$ or inequality (5) is satisfied if $3b + a^2 > 0$.

Proof. It is obvious that $\mathcal{M}_{T'} \subseteq \mathcal{M}_T$ if $T < T'$.

For any $T > 0$, the set \mathcal{M}_T is a finite-dimensional vector space. In fact, if u_{0n} is a sequence in a unit ball $\{y \in \mathcal{M}_T : \|y\|_{L_2(I)} \leq 1\}$, it follows from (12) that the corresponding sequence of weak solutions $\{u_n\}$ is bounded in $L_2(0, T; H^1(I))$ and, therefore, the set

$$u_{nt} = -u_{nxxx} + ia u_{nxx} - b u_{nx} \tag{17}$$

is bounded in $L_2(0, T; H^{-2}(I))$. With the use of the continuous embeddings $H^1(I) \subset L_2(I) \subset H^{-2}(I)$, where the first one is compact, by the standard argument (see [23]) we obtain that the set u_n is relatively compact in $L_2(Q_T)$. Extracting the subsequence, we derive that it is convergent in $L_2(Q_T)$, whence it follows from (16) that the corresponding subsequence of u_{0n} is convergent in $L_2(I)$. It means that the considered unit ball is compact, and the Riesz theorem (see [24]) implies that the space \mathcal{M}_T has a finite dimension.

$T' > 0$ is given. To prove that $\mathcal{M}_{T'} = \{0\}$, it is sufficient to find $T \in (0, T')$ such that $\mathcal{M}_T = \{0\}$. Since the map $T \mapsto \dim(\mathcal{M}_T)$ is nonincreasing and step-like, there exists $T, \epsilon > 0$ such that $T < T + \epsilon < T'$ and $\dim \mathcal{M}_T = \dim \mathcal{M}_{T+\epsilon}$. Let $u_0 \in \mathcal{M}_T$ and $t \in (0, \epsilon)$. Since $e^{tA}e^{\tau A}u_0 = e^{(t+\tau)A}u_0$ for $\tau \geq 0$ and $u_0 \in \mathcal{M}_{T+\epsilon}$, then

$$\frac{e^{tA}u_0 - u_0}{t} \in \mathcal{M}_T. \tag{18}$$

Let $\mathcal{G}_T = \{u = e^{\tau A}u_0 : \tau \in [0, T], u_0 \in \mathcal{M}_T\} \subset C([0, T]; L_2(I))$. Since $u \in H^1(0, T + \epsilon; H^{-2}(I))$, there exists

$$\lim_{t \rightarrow +0} \frac{u(\tau + t) - u(\tau)}{t} = u'(\tau) \quad \text{in } L_2(0, T; H^{-2}(I)).$$

On the other hand, by (18).

$$\frac{u(\tau + t) - u(\tau)}{t} = e^{\tau A} \frac{e^{tA}u_0 - u_0}{t} \in \mathcal{G}_T$$

for $t \in (0, \epsilon)$ and \mathcal{G}_T is closed in $L_2(0, T; H^{-2}(I))$ since $\dim \mathcal{G}_T < \infty$. Therefore, $u' \in C([0, T]; L_2(I))$ and $u \in C^1([0, T]; L_2(I))$. In particular,

$$u'(0) = \lim_{t \rightarrow +0} \frac{e^{tA}u_0 - u_0}{t} \quad \text{in } L_2(I).$$

Therefore,

$$u_0 \in D(A), \quad Au_0 = u'(0) \in \mathcal{M}_T, \quad Pu_0 = u_x|_{x=0} \in C[0, T]$$

(the last property holds since $u \in C([0, T]; H^3(I))$). Hence,

$$u'_0(0) = u_x(0, 0) = 0.$$

Since $\dim \mathcal{M}_T < \infty$, if $\mathcal{M}_T \neq \{0\}$, the map $u_0 \in \mathcal{M}_T \mapsto Au_0 \in \mathcal{M}_T$ has at least one nontrivial eigenfunction, which contradicts Lemma 3. \square

Lemma 5. *Let either $3b + a^2 \leq 0$ or $3b + a^2 > 0$ and inequality (5) be satisfied. Then, for any $T > 0$, there exists a constant $c = c(T, R)$ such that for any $u_0 \in L_2(I)$,*

$$\|u_0\|_{L_2(I)} \leq c \|Pu_0\|_{L_2(0, T)}. \tag{19}$$

Proof. We argue by contradiction. If (19) is not verified, there exists a sequence $\{u_{0n}\}_{n \in \mathbb{N}}$ such that $\|u_{0n}\|_{L_2(I)} = 1 \ \forall n$ and $\|Pu_{0n}\|_{L_2(0, T)} \rightarrow 0$ when $n \rightarrow +\infty$. As in the proof of the previous lemma, the corresponding sequence of weak solutions $\{u_n\}$ is bounded in $L_2(0, T; H^1(I))$, and according to (17), the sequence u_{nt} is bounded in $L_2(0, T; H^{-2}(I))$. Again as in the proof of the previous lemma, extract a subsequence of $\{u_n\}$, for simplicity also denoted as $\{u_n\}$, such that it is convergent in $L_2(Q_T)$. Then, by (16), $\{u_{0n}\}$ converges in $L_2(I)$ to the certain function u_0 . Inequality (15) implies that $Pu_{0n} \rightarrow Pu_0$ in $L_2(0, T)$. Then, $\|u_0\|_{L_2(I)} = 1$ and $\|Pu_0\|_{L_2(0, T)} = 0$, which contradicts Lemma 4. \square

Now consider the nonhomogeneous linear equation

$$iu_t + au_{xx} + ibu_x + iu_{xxx} = f_0(t, x) - f_{1x}(t, x). \tag{20}$$

The notion of a weak solution to the corresponding initial-boundary-value problem with initial and boundary conditions (2) and (3) is similar to Definition 1. In particular, the corresponding integral identity (for the same test functions as in Definition 1) is written as follows:

$$\begin{aligned} \iint_{Q_T} [iu\phi_t + au_x\phi_x + ibu\phi_x - iu_x\phi_{xx} + f_0\phi + f_1\phi_x] dxdt \\ + i \int_0^R u_0\phi|_{t=0} dx + i \int_0^T h\phi_x|_{x=R} dt = 0. \end{aligned} \tag{21}$$

The following result is established in [18].

Lemma 6. *Let $u_0 \in L_2(I)$, $\mu, \nu \in H^{1/3}(0, T)$, $h \in L_2(0, T)$, $f_0 \in L_1(0, T; L_2(I))$, $f_1 \in L_2(Q_T)$. Then, there exists a unique weak solution $u = S(u_0, \mu, \nu, h, f_0, f_1) \in X(Q_T)$ to problem (20), (2), (3) and*

$$\begin{aligned} \|u\|_{X(Q_T)} \leq c(T) [\|u_0\|_{L_2(I)} + \|\mu\|_{H^{1/3}(0, T)} + \|\nu\|_{H^{1/3}(0, T)} + \|h\|_{L_2(0, T)} \\ + \|f_0\|_{L_1(0, T; L_2(I))} + \|f_1\|_{L_2(Q_T)}], \end{aligned} \tag{22}$$

for a certain constant $c(T)$, nondecreasing with respect to T .

Remark 5. *Let*

$$S_T(u_0, \mu, \nu, h, f_0, f_1) \equiv S(u_0, \mu, \nu, h, f_0, f_1)|_{t=T}$$

Then, it follows from (22) that

$$\|S_T(u_0, \mu, \nu, h, f_0, f_1)\|_{L_2(0,R)} \leq c(T) \left[\|u_0\|_{L_2(I)} + \|\mu\|_{H^{1/3}(0,T)} + \|\nu\|_{H^{1/3}(0,T)} + \|h\|_{L_2(0,T)} + \|f_0\|_{L_1(0,T;L_2(I))} + \|f_1\|_{L_2(Q_T)} \right]. \tag{23}$$

Note also that $S(u_0, 0, 0, 0, 0, 0) = \{e^{tA}u_0 : t \in [0, T]\}$.

Corollary 2. Let the hypothesis of Lemma 6 be satisfied; then, for $u = S(u_0, \mu, \nu, h, f_0, f_1)$ and any function $\phi \in C^1([0, T]; L_2(I)) \cap C([0, T]; (H^3 \cap H_0^1)(I))$, $\phi_x|_{x=0} \equiv 0$, the following identity holds:

$$\iint_{Q_T} \left[u(i\phi_t - a\phi_{xx} + ib\phi_x + i\phi_{xxx}) + f_0\phi + f_1\phi_x \right] dxdt + i \int_0^R u_0\phi|_{t=0} dx - i \int_0^R (u\phi)|_{t=T} dx + \int_0^T \mu(i\phi_{xx} - a\phi_x)|_{x=0} dt + i \int_0^T (h\phi_x - \nu\phi_{xx})|_{x=R} dt = 0. \tag{24}$$

Proof. Let $\eta(x)$ be a cut-off function; namely, η is an infinitely smooth nondecreasing function on \mathbb{R} such that $\eta(x) = 0$ for $x \leq 0$, $\eta(x) = 1$ for $x \geq 1$ and $\eta(x) + \eta(1 - x) \equiv 1$. Denote $\phi_\varepsilon(t, x) \equiv \phi(t, x)\eta((T - t)/\varepsilon)$; then, ϕ_ε satisfies the assumptions on the test functions from Definition 1. Write the corresponding equality (21):

$$\iint_{Q_T} \left[iu\phi_{\varepsilon t} + au_x\phi_{\varepsilon x} + ibu\phi_{\varepsilon x} - iu_x\phi_{\varepsilon xx} + f_0\phi_\varepsilon + f_1\phi_{\varepsilon x} \right] dxdt + i \int_0^R u_0\phi_\varepsilon|_{t=0} dx + i \int_0^T h\phi_{\varepsilon x}|_{x=R} dt = 0.$$

Here,

$$\phi_{\varepsilon t}(t, x) = \phi_t(t, x)\eta\left(\frac{T-t}{\varepsilon}\right) - \frac{1}{\varepsilon}\phi(t, x)\eta'\left(\frac{T-t}{\varepsilon}\right).$$

Since $u\phi \in C([0, T]; L_1(I))$,

$$-\frac{1}{\varepsilon} \iint_{Q_T} u\phi\eta'\left(\frac{T-t}{\varepsilon}\right) dxdt \rightarrow - \int_0^R (u\phi)|_{t=T} dx$$

when $\varepsilon \rightarrow +0$. Therefore, passing to the limit when $\varepsilon \rightarrow +0$ and integrating by parts, we derive equality (24). \square

Now we are able to establish a result on boundary controllability in the linear case.

Theorem 2. Let $u_0, u_T \in L_2(I)$, $\mu, \nu \in H^{1/3}(0, T)$, $f_0 \in L_1(0, T; L_2(I))$, $f_1 \in L_2(Q_T)$. Assume also that the $3b + a^2 > 0$ condition (5) is satisfied. Then, there exists a function $h \in L_2(0, T)$ and the corresponding unique solution to problem (20), (2), (3) $u \in X(Q_T)$, verifying condition (4).

Proof. Assume first that $u_0 \equiv 0$, $\mu = \nu \equiv 0$, $f_0 = f_1 \equiv 0$. For $h \in L_2(0, T)$, consider the solution $u = S(0, 0, 0, h, 0, 0) \equiv S_0h \in X(Q_T)$ of the corresponding problem (20), (2), (3); let $S_{0T}h \equiv S_0h|_{t=T}$. Then, estimate (23) implies that S_{0T} is the linear bounded operator from $L_2(0, T)$ to $L_2(I)$.

Consider the backward problem in Q_T :

$$i\phi_t - a\phi_{xx} + ib\phi_x + i\phi_{xxx} = 0, \tag{25}$$

$$\phi|_{t=T} = \phi_0(x), \quad \phi|_{x=0} = \phi_x|_{x=0} = \phi|_{x=R} = 0. \tag{26}$$

Then, this problem is equivalent to the problem for the function $\tilde{\phi}(t, x) \equiv \phi(T - t, R - x)$:

$$i\tilde{\phi}_t + a\tilde{\phi}_{xx} + ib\tilde{\phi}_x + i\tilde{\phi}_{xxx} = 0, \\ \tilde{\phi}|_{t=0} = \tilde{\phi}_0(x) \equiv \phi_0(R - x), \quad \tilde{\phi}|_{x=0} = \tilde{\phi}|_{x=R} = \tilde{\phi}_x|_{x=R} = 0.$$

Let

$$(\Lambda\phi_0)(t) \equiv -(P_0\tilde{\phi}_0)(T - t).$$

Then, it follows from Corollary 1 that $\Lambda\phi_0 = \phi_x|_{x=R}$ if $\phi_0 \in D(A^*)$ and from inequalities (15) and (19) that

$$\|\Lambda\phi_0\|_{L_2(0,T)} \leq \|\phi_0\|_{L_2(I)} \leq c\|\Lambda\phi_0\|_{L_2(0,T)}. \tag{27}$$

In the case $\phi_0 \in D(A^*)$, the corresponding solution to problem (25) and (26) satisfies the assumptions on the functions ϕ from Corollary 2. Write equality (24) for $u = S_0h$ and $\bar{\phi}$, then

$$\int_0^R S_{0T}h \cdot \bar{\phi}_0 \, dx = \int_0^T h \cdot \overline{\Lambda\phi_0} \, dt. \tag{28}$$

By continuity, this equality can be extended to the case $h \in L_2(0, T)$, $\phi_0 \in L_2(I)$. Let $B \equiv S_{0T} \circ \Lambda$; then, according to (27) and the aforementioned properties of the operator S_{0T} , the operator B is bounded in $L_2(I)$. Moreover, (27) and (28) provide that

$$(B\phi_0, \phi_0)_{L_2(I)} = \int_0^R (S_{0T} \circ \Lambda)\phi_0 \cdot \bar{\phi}_0 \, dx = \int_0^T |\Lambda\phi_0|^2 \, dt \geq \frac{1}{c^2} \|\phi_0\|_{L_2(I)}^2.$$

The application of the Lax–Milgram theorem (see, [24]) implies that the operator B is invertible and B^{-1} is bounded in $L_2(I)$. Let

$$\Gamma \equiv \Lambda \circ B^{-1}. \tag{29}$$

This operator is bounded from $L_2(I)$ to $L_2(0, T)$. Then, $h \equiv \Gamma u_T$ ensures the desired result in the considered case since

$$(S_{0T} \circ \Gamma)u_T = (S_{0T} \circ \Lambda \circ B^{-1})u_T = u_T.$$

In the general case, the desired solution is constructed by formulas

$$h \equiv \Gamma(u_T - S_T(u_0, \mu, \nu, 0, f_0, f_1)), \quad u \equiv S(u_0, \mu, \nu, 0, f_0, f_1) + S_0h. \tag{30}$$

□

Remark 6. Note that the function h can not be defined in a unique way. Indeed, choose $h \neq 0$ in $L_2(0, T/2)$. Move the time origin to the point $T/2$, and for $u_0 \equiv S_{T/2}(0, 0, 0, h, 0, 0)$ and $u_T \equiv 0$, construct the solution to the corresponding boundary-controllability problem, which is, of course, nontrivial. However, $h \equiv 0$ and $u \equiv 0$ solve the same problem.

3. Nonlinear Problem

Now we pass to the nonlinear equation and first of all establish three auxiliary estimates.

Lemma 7. Let $p \in [1, 4]$; then, for any functions $u, v, \in X(Q_T)$,

$$\| |u|^p v \|_{L_1(0,T;L_2(I))} \leq c(T^{(4-p)/4} + T) \|u\|_{X(Q_T)}^p \|v\|_{X(Q_T)}. \tag{31}$$

Proof. Applying the interpolating inequality (8), we find that

$$\| |u|^p v \|_{L_2(I)} \leq \|u\|_{L_\infty(I)}^p \|v\|_{L_2(I)} \leq c \left(\|u_x\|_{L_2(I)}^{p/2} \|u\|_{L_2(I)}^{p/2} + \|u\|_{L_2(I)}^p \right) \|v\|_{L_2(I)}, \tag{32}$$

and, applying the Hölder inequality, we find that

$$\begin{aligned} & \| \|u_x\|_{L_2(I)}^{p/2} \|v\|_{L_2(I)} \|_{L_1(0,T)} \\ & \leq T^{(4-p)/4} \operatorname{ess\,sup}_{t \in [0,T]} \left[\|u(t, \cdot)\|_{L_2(I)}^{p/2} \|v(t, \cdot)\|_{L_2(I)} \right] \|u_x\|_{L_2(Q_T)}^{p/2} \\ & \leq T^{(4-p)/4} \|u\|_{X(Q_T)}^p \|v\|_{X(Q_T)}. \end{aligned}$$

Finally,

$$\begin{aligned} \| \|u\|_{L_2(I)}^p \|v\|_{L_2(I)} \|_{L_1(0,T)} & \leq T \operatorname{ess\,sup}_{t \in [0,T]} \left[\|u(t, \cdot)\|_{L_2(I)}^p \|v(t, \cdot)\|_{L_2(I)} \right] \\ & \leq T \|u\|_{X(Q_T)}^p \|v\|_{X(Q_T)}. \end{aligned}$$

□

Lemma 8. Let $p \in [1, 2]$; then, for any functions $u, v \in X(Q_T)$,

$$\| |u|^p v \|_{L_2(Q_T)} \leq c(T^{(2-p)/4} + T^{1/2}) \|u\|_{X(Q_T)}^p \|v\|_{X(Q_T)}. \tag{33}$$

Proof. Applying estimate (32) and the Hölder inequality, we find that

$$\begin{aligned} & \| \|u_x\|_{L_2(I)}^{p/2} \|v\|_{L_2(I)} \|_{L_2(0,T)} \\ & \leq T^{(2-p)/4} \operatorname{ess\,sup}_{t \in [0,T]} \left[\|u(t, \cdot)\|_{L_2(I)}^{p/2} \|v(t, \cdot)\|_{L_2(I)} \right] \|u_x\|_{L_2(Q_T)}^{p/2} \\ & \leq T^{(2-p)/4} \|u\|_{X(Q_T)}^p \|v\|_{X(Q_T)}. \end{aligned}$$

Finally,

$$\begin{aligned} \| \|u\|_{L_2(I)}^p \|v\|_{L_2(I)} \|_{L_2(0,T)} & \leq T^{1/2} \operatorname{ess\,sup}_{t \in [0,T]} \left[\|u(t, \cdot)\|_{L_2(I)}^p \|v(t, \cdot)\|_{L_2(I)} \right] \\ & \leq T^{1/2} \|u\|_{X(Q_T)}^p \|v\|_{X(Q_T)}. \end{aligned}$$

□

Lemma 9. Let $p \in [1, 2]$; then, for any functions $u, v, w \in X(Q_T)$,

$$\| |u|^{p-1} v w_x \|_{L_1(0,T;L_2(I))} \leq c(T^{(2-p)/4} + T^{1/2}) \|u\|_{X(Q_T)}^{p-1} \|v\|_{X(Q_T)} \|w\|_{X(Q_T)}. \tag{34}$$

Proof. Applying interpolating inequality (8), we find that

$$\begin{aligned} \| |u|^{p-1} v w_x \|_{L_2(I)} & \leq \|u\|_{L_\infty(I)}^{p-1} \|v\|_{L_\infty(I)} \|w_x\|_{L_2(I)} \\ & \leq c \left(\|u_x\|_{L_2(I)}^{(p-1)/2} \|u\|_{L_2(I)}^{(p-1)/2} + \|u\|_{L_2(I)}^{p-1} \right) \left(\|v_x\|_{L_2(I)}^{1/2} \|v\|_{L_2(I)}^{1/2} + \|v\|_{L_2(I)} \right) \|w_x\|_{L_2(I)}. \end{aligned}$$

Here, because of the restriction on p ,

$$1 - \frac{p-1}{4} - \frac{1}{4} - \frac{1}{2} = \frac{2-p}{4} \geq 0$$

and, applying the Hölder inequality, we find that

$$\begin{aligned} & \left\| \|u_x\|_{L_2(I)}^{(p-1)/2} \|v_x\|_{L_2(I)}^{1/2} \|w_x\|_{L_2(I)} \|u\|_{L_2(I)}^{(p-1)/2} \|v\|_{L_2(I)}^{1/2} \right\|_{L_1(0,T)} \\ & \leq T^{(2-p)/4} \operatorname{ess\,sup}_{t \in [0,T]} \left[\|u(t, \cdot)\|_{L_2(I)}^{(p-1)/2} \|v(t, \cdot)\|_{L_2(I)}^{1/2} \right] \|u_x\|_{L_2(Q_T)}^{(p-1)/2} \|v_x\|_{L_2(Q_T)}^{1/2} \|w_x\|_{L_2(Q_T)} \\ & \leq T^{(2-p)/4} \|u\|_{X(Q_T)}^{p-1} \|v\|_{X(Q_T)} \|w\|_{X(Q_T)}. \end{aligned}$$

Finally,

$$\begin{aligned} & \left\| \|u\|_{L_2(I)}^{p-1} \|v\|_{L_2(I)} \|w_x\|_{L_2(I)} \right\|_{L_1(0,T)} \\ & \leq T^{1/2} \operatorname{ess\,sup}_{t \in [0,T]} \left[\|u(t, \cdot)\|_{L_2(I)}^{p-1} \|v(t, \cdot)\|_{L_2(I)} \right] \|w_x\|_{L_2(Q_T)} \\ & \leq T^{1/2} \|u\|_{X(Q_T)}^{p-1} \|v\|_{X(Q_T)} \|w\|_{X(Q_T)}. \end{aligned}$$

□

Proof of existence part of Theorem 1. For a function $v \in X(Q_T)$, set

$$\begin{aligned} f_{00}(t, x; v) & \equiv f(t, x) - \lambda |v|^{p_0} v, & f_{01}(t, x; v) & \equiv i\gamma |v|^{p_1} v_x, & (35) \\ f_0(t, x; v) & \equiv f_{00}(t, x; v) + f_{01}(t, x; v), & f_1(t, x; v) & \equiv i(\beta + \gamma) |v|^{p_1} v & (36) \end{aligned}$$

and consider the corresponding controllability problem for Equation (20). Lemmas 7–9 provide that $f_0 \in L_1(0, T; L_2(I))$, $f_1 \in L_2(Q_T)$. Then, Theorem 2 implies that there exists a function $h \in L_2(0, T)$ and the corresponding unique solution $u \in X(Q_T)$ to problem (20), (2), (3), verifying condition (4). Therefore, on the space $X(Q_T)$, one can define a map Θ , where $u = \Theta v$ is given by formula (30). Moreover, according to (31),

$$\|f_{00}(\cdot, \cdot; v)\|_{L_1(0,T;L_2(I))} \leq \|f\|_{L_1(0,T;L_2(I))} + c(T) \|v\|_{X(Q_T)}^{p_0+1} \tag{37}$$

and according to (33) and (34),

$$\|f_{01}(\cdot, \cdot; v)\|_{L_1(0,T;L_2(I))}, \|f_1(\cdot, \cdot; v)\|_{L_2(Q_T)} \leq c(T) \|v\|_{X(Q_T)}^{p_1+1}. \tag{38}$$

Apply Lemma 6; then, inequality (22) and formula (30) imply that

$$\|\Theta v\|_{X(Q_T)} \leq c(T)c_0 + c(T) \left(\|v\|_{X(Q_T)}^{p_0+1} + \|v\|_{X(Q_T)}^{p_1+1} \right), \tag{39}$$

where the value of c_0 is given by (7).

Next, for any functions $v_1, v_2 \in X(Q_T)$,

$$|f_{00}(t, x; v_1) - f_{00}(t, x; v_2)| \leq c(|v_1|^{p_0} + |v_2|^{p_0}) |v_1 - v_2|, \tag{40}$$

$$\begin{aligned} |f_{01}(t, x; v_1) - f_{01}(t, x; v_2)| & \leq c(|v_1|^{p_1} + |v_2|^{p_1}) |v_{1x} - v_{2x}| \\ & \quad + c(|v_1|^{p_1-1} + |v_2|^{p_1-1}) (|v_{1x}| + |v_{2x}|) |v_1 - v_2|, \end{aligned} \tag{41}$$

$$|f_1(t, x; v_1) - f_1(t, x; v_2)| \leq c(|v_1|^{p_1} + |v_2|^{p_1}) |v_1 - v_2|, \tag{42}$$

therefore, similarly to (37)

$$\|f_{00}(\cdot, \cdot; v_1) - f_{00}(\cdot, \cdot; v_2)\|_{L_1(0,T;L_2(I))} \leq c(T) (\|v_1\|_{X(Q_T)}^{p_0} + \|v_2\|_{X(Q_T)}^{p_0}) \|v_1 - v_2\|_{X(Q_T)},$$

and similarly to (38)

$$\|f_{01}(\cdot, \cdot; v_1) - f_{01}(\cdot, \cdot; v_2)\|_{L_1(0,T;L_2(I))}, \|f_1(\cdot, \cdot; v_1) - f_1(\cdot, \cdot; v_2)\|_{L_2(Q_T)} \leq c(T) (\|v_1\|_{X(Q_T)}^{p_1} + \|v_2\|_{X(Q_T)}^{p_1}) \|v_1 - v_2\|_{X(Q_T)}.$$

Since

$$\Theta v_1 - \Theta v_2 = S(0, 0, 0, 0, f_0(t, x; v_1) - f_0(t, x; v_2), f_1(t, x; v_1) - f_1(t, x; v_2)) - (S_0 \circ \Gamma) \left(S(0, 0, 0, 0, f_0(t, x; v_1) - f_0(t, x; v_2), f_1(t, x; v_1) - f_1(t, x; v_2)) \right),$$

it follows similarly to (39) that

$$\|\Theta v_1 - \Theta v_2\|_{X(Q_T)} \leq c(T) \left(\|v_1\|_{X(Q_T)}^{p_0} + \|v_1\|_{X(Q_T)}^{p_1} + \|v_2\|_{X(Q_T)}^{p_0} + \|v_2\|_{X(Q_T)}^{p_1} \right) \|v_1 - v_2\|_{X(Q_T)}. \tag{43}$$

Now choose $r > 0$ such that

$$r^{p_0} + r^{p_1} \leq \frac{1}{4c(T)}$$

and then $\delta > 0$ such that

$$\delta \leq \frac{r}{2c(T)}.$$

Then it follows from (39) and (43) that on the ball $\bar{X}_r(Q_T)$, the map Θ is a contraction. Its unique fixed point $u \in X(Q_T)$ is the desired solution. \square

The contraction principle used in the previous proof ensures the uniqueness of the solution u only in the ball $\bar{X}_r(Q_T)$. The next lemma provides uniqueness in the whole space $X(Q_T)$, which finishes the proof of Theorem 1.

Lemma 10. *A weak solution to problem (1)–(3) is unique in the space $X(Q_T)$ if $p_0 \in [1, 4]$, $p_1 \in [1, 2]$.*

Proof. Let $u, \tilde{u} \in X(Q_T)$ be two weak solutions to the same problem (1)–(3). Denote $w \equiv u - \tilde{u}$; then, the function $w \in X(Q_T)$ is the weak solution to the problem of the (9) and (10) type for $f \equiv f_0 - f_{1x}$, where

$$f_0(t, x) \equiv f_{00}(t, x; u) - f_{00}(t, x; \tilde{u}) + f_{01}(t, x; u) - f_{01}(t, x; \tilde{u}),$$

$$f_1(t, x) \equiv f_1(t, x; u) - f_1(t, x; \tilde{u}),$$

given by formulas (35) and (36), similarly to the previous proof $f_0 \in L_1(0, T; L_2(I))$, $f_1 \in L_2(Q_T)$. Then, the corresponding equality (13) in the case $\rho(x) \equiv 1 + x$ yields that

$$\frac{d}{dt} \int_0^R (1+x)|w(t, x)|^2 dx + 3 \int_0^R |w_x|^2 dx = b \int_0^R |w|^2 dx + 2a \Im \int_0^R w_x \bar{w} dx + 2 \Im \int_0^R (1+x)f_0 \bar{w} dx + 2 \Im \int_0^R f_1 ((1+x)\bar{w}_x + \bar{w}) dx. \tag{44}$$

To estimate the last two integrals on the right-hand side of (44), apply inequalities (40)–(42). Then,

$$\left| \int_0^R (1+x)(f_{00}(t,x;u) - f_{00}(t,x;\tilde{u}))\bar{w} dx \right| \leq c \sup_{x \in I} (|u|^{p_0} + |\tilde{u}|^{p_0}) \int_0^R |w|^2 dx,$$

$$\begin{aligned} \left| \int_0^R (1+x)(f_1(t,x;u) - f_1(t,x;\tilde{u}))\bar{w}_x dx \right| &\leq c \sup_{x \in I} (|u|^{p_1} + |\tilde{u}|^{p_1}) \int_0^R |w w_x| dx \\ &\leq \varepsilon \int_0^R |w_x|^2 dx + c(\varepsilon) \sup_{x \in I} (|u|^{2p_1} + |\tilde{u}|^{2p_1}) \int_0^R |w|^2 dx, \end{aligned}$$

where $\varepsilon > 0$ can be chosen arbitrarily small,

$$\begin{aligned} \left| \int_0^R (1+x)(f_{01}(t,x;u) - f_{01}(t,x;\tilde{u}))\bar{w} dx \right| &\leq c \sup_{x \in I} (|u|^{p_1} + |\tilde{u}|^{p_1}) \int_0^R |w w_x| dx \\ &\quad + c \sup_{x \in I} [(|u|^{p_1-1} + |\tilde{u}|^{p_1-1}) |w|] \int_0^R (|u_x| + |\tilde{u}_x|) |w| dx, \end{aligned}$$

where the first term on the right-hand side is already estimated above, while the second one does not exceed

$$\begin{aligned} &c \left(\int_0^R |w_x|^2 dx \right)^{1/4} \operatorname{ess\,sup}_{x \in I} (|u|^{p_1-1} + |\tilde{u}|^{p_1-1}) \left(\int_0^R (|u_x|^2 + |\tilde{u}_x|^2) dx \right)^{1/2} \\ &\quad \times \left(\int_0^R |w|^2 dx \right)^{3/4} \leq \varepsilon \int_0^R |w_x|^2 dx + c(\varepsilon) \left[\operatorname{ess\,sup}_{x \in I} (|u|^{4(p_1-1)} + |\tilde{u}|^{4(p_1-1)}) \right. \\ &\quad \left. + \int_0^R (|u_x|^2 + |\tilde{u}_x|^2) dx \right] \int_0^R |w|^2 dx \end{aligned}$$

(here, estimate (8) is used in the case of the space $H_0^1(I)$). Note that according to (8), $u, \tilde{u} \in L_4(0, T; L_\infty(I))$. Then, since $p_0, 2p_1, 4(p_1 - 1) \leq 4$, it follows from (44) that

$$\frac{d}{dt} \int_0^R (1+x)|w(t,x)|^2 dx \leq \omega(t) \int_0^R (1+x)|w(t,x)|^2 dx,$$

for a certain function $\omega \in L_1(0, T)$. The application of the Gronwall lemma yields that $w \equiv 0$. \square

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