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Fixed Point Results in Generalized Menger Probabilistic Metric Spaces with Applications to Decomposable Measures

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Abstract: The aim of this paper is to give some fixed point results in generalized Menger probabilistic metric spaces. Moreover, some nontrivial examples are presented to illustrate the superiority of the obtained results. In addition, several interesting applications are given to show that our results are meaningful and valuable.

Keywords: fixed point; probabilistic metric space; probabilistic Banach q-contraction; triangular norm; decomposable measure

MSC: 47H10; 54E25

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1. Introduction and Preliminaries

Fixed point theory is a vital branch of nonlinear analysis. It has also been used extensively in the study of all kinds of scientific problems, such as fractional differential equations, stochastic operator theory, engineering mathematics, dynamical systems, physics, computer science, models in economy and related areas (see [1,2]). One of the most important results in fixed point theory is the Banach contraction principle [3], which is used in metric spaces. Nowadays, with the indefatigable efforts of several generations, it has been generalized to many other spaces, such as fuzzy metric space, Menger space, b-metric space, probabilistic metric space and so on (see [2,4–12]). It is worth mentioning that a generalization of this principle in the context of probabilistic metric spaces was performed by Ćirić [13], where quasi-contractive mappings were introduced and the triangle norm τ_m was used. In recent years, the pioneering fixed point theorem of Sehgal and Bharucha-Reid [14,15] is a strong incentive and motivation for the further development of the principle on probabilistic metric spaces. There are plenty of papers (see [12,16–21]) motivated by the above results. On the other hand, the theory of probabilistic metric spaces is the first area where the triangular norm plays a significant role. Therein, the concept of triangle norm was first introduced by Menger in [22] by initiating it from the basic triangle inequality. The original set of axioms is the content and core known today as triangle conorms, and it is necessary to make some changes. Schweeizer and Sklar [23] gave the final definition of triangular norms. In modern society, triangular norms have been affirmed to be an important operation in several fields as well, such as fuzzy logic theory, general measure theory, differential equation theory and so on.

Generally speaking, fixed point theorems in the framework of probabilistic metric spaces are interesting for two reasons: how much we can "relax" the contractive condition without "narrowing" the class of triangular norms too much, and vice versa. It is well-known that when using the minimum triangular norm, the contractive condition is the most "relaxed". However, because the minimum triangular norm is the strongest, fixed point theorem with such a strong condition on the triangular norm is the least interesting. In addition, most fixed point theorems that have been proven in metric spaces can be "translated" to probabilistic metric spaces if the triangular norm minimum is used. That is why it is a huge challenge for finding the "optimal relationship" between the contractive condition and the choice of the triangular norm with a potential additional condition on the probabilistic distribution function itself.

Based on the above statements, in this paper, we propose an additional condition both on the metric itself and on the triangular norm in probabilistic metric spaces. Many corollaries, examples and applications are also shown.

First of all, for the sake of readers, in what follows we recall some notions and known results.

Definition 1 ([24])**.** *The mapping $\tau : [0, 1]^2 \to [0, 1]$ is a triangular norm if, for all $a, b, c, d \in [0, 1]$, the following conditions are satisfied:*
(τ_1) $\tau(a, 1) = a$;
(τ_2) $\tau(a, b) = \tau(b, a)$;
(τ_3) $a \geq c, b \geq d \Rightarrow \tau(a, b) \geq \tau(c, d)$.

Basic examples of triangular norms are as follows:

$$\tau_m(a, b) = \min\{a, b\},\ \tau_p(a, b) = a \cdot b,\ \tau_l(a, b) = \max\{a + b - 1, 0\}.$$

In 1983, Schweizer and Sklar introduced the concept of triangular conorm as dual operations to the triangular norm.

Definition 2 ([23])**.** *The mapping $\zeta : [0, 1]^2 \to [0, 1]$ is a triangular conorm if, for all $a, b, c \in [0, 1]$, the following conditions are satisfied:*
(ζ_1) $\zeta(a, b) = \zeta(b, a)$;
(ζ_2) $\zeta(a, \zeta(b, c)) = \zeta(\zeta(a, b), c)$;
(ζ_3) $\zeta(a, b) \leq \zeta(a, c)$ *for* $b \leq c$.

The connection between triangular norm and triangular conorm is given by the following result.

Proposition 1 ([24])**.** *The function $\zeta : [0, 1]^2 \to [0, 1]$ is a triangular conorm if and only if there is a triangular norm τ such that for each $(a, b) \in [0, 1]^2$, $\zeta(a, b) = 1 - \tau(1 - a, 1 - b)$.*

The opposite statement is also true. Basic examples of triangular conorms are the following ones:

$$\zeta_m(a, b) = \max\{a, b\},\ \zeta_p(a, b) = a + b - ab,\ \zeta_l(a, b) = \min\{a + b, 1\}.$$

Triangular norms of h-type (see [5]) represent a very important class, especially in the theory of fixed point.

Definition 3. *Let τ be a triangular norm and $\tau_n : [0, 1] \to [0, 1]$ a mapping defined in the following way:*

$$\tau_1(a) = \tau(a, a),\ \tau_{n+1}(a) = \tau(\tau_n(a), a),\ n \in \mathbb{N},\ a \in [0, 1].$$

A triangular norm τ is h-type if the family $\{\tau_n(a)\}_{n\in\mathbb{N}}$ is equi-continuous at the point $a = 1$, that is, if for every $\theta \in (0, 1)$ there exists $\rho(\theta) \in (0, 1)$ such that $a > 1 - \rho(\theta)$ implies $\tau_n(a) > 1 - \theta$ for every $n \in \mathbb{N}$.

Using Definition 3, for every $(a_1, a_2, \ldots, a_n) \in [0, 1]^n$, we have

$$\tau_{i=1}^{1} a_i = a_1,\ \tau_{i=1}^{n} a_i = \tau(\tau_{i=1}^{n-1} a_i, a_n) = \tau(a_1, \ldots, a_n).$$

Because the sequence $\{\tau_{i=1}^{n} a_i\}_{n\in\mathbb{N}}$ is non-decreasing and bounded from below, we obtain

$$\tau_{i=1}^{\infty} a_i = \lim_{n\to\infty} \tau_{i=1}^{n} a_i$$

for every $\{a_i\}_{i\in\mathbb{N}} \in [0, 1]$. The analogous case could be applied to triangular conorms.

Proposition 2 ([11])**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$ such that $\lim_{n\to\infty} a_n = 1$ and let the triangular norm τ be h-type. Then,*

$$\lim_{n\to\infty} \tau_{i=n}^{\infty} a_i = \lim_{n\to\infty} \tau_{i=1}^{\infty} a_{n+i} = 1.$$

For some families of triangular norms τ, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} \tau_{i=n}^{\infty} a_i = 1$.

Definition 4 ([11])**.** *The triangular norm τ is called geometrically convergent if for some $a_0 \in (0, 1)$, it satisfies*

$$\lim_{n\to\infty} \tau_{i=n}^{\infty} (1 - a_0^i) = 1. \tag{1}$$

It is proved from [2] that (1) implies $\lim_{n\to\infty} \tau_{i=n}^{\infty} (1 - a^i) = 1$ for every $a \in (0, 1)$.

Proposition 3 ([11])**.** *Let τ be a triangular norm and $\psi : (0, 1] \to [0, \infty)$ a mapping. If, for some $\delta \in (0, 1)$ and all $a \in [0, 1]$, $b \in [1 - \delta, 1]$, the following is satisfied,*

$$|\tau(a, b) - \tau(a, 1)| \leq \psi(b),$$

then for every sequence $\{a_n\}_{n\in\mathbb{N}}$ in $[0, 1]$ such that $\lim_{n\to\infty} a_n = 1$, the following implication is valid:

$$\sum_{n=1}^{\infty} \psi(a_n) < \infty \Rightarrow \lim_{n\to\infty} \tau_{i=n}^{\infty} (a_i - a_n) = 0.$$

Definition 5 ([11])**.** *The triangular norm τ is strict if it is continuous and strictly monotone, i.e., if $\tau(a, b) < \tau(a, c)$ whenever $a, b, c \in (0, 1)$ and $b < c$.*

A simple example of a strict triangle norm is $\tau = \tau_p$.

Example 1. *The Dombi, Aczél–Alsina and Sugeno–Weber families of triangular norms are defined as follows:*

$$\text{(i)}\quad \tau_\lambda^D(a, b) = \begin{cases} \tau_d(a, b), & \lambda = 0; \\ \tau_m(a, b), & \lambda = \infty; \\ \dfrac{1}{1 + \left(\left(\frac{1-a}{a}\right)^\lambda + \left(\frac{1-b}{b}\right)^\lambda\right)^{1/\lambda}}, & \lambda \in (0, \infty). \end{cases}$$

$$\text{(ii)}\quad \tau_\lambda^{AA}(a, b) = \begin{cases} \tau_d(a, b), & \lambda = 0; \\ \tau_m(a, b), & \lambda = \infty; \\ e^{-((-\log a)^\lambda + (-\log b)^\lambda)^{1/\lambda}}, & \lambda \in (0, \infty). \end{cases}$$

$$\text{(iii)}\quad \tau_\lambda^{SW}(a, b) = \begin{cases} \tau_d(a, b), & \lambda = -1; \\ \tau_p(a, b), & \lambda = \infty; \\ \max(0, \frac{a+b-1+\lambda ab}{1+\lambda}), & \lambda \in (-1, \infty). \end{cases}$$

where $\tau_d(a, b) = \tau_m(a, b)$ if $\max(a, b) = 1$ and $\tau_d(a, b) = 0$, otherwise.

The following proposition is given in [11].

Proposition 4. *Let $(\tau_\lambda^D)_{\lambda\in(0,\infty)}$, $(\tau_\lambda^{AA})_{\lambda\in(0,\infty)}$ and $(\tau_\lambda^{SW})_{\lambda\in(-1,\infty]}$ be Dombi, Aczél–Alsina and Sugeno–Weber families of triangular norms, respectively, and $\{a_n\}_{n\in\mathbb{N}}$ a sequence in $(0, 1]$ such that $\lim_{n\to\infty} a_n = 1$. Then, the following equivalences hold:*

(a) $\sum_{i=1}^{\infty}(1-a_i)^\lambda < \infty \Longleftrightarrow \lim_{n\to\infty}(\tau_\lambda^\star)_{i=n}^\infty a_i = 1$, *where* $\star \in \{D, AA\}$;

(b) $\sum_{i=1}^{\infty}(1-a_i) < \infty \Longleftrightarrow \lim_{n\to\infty}(\tau_\lambda^{SW})_{i=n}^\infty a_i = 1.$

Definition 6 ([5])**.** *The continuous triangular norm $\tau : [0, 1]^2 \to [0, 1]$ is said to be an Archimedean triangle norm if $\tau(a, a) < a$, for every $a \in (0, 1)$.*

Theorem 1 ([11])**.** *(a) The function $\tau : [0,1]^2 \to [0,1]$ is a continuous Archimedean triangular norm if and only if there exists a continuous and strictly decreasing function $\mathbf{a} :[0,1] \to [0,\infty)$ called an additive generator of τ with $\mathbf{a}(1) = 0$ and $\tau(a,b) = \mathbf{a}^{-1}(\min\{\mathbf{a}(a)+\mathbf{a}(b),\mathbf{a}(0)\})$, for any $a, b \in [0,1]$.*

(b) The function $\tau : [0,1]^2 \to [0,1]$ is a continuous Archimedean triangular norm if and only if there exists a continuous and strictly increasing function $\mathbf{m} :[0,1] \to [0,1]$ called a multiplicative generator of τ with $\mathbf{m}(1) = 1$ and $\tau(a,b) = \mathbf{m}^{-1}(\max\{\mathbf{m}(a)\cdot\mathbf{m}(b),\mathbf{m}(0)\})$, for any $a, b \in [0,1]$.

Remark 1. *Triangular norm τ is strict if and only if $\mathbf{m}(0) = 0$.*

Proposition 5 ([11])**.** *Let τ be a strict triangular norm with an additive generator $\mathbf{a}$ and the corresponding multiplicative generator θ. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} a_n = 1$. Then,*

(a) $\lim_{n\to\infty}\sum_{i=n}^\infty \mathbf{a}(a_i) = 0,$

(b) $\lim_{n\to\infty}\prod_{i=n}^\infty \mathbf{m}(a_i) = 1$

hold if and only if $\lim_{n\to\infty}\tau_{i=n}^\infty a_i = 1$.

Definition 7 ([23])**.** *Let Ω be a nonempty set and Δ^+ be the set of all distribution functions. Suppose that $\mathcal{F}_{pm} : \Omega\times\Omega \to \Delta^+$ is a mapping and $\mathcal{F}_{pm}(p, q) = f_{p,q}$ for each $(p, q) \in \Omega\times\Omega$. The ordered pair $(\Omega, \mathcal{F}_{pm})$ is called a probabilistic metric space if the following conditions are satisfied:*

(pm_0) $f_{p,q}(0) = 0$, *for all* $p, q \in \Omega$;
(pm_1) $f_{p,q}(t) = f_{q,p}(t)$, *for all* $p, q \in \Omega,\ t > 0$;
(pm_2) $f_{p,q}(t) = 1$ *if and only if* $p = q$ *for all* $p, q \in \Omega,\ t > 0$;
(pm_3) $f_{p,q}(t) = 1$ *and* $f_{q,r}(s) = 1$ *imply* $f_{p,r}(t+s) = 1$, *for all* $p, q, r \in \Omega$ *and all* $t, s > 0$.

Definition 8. *Let $(\Omega, \mathcal{F}_{pm})$ be a probabilistic metric space. The sequence $\{p_n\}_{n\in\mathbb{N}}$ from Ω is called a Cauchy sequence if for every $t > 0$ and $\omega \in (0, 1)$, there exists $n_0(t, \omega) \in \mathbb{N}$ such that $f_{p_{n+m}, p_n}(t) > 1-\omega$, for each $n \geq n_0(t, \omega)$ and each $m \in \mathbb{N}$.*

If the probabilistic metric space $(\Omega, \mathcal{F}_{pm})$ is such that every Cauchy sequence in Ω converges to Ω, then $(\Omega, \mathcal{F}_{pm})$ is called a complete space.

Definition 9. *Let $(\Omega, \mathcal{F}_{pm})$ be a probabilistic metric space and τ a triangular norm. The ordered triple $(\Omega, \mathcal{F}_{pm}, \tau)$ is called a generalized Menger probabilistic metric space if the following inequality is satisfied:*

(pm_4) $f_{p,r}(t+s) \geq \tau(f_{p,q}(t), f_{q,r}(s))$ *for all* $p, q, r \in \Omega$ *and all* $t, s > 0$.

Definition 10. *If $(\Omega, \mathcal{F}_{pm}, \tau)$ is a complete Menger probabilistic metric space with a continuous triangular norm τ, then Ω is called a Hausdorff topological space with topology induced by the family (ε, λ)-environment*

$$\mathcal{O} = \{O_p(\varepsilon, \lambda) : p \in \Omega, \varepsilon > 0, \lambda \in (0, 1)\},$$

wherein

$$O_p(\varepsilon, \lambda) = \{x \in S : f_{x,p}(\varepsilon) > 1 - \lambda\}.$$

Remark 2. *If $\sup\limits_{a<1} \tau(a, a) = 1$, then the family $\{\mathcal{O}\}$ defined on Ω is a metrizable topology.*

One of the most important generalizations of probabilistic metric spaces is represented by fuzzy metric space which was introduced by Kramosil and Michalek [25]. They defined the notion of fuzzy metric space using the notion of a fuzzy number and gave a connection between probabilistic metric spaces and fuzzy metric spaces.

The fuzzy number u is the mapping of $u : \mathbb{R} \to [0, 1]$. We say that the fuzzy number u is normal if there exists $t_0 \in \mathbb{R}$ such that $u(t_0) = 1$, and it is convex if for each $t_1, t_2 \in \mathbb{R}$ and for each $\mu \in [0, 1]$ the following is satisfied:

$$u(\mu t_1 + (1 - \mu)t_2) \geq \min(u(t_1), u(t_2)).$$

By $\mathcal{E}$, we denote all fuzzy numbers that satisfy the condition that they are semi-continuous, normal and convex from above, where

$$\mathcal{E}^+ = \{u : u \in \mathcal{E}, u(t) = 0, \text{ for all } t < 0\}.$$

For the fuzzy number u, the α-cutting level is defined as follows, $[u]_\alpha = \{t : t \in \mathbb{R}, u(t) \geq \alpha\}$ $(\alpha \in (0, 1])$, where $[a^\alpha, b^\alpha]$ is a nonempty closed interval if $a^\alpha, b^\alpha \in \mathbb{R}$ and semi-open interval $(-\infty, b^\alpha]$, $[a^\alpha, \infty)$ if $a^\alpha = -\infty$ or $b^\alpha = +\infty$.

Let $L, R : [0, 1] \times [0, 1] \to [0, 1]$ be symmetric, non-decreasing in both argument functions, satisfying $L(0, 0) = 0$ and $R(1, 1) = 1$. Let Ω be a nonempty set, $d : \Omega^2 \to \mathcal{E}^+$ a mapping satisfying for each $\alpha \in (0, 1]$, and for $(k, i) \in \Omega^2$, one has

$$[d(k, i)]_\alpha = [\lambda_\alpha(k, i), \rho_\alpha(k, i)], \alpha \in (0, 1].$$

Fuzzy metric space is defined as an ordered quadruple (Ω, d, L, R) where d is the fuzzy metric if the following conditions are satisfied:

(f_1) $d(k, i) = I_{\{0\}} \iff k = i$, for each $k, i \in \Omega$;
(f_2) $d(k, i) = d(i, k)$, for each $k, i \in \Omega$;
(f_{2a}) $d(k, z)(s + t) \geq L(d(k, i)(s), d(i, z)(t))$, whenever $s \leq \lambda_1(k, i)$, $t \leq \lambda_1(i, z)$ and $s + t \leq \lambda_1(k, ,z)$, for each $k, i, z \in \Omega$;
(f_{2b}) $d(k, z)(s + t) \leq R(d(k, i)(s), d(i, z)(t))$, whenever $s \geq \lambda_1(k, i)$, $t \geq \lambda_1(i, z)$ and $s + t \geq \lambda_1(k, z)$, for each $k, i, z \in \Omega$.

Every general Menger space $(\Omega, \mathcal{F}_{pm}, \tau)$ is also a fuzzy metric space (Ω, d, L, R) if

$$d(k, i)(u) = \begin{cases} 0, & u < \sup\{s : F_{k,i}(s) = 0\} = u_{k,i}, \\ 1 - F_{k,i}(u), & u \geq u_{k,i}. \end{cases}$$

The functions R and L are defined as follows:

$$L \equiv 0, \quad R(a, b) = 1 - \tau(1 - a, 1 - b) \; (a, b \in [0, 1]).$$

If

$$\lim_{u \to \infty} d(k, i)(u) = 0 \text{ for every } k, i \in \Omega,$$

then $(\Omega, \mathcal{F}_{pm}, \tau)$ is a Menger space, where $\tau(a, b) = 1 - R(1 - a, 1 - b)$, for each $a, b \in [0, 1]$, and the mapping $\mathcal{F}_{pm}$ is defined by

$$f_{k,i}(s) = \begin{cases} 0, & s < \lambda_1(k, i), \\ 1 - d(k, i)(s), & s \geq \lambda_1(k, i), \end{cases}$$

where $k, i \in \Omega$, $s \in \mathbb{R}$.

The following lemma from [10] gives the connection between fuzzy metric spaces and probabilistic metric spaces.

Lemma 1. *Let (Ω, d, L, R) be a fuzzy metric space, $\nu : [0, 1] \to [0, 1]$ a continuous, monotonically decreasing function such that $\nu(1) = 0$ and $\nu(0) = 1$ and d and R satisfy the following conditions:*

(a) $\lim_{s\to\infty} d(k, i)(s) = 0$, *for each $k, i \in \Omega$;*
(b) $R(a, 1) = 1$, $R(a, 0) = a$, *for each $a \in [0, 1]$;*
(c) *R is associative.*

Then, $(\Omega, \mathcal{F}_{pm}, \tau)$ is a generalized Menger space, where $\mathcal{F}_{pm}$ and τ are defined as follows:

$$f(k, i)(s) = \begin{cases} 0, & \text{if } s < \lambda_1(k, i), \\ \nu^{-1}[d(k, i)(s)], & \text{if } s \geq \lambda_1(k, i), \end{cases}$$

$$\tau(a, b) = \nu^{-1}[R(f(a), f(b))], \quad (a, b \in [0, 1]).$$

As we know, one of the most important results from fixed point theory is Banach's contraction principle in metric spaces:

Each Banach q-contraction $\xi : \mathcal{M} \to \mathcal{M}$ in the complete metric space $(\mathcal{M}, d_m)$ has a unique fixed point.

Sehgal and Bharucha-Reid generalized the concept of the Banach q-contraction in the framework of probabilistic metric spaces.

Definition 11 ([15])**.** *Let $(\Omega, \mathcal{F}_{pm})$ be a probabilistic metric space. The mapping $\xi : \Omega \to \Omega$ is said to be a probabilistic Banach q-contraction if there exists $q \in (0, 1)$ such that*

$$f_{\xi p, \xi q}(t) \geq f_{p,q}\left(\frac{t}{q}\right) \tag{2}$$

for each $p, q \in \Omega$ and $t > 0$.

2. Main Results

Once more, we emphasize the fixed point theorem of Sehgal and Bharucha-Reid [15] and its importance for fixed point investigations in the framework of probabilistic metric spaces. In the rest of this paper, it is not necessary to suppose that a triangle norm is Archimedean. It is a big challenge even today to find a weaker condition for the triangular norm than the triangular norm minimum, so that the Banach q-contraction, as well as its generalizations, are valid. In the following theorems, we give an additional condition that enables this statement.

Theorem 2. *Let $(\Omega, \mathcal{F}_{pm}, \tau)$ be a complete generalized Menger probabilistic metric space such that $\sup_{a<1} \tau(a, a) = 1$ and $\xi : \Omega \to \Omega$ be a probabilistic Banach q-contraction such that for some $p_0 \in \Omega$ and some $k > 0$, it satisfies*

$$\psi \circ f_{p_0, \xi p_0}(t) = O\left(\frac{1}{t^k}\right), \; t > 1, \tag{3}$$

where $\psi : [0, 1] \to [0, s]$ is a continuous, decreasing function such that $\psi(1) = 0$. If the triangular norm τ satisfies the condition

$$\lim_{n\to\infty} \tau_{i=n}^{\infty} \psi^{-1}((\varsigma^i)^k) = 1, \tag{4}$$

where $\varsigma \in (0, 1)$, then there is a unique fixed point z such that $z = \lim_{n\to\infty} \xi^n p_0$.

Proof. Let $p_0 \in \Omega$ satisfy condition (3) and define a sequence $p_{n+1} = \xi p_n$, $n \in \mathbb{N}_0$. Then, by (2), we have

$$f_{p_{n+1}, p_n}(t) \geq f_{p_n, p_{n-1}}\left(\frac{t}{q}\right) \geq \cdots \geq f_{\xi p_0, p_0}\left(\frac{t}{q^n}\right), \; t > 0, \; n \in \mathbb{N}.$$

Next, it is necessary to prove that $\{\xi^n p_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\delta \in (q,1)$. Because the series $\sum_{i=1}^{\infty} \delta^i$ is convergent, it follows that there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $\sum_{i=n_1}^{\infty} \delta^i \leq \varepsilon$. Then, for each $n \geq n_1$ and $m \in \mathbb{N}$, one has

$$\begin{aligned} f_{\xi^n p_0,\xi^{n+m} p_0}(\varepsilon) &\geq f_{\xi^n p_0,\xi^{n+m} p_0}\Big(\sum_{i=n}^{\infty} \delta^i\Big) \\ &\geq f_{\xi^n p_0,\xi^{n+m} p_0}\Big(\sum_{i=n}^{n+m-1} \delta^i\Big) \\ &\geq \underbrace{\tau(\tau(\dots\tau}_{(m-1)-times}(f_{\xi^n p_0,\xi^{n+1} p_0}(\delta^n),\dots,f_{\xi^{n+m-1} p_0,\xi^{n+m} p_0}(\delta^{n+m-1})) \\ &\geq \underbrace{\tau\Big(\tau\Big(\dots\tau}_{(m-1)-times}\Big(f_{p_0,\xi p_0}\Big(\frac{\delta^n}{q^n}\Big),\dots,f_{p_0,\xi p_0}\Big(\frac{\delta^{n+m-1}}{q^{n+m-1}}\Big)\Big). \end{aligned}$$

Let $\varsigma = \frac{q}{\delta} \in (0,1)$, and then

$$f_{\xi^n p_0,\xi^{n+m} p_0}(\varepsilon) \geq \underbrace{\tau\Big(\tau\Big(\dots\tau}_{(m-1)-times}\Big(f_{p_0,\xi p_0}\Big(\frac{1}{\varsigma^n}\Big),\dots f_{p_0,\xi p_0}\Big(\frac{1}{\varsigma^{n+m-1}}\Big)\Big)\Big), \ n \geq n_1, \ m \in \mathbb{N}. \tag{5}$$

By (3), there is $G > 0$ and $k > 0$ such that $\psi(f_{p_0,\xi p_0}(t)) \leq G \cdot \frac{1}{t^k}$, $t > 1$, i.e., $f_{p_0,\xi p_0}(t) \geq \psi^{-1}(G \cdot \frac{1}{t^k})$, $t > 1$. Concretely, for $t = \frac{1}{\varsigma^n} > 1$, we have

$$f_{p_0,\xi p_0}\Big(\frac{1}{\varsigma^n}\Big) \geq \psi^{-1}(G(\varsigma^n)^k), \ n \in \mathbb{N}. \tag{6}$$

Choose $n_2 \in \mathbb{N}$ such that $G\varsigma^n \in [0, s), \ n \geq n_2$. Using (5) and (6) for $n \geq \max\{n_1, n_2\}$ and $m \in \mathbb{N}$, it follows that

$$f_{\xi^n p_0,\xi^{n+m} p_0}(\varepsilon) \geq \underbrace{\tau(\tau(\dots(\tau}_{(m-1)-times}(\psi^{-1}(G\varsigma^{kn}),\ \psi^{-1}(G\varsigma^{k(n+1)}),\dots,\psi^{-1}(G\varsigma^{k(n+m-1)}))).$$

Let s_0 be a constant such that $G\varsigma^{k\cdot s_0} < \varsigma$. Then, using (5) for $n \geq \max\{n_1, n_2\}$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} f_{\xi^{n+s_0} p_0,\xi^{n+s_0+m} p_0}(\varepsilon) &\geq \underbrace{\tau(\tau(\dots(\tau}_{(m-1)-times}(\psi^{-1}(G\,\varsigma^{k(n+s_0)})),\ (\psi^{-1}(G\,\varsigma^{k(n+s_0+1)})),\dots,(\psi^{-1}(G\,\varsigma^{k(n+s_0+m-1)})) \\ &\geq \tau_{i=n+1}^{\infty}\psi^{-1}((\varsigma^i)^k). \end{aligned}$$

Based on the condition (4), we conclude that $\{\xi^n p_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $z = \lim_{n\to\infty} \xi^n p_0$. We want to show that $z = \xi z$. Using (2), we have

$$f_{\xi p_n,\xi z}(t) \geq f_{p_n,z}\Big(\frac{t}{q}\Big).$$

Letting $n \to \infty$, we conclude that $f_{z,\xi z}(t) \geq f_{\xi z,z}(\frac{t}{q})$, and so $z = \xi z$.

Finally, we prove the uniqueness of fixed point. Indeed, suppose that there exists $v \neq z$ such that $v = \xi v$. Then, from (2), we have

$$f_{z,v}(t) = f_{\xi z, \xi v}(t) \geq f_{z,v}\left(\frac{t}{q}\right),$$

which follows that $z = v$. $\square$

Remark 3. *In [26], the following statement was made: if condition (4) holds for some $\varsigma_0 \in (0,1)$, then it holds for every $\varsigma \in (0,1)$.*

Remark 4. *Theorem 2 follows via the following condition (which treats triangular norms and distribution functions mutually),*

$$\lim_{n\to\infty} \tau_{i=n}^{\infty} f_{p,r}(t_n) = 1,$$

for every s-increasing sequence $\{t_n\}$ and every $p, r \in \Omega$, instead of conditions (3) and (4). It seems that condition (4) is more appropriate to deal with different types of triangular norms which is one of the main goals of the current paper.

Let $\psi(x) = 1 - x$. It is easy to check that condition (3) holds for distribution functions of half-normally or exponentially distributed random variables.

Example 2. *Let $\Omega = [0,1]$, $\xi(p) = \dfrac{p}{a}$, $a > 1$, $f_{p,r}(t) = \dfrac{t}{t + |p - r|}$, $p, r \in \Omega$, $t > 0$, $\tau = \tau_p$ and $\psi(x) = 1 - x$. In view of*

$$f_{\xi p, \xi r}(t) = \frac{t}{t + \frac{1}{a}|p - r|} \geq \frac{t}{t + q|p - r|} = f_{p,r}\left(\frac{t}{q}\right),\ p,\ r \in \Omega,\ t > 0,$$

it follows that ξ is a probabilistic Banach q-contraction with $q \in [\frac{1}{a}, 1)$.

Condition (3) is fulfilled because of

$$\psi(f_{p,\xi p}(t)) = \frac{p}{\frac{a}{a-1}\, t^k + p} \leq \frac{1}{t^k},\ t > 1,\ k > 0.$$

Let $a_i = \psi^{-1}((\varsigma^i)^k) = 1 - (\varsigma^i)^k$. Then, $\sum\limits_{i=1}^{\infty}(1 - a_i) < \infty$, and because the following equivalence holds,

$$\prod_{i=1}^{\infty} a_i > 0 \Longleftrightarrow \lim_{n\to\infty} \prod_{i=n}^{\infty} a_i = 1 \Longleftrightarrow \sum_{i=1}^{\infty}(1 - a_i) < \infty,$$

we conclude that all conditions of the previous theorem are satisfied and hence $p = 0$ is the unique fixed point of ξ.

Corollary 1. *Let $(\Omega, \mathcal{F}_{pm}, \tau)$ be a complete generalized Menger probabilistic metric space such that $\sup_{a<1} \tau(a, a) = 1$. Let $\xi : \Omega \to \Omega$ be a probabilistic Banach q-contraction such that for some $p_0 \in \Omega$ and some $k > 0$, it satisfies*

$$\psi \circ f_{p_0, \xi p_0}(t) = O\left(\frac{1}{t^k}\right),\ t > 1,$$

where $\psi : [0,\ 1] \to [0,\ s]$ is a continuous, decreasing function such that $\psi(1) = 0$. If $\phi : (0,\ 1] \to [0,\ \infty)$ is a function such that for some $\delta \in (0,\ 1)$ the following is satisfied,

$$|\tau(a,\ b) - \tau(a,\ 1)| \leq \phi(b),\ a \in [0,\ 1],\ b \in [1 - \delta,\ 1],$$

and $\sum_{n=1}^{\infty} \phi(\psi^{-1}((\varsigma^i)^k)) < \infty$, *for some* $\varsigma \in (0, 1)$, *then there is a unique fixed point* z *of mapping* ξ *and* $z = \lim_{n\to\infty} \xi^n p_0$.

Proof. Based on Proposition 3, condition $\sum_{n=1}^{\infty} \phi(\psi^{-1}((\varsigma^i)^k)) < \infty$ implies $\lim_{n\to\infty} \tau_{i=n}^{\infty} \psi^{-1}((\varsigma^i)^k) = 1$, and hence all conditions of the previous theorem are satisfied. □

The following corollary is a fuzzy metric version of Theorem 2.

Corollary 2. *Let* (Ω, d, L, R) *be a complete fuzzy metric space such that* $\lim_{t\to\infty} d(p, r)(t) = 0$, $p, r \in \Omega$, $R(a, 1) = 1$, $a \in [0, 1]$, $R(a, 0) = a$, $a \in [0, 1]$, *and* R *is continuous at* $(0, 0)$. *Let* $\xi : \Omega \to X$ *be a probabilistic Banach q-contraction such that for some* $p_0 \in \Omega$ *and some* $k > 0$, *it satisfies*

$$d(p_0, \xi p_0)(t) = O\left(\frac{1}{t^k}\right), \ t > 1.$$

If $\lim_{n\to\infty} R_{i=n}^{\infty}((\varsigma^k)^i) = 0$ *for* $\varsigma \in (0, 1)$, *then there exists a unique fixed point* z *of* ξ *and* $z = \lim_{n\to\infty} \xi^n p_0$.

Proof. If we choose $\psi(x) = 1 - x$, together with Lemma 1, then all conditions of Theorem 2 are satisfied. The proof is completed. □

As it is pointed out, we deal with triangular norms via condition (4) and, therefore, in the following corollaries it will be relaxed (or omitted). By Theorem 2 and Proposition 2, it follows the subsequent corollary where the triangular norms of h-type are used.

Corollary 3. *Let* $(\Omega, \mathcal{F}_{pm}, \tau)$ *be a complete generalized Menger probabilistic metric space such that* $\sup_{a<1} \tau(a, a) = 1$ *and* $\xi : \Omega \to \Omega$ *be a probabilistic Banach q-contraction such that for some* $p_0 \in \Omega$ *and some* $k > 0$, *it satisfies*

$$\psi \circ f_{p_0, \xi p_0}(t) = O\left(\frac{1}{t^k}\right), \ t > 1,$$

where $\psi : [0, 1] \to [0, s]$ *is a continuous, decreasing function such that* $\psi(1) = 0$. *If the triangular norm* τ *is of* h*-type, then there is a unique fixed point* z *of* ξ *and* $z = \lim_{n\to\infty} \xi^n p_0$.

So, in the previous corollary we deal with a triangular norm of h-type, while in the next one τ is a strict triangular norm. Note that, for example, τ_m is not strict, while τ_p, τ_l and τ^{SW} are strict triangular norms.

Corollary 4. *Let* $(\Omega, \mathcal{F}_{pm}, \tau)$ *be a complete generalized Menger probabilistic metric space such that* $\sup_{a<1} \tau(a, a) = 1$ *and the triangular norm* τ *is strict with additive generator* **a** *(multiplicative generator* **m***). Let* $\xi : \Omega \to \Omega$ *be a probabilistic Banach q-contraction and* $\psi : [0, 1] \to [0, s]$ *be a continuous, decreasing function with* $\psi(1) = 0$ *such that for some* $p_0 \in \Omega$ *and some* $k > 0$, *it satisfies*

$$\psi \circ f_{p_0, \xi p_0}(t) = O\left(\frac{1}{t^k}\right), \ t > 1.$$

If there exits $\varsigma \in (0, 1)$ *such that*

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} \mathbf{a}(\psi^{-1}((\varsigma^i)^k) = 0 \quad (\lim_{n\to\infty} \prod_{i=n}^{\infty} \mathbf{m}(\psi^{-1}((\varsigma^i)^k)) = 1),$$

then there is a unique fixed point z *of* ξ *and* $z = \lim_{n\to\infty} \xi^n p_0$.

Proof. Using Proposition 5, we conclude that all conditions of Theorem 2 are satisfied. □

Further, different contractive conditions are suggested instead of the Banach q-contraction under the same class of triangular norms. The following conditions introduced in Theorem 2 are used, with the contractive condition in the spirit of one suggested in [27].

Theorem 3. *Let $(\Omega, \mathcal{F}_{pm}, \tau)$ be a complete generalized Menger probabilistic metric space such that $\sup_{a<1} \tau(a, a) = 1$ and the mapping $\xi : \Omega \to \Omega$ satisfy the following contractive condition*

$$f_{\xi p, \xi r}(t) \geq \min\left\{ f_{p,r}\left(\frac{t}{q}\right), f_{\xi p,p}\left(\frac{t}{q}\right), f_{\xi r,r}\left(\frac{t}{q}\right) \right\},\ p,\ r \in \Omega,\ t > 0, \tag{7}$$

for some $q \in (0,1)$. Suppose that for some $p_0 \in \Omega$ and some $k > 0$, condition (3) is satisfied, where $\psi : [0, 1] \to [0, s]$ is a continuous, decreasing function such that $\psi(1) = 0$. If the triangular norm τ satisfies condition (4), then there is a unique fixed point z of mapping ξ and $z = \lim_{n\to\infty} \xi^n p_0$.

Proof. Take $p_0 \in \Omega$ determined in (3) and define a sequence $p_{n+1} = \xi p_n,\ n \in \mathbb{N}$. Then, by (7), we have

$$f_{p_{n+1},p_n}(t) \geq \min\left\{ f_{p_n,p_{n-1}}\left(\frac{t}{q}\right), f_{p_{n+1},p_n}\left(\frac{t}{q}\right), f_{p_n,p_{n-1}}\left(\frac{t}{q}\right) \right\},\ n \in \mathbb{N},\ t > 0.$$

Since $f_{p_{n+1},p_n}(t) \geq f_{p_{n+1},p_n}(\frac{t}{q}) > f_{p_{n+1},p_n}(t)$ leads to a contradiction, then we conclude that

$$f_{p_{n+1},p_n}(t) \geq f_{p_n,p_{n-1}}\left(\frac{t}{q}\right) \geq \cdots \geq f_{p_0,\xi p_0}\left(\frac{t}{q^n}\right),\ n \in \mathbb{N},\ t > 0.$$

Further, the proof that $\{\xi^n p_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence is analogous as in Theorem 2 due to the conditions (3) and (4).

Let $z = \lim_{n\to\infty} \xi^n p_0$. Suppose that $z \neq \xi z$. Using (7), we have

$$f_{\xi p_n,\xi z}(t) \geq \min\left\{ f_{p_n,z}\left(\frac{t}{q}\right), f_{p_{n+1},p_n}\left(\frac{t}{q}\right), f_{\xi z,z}\left(\frac{t}{q}\right) \right\},\ n \in \mathbb{N},\ t > 0.$$

Letting $n \to \infty$, we obtain a contradiction on account of $f_{z,\xi z}(t) \geq f_{\xi z,z}(\frac{t}{q}) > f_{z,\xi z}(t)$ and so $z = \xi z$.

Finally, we prove the uniqueness of fixed point. To this end, suppose that there exists $v \neq z$ such that $v = \xi v$. Then, from (7), we have

$$f_{z,v}(t) = f_{\xi z,\xi v}(t) \geq \min\left\{ f_{z,v}\left(\frac{t}{q}\right), f_{z,z}\left(\frac{t}{q}\right), f_{v,v}\left(\frac{t}{q}\right) \right\},\ t > 0,$$

i.e., $f_{z,v}(t) \geq f_{z,v}(\frac{t}{q})$ and so $z = v$. □

Example 3. *Let $\Omega = [0,1]$, $\xi(p) = \dfrac{p}{2}$, $f_{p,r}(t) = \mathrm{e}^{-\frac{|p-r|}{t}}$, $p, r \in \Omega, t > 0, \tau = \tau_p$ and $\psi(x) = 1 - x$. We need to check condition (7), i.e., the relation is written as*

$$\mathrm{e}^{-\frac{|p-r|}{2t}} \geq \min\left\{ \mathrm{e}^{-\frac{q|p-r|}{t}}, \mathrm{e}^{-\frac{qp}{2t}}, \mathrm{e}^{-\frac{qr}{2t}} \right\},\ p, r \in \Omega,\ t > 0.$$

Due to the symmetric role of p and r without loss of generality, we suppose that $p > r$ and split the discussion into two cases, $p \geq 2r$ and $r < p < 2r$. If $p \geq 2r$, we have

$$\mathrm{e}^{-\frac{|p-r|}{2t}} \geq \mathrm{e}^{-\frac{q|p-r|}{t}},\ t > 0,$$

while for $r < p < 2r$, it follows that

$$\mathrm{e}^{-\frac{|p-r|}{2t}} \geq \mathrm{e}^{-\frac{qp}{2t}},\ t > 0,$$

and both inequalities are correct when $q \in [\frac{1}{2}, 1)$.

Condition (3) is fulfilled because for arbitrary $p_0 \in (0,1)$ and $k > 0$, one has

$$\psi(f_{p_0,\xi p_0}(t)) = 1 - \mathrm{e}^{-\frac{p_0}{2t^k}} \leq \frac{1}{t^k},\ t > 1.$$

Let $a_i = \psi^{-1}((\varsigma^i)^k) = 1 - (\varsigma^i)^k$. Then, $\sum_{i=1}^{\infty}(1 - a_i) < \infty$, and because the following equivalence holds,

$$\prod_{i=1}^{\infty} a_i > 0 \iff \lim_{n\to\infty}\prod_{i=n}^{\infty} a_i = 1 \iff \sum_{i=1}^{\infty}(1 - a_i) < \infty,$$

we conclude that all conditions of Theorem 3 are satisfied and $p = 0$ is the unique fixed point of ξ.

Remark 5. *Condition (7) could be extended with term $f_{p,\xi r}\left(\frac{t}{q}\right)$ without changing the triangular norm, but if we want to add the symmetric term $f_{\xi p,r}\left(\frac{t}{q}\right)$ too, then the class of triangular norm must be narrowed.*

Theorem 4. *Let $(\Omega, \mathcal{F}_{pm}, \tau_m)$ be a complete Menger probabilistic metric space such that $\sup_{a<1}\tau(a,a) = 1$ and $\xi : \Omega \to \Omega$ be a mapping satisfying*

$$f_{\xi p,\xi r}(t) \geq \min\left\{f_{p,r}\left(\frac{t}{q}\right), f_{\xi p,p}\left(\frac{t}{q}\right), f_{\xi r,r}\left(\frac{t}{q}\right), f_{\xi p,r}\left(\frac{2t}{q}\right), f_{p,\xi r}\left(\frac{t}{q}\right)\right\}, \tag{8}$$

for some $q \in (0,1)$ and every $p, r \in \Omega$, $t > 0$. Suppose that for some $p_0 \in \Omega$ and some $k > 0$ condition (3) is satisfied, where $\psi : [0, 1] \to [0, s]$ is a continuous, decreasing function such that $\psi(1) = 0$. Then, there is a unique fixed point z of ξ and $z = \lim_{n\to\infty}\xi^n p_0$.

Theorem 5. *Let $(\Omega, \mathcal{F}_{pm}, \tau_p)$ be a complete Menger probabilistic metric space such that $\sup_{a<1}\tau(a,a) = 1$ and $\xi : \Omega \to \Omega$ be a mapping satisfying the following contractive condition,*

$$f_{\xi p,\xi r}(t) \geq \min\left\{f_{p,r}\left(\frac{t}{q}\right), f_{\xi p,p}\left(\frac{t}{q}\right), f_{\xi r,r}\left(\frac{t}{q}\right), f_{p,\xi r}\left(\frac{t}{q}\right), \sqrt{f_{r,\xi p}\left(\frac{2t}{q}\right)}\right\}, \tag{9}$$

for some $q \in (0,1)$ and every $p, r \in \Omega$, $t > 0$. Suppose that for some $p_0 \in \Omega$ and some $k > 0$ condition (3) is satisfied where $\psi : [0, 1] \to [0, s]$ is a continuous, decreasing function such that $\psi(1) = 0$. If the triangular norm τ satisfies

$$\lim_{n\to\infty}\sum_{i=n}^{\infty}(1 - \psi^{-1}((\varsigma^i)^k)) < \infty,$$

where $\varsigma \in (0, 1)$, then there is a unique fixed point z of ξ and $z = \lim_{n\to\infty}\xi^n p_0$.

Theorem 6. *Let $(\Omega, \mathcal{F}_{pm}, \tau)$ be a complete Menger probabilistic metric space such that $\sup_{a<1}\tau(a,a) = 1$ and $\xi : \Omega \to \Omega$ be a mapping satisfying*

$$f_{\xi p,\xi r}(t) \geq \min\left\{f_{p,r}\left(\frac{t}{q}\right), f_{\xi p,p}\left(\frac{t}{q}\right), f_{\xi r,r}\left(\frac{t}{q}\right), f_{p,\xi r}\left(\frac{t}{q}\right), \frac{1 - f_{\xi p,r}\left(\frac{t}{q}\right) + f_{p,\xi r}\left(\frac{t}{q}\right)}{f_{\xi p,r}\left(\frac{t}{q}\right)\cdot f_{p,\xi r}\left(\frac{t}{q}\right)}\right\}, \tag{10}$$

for some $q \in (0,1)$ and every $p, r \in \Omega$, $t > 0$. Suppose that for some $p_0 \in \Omega$ and some $k > 0$ condition (3) is satisfied, where $\psi : [0, 1] \to [0, s]$ is a continuous, decreasing function such that $\psi(1) = 0$. If the triangular norm τ satisfies (4), then there is a unique fixed point z of ξ and $z = \lim_{n\to\infty}\xi^n p_0$.

Proof. Let $p_0 \in \Omega$ satisfy condition (3) and let $p_{n+1} = \xi p_n$, $n \in \mathbb{N}$. By (10), for $n \in \mathbb{N}$, $t > 0$, we have

$$f_{p_{n+1},p_n}(t) \geq \min\left\{f_{p_n,p_{n-1}}\left(\frac{t}{q}\right), f_{p_{n+1},p_n}\left(\frac{t}{q}\right), f_{p_n,p_{n-1}}\left(\frac{t}{q}\right), f_{p_n,p_n}\left(\frac{t}{q}\right), \frac{2 - f_{p_{n+1},p_{n-1}}\left(\frac{t}{q}\right)}{f_{p_{n+1},p_{n-1}}\left(\frac{t}{q}\right)}\right\},$$

which implies that

$$f_{p_{n+1},p_n}t \geq f_{p_n,p_{n-1}}\left(\frac{t}{q}\right) \geq \cdots \geq f_{p_0,\xi p_0}\left(\frac{t}{q^n}\right),\ n \in \mathbb{N},\ t > 0.$$

Using conditions (3) and (4) as in Theorem 2, one could prove that $\{\xi^n p_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Let $z = \lim_{n\to\infty} \xi^n p_0$ and suppose that $z \neq \xi z$. By (10), we arrive at

$$f_{\xi p_n,\xi z}(t) \geq \min\left\{f_{p_n,z}\left(\frac{t}{q}\right), f_{p_{n+1},p_n}\left(\frac{t}{q}\right), f_{\xi z,z}\left(\frac{t}{q}\right), f_{p_n,\xi z}\left(\frac{t}{q}\right), \frac{1 - f_{p_{n+1},z}\left(\frac{t}{q}\right) + f_{p_n,\xi z}\left(\frac{t}{q}\right)}{f_{p_{n+1},z}\left(\frac{t}{q}\right)\cdot f_{p_n,\xi z}\left(\frac{t}{q}\right)}\right\},$$

for every $n \in \mathbb{N}$ and $t > 0$. If we take $n \to \infty$ in the last inequality, it follows that

$$f_{z,\xi z}(t) \geq \min\left\{1, 1, f_{\xi z,z}\left(\frac{t}{q}\right), f_{z,\xi z}\left(\frac{t}{q}\right), 1\right\} = f_{\xi z,z}\left(\frac{t}{q}\right) > f_{z,\xi z}(t),$$

that is, $z = \xi z$.

Finally, we start to prove the uniqueness of fixed point. As a matter of fact, suppose that there exists $v \neq z$ such that $v = \xi v$. By (7), we obtain

$$f_{z,v}(t) = f_{\xi z,\xi v}(t) \geq \min\left\{f_{z,v}\left(\frac{t}{q}\right), f_{z,z}\left(\frac{t}{q}\right), f_{v,v}\left(\frac{t}{q}\right), f_{z,v}\left(\frac{t}{q}\right), \frac{1}{f^2_{z,v}\left(\frac{t}{q}\right)}\right\} = f_{z,v}\left(\frac{t}{q}\right),\ t > 0,$$

which means that $z = v$. □

Corollary 5. *Let $(\Omega, \mathcal{F}_{pm}, (\tau^\star_\lambda)_{\lambda\in(0,\infty)})$ be a complete Menger probabilistic metric space such that $\sup_{a<1} \tau^\star_\lambda(a,a) = 1$ and $\xi : \Omega \to \Omega$ be a probabilistic Banach q-contraction, where $\star \in \{D, AA\}$. Suppose that, for some $p_0 \in \Omega$ and some $k > 0$, condition (3) is satisfied, where $\psi : [0, 1] \to [0, s]$ is a continuous, decreasing function such that $\psi(1) = 0$. If $\sum_{i=1}^{\infty}(1 - \psi^{-1}((\varsigma^k)^i))^\lambda$ converges for $\varsigma \in (0,1)$, then there is a unique fixed point z of ξ such that $z = \lim_{n\to\infty} \xi^n p_0$.*

Proof. Let $\varsigma \in (0,1)$ and $\lambda \in (0,\infty)$. By Proposition 4, $\sum_{i=1}^{\infty}(1 - \psi^{-1}((\varsigma^k)^i))^\lambda < \infty$ and $\lim_{n\to\infty}(\tau^\star_\lambda)_{i=1}^{\infty}\psi^{-1}((\varsigma^k)^i) = 1$ are equivalent, and then the assertion holds by Theorem 2. □

Remark 6. *If $\psi(x) = \psi^{-1}(x) = 1 - x$, then the series $\sum_{i=1}^{\infty}(1 - \psi^{-1}(\varsigma^i))^\lambda = \sum_{i=1}^{\infty}(\varsigma^i)^\lambda$ converges to $\varsigma \in (0,1)$ and the condition could be omitted by Corollary 5.*

Theorem 7. *Let $(\Omega, \mathcal{F}_{pm}, \tau)$ be a complete Menger probabilistic metric space such that $\sup_{a<1} \tau(a,a) = 1$ and $\xi : \Omega \to \Omega$ be a probabilistic Banach q-contraction such that for some $p_0 \in \Omega$ and some $k > 0$, it satisfies*

$$\psi \circ f_{p_0,\xi p_0}(t) = O\left(\frac{1}{(g(t))^k}\right),\ t > 1, \tag{11}$$

where $\psi : [0, 1] \to [0, s]$ is a continuous, decreasing function such that $\psi(1) = 0$, and $g : (0,\infty) \to \mathbb{R}$ is a function such that $g(x) + g(y) = g(x \cdot y)$. If, for some $Q > 0$, the triangular norm τ satisfies

$$\lim_{n\to\infty} \tau_{i=n}^{\infty}\, \psi^{-1}\left(\frac{Q}{i^k}\right) = 1, \tag{12}$$

then there is a unique fixed point z of ξ such that $z = \lim_{n\to\infty} \xi^n p_0$.

Proof. Let $\mu \in (q, 1)$ and $\delta = \frac{q}{\mu}$. Then, $\delta \in (q, 1)$ and $\sum_{i=1}^{\infty} \delta^i < \infty$. So, for some $n_1 = n_1(\varepsilon) \in \mathbb{N}$, we obtain $\sum_{i=n_1}^{\infty} \delta^i \leq \varepsilon$.

Take $p_0 \in \Omega$ such that (11) is fulfilled and, similar as in the proof of Theorem 2, for $n \geq n_1,\ m \in \mathbb{N}$, one has

$$\begin{aligned} f_{\xi^n p_0, \xi^{n+m} p_0}(\varepsilon) &\geq f_{\xi^n p_0, \xi^{n+m} p_0}\left(\sum_{i=n}^{\infty} \delta^i\right) \\ &\geq f_{\xi^n p_0, \xi^{n+m} p_0}\left(\sum_{i=n}^{n+m-1} \delta^i\right) \\ &\geq \underbrace{\tau(\tau(\ldots\tau}_{(m-1)-times}\left(f_{\xi^n p_0, \xi^{n+1} p_0}(\delta^n), f_{\xi^{n+1} p_0, \xi^{n+2} p_0}(\delta^{n+1}), \ldots, f_{\xi^{n+m-1} p_0, f^{n+m} p_0}(\delta^{n+m-1})\right) \\ &\geq \underbrace{\tau\Big(\tau\Big(\ldots\tau}_{(m-1)-times}\left(f_{p_0, \xi p_0}\left(\frac{1}{\mu^n}\right), f_{p_0, \xi p_0}\left(\frac{1}{\mu^{n+1}}\right), \ldots f_{p_0, \xi p_0}\left(\frac{1}{\mu^{n+m-1}}\right)\right)\Big). \end{aligned}$$

By (11), there exists $M > 0$ such that

$$f_{p_0, \xi p_0}(t) \geq \psi^{-1}\Big(\frac{M}{(g(t))^k}\Big),\ t > 1,$$

Let $t = \dfrac{1}{\mu^n} > 1$. Because $g(t^n) = ng(t),\ n \in \mathbb{N}$, for $Q = \dfrac{M}{(g(\frac{1}{\mu}))^k} > 0$, we have that

$$f_{p_0, \xi p_0}\left(\frac{1}{\mu^n}\right) \geq \psi^{-1}\Big(\frac{M}{n^k(g(\frac{1}{\mu}))^k}\Big) = \psi^{-1}\Big(\frac{Q}{n^k}\Big),\ n \in \mathbb{N}.$$

Now,

$$\begin{aligned} f_{\xi^n p_0, \xi^{n+m} p_0}(\varepsilon) &\geq \underbrace{\tau\Big(\tau\Big(\ldots\Big(\tau}_{(m-1)-times}\left(\psi^{-1}\Big(\frac{Q}{n^k}\Big), \psi^{-1}\Big(\frac{Q}{(n+1)^k}\Big), \cdots, \psi^{-1}\Big(\frac{Q}{(n+m-1)^k}\Big)\right)\Big) \\ &\geq \tau_{i=n}^{\infty}\left(\psi^{-1}\Big(\frac{Q}{i^k}\Big)\right) \\ &> 1 - \lambda, \end{aligned}$$

for $n \geq n_0(\varepsilon, \lambda)$. So, $\{\xi^n p_0\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

Because ξ is a probabilistic Banach q-contraction (and consequently it is a continuous mapping), analogous as in the proof of Theorem 2, it follows that ξ exists a unique fixed point $z = \lim_{n \to \infty} \xi^n p_0$. □

Remark 7. *If we take that $g(x) = \ln x$, then by Theorem 7 we obtain Theorem 3 from [26].*

Corollary 6. *Let $(\Omega, \mathcal{F}_{pm}, (T_\lambda^{SW})_{\lambda \in (-1, \infty]})$ be a complete Menger probabilistic metric space such that $\sup_{a<1} \tau_\lambda^\star(a, a) = 1$ and $\xi : \Omega \to \Omega$ be a probabilistic Banach q-contraction such that, for some $p_0 \in \Omega$ and some $k > 0$, it satisfies*

$$\psi(f_{p_0, \xi p_0}(t)) = O\Big(\frac{1}{(\log_a t)^k}\Big),\ a > 1,\ t > 1,$$

where $\psi : [0, 1] \to [0, s]$ is a continuous, decreasing function such that $\psi(1) = 0$. If $\sum_{i=1}^{\infty}(1 - \psi^{-1}(\frac{Q}{i^k}))$ converges for some $Q > 0$, then there is a unique fixed point z of ξ such that $z = \lim_{n \to \infty} \xi^n p_0$.

Proof. Let $T = T_\lambda^{SW}$. By Proposition 4, $\sum_{i=1}^{\infty}(1-\psi^{-1}(\frac{Q}{i^k}))$ is equivalent to $\lim_{n\to\infty}(T_\lambda^{SW})_{i=n}^{\infty}(\psi^{-1}(\frac{Q}{i^k}))=1$, and hence the assertion follows by Theorem 7. □

Remark 8. *If $\psi(x)=1-x$, then the series $\sum_{i=1}^{\infty}(1-\psi^{-1}(\frac{Q}{i^k}))=\sum_{i=1}^{\infty}\frac{Q}{i^k}$ converges when $k>1$, so it is unnecessary to impose the condition in the previous corollary.*

3. Applications to Decomposable Measures

The following definitions are given in [11]. Let $\mathcal{A}$ be the σ-algebra of subsets of the given set Ω. The classical measure is a set function $m:\mathcal{A}\to[0,+\infty]$ such that $m(\emptyset)=0$ and

$$m\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}m(A_i),$$

for each sequence $\{A_i\}_{i\in\mathbb{N}}$ of mutually disjoint sets from $\mathcal{A}$. In the special case when $m:\mathcal{A}\to[0,1]$ and $m(\Omega)=1$, m is a probability measure and is marked with P.

Definition 12. *Let ζ be a triangular conorm. We say that the ζ-decomposable measure m is a set function $m:\mathcal{A}\to[0,1]$ such that $m(\emptyset)=0$ and $m(A\cup B)=\zeta(m(A),m(B))$, where $A,B\in\mathcal{A}$ and $A\cap B=\emptyset$.*

Definition 13. *Let ζ be a left continuous triangular conorm. The collective function $m:\mathcal{A}\to[0,1]$ is a ζ-decomposable measure if $m(\emptyset)=0$ and*

$$m\left(\bigcup_{i=1}^{\infty}A_i\right)=\zeta_{i=1}^{\infty}m(A_i),$$

for each sequence$\{A_i\}_{i\in\mathbb{N}}$ from $\mathcal{A}$ whose elements are mutually disjoint gatherings.

The set function from the previous example is a σ-ζ_l-decomposable measure. We say that a ζ-decomposable measure m is monotone if for $A,B\in\mathcal{A}$, $A\subseteq B$ implies $m(A)<m(B)$. A measure m is an (NSA)-type if and only if $s\circ m$ is a finite additive measure, where s is an additive generator for the triangular conorm ζ, which is continuous, not strict, Archimedean and in relation to which m is decomposable ($s(l)=1$).

Proposition 6 ([11])**.** *Let $(\Omega,\mathcal{A},m)$ be a measurable space, where m is a continuous, ζ-decomposable measure (NSA)-type with a monotonically increasing generator s. Then, $(\zeta,\mathcal{F}_{pm},\tau)$ is a Menger space, where $\mathcal{F}_{pm}$ and the triangular norm τ are defined as follows:*

$$(\mathcal{F}_{pm}(\widehat{X},\widehat{Y})=f_{\widehat{X},\widehat{Y}}:f_{\widehat{X},\widehat{Y}}(u)=m\{\omega,\,\omega\in\Omega,\,d(X(\omega),Y(\omega))<u\},\ \widehat{X},\widehat{Y}\in\zeta,\ u\in\mathbb{R},$$

and $\tau(a,\,b)=s^{-1}(\max(0,s(a)+s(b)-1)),\ a,b\in[0,1].$

The following is a straightforward application of Theorem 2 to a decomposable measure.

Corollary 7. *Let $(\Omega,\,\mathcal{A},\,m)$ be a measurable space, where m is a continuous, s-decomposable measure (NSA)-type with monotone increasing additive generator s. Let $(\mathcal{M},\,d)$ be a complete, separable metric space and $\xi:\Omega\times\mathcal{M}\to\mathcal{M}$ be a continuous random operator such that for some $q\in(0,\,1)$, it satisfies*

$$m(\{\omega\,|\,\omega\in\Omega,\,d(\hat{f}\,\hat{X}(\omega),\,\hat{f}\,\hat{Y}(\omega))<u\})\geq m(\{\omega\,|\,\omega\in\Omega,\,d(X(\omega),\,Y(\omega))<\frac{u}{q}\})$$

for all measurable mappings $X, Y : \Omega \to \mathcal{M}$ and each $u > 0$. If there is a measurable mapping $U : \Omega \to \mathcal{M}$ such that for some $k > 0$, it satisfies

$$\psi(m(d(\hat{U}, \xi\hat{U}) < t)) = O\Big(\frac{1}{t^k}\Big),\ t > 1,$$

where $\psi : [0, 1] \to [0, b]$ $(b > 0)$ is a continuous, decreasing function such that $\psi(1) = 0$ and the triangular norm τ defined by

$$\tau(u, v) = s^{-1}(\max(0, s(u) + s(v) - 1)),\quad u, v \in [0, 1]$$

satisfies the condition $\lim_{n\to\infty} \tau_{i=n}^{\infty} \psi^{-1}((\varsigma^k)^i) = 1$, $\varsigma \in (0, 1)$, then there is a random fixed point for the operator ξ.

Remark 9. *Let $(\Omega, \mathcal{A}, m)$, $(\mathcal{M}, d)$ and ξ be the same as in Corollary 7 for some t-conorm σ_λ^{SW}, $\lambda \in (-1, \infty)$, where σ_λ^{SW} is a Sugeno–Weber family of t-conorms as follows:*

$$\sigma_\lambda^{SW}(a, b) = \sigma_m(a, b),\ \text{if } \lambda = \infty \text{ and } \sigma_\lambda^{SW}(a, b) = \min(1, a + b + \lambda ab),\ \text{if } \lambda \in (-1, \infty).$$

Put $\xi(x) = 1 - x$. Because the triangular norm τ_λ^{SW}, $\lambda \in (-1, \infty)$ is geometrically convergent, it implies the existence of the random fixed point of the random operator $\xi : \Omega \times \mathcal{M} \to \mathcal{M}$.

4. Conclusions

For more than a century, fixed point theory has widespread and significant applications in many fields at the core of many branches of pure and applied mathematics, including convex analysis, variational analysis, nonlinear ordinary and partial differential equations, critical point theory, nonlinear optimization, fractional calculus and so on. It is known that plenty of problems caused by the real world are often due to seeking to find a fixed point and then using different mathematical techniques. In this work, a technique is furnished, based on nontrivial results in generalized Menger probabilistic metric spaces. We establish several fixed point theorems with illustrative examples in the framework of such spaces. We make a conclusion that our results in this paper generalize and improve many known results in the existing literature. Additionally, we apply our results to cope with the decomposable measures. We make sure that the idea of further elaborating our method, which is presented throughout this paper, is quite important and can be applied to probability theory and nonlinear fractional differential equations in the upcoming future.

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