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Bayesian and Non-Bayesian Estimation for a New Extension of Power Topp–Leone Distribution under Ranked Set Sampling with Applications

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Abstract: In this article, we intend to introduce and study a new two-parameter distribution as a new extension of the power Topp–Leone (PTL) distribution called the Kavya–Manoharan PTL (KMPTL) distribution. Several mathematical and statistical features of the KMPTL distribution, such as the quantile function, moments, generating function, and incomplete moments, are calculated. Some measures of entropy are investigated. The cumulative residual Rényi entropy (CRRE) is calculated. To estimate the parameters of the KMPTL distribution, both maximum likelihood and Bayesian estimation methods are used under simple random sample (SRS) and ranked set sampling (RSS). The simulation study was performed to be able to verify the model parameters of the KMPTL distribution using SRS and RSS to demonstrate that RSS is more efficient than SRS. We demonstrated that the KMPTL distribution has more flexibility than the PTL distribution and the other nine competitive statistical distributions: PTL, unit-Gompertz, unit-Lindley, Topp–Leone, unit generalized log Burr XII, unit exponential Pareto, Kumaraswamy, beta, Marshall-Olkin Kumaraswamy distributions employing two real-world datasets.

Keywords: power Topp–Leone distribution; cumulative residual Rényi entropy; maximum likelihood estimation; ranked set sampling; Bayesian statistics; Kavya–Manoharan class

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1. Introduction

The Topp–Leone (TL) distribution, developed by [1], is one of the most helpful of the known distributions, with support across the unit interval. It has the following distribution function (CDF) and density function (PDF):

$$G(t;\delta) = t^{\delta}(2-t)^{\delta}, \quad 0 < t < 1, \quad \delta > 0, \tag{1}$$

and

$$g(t;\delta) = 2\delta t^{\delta-1}(1-t)(2-t)^{\delta-1}, \quad 0 < t < 1, \quad \delta > 0, \tag{2}$$

where δ is a shape parameter.

Generalizations of the TL distribution have been shown to be beneficial in a variety of disciplines. Many notable academics have made significant attempts to suggest new generalized and developed modifications of the TL model, for example, TL inverse Weibull distribution [2], Type I half-Logistic TL distribution [3], TL geometric distribution [4], truncated Cauchy power inverted TL distribution [5], TL Nadarajah-Haghighi distribution [6], power inverted TL distribution [7], TL linear exponential distribution [8], Kumaraswamy inverted TL distribution [9], new power TL family [10], alpha power inverted TL distribution [11], half-logistic inverted TL distribution [12], type II Power TL family [13], odd log-logistic TL family [14], TL odd Fréchet distribution [15], Burr III-TL family [16], inverse TL distribution [17], exponentiated generalized TL family [18], transmuted TL power function distribution [19], transmuted TL family [20], sine TL family [21], weighted TL family [22], TL inverse Lomax distribution [23], and new weighted TL family [24], among others.

Recently, Ref. [25] used this transformation, $Z = T^{1/\lambda}$, to introduce the PTL distribution as a generalization of the TL distribution, and it has the following CDF, PDF, and quantile function (qf)

$$G(z;\lambda,\delta) = z^{\lambda\delta}\left(2 - z^{\lambda}\right)^{\delta}, \quad 0 < z < 1, \quad \lambda,\delta > 0, \tag{3}$$

$$g(z;\lambda,\delta) = 2\lambda\delta z^{\lambda\delta-1}\left(1 - z^{\lambda}\right)\left(2 - z^{\lambda}\right)^{\delta-1}, \quad 0 < z < 1, \quad \lambda,\delta > 0, \tag{4}$$

and

$$Q(q) = \left[1 - \left(1 - q^{\frac{1}{\delta}}\right)^{0.5}\right]^{\frac{1}{\lambda}}, \tag{5}$$

where λ and δ are two shape parameters and $q \in (0,\ 1)$.

In recent years, many authors used the power transformation technique to obtain statistical distributions that are more flexible due to the shape parameter. Several of the potential power distributions are as follows: the power Zeghdoudi distribution [26], power modified Kies distribution [27], power Darna distribution [28], power Burr X distribution [29], power transmuted inverse Rayleigh distribution [30], power Rama distribution [31], power binomial exponential distribution [32], power Aradhana distribution [33], power Lomax distribution [34], power half logistic distribution [35], power Shanker distribution [36], power Lindley distribution [37], and power Cauchy distribution [38], among others.

Adding parameters improves versatility, but they also increase the problem of the complexity estimation parameters. In recent times, Ref. [39] created the Kavya–Manoharan (KM) class in order to acquire new parsimonious distribution families, and it has the following CDF:

$$F(z;\Omega) = \frac{e}{e-1}\left(1 - e^{-G(z;\Omega)}\right), \quad z \in R. \tag{6}$$

The corresponding PDF and qf of the KM family are

$$f(z;\Omega) = \frac{e}{e-1} g(z;\Omega) e^{-G(z;\Omega)}, \tag{7}$$

and

$$Q(q) = G^{-1}\left(-\ln\left[1 - q\left(1 - e^{-1}\right)\right]\right), \tag{8}$$

where $G(z;\Omega)$ and $g(z;\Omega)$ are the CDF and the PDF of the parent distribution.

Ranked set sampling (RSS) is an efficient and successful alternative to simple random sampling (SRS). When sampling units are expensive and difficult to quantify, this method is typically used to generate samples that are more representative of the underlying population, as well as samples that are straightforward and affordable to arrange in line with the variable of interest. Plenty of studies on RSS technique changes have been undertaken. Additional details about the RSS system may be found in, for example, [40–43]. The study of RSS-based estimates for several parametric models has lately attracted a significant amount of interest. These studies repeatedly indicate that RSS outperforms SRS and other traditional sampling strategies. Here are several instances: Ref. [44] for the Lomax distribution, Ref. [45] for the exponentiated exponential distribution, Ref. [46] for the xgamma distribution, Refs. [47,48] for exponentiated Pareto distribution, Ref. [49,50] for the Kumaraswamy distribution, Ref. [51] for the exponential-Poisson distribution, Ref. [52] for the generalized inverse Lindley distribution, Ref. [53] for the Birnbaum–Saunders distribution, Ref. [54] for the generalized quasi-Lindley distribution, Ref. [55] for the Weibull distribution, and Ref. [56] for the Zubair Lomax distribution.

Although [25] introduced a new flexible two-shape parameter distribution called the PTL distribution, the limitation of this study is that the PTL distribution does not give more fitting for different types of data. Because of this, we aim in this paper to introduce a new extension of the PTL distribution by the same number of parameters, called the KMPTL distribution. The new suggested distribution gives a better fit than the PTL distribution and some well-known distribution, and we supported that in the application section. Here, we aim to profit from the combined KM-class of distributions and the PTL distribution by investigating a new two-parameter distribution named the KMPTL distribution. The major motivations behind the KMPTL distribution are as follows:

1. The KMPTL distribution is very simple and it has more flexibility than the PTL distribution, and both distributions have two parameters.
2. We demonstrate that the KMPTL distribution may provide desirable features in both practical and theoretical terms.
3. Numerous broad statistical and mathematical aspects of the PTL distribution were studied.
4. Four different measures of entropy are calculated.
5. The cumulative residual Rényi entropy (CRRE) is calculated.
6. Classical and Bayesian approaches of estimation were utilized to compute the estimate of parameters for the KMPTL distribution, under SRS and RSS.
7. For modeling, we used two genuine datasets from economics and physics; the KMPTL distribution provides a better fit than the PTL distribution and the other nine competitive statistical distributions, the PTL, unit-Gompertz (UG), unit-Lindley (UL), TL, unit generalized log Burr XII (UGLBXII), unit exponential Pareto distribution (UEPD), Kumaraswamy (Kw), beta, and Marshall-Olkin Kumaraswamy (MOK) distributions, and we suggested this in the application section.

The following is an outline of the contents of this article. In Section 2, the formulation of the KMPTL distribution is discussed. Numerous mathematical characteristics of the KMPTL distribution are proposed in Section 3. Two Sections, Sections 4 and 5, are devoted to four different measures of entropy and the CRRE from the KMPTL distribution. Classical and Bayesian estimation of the two shape unknown parameters from the KMPTL distribution, utilizing a thorough simulation experiment, are discussed in Sections 6 and 7, respectively. Section 8 examines two real datasets and provides a clear comparison of nine competitive distributions. Section 9 presents a conclusion to the topic.

2. The Kavya–Manoharan Power Topp–Leone Distribution

Here, in this section, we intend to create the KMPTL distribution. Substituting Equations (3) and (4) into Equations (6) and (7) to obtain the CDF and PDF of the KMPTL distribution as below:

$$F(z;\lambda,\delta) = \frac{e}{e-1}\left(1 - e^{-z^{\lambda\delta}\left(2-z^{\lambda}\right)^{\delta}}\right), \quad 0 < z < 1, \quad \lambda,\delta > 0, \tag{9}$$

and

$$f(z;\lambda,\delta) = \frac{2\lambda\delta e}{e-1} z^{\lambda\delta-1}\left(1 - z^{\lambda}\right)\left(2 - z^{\lambda}\right)^{\delta-1} e^{-z^{\lambda\delta}\left(2-z^{\lambda}\right)^{\delta}}. \tag{10}$$

In addition, the survival function and the hazard rate function (HRF) are provided via

$$S(z;\lambda,\delta) = 1 - \frac{e}{e-1}\left(1 - e^{-z^{\lambda\delta}\left(2-z^{\lambda}\right)^{\delta}}\right),$$

and

$$h(z;\lambda,\delta) = \frac{2\lambda\delta e z^{\lambda\delta-1}\left(1-z^{\lambda}\right)\left(2-z^{\lambda}\right)^{\delta-1} e^{-z^{\lambda\delta}\left(2-z^{\lambda}\right)^{\delta}}}{e^{1-z^{\lambda\delta}\left(2-z^{\lambda}\right)^{\delta}}-1}.$$

The reversed and cumulative HRFs are

$$\tau(z;\lambda,\delta) = \frac{2\lambda\delta z^{\lambda\delta-1}\left(1-z^{\lambda}\right)\left(2-z^{\lambda}\right)^{\delta-1}}{e^{z^{\lambda\delta}\left(2-z^{\lambda}\right)^{\delta}}-1},$$

and

$$H(z;\lambda,\delta) = -\ln\left[1-\frac{e}{e-1}\left(1-e^{-z^{\lambda\delta}\left(2-z^{\lambda}\right)^{\delta}}\right)\right].$$

Figures 1 and 2 show some curves of the PDF and the HRF for the KMPTL distribution. We can note from Figure 1 that the curves of PDF for the KMPTL distribution can be unimodal, right-skewed, left-skewed, symmetric, heavy-tailed, and decreasing, but from Figure 2 we can note that the curves of HRF for the KMPTL distribution can be increasing, J-shaped, U-shaped and, bathtub.

The HRF is complex in definition, but an asymptotic study is possible. The limit gives important information on its behavior.

When $z \to 0$, we have $h(z;\lambda,\delta) \sim \frac{2^{\delta}\lambda\delta e}{1-e} z^{\lambda\delta-1}$. As a result, when $\lambda\delta < 1$, we have $h(z;\lambda,\delta) \to +\infty$, when $\lambda\delta = 1$, we have a constant limit: $h(z;\lambda,\delta) = \frac{2^{\delta}\lambda\delta e}{1-e}$, and when $\lambda\delta > 1$, we have $h(z;\lambda,\delta) \to 0$.

On the other hand, when $z \to 1$, there is no parameter nuance; we have $h(z;\lambda,\delta) \to +\infty$. This asymptotic analysis is completed by a graphic analysis in Figure 2. In addition, Figure 3 shows a monotone plot of the HRF for the KMPTL distribution to confirm the HRF has more shapes under the different values of parameter.

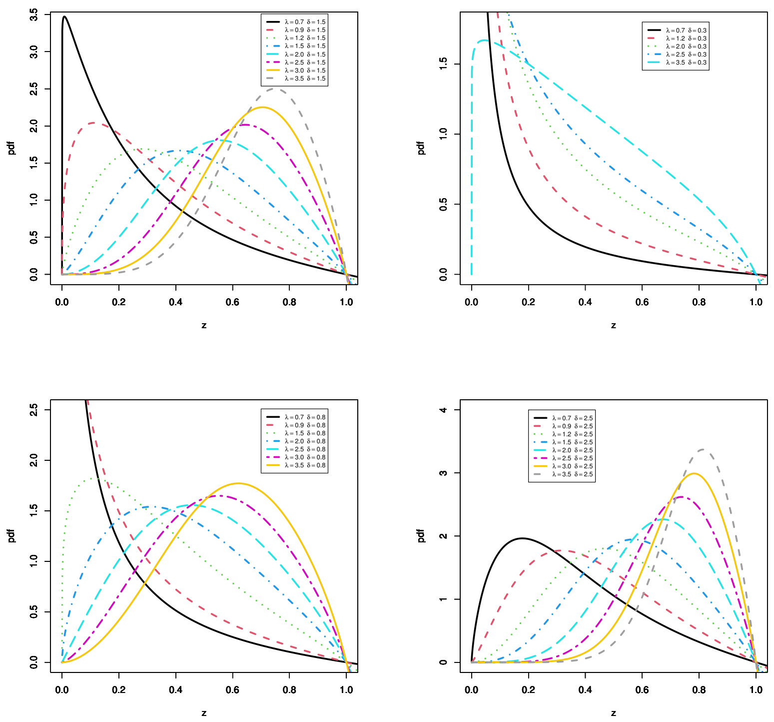

Figure 1. Some curves of the PDF for the KMPTL distribution.

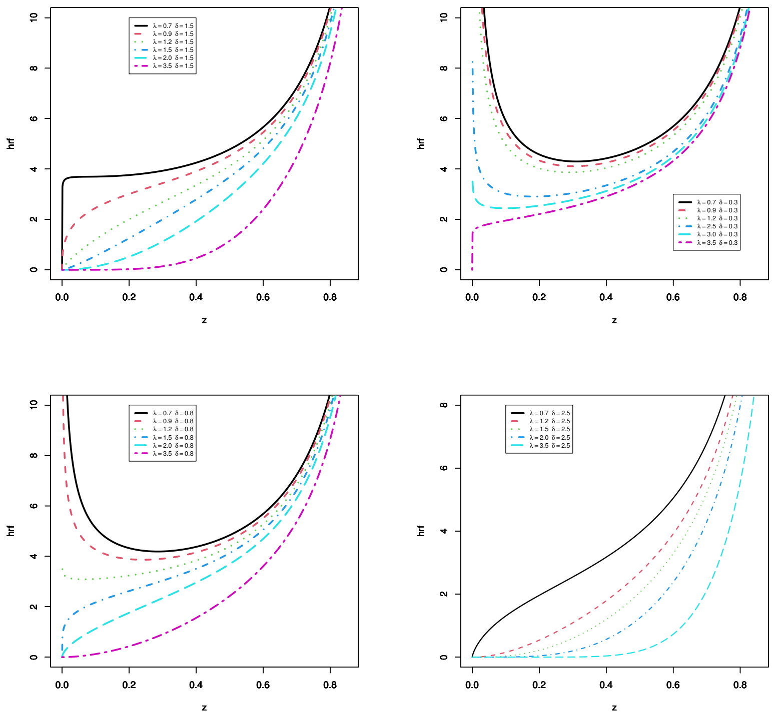

Figure 2. Some curves of the HRF for the KMPTL distribution.

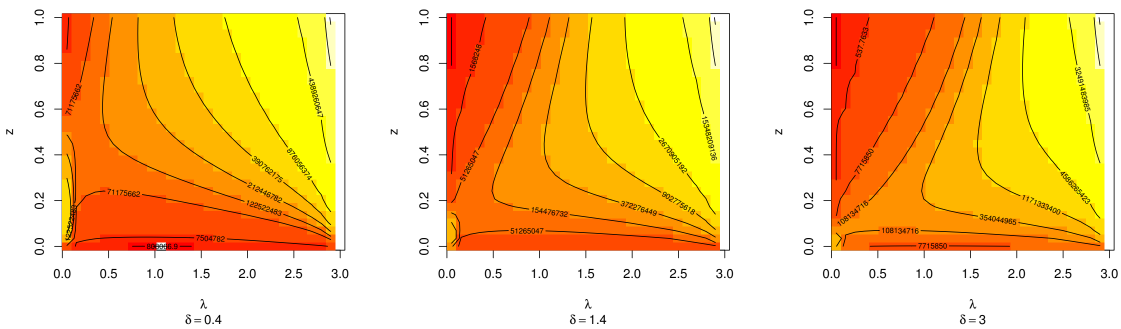

Figure 3. Monotone plot of the HRF for the KMPTL distribution.

3. Some Basic Statistical Properties

In this part, we will look at some basic structural aspects of the KMPTL distribution.

3.1. Quantile Function

The qf, defined as $Q(q; \lambda, \delta) = F^{-1}(q; \lambda, \delta)$, is computed by inverting Equation (9) as

$$Q(q; \lambda, \delta) = \left(1 - \left(1 - \left[-\ln\left(1 - q\left(1 - e^{-1}\right)\right)\right]^{\frac{1}{\delta}}\right)^{0.5}\right)^{\frac{1}{\lambda}}. \tag{11}$$

We derive the first, second (median), and third quantiles by inserting $q = 0.25, 0.5$, and 0.75. Furthermore, Bowley's skewness (γ_1) and Moor's kurtosis (γ_2) are calculated using the quantiles as below

$$\gamma_1 = \frac{Q(0.75;\, \lambda,\, \delta) - 2Q(0.5;\, \lambda,\, \delta) + Q(0.25;\, \lambda,\, \delta)}{Q(0.75;\, \lambda,\, \delta) - Q(0.25;\, \lambda,\, \delta)},$$

and

$$\gamma_2 = \frac{Q(0.875;\,\lambda,\,\delta) - Q(0.625;\,\lambda,\,\delta) - Q(0.375;\,\lambda,\,\delta) + Q(0.125;\,\lambda,\,\delta)}{Q(0.75;\,\lambda,\,\delta) - Q(0.25;\,\lambda,\,\delta)}.$$

Table 1 shows some numerical values of $Q1$, $Q2$, $Q3$, γ_1, and γ_2 for the KMPTL distribution with numerous numerical combinations of the two shape parameters, λ and δ.

Table 1. Results of some numerical values of $Q1$, $Q2$, $Q3$, γ_1, and γ_2 for the KMPTL distribution.

δ	λ	$Q1$	$Q2$	$Q3$	γ_1	γ_2
1.5	1.8	0.372	0.522	0.676	0.015	1.159
	2.3	0.462	0.601	0.736	−0.017	1.17
	2.8	0.53	0.659	0.778	−0.038	1.18
	3.3	0.583	0.702	0.808	−0.052	1.187
	3.8	0.626	0.735	0.831	−0.063	1.194
	4.3	0.661	0.762	0.849	−0.071	1.199
	4.8	0.69	0.784	0.864	−0.078	1.204
	5.3	0.715	0.802	0.876	−0.083	1.208
	5.8	0.736	0.817	0.886	−0.087	1.211
	6.3	0.754	0.831	0.894	−0.091	1.214
2	1.8	0.447	0.585	0.721	−0.005	1.172
	2.3	0.533	0.657	0.774	−0.031	1.182
	2.8	0.596	0.708	0.81	−0.047	1.19
	3.3	0.645	0.746	0.836	−0.059	1.197
	3.8	0.683	0.776	0.856	−0.067	1.202
	4.3	0.714	0.799	0.872	−0.074	1.206
	4.8	0.74	0.818	0.884	−0.079	1.21
	5.3	0.761	0.833	0.895	−0.083	1.213
	5.8	0.779	0.847	0.903	−0.087	1.215
	6.3	0.795	0.858	0.911	−0.089	1.218
3.5	1.8	0.577	0.687	0.791	−0.028	1.19
	2.3	0.65	0.745	0.832	−0.045	1.198
	2.8	0.702	0.785	0.86	−0.056	1.204
	3.3	0.741	0.815	0.88	−0.064	1.208
	3.8	0.771	0.837	0.895	−0.069	1.211
	4.3	0.794	0.854	0.906	−0.074	1.214
	4.8	0.814	0.869	0.916	−0.077	1.216
	5.3	0.83	0.88	0.923	−0.08	1.218
	5.8	0.843	0.89	0.93	−0.082	1.22
	6.3	0.854	0.898	0.935	−0.084	1.221
5	1.8	0.646	0.739	0.826	−0.036	1.198
	2.3	0.71	0.789	0.861	−0.049	1.204
	2.8	0.755	0.823	0.884	−0.058	1.208
	3.3	0.788	0.848	0.901	−0.064	1.212
	3.8	0.813	0.867	0.913	−0.068	1.214
	4.3	0.833	0.881	0.923	−0.071	1.216
	4.8	0.849	0.893	0.931	−0.074	1.218
	5.3	0.862	0.902	0.937	−0.076	1.219
	5.8	0.873	0.91	0.942	−0.078	1.22
	6.3	0.883	0.917	0.947	−0.079	1.221

3.2. Moments

The w_{th} moment of a random variable Z with the KMPTL distribution is

$$\mu'_w = E(Z^w) = \int_0^1 z^w f(z;\lambda,\delta)dz = \frac{2\lambda\delta e}{e-1}\int_0^1 z^{w+\lambda\delta-1}\left(1-z^\lambda\right)\left(2-z^\lambda\right)^{\delta-1} e^{-z^{\lambda\delta}\left(2-z^\lambda\right)^\delta}dz, \quad (12)$$

by using the exponential expansion

$$e^{-z^{\lambda\delta}\left(2-z^{\lambda}\right)^{\delta}}=\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}z^{\lambda\delta i}\left(2-z^{\lambda}\right)^{\delta i}, \tag{13}$$

by inserting (13) in (12), we have

$$\mu_w'=\frac{2\lambda\delta e}{e-1}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\int_0^1 z^{w+\lambda\delta(i+1)-1}\left(1-z^{\lambda}\right)\left(2-z^{\lambda}\right)^{\delta(i+1)-1}dz.$$

Let $x=z^{\lambda}$, then

$$\mu_w'=\frac{2\delta e}{e-1}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\int_0^1 x^{\frac{w}{\lambda}+\delta(i+1)-1}(1-x)(2-x)^{\delta(i+1)-1}dx.$$

Let $v=1-x$, then

$$\mu_w'=\frac{2\delta e}{e-1}\sum_{i=0}^{\infty}\frac{(-1)^{i}}{i!}\int_0^1 v(1+v)^{\delta(i+1)-1}(1-v)^{\frac{w}{\lambda}+\delta(i+1)-1}dv. \tag{14}$$

Employing the binomial expansion

$$(1+v)^{\delta(i+1)-1}=\sum_{j=0}^{\infty}\binom{\delta(i+1)-1}{j}v^{j}, \tag{15}$$

by inserting (15) in (14), we have

$$\mu_w'=\frac{2\delta e}{e-1}\sum_{i,j=0}^{\infty}\frac{(-1)^{i}}{i!}\binom{\delta(i+1)-1}{j}\int_0^1 v^{j+1}(1-v)^{\frac{w}{\lambda}+\delta(i+1)-1}dv,$$

then

$$\mu_w'=\frac{2\delta e}{e-1}\sum_{i,j=0}^{\infty}\frac{(-1)^{i}}{i!}\binom{\delta(i+1)-1}{j}B\left(j+2,\,\frac{w}{\lambda}+\delta(i+1)\right). \tag{16}$$

The moment-generating function the KMPTL distribution is

$$M_t(z)=\sum_{w=0}^{\infty}\frac{t^{w}}{w!}\mu'_w=\frac{2\delta e}{e-1}\sum_{i,j,w=0}^{\infty}\frac{(-1)^{i}t^{w}}{i!w!}\binom{\delta(i+1)-1}{j}B\left(j+2,\,\frac{w}{\lambda}+\delta(i+1)\right).$$

Table 2 shows some numerical values of moments, variance (σ^2), skewness (α_1), kurtosis (α_2), and coefficient of variation (CV) for the KMPTL distribution. Figure 4 shows 3D Plots of moments for the KMPTL distribution. Figure 4 shows that when the value of δ increases then the values of variance, skewness, kurtosis, and CV decrease, but the values of mean and index of dispersion (ID) increase.

Table 2. Results of some moments, σ^2, α_1, α_2, and CV for the KMPTL distribution.

δ	λ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	α_1	α_2	CV
1.5	1.8	0.524	0.316	0.209	0.148	0.041	−0.012	2.283	0.387
	2.3	0.595	0.389	0.271	0.199	0.035	−0.182	2.407	0.313
	2.8	0.648	0.448	0.327	0.247	0.029	−0.322	2.569	0.264
	3.3	0.688	0.498	0.375	0.291	0.025	−0.427	2.736	0.228
	3.8	0.721	0.54	0.418	0.332	0.021	−0.511	2.894	0.201

Table 2. *Cont.*

δ	λ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	α_1	α_2	CV
	4.3	0.747	0.576	0.456	0.368	0.018	−0.578	3.04	0.179
	4.8	0.769	0.607	0.489	0.401	0.016	−0.634	3.175	0.162
	5.3	0.787	0.633	0.519	0.432	0.014	−0.682	3.297	0.148
	5.8	0.803	0.657	0.545	0.459	0.012	−0.722	3.409	0.136
	6.3	0.816	0.677	0.569	0.484	0.011	−0.757	3.51	0.126
2	1.8	0.581	0.372	0.256	0.185	0.034	−0.118	2.388	0.319
	2.3	0.648	0.447	0.324	0.244	0.028	−0.289	2.565	0.257
	2.8	0.696	0.507	0.383	0.298	0.022	−0.411	2.748	0.215
	3.3	0.733	0.556	0.434	0.346	0.018	−0.502	2.92	0.185
	3.8	0.762	0.596	0.477	0.389	0.015	−0.573	3.075	0.163
	4.3	0.786	0.63	0.515	0.427	0.013	−0.63	3.213	0.145
	4.8	0.805	0.659	0.547	0.461	0.011	−0.677	3.335	0.131
	5.3	0.821	0.683	0.576	0.491	0.0096	−0.717	3.445	0.119
	5.8	0.834	0.705	0.601	0.518	0.0084	−0.75	3.543	0.11
	6.3	0.846	0.723	0.624	0.543	0.0074	−0.779	3.631	0.101
3.5	1.8	0.679	0.483	0.357	0.273	0.022	−0.299	2.64	0.22
	2.3	0.735	0.557	0.433	0.344	0.017	−0.429	2.847	0.175
	2.8	0.775	0.613	0.494	0.405	0.013	−0.518	3.024	0.146
	3.3	0.804	0.657	0.544	0.456	0.01	−0.584	3.173	0.125
	3.8	0.827	0.692	0.585	0.5	0.0082	−0.634	3.298	0.109
	4.3	0.845	0.721	0.62	0.537	0.0068	−0.674	3.404	0.097
	4.8	0.859	0.744	0.649	0.57	0.0057	−0.706	3.495	0.087
	5.3	0.872	0.764	0.674	0.598	0.0048	−0.733	3.574	0.08
	5.8	0.882	0.781	0.696	0.623	0.0041	−0.756	3.642	0.073
	6.3	0.89	0.796	0.715	0.645	0.0036	−0.775	3.702	0.067
5	1.8	0.731	0.55	0.425	0.336	0.016	−0.373	2.784	0.174
	2.3	0.78	0.621	0.502	0.412	0.012	−0.479	2.978	0.138
	2.8	0.815	0.672	0.561	0.474	0.0087	−0.551	3.134	0.115
	3.3	0.839	0.712	0.608	0.524	0.0068	−0.604	3.259	0.098
	3.8	0.859	0.743	0.647	0.566	0.0054	−0.644	3.361	0.086
	4.3	0.874	0.768	0.678	0.602	0.0044	−0.675	3.446	0.076
	4.8	0.886	0.788	0.704	0.632	0.0037	−0.7	3.517	0.068
	5.3	0.896	0.805	0.727	0.658	0.0031	−0.721	3.577	0.062
	5.8	0.904	0.82	0.746	0.681	0.0026	−0.739	3.629	0.057
	6.3	0.911	0.833	0.763	0.701	0.0023	−0.753	3.675	0.052

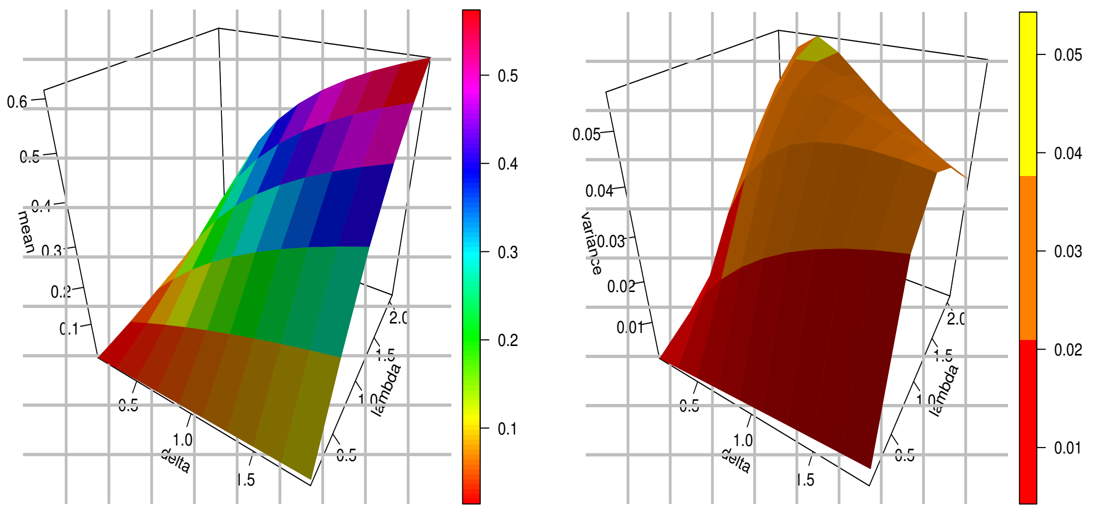

Figure 4. *Cont.*

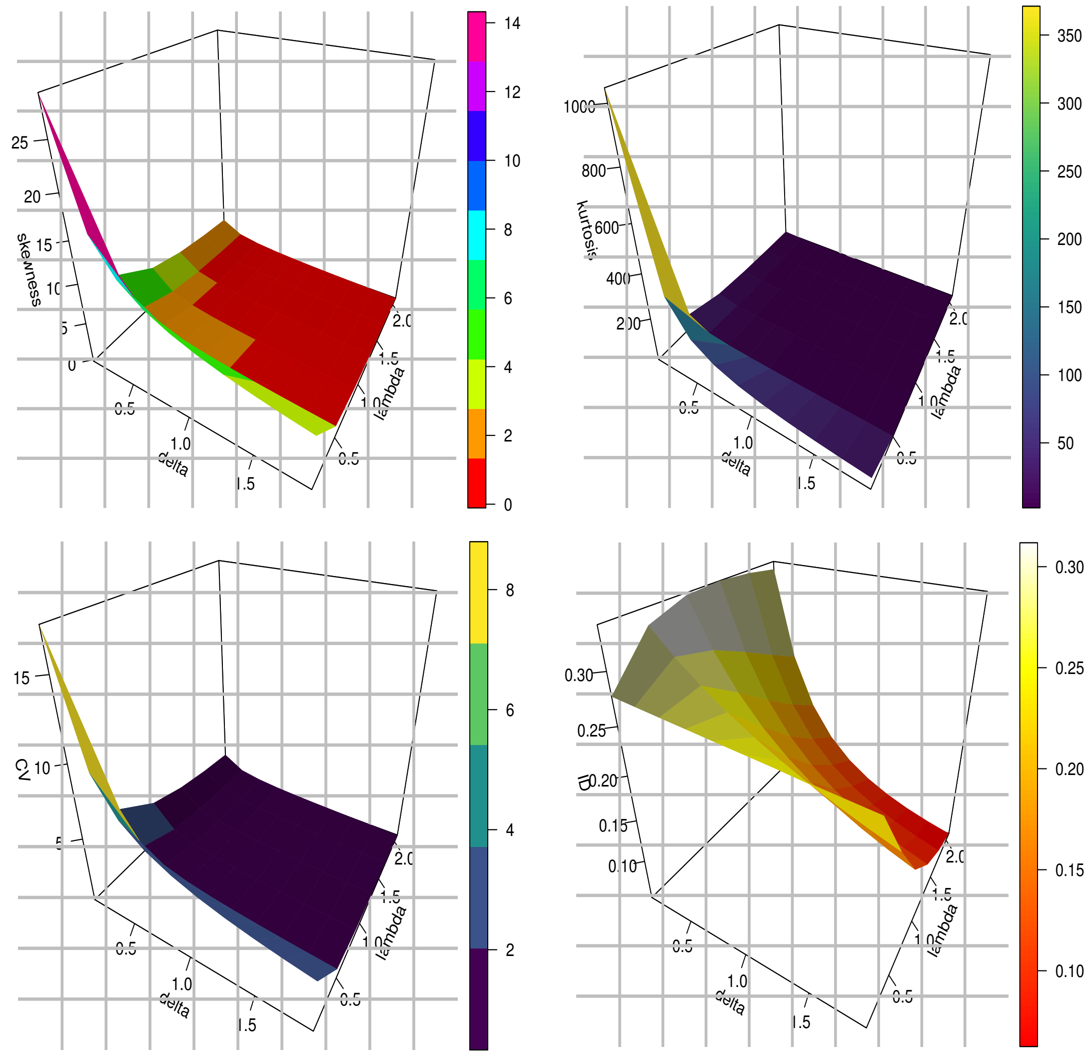

Figure 4. 3D Plots of moments for the KMPTL distribution.

3.3. Incomplete Moments

The q_{th} lower incomplete moment of a random variable Z having the KMPTL distribution is calculated from the next equation:

$$\varpi_q(t) = \frac{2\lambda\delta e}{e-1} \int_0^t z^{q+\lambda\delta-1} \left(1 - z^\lambda\right) \left(2 - z^\lambda\right)^{\delta-1} e^{-z^{\lambda\delta} \left(2-z^\lambda\right)^\delta} dz. \tag{17}$$

Employing (13) in (17), we get

$$\varpi_q(t) = \frac{2\lambda\delta e}{e-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_0^t z^{q+\lambda\delta(i+1)-1} \left(1 - z^\lambda\right) \left(2 - z^\lambda\right)^{\delta(i+1)-1} dz.$$

Let $x = z^\lambda$, then

$$\varpi_q(t) = \frac{2\delta e}{e-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_0^{t^\lambda} x^{\frac{q}{\lambda}+\delta(i+1)-1} (1-x)(2-x)^{\delta(i+1)-1} dx.$$

Let $v = 1 - x$, then

$$\varpi_q(t) = \frac{2\delta e}{e-1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\int_0^{1-t^\lambda} v(1+v)^{\delta(i+1)-1}(1-v)^{\frac{q}{\lambda}+\delta(i+1)-1}dv,$$

by inserting (15) in the above equation, we have

$$\varpi_q(t) = \frac{2\delta e}{e-1}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\binom{\delta(i+1)-1}{j}\int_0^{1-t^\lambda} v^{j+1}(1-v)^{\frac{q}{\lambda}+\delta(i+1)-1}dv,$$

then,

$$\varpi_q(t) = \frac{2\delta e}{e-1}\sum_{i,j=0}^{\infty}\frac{(-1)^i}{i!}\binom{\delta(i+1)-1}{j}B_{1-t^\lambda}\left(j+2, \frac{q}{\lambda}+\delta(i+1)\right), \tag{18}$$

where $B_z(.,.)$ is the incomplete beta function.

4. Entropy

Entropy is one of the most significant measurements of uncertainty. Many kinds of entropy can be used to determine risk and dependability.

4.1. Rényi Measure of Entropy

The Rényi entropy [57] measure is calculated from the next equation:

$$R_\zeta = \frac{1}{1-\zeta}\log\left[E_\zeta(\lambda,\delta)\right],\ \zeta > 0, \zeta \neq 1, \tag{19}$$

where $E_\zeta(\lambda,\delta) = \int_0^1 [f(z;\lambda,\delta)]^\zeta dz$. Now, we need to calculate $E_\zeta(\lambda,\delta)$. Then,

$$E_\zeta(\lambda,\delta) = \int_0^1 [f(z;\lambda,\delta)]^\zeta dz = \left(\frac{2\lambda\delta e}{e-1}\right)^\zeta \int_0^1 z^{\zeta(\lambda\delta-1)}\left(1-z^\lambda\right)^\zeta\left(2-z^\lambda\right)^{(\delta-1)\zeta} e^{-\zeta z^{\lambda\delta}\left(2-z^\lambda\right)^\delta}dz.$$

Applying (13) in the previous equation, we obtain

$$E_\zeta(\lambda,\delta) = \left(\frac{2\lambda\delta e}{e-1}\right)^\zeta \sum_{i=0}^{\infty}\frac{(-\zeta)^i}{i!}\int_0^1 z^{\lambda\delta(i+\zeta)-\zeta}\left(1-z^\lambda\right)^\zeta\left(2-z^\lambda\right)^{\delta(i+\zeta)-\zeta}dz. \tag{20}$$

Let $x = z^\lambda$, then

$$E_\zeta(\lambda,\delta) = \left(\frac{2\lambda\delta e}{e-1}\right)^\zeta \sum_{i=0}^{\infty}\frac{(-\zeta)^i}{\lambda i!}\int_0^1 x^{\delta(i+\zeta)-\frac{\zeta}{\lambda}+\frac{1}{\lambda}-1}(1-x)^\zeta(2-x)^{\delta(i+\zeta)-\zeta}dx.$$

Let $v = 1 - x$, then

$$E_\zeta(\lambda,\delta) = \left(\frac{2\lambda\delta e}{e-1}\right)^\zeta \sum_{i=0}^{\infty}\frac{(-\zeta)^i}{\lambda i!}\int_0^1 v^\zeta(1+v)^{\delta(i+\zeta)-\zeta}(1-v)^{\delta(i+\zeta)-\frac{\zeta}{\lambda}+\frac{1}{\lambda}-1}dv.$$

By employing the binomial expansion (15) in the above equation, we get

$$E_\zeta(\lambda,\delta) = \sum_{i,j=0}^{\infty}\Psi_{i,j}\int_0^1 v^{\zeta+j}(1-v)^{\delta(i+\zeta)-\frac{\zeta}{\lambda}+\frac{1}{\lambda}-1}dv,$$

then

$$E_\zeta(\lambda,\delta) = \sum_{i,j=0}^{\infty} \Psi_{i,j} B\left(\zeta + j + 1,\, \delta(i+\zeta) - \frac{\zeta}{\lambda} + \frac{1}{\lambda}\right), \tag{21}$$

where $\Psi_{i,j} = \left(\frac{2\lambda\delta e}{e-1}\right)^\zeta \frac{(-\zeta)^i}{\lambda i!} \begin{pmatrix} \delta(i+\zeta) - \zeta \\ j \end{pmatrix}$.

Inserting (21) into (19), then the Rényi entropy of the KMPTL distribution is

$$R_\zeta = \frac{1}{1-\zeta} \log\left[\sum_{i,j=0}^{\infty} \Psi_{i,j} B\left(\zeta + j + 1,\, \delta(i+\zeta) - \frac{\zeta}{\lambda} + \frac{1}{\lambda}\right)\right],\ \zeta > 0, \zeta \neq 1. \tag{22}$$

4.2. Arimoto Measure of Entropy

The Arimoto entropy [58] measure is calculated from the next equation:

$$A_\zeta = \frac{\zeta}{1-\zeta}\left[(E_\zeta(\lambda,\delta))^{\frac{1}{\zeta}} - 1\right],\ \zeta > 0, \zeta \neq 1. \tag{23}$$

Inserting (21) into (23), then the Arimoto entropy of the KMPTL distribution is

$$A_\zeta = \frac{\zeta}{1-\zeta}\left[\left(\sum_{i,j=0}^{\infty} \Psi_{i,j} B\left(\zeta + j + 1,\, \delta(i+\zeta) - \frac{\zeta}{\lambda} + \frac{1}{\lambda}\right)\right)^{\frac{1}{\zeta}} - 1\right],\ \zeta > 0, \zeta \neq 1. \tag{24}$$

4.3. Tsallis Measure of Entropy

The Tsallis entropy [59] measure is calculated from the next equation:

$$T_\zeta = \frac{1}{\zeta - 1}\left[1 - E_\zeta(\lambda,\delta)\right],\ \zeta > 0, \zeta \neq 1. \tag{25}$$

Inserting (21) into (25), then the Tsallis entropy of the KMPTL distribution is

$$T_\zeta = \frac{1}{\zeta - 1}\left[1 - \sum_{i,j=0}^{\infty} \Psi_{i,j} B\left(\zeta + j + 1,\, \delta(i+\zeta) - \frac{\zeta}{\lambda} + \frac{1}{\lambda}\right)\right],\ \zeta > 0, \zeta \neq 1. \tag{26}$$

4.4. Havrda and Charvat Measure of Entropy

The Havrda and Charvat entropy [60] measure is calculated from the next equation:

$$HC_\zeta = \frac{1}{2^{1-\zeta} - 1}\left[(E_\zeta(\lambda,\delta))^{\frac{1}{\zeta}} - 1\right],\ \zeta > 0, \zeta \neq 1. \tag{27}$$

Inserting (21) into (27), then the Havrda and Charvat entropy of the KMPTL distribution is

$$HC_\zeta = \frac{1}{2^{1-\zeta} - 1}\left[\left(\sum_{i,j=0}^{\infty} \Psi_{i,j} B\left(\zeta + j + 1,\, \delta(i+\zeta) - \frac{\zeta}{\lambda} + \frac{1}{\lambda}\right)\right)^{\frac{1}{\zeta}} - 1\right],\ \zeta > 0, \zeta \neq 1. \tag{28}$$

Tables 3 and 4 show some numerical values of R_ζ, A_ζ, T_ζ, and HC_ζ for the KMPTL distribution.

Table 3. Some numerical values of R_ζ, A_ζ, T_ζ, and HC_ζ for the KMPTL distribution at ζ = 0.5 and 0.8.

δ	λ	ζ = 0.5				ζ = 0.8			
		R_ζ	HC_ζ	A_ζ	T_ζ	R_ζ	HC_ζ	A_ζ	T_ζ
1.5	1.8	−0.133	−0.155	−0.125	−0.129	−0.18	−0.237	−0.176	−0.177
	2.3	−0.2	−0.23	−0.182	−0.191	−0.259	−0.34	−0.251	−0.252
	2.8	−0.277	−0.313	−0.242	−0.259	−0.348	−0.452	−0.333	−0.336
	3.3	−0.356	−0.394	−0.299	−0.326	−0.436	−0.562	−0.413	−0.418
	3.8	−0.432	−0.469	−0.351	−0.389	−0.521	−0.665	−0.488	−0.495
	4.3	−0.505	−0.539	−0.397	−0.446	−0.601	−0.762	−0.558	−0.566
	4.8	−0.575	−0.603	−0.437	−0.499	−0.677	−0.851	−0.623	−0.633
	5.3	−0.64	−0.662	−0.473	−0.548	−0.748	−0.934	−0.682	−0.694
	5.8	−0.703	−0.715	−0.505	−0.593	−0.815	−1.011	−0.737	−0.752
	6.3	−0.762	−0.765	−0.533	−0.634	−0.878	−1.083	−0.788	−0.805
2	1.8	−0.201	−0.231	−0.182	−0.191	−0.26	−0.34	−0.251	−0.253
	2.3	−0.297	−0.333	−0.257	−0.276	−0.368	−0.478	−0.352	−0.355
	2.8	−0.394	−0.431	−0.325	−0.357	−0.477	−0.611	−0.449	−0.455
	3.3	−0.487	−0.522	−0.386	−0.432	−0.579	−0.735	−0.539	−0.547
	3.8	−0.575	−0.603	−0.437	−0.5	−0.674	−0.848	−0.62	−0.63
	4.3	−0.657	−0.676	−0.482	−0.56	−0.762	−0.951	−0.694	−0.707
	4.8	−0.734	−0.741	−0.52	−0.614	−0.844	−1.045	−0.761	−0.777
	5.3	−0.806	−0.801	−0.553	−0.663	−0.92	−1.13	−0.822	−0.84
	5.8	−0.873	−0.854	−0.582	−0.708	−0.991	−1.21	−0.878	−0.899
	6.3	−0.937	−0.903	−0.608	−0.748	−1.058	−1.283	−0.93	−0.954
3.5	1.8	−0.392	−0.429	−0.324	−0.356	−0.471	−0.605	−0.445	−0.45
	2.3	−0.531	−0.563	−0.412	−0.466	−0.622	−0.787	−0.576	−0.585
	2.8	−0.658	−0.677	−0.482	−0.561	−0.758	−0.946	−0.69	−0.703
	3.3	−0.774	−0.774	−0.539	−0.642	−0.88	−1.085	−0.79	−0.807
	3.8	−0.878	−0.858	−0.584	−0.711	−0.989	−1.207	−0.876	−0.897
	4.3	−0.973	−0.93	−0.622	−0.771	−1.089	−1.316	−0.953	−0.978
	4.8	−1.061	−0.994	−0.654	−0.823	−1.179	−1.413	−1.021	−1.051
	5.3	−1.142	−1.05	−0.681	−0.87	−1.263	−1.501	−1.083	−1.116
	5.8	−1.216	−1.1	−0.704	−0.911	−1.34	−1.581	−1.139	−1.175
	6.3	−1.286	−1.145	−0.724	−0.949	−1.412	−1.654	−1.19	−1.23
5	1.8	−0.541	−0.572	−0.418	−0.474	−0.63	−0.797	−0.583	−0.592
	2.3	−0.702	−0.714	−0.504	−0.592	−0.801	−0.995	−0.726	−0.74
	2.8	−0.843	−0.831	−0.57	−0.688	−0.949	−1.162	−0.845	−0.864
	3.3	−0.969	−0.927	−0.62	−0.768	−1.079	−1.306	−0.946	−0.971
	3.8	−1.081	−1.008	−0.661	−0.835	−1.195	−1.43	−1.033	−1.063
	4.3	−1.182	−1.077	−0.693	−0.892	−1.3	−1.539	−1.11	−1.145
	4.8	−1.274	−1.137	−0.72	−0.942	−1.394	−1.637	−1.177	−1.217
	5.3	−1.359	−1.19	−0.743	−0.986	−1.481	−1.724	−1.238	−1.282
	5.8	−1.437	−1.237	−0.762	−1.025	−1.561	−1.803	−1.292	−1.341
	6.3	−1.509	−1.279	−0.779	−1.06	−1.635	−1.876	−1.342	−1.395

Table 4. Some numerical values of R_ζ, A_ζ, T_ζ, and HC_ζ for the KMPTL distribution at ζ = 1.2 and 2.0.

δ	λ	ζ = 1.2				ζ = 2			
		R_ζ	HC_ζ	A_ζ	T_ζ	R_ζ	HC_ζ	A_ζ	T_ζ
1.5	1.8	−0.225	−0.355	−0.229	−0.23	−0.284	−0.656	−0.305	−0.328
	2.3	−0.312	−0.498	−0.321	−0.322	−0.378	−0.92	−0.417	−0.46
	2.8	−0.409	−0.658	−0.423	−0.426	−0.481	−1.237	−0.544	−0.618
	3.3	−0.503	−0.818	−0.525	−0.53	−0.582	−1.578	−0.675	−0.789
	3.8	−0.593	−0.974	−0.624	−0.63	−0.676	−1.932	−0.804	−0.966
	4.3	−0.678	−1.122	−0.718	−0.726	−0.764	−2.295	−0.931	−1.148
	4.8	−0.757	−1.263	−0.807	−0.818	−0.847	−2.663	−1.054	−1.332
	5.3	−0.831	−1.397	−0.892	−0.905	−0.923	−3.035	−1.173	−1.518
	5.8	−0.901	−1.525	−0.972	−0.987	−0.995	−3.41	−1.289	−1.705
	6.3	−0.966	−1.647	−1.048	−1.066	−1.062	−3.786	−1.402	−1.893

Table 4. *Cont.*

δ	λ	ζ = 1.2				ζ = 2			
		R_ζ	HC_ζ	A_ζ	T_ζ	R_ζ	HC_ζ	A_ζ	T_ζ
2	1.8	−0.313	−0.499	−0.321	−0.323	−0.38	−0.924	−0.418	−0.462
	2.3	−0.43	−0.694	−0.446	−0.449	−0.504	−1.31	−0.573	−0.655
	2.8	−0.545	−0.89	−0.571	−0.576	−0.625	−1.735	−0.733	−0.868
	3.3	−0.653	−1.078	−0.69	−0.697	−0.737	−2.179	−0.891	−1.089
	3.8	−0.752	−1.254	−0.802	−0.812	−0.84	−2.632	−1.044	−1.316
	4.3	−0.844	−1.421	−0.906	−0.92	−0.935	−3.092	−1.191	−1.546
	4.8	−0.929	−1.577	−1.005	−1.021	−1.022	−3.556	−1.334	−1.778
	5.3	−1.008	−1.725	−1.097	−1.116	−1.103	−4.023	−1.471	−2.012
	5.8	−1.081	−1.865	−1.185	−1.207	−1.178	−4.493	−1.603	−2.246
	6.3	−1.15	−1.997	−1.267	−1.292	−1.247	−4.963	−1.732	−2.482
3.5	1.8	−0.538	−0.877	−0.563	−0.568	−0.616	−1.702	−0.721	−0.851
	2.3	−0.695	−1.153	−0.737	−0.746	−0.779	−2.358	−0.952	−1.179
	2.8	−0.836	−1.406	−0.897	−0.91	−0.924	−3.036	−1.174	−1.518
	3.3	−0.962	−1.638	−1.043	−1.06	−1.052	−3.726	−1.384	−1.863
	3.8	−1.074	−1.851	−1.176	−1.198	−1.167	−4.422	−1.584	−2.211
	4.3	−1.176	−2.048	−1.299	−1.326	−1.27	−5.123	−1.774	−2.561
	4.8	−1.268	−2.231	−1.413	−1.444	−1.364	−5.826	−1.956	−2.913
	5.3	−1.354	−2.402	−1.518	−1.554	−1.451	−6.531	−2.131	−3.266
	5.8	−1.432	−2.562	−1.617	−1.658	−1.53	−7.238	−2.298	−3.619
	6.3	−1.505	−2.713	−1.711	−1.756	−1.604	−7.945	−2.46	−3.973
5	1.8	−0.703	−1.166	−0.746	−0.755	−0.785	−2.387	−0.962	−1.193
	2.3	−0.878	−1.483	−0.946	−0.96	−0.965	−3.251	−1.241	−1.625
	2.8	−1.03	−1.768	−1.124	−1.144	−1.12	−4.132	−1.502	−2.066
	3.3	−1.164	−2.024	−1.284	−1.31	−1.256	−5.022	−1.747	−2.511
	3.8	−1.282	−2.258	−1.429	−1.461	−1.376	−5.917	−1.979	−2.959
	4.3	−1.388	−2.472	−1.562	−1.6	−1.483	−6.815	−2.199	−3.408
	4.8	−1.484	−2.67	−1.684	−1.728	−1.581	−7.716	−2.408	−3.858
	5.3	−1.572	−2.854	−1.797	−1.847	−1.669	−8.618	−2.608	−4.309
	5.8	−1.653	−3.027	−1.903	−1.959	−1.751	−9.521	−2.8	−4.761
	6.3	−1.728	−3.189	−2.002	−2.064	−1.827	−10.425	−2.985	−5.213

5. Cumulative Residual Rényi Entropy

Plenty of authors have become interested in alternative metrics of uncertainty for probability distributions in recent years, particularly in survival and reliability analysis investigations. As a result, Ref. [61] proposed the cumulative residual Rényi entropy as

$$\rho_\zeta = \frac{1}{1-\zeta} \log\left[D_\zeta(\lambda, \delta)\right],\ \zeta > 0, \zeta \neq 1, \tag{29}$$

where $D_\zeta(\lambda,\delta) = \int_0^1 [s(z;\lambda,\delta)]^\zeta dz$. Now, we need to calculate $D_\zeta(\lambda,\delta)$. Then,

$$D_\zeta(\lambda,\delta) = \int_0^1 (S(z;\lambda,\delta))^\zeta dz = \int_0^1 \left[1 - \frac{e}{e-1}\left(1 - e^{-z^{\lambda\delta}\left(2-z^\lambda\right)^\delta}\right)\right]^\zeta dz.$$

By applying the binomial expansion in the previous equation two times, we get

$$D_\zeta(\lambda,\delta) = \sum_{i,j=0}^{\infty} (-1)^{i+j} \binom{\zeta}{i} \binom{i}{j} \left(\frac{e}{e-1}\right)^i \int_0^1 e^{-jz^{\lambda\delta}\left(2-z^\lambda\right)^\delta} dz.$$

By applying the exponential expansion (13) in the previous equation, we get

$$D_\zeta(\lambda,\delta) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k} j^k}{k!} \binom{\zeta}{i} \binom{i}{j} \left(\frac{e}{e-1}\right)^i \int_0^1 z^{k\lambda\delta}\left(2-z^\lambda\right)^{k\delta} dz.$$

Let $x = z^{\lambda}$, then

$$D_{\zeta}(\lambda,\delta) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k} j^k}{\lambda k!} \binom{\zeta}{i} \binom{i}{j} \left(\frac{e}{e-1}\right)^i \int_0^1 x^{k\delta+\frac{1}{\lambda}-1} (2-x)^{k\delta} dx.$$

Let $v = 1 - x$, then

$$D_{\zeta}(\lambda,\delta) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k} j^k}{\lambda k!} \binom{\zeta}{i} \binom{i}{j} \left(\frac{e}{e-1}\right)^i \int_0^1 (1-v)^{k\delta+\frac{1}{\lambda}-1} (1+v)^{k\delta} dv.$$

By applying the binomial theory (15) in the previous equation, we get

$$D_{\zeta}(\lambda,\delta) = \sum_{i,j,k,m=0}^{\infty} \Phi_{i,j,k,m} \int_0^1 v^m (1-v)^{k\delta+\frac{1}{\lambda}-1} dv.$$

Then,

$$D_{\zeta}(\lambda,\delta) = \sum_{i,j,k,m=0}^{\infty} \Phi_{i,j,k,m} B\left(m+1,\, k\delta+\frac{1}{\lambda}\right), \tag{30}$$

where, $\Phi_{i,j,k,m} = \frac{(-1)^{i+j+k} j^k}{\lambda k!} \binom{\zeta}{i} \binom{i}{j} \binom{k\delta}{m} \left(\frac{e}{e-1}\right)^i$. Inserting (30) into (29), then the cumulative residual Rényi entropy of the KMPTL distribution is

$$\rho_{\zeta} = \frac{1}{1-\zeta} \log\left[\sum_{i,j,k,m=0}^{\infty} \Phi_{i,j,k,m} B\left(m+1,\, k\delta+\frac{1}{\lambda}\right)\right],\ \zeta > 0, \zeta \neq 1. \tag{31}$$

6. Estimation Methods Based on Sampling Approach

In this section, the MLE and Bayesian have been discussed. The procedures used to draw conclusions or make generalizations about a population parameter make up the estimate theory. Differentiating between the maximum likelihood approach and the Bayesian method of estimate is the current fashion. The Bayesian approach makes use of sample data information as well as previously acquired subjective knowledge about the probability distribution of the unknown parameters. In addition, the most popular technique for gathering data is the SRS approach. RSS approaches may be employed in these scenarios to study greater proportions of samples from the population under consideration and enhance the accuracy of statistical inference. This approach was first discussed and introduced by [62], which originally proposed the RSS approach. Numerous investigations have demonstrated that RSS statistical processes are numerically and theoretically superior to their SRS competitors.

The next step outlines the technique for choosing the samples:

- Choose observations from an r-set, each of which contains r randomly chosen units.
- Using an accessible auxiliary variable or individual assessment, the units are rated.
- The lowest unit in the first set should be chosen, followed by the second-lowest unit in the second set, and so on until the fourth unit in the fourth set is chosen.

$$\begin{array}{lllllll}
1. & \underline{z^{(1)}_{(1:r)}} & z^{(1)}_{(2:r)} & \ldots & z^{(1)}_{(r:r)} & \rightarrow z_{(1)c} = z^{(1)}_{(1:r)} \\
2. & z^{(2)}_{(1:r)} & \underline{z^{(2)}_{(2:r)}} & \ldots & z^{(2)}_{(r:r)} & \rightarrow z_{(2)c} = z^{(2)}_{(2:r)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \qquad \vdots \\
n. & z^{(r)}_{(1:r)} & z^{(r)}_{(2:r)} & \ldots & \underline{z^{(r)}_{(r:r)}} & \rightarrow z_{(r)c} = z^{(r)}_{(r:r)}
\end{array}$$

where $c = 1, 2, \ldots, m$ as a cycle iteration.

- The process returns ranked set samples with an r. The operation will therefore be carried out m times in order to obtain the necessary sample n, where $n = mr$, $c = 1, 2, \ldots, m$, and $h = 1, 2, \ldots, r$ samples. According to the foundation of flawless judgement ranking, $z_{(h)c}$ has the same distribution; see citation [63].

$$g_{(h)c}(z) = \frac{r!}{(h-1)!(r-h)!}[F(z_{(h)c})]^{h-1}[1-F(z_{(h)c})]^{r-h}f(z_{(h)c}), 0 < z_{(h)c} < \infty. \tag{32}$$

For more information, see recent papers such as [64–66]. In this section, maximum likelihood (ML) and Bayesian using SRS and RSS have been discussed.

6.1. Maximum Likelihood Estimation under SRS

In this subsection, first, we need to look into the ML estimates (MLEs) of λ and δ according to SRS. Assume that Z_i, where $i = (1, \ldots, n)$. The likelihood function (LFN) using SRS is:

$$L(z; \Delta) = \frac{2^n \lambda^n \delta^n e^n}{(e-1)^n} \prod_{i=1}^{n} z_i^{\lambda\delta-1}\left(1-z_i^{\lambda}\right)\left(2-z_i^{\lambda}\right)^{\delta-1} e^{-\sum_{i=1}^{n} z_i^{\lambda\delta}\left(2-z_i^{\lambda}\right)^{\delta}}. \tag{33}$$

The ln-LFN under SRS is:

$$\begin{aligned}\ell(z; \Delta) =& n[\ln(2) + \ln(\lambda) + \ln(\delta) + 1 - \ln(e-1)] + (\lambda\delta - 1)\sum_{i=1}^{n} \ln(z_i) + \sum_{i=1}^{n} \ln\left(1 - z_i^{\lambda}\right) + \\ &(\delta - 1)\sum_{i=1}^{n} \ln\left(2 - z_i^{\lambda}\right) - \sum_{i=1}^{n} z_i^{\lambda\delta}\left(2 - z_i^{\lambda}\right)^{\delta}.\end{aligned} \tag{34}$$

In order to solve the non-linear log-LFNs that are produced by differentiating Equation (34) (with respect to the distribution parameters $\Delta = (\lambda, \delta)$) and equating it to zero, Equation (34) can be explicitly maximized using the R program by a "mle2" function, which implements the NelderMead (NM) maximization procedure for MLE calculations.

6.2. Maximum Likelihood Estimation under RSS

Assume that c-cycle of RSS is $z_{(h)c}; h = 1, \ldots, r, \;\; c = 1, 2, \ldots, m$, which has KMPTL distribution with vector parameters Δ. Equation (32) yields the next function under RSS:

$$\begin{aligned}g_{(h)c}(z|\Delta) =& \frac{r!}{(h-1)!(r-h)!}\left[\frac{e}{e-1}\left(1 - e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]^{h-1} \frac{2\lambda\delta e}{e-1} z_{(h)c}^{\lambda\delta-1} \\ &\left[1 - \frac{e}{e-1}\left(1 - e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]^{r-h}\left(1 - z_{(h)c}^{\lambda}\right)\left(2 - z_{(h)c}^{\lambda}\right)^{\delta-1} e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}.\end{aligned} \tag{35}$$

By utilizing Equation (35), the LF under RSS is provided via

$$\begin{aligned}L_{RSS}(z|\Delta) =& \prod_{c=1}^{m}\prod_{h=1}^{r} \frac{r!}{(h-1)!(r-h)!}\left[\frac{e}{e-1}\left(1 - e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]^{h-1} z_{(h)c}^{\lambda\delta-1}\frac{2^n\lambda^n\delta^n e^n}{(e-1)^n} \\ &\prod_{c=1}^{m}\prod_{h=1}^{r}\left[1 - \frac{e}{e-1}\left(1 - e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]^{r-h}\left(1 - z_{(h)c}^{\lambda}\right)\left(2 - z_{(h)c}^{\lambda}\right)^{\delta-1} e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}.\end{aligned} \tag{36}$$

In this case, the ln-LF of the KMPTL distribution is supplied by

$$
\begin{aligned}
\ell_{RSS}(z|\Delta) =& n[\ln(2)+\ln(\lambda)+\ln(\delta)+1-\ln(e-1)]+\sum_{c=1}^{m}\sum_{h=1}^{r}\ln\left[\frac{r!}{(h-1)!(r-h)!}\right]+\\
&\sum_{c=1}^{m}\sum_{h=1}^{r}(h-1)\ln\left[\frac{e}{e-1}\right]+\sum_{c=1}^{m}\sum_{h=1}^{r}(h-1)\ln\left[1-e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right]+(\lambda\delta-1)\sum_{c=1}^{m}\sum_{h=1}^{r}\ln(z_{(h)c})+\\
&\sum_{c=1}^{m}\sum_{h=1}^{r}(r-h)\ln\left[1-\frac{e}{e-1}\left(1-e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]+\sum_{c=1}^{m}\sum_{h=1}^{r}\ln\left(1-z_{(h)c}^{\lambda}\right)+\\
&(\delta-1)\sum_{c=1}^{m}\sum_{h=1}^{r}\ln\left(2-z_{(h)c}^{\lambda}\right)-\sum_{c=1}^{m}\sum_{h=1}^{r}z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}.
\end{aligned} \tag{37}
$$

In order to solve the non-linear RSS of log-LFNs that are created by differentiating Equation (37) (with regards to the distribution parameters Δ) and equating it to 0, Equation (37) can be explicitly maximized employing the R program by a "mle2" function, which implements the NM maximization procedure for MLE calculations; see [67].

6.3. Bayesian Estimation

In many different contexts and fields, such as physics (see [68]), food chain (see [69]), epidemiology (see [70]), environmental (see [71]), COVID-19 (see [72]), and econometrics (see [73]), Bayesian approaches for parameter estimation have been successfully used. One benefit of using Bayesian approaches is that they enable model estimation when more complex models prevent standard estimation from succeeding [74].

The Bayes estimate based on ranked set sampling of the Δ function under the square error loss (SEL) function will now be covered. The posterior distributions given the data are used to construct the Bayes estimates (BEs) of the parameters in the Δ BE vector. The suggested method is described in brief below. Assume that I and K are the iterations and sample burn, respectively. The posterior mean is then used to minimize the SEL function for the presumptive prior distribution as follows:

$$
\tilde{\Delta} = \frac{1}{I-K}\sum_{j=K+1}^{I}\tilde{\Delta}^{(j)}
$$

where $\tilde{\Delta}^{(j)}$ is the Bayesian estimation vector for Δ at iteration (j). Furthermore, using the R software's coda and HDInterval package, developed in [75] and [76], respectively, we can determine MCMC results and the highest posterior density (HPD) credible interval for Δ, respectively.

Let $z_1, z_2, \ldots, z_n$ be associated observations, and $z_1, z_2, \ldots, z_n$ be a random sample of size n taken from a general distribution of the KMPTL distribution with unknown parameters, assumed to have independent rvs that adhere to the gamma distribution with the PDF specified by

$$
\Pi(\Delta) \propto \lambda^{a_1-1}\delta^{a_2-1}e^{-(b_1\lambda+b_2\delta)}. \tag{38}
$$

To determine the proper hyper-parameters for the independent joint prior, use the estimate and variance-covariance matrix of the MLE technique. By equating the gamma priors' mean and variance, the estimated hyper-parameters can be written as follows:

$$
a_1 = \frac{\left[\frac{1}{L}\sum_{j=1}^{L}\hat{\lambda}^j\right]^2}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\lambda}^j-\frac{1}{L}\sum_{i=1}^{L}\hat{\lambda}^j\right]^2};\quad a_2 = \frac{\left[\frac{1}{L}\sum_{j=1}^{L}\hat{\delta}^j\right]^2}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\delta}^j-\frac{1}{L}\sum_{i=1}^{L}\hat{\delta}^j\right]^2}
$$

$$
b_1 = \frac{\frac{1}{L}\sum_{i=1}^{L}\hat{\lambda}^j}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\lambda}^j-\frac{1}{L}\sum_{i=1}^{L}\hat{\lambda}^j\right]^2};\quad b_2 = \frac{\frac{1}{L}\sum_{i=1}^{L}\hat{\delta}^j}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\delta}^j-\frac{1}{L}\sum_{i=1}^{L}\hat{\delta}^j\right]^2}.
$$

where L denotes the quantity of MLE iterations.

The parameters' posterior distribution can be stated as follows:

$$\pi^*(\Delta \mid data) = \frac{\Pi(\Delta)\, L(\Delta \mid data)}{\int_0^\infty \int_0^\infty \Pi(\Delta)\, L(\Delta \mid data)\, d\lambda d\delta}. \tag{39}$$

where *data* is the generated sample by SRS or RSS. It is possible to express the joint posterior to the proportionality as an equation, as shown in Equation (40).

$$\begin{aligned} \pi^*(\Delta \mid data) \propto & \prod_{c=1}^{m}\prod_{h=1}^{r} \frac{r!}{(h-1)!(r-h)!}\left[\frac{e}{e-1}\left(1-e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]^{h-1} z_{(h)c}^{\lambda\delta-1}\lambda^{n+a_1-1}\delta^{n+a_2-1}e^{-(b_1\lambda+b_2\delta)} \\ & \prod_{c=1}^{m}\prod_{h=1}^{r}\left[1-\frac{e}{e-1}\left(1-e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]^{r-h}\left(1-z_{(h)c}^{\lambda}\right)\left(2-z_{(h)c}^{\lambda}\right)^{\delta-1}e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}. \end{aligned} \tag{40}$$

The complete conditionals for λ and δ can be written, up to proportionality, as

$$\begin{cases} \pi^*(\lambda \mid \delta, data) \propto \prod_{c=1}^{m}\prod_{h=1}^{r}\left(1-e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)^{h-1} z_{(h)c}^{\lambda\delta-1}\lambda^{n+a_1-1}e^{-b_1\lambda} \\ \quad \prod_{c=1}^{m}\prod_{h=1}^{r}\left[1-\frac{e}{e-1}\left(1-e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]^{r-h}\left(1-z_{(h)c}^{\lambda}\right)\left(2-z_{(h)c}^{\lambda}\right)^{\delta-1}e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}. \\ \pi^*(\delta \mid \lambda data) \propto \prod_{c=1}^{m}\prod_{h=1}^{r}\left[1-e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right]^{h-1} z_{(h)c}^{\lambda\delta-1}\delta^{n+a_2-1}e^{-b_2\delta} \\ \quad \prod_{c=1}^{m}\prod_{h=1}^{r}\left[1-\frac{e}{e-1}\left(1-e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}\right)\right]^{r-h}\left(2-z_{(h)c}^{\lambda}\right)^{\delta-1}e^{-z_{(h)c}^{\lambda\delta}\left(2-z_{(h)c}^{\lambda}\right)^{\delta}}. \end{cases} \tag{41}$$

We can use the Metropolis–Hastings algorithm (MHA) and the Gibbs sampling technique to produce samples from the posterior distributions as none of these posterior PDFs relate to a single common distribution. References [77–79] provide additional information on this topic. The normal distribution is the proposal distribution used by the MHA. The detailed steps are given below, taking into account the Gibbs sampling method:

- Begin with initial values $\lambda^{(0)} = \hat{\lambda}$, $\delta^{(0)} = \hat{\delta}$;
- Put t = 1;
- Utilize the MHA to create $\lambda^{(t)}$;
- Utilize the MHA to create $\delta^{(t)}$;
- Set t = t + 1;
- Repeat the procedures items I times.

Based on the SEL function, $\tilde{\Delta}$ can be determined for a large enough I, which make convergence of MCMC results. Furthermore, the method described in [80] can be used to generate a roughly $100(1-\alpha)\%$ HPD credible interval of Δ.

7. Simulation

Simulation studies has been used to produce random sampling of the proposed model. Because of some parameter factors, simulation studies are characterized by understanding the behavior of statistical approaches. This enables us to take into account technique characteristics such as bias, MSE, standard deviation, etc.; see also [81].

Using the R-4.3.0 program, a simulation research was carried out to compare the effectiveness of the ML and BE approaches under the SRS and RSS sampling strategies, as stated above for the unique KMPTL distribution. In this regard, the KMPTL distribution is used to generate M = 1000 moderate samples for each techniques, each with size n = (8,12,17). The parameter $\Delta = (\lambda, \delta)$ values have been selected as:

In Table 5: $\lambda = 0.5$ and $\delta = 0.5, 2, 3$.
In Table 6: $\lambda = 2$ and $\delta = 0.5, 2, 3$.
In Table 7: $\lambda = 3$ and $\delta = 0.5, 2, 3$.

We utilise $I = 12,000$ iterations and the first 16.67% as burn-in samples for the Bayes estimation. The bias and mean square error (MSE) of the estimates are examined. The table results show the simulation findings, including the length of confidence intervals (LCI) with converge probability (CP). In LCI, the asymptotic CI (LACI) has been obtained for MLE, and credible CI (LCCI) for Bayesian has been obtained. In addition, the relative efficiency (RE) of sampling techniques were

$$RE1(\Delta) = \frac{MSE_{SRS}(\Delta)}{MSE_{RSS1}(\Delta)} \times 100\%,\ RE2(\Delta) = \frac{MSE_{SRS}(\Delta)}{MSE_{RSS2}(\Delta)} \times 100\%, RE3(\Delta) = \frac{MSE_{RSS1}(\Delta)}{MSE_{RSS2}(\Delta)} \times 100\% .$$

Below, we provide a few crucial procedures for the numerical simulation:

- Select the replication number M & I, the sample size n, and the parameter values;
- Select the replication number cycle m, and the sample size of each cycle r;
- Generate SRS and RSS from LMPTL by using the "rss" package in the R-4.3.0 program and qf (11) of the KMPTL distribution.
- Using the simulated data, determine the MLEs and BEs of the KMPTL distribution's parameters;
- Repeat the above steps, M times;
- Determine the mean of bias, average MSE, RE, LCI, and CP for each parameter.

From Tables 5–7, we can conclude the following:

- In every calculation, the bias, MSE and LCI become smaller as n is increased.
- The relevance of the Bias, MSE, and LCI for KMPL distribution parameters decreases as the number of cycles (r) in the RSS sampling rises, but the RE rises.
- As the sample size for each cycle (m) rises in the RSS sampling, the significance of the Bias, MSE, and LCI for KMPTL distribution parameters declines, while the RE increases.
- Comparatively speaking, Bayesian estimates are substantially more efficient than MLE for the majority of KMPTL distribution parameters.
- Comparatively speaking, RSS techniques are substantially more efficient than SRS for the majority of KMPTL distribution parameters.
- The CP rises when the sample size increases.
- Comparatively speaking, credible CI of HPD is substantially more efficient than asymptotic CI for the majority of the KMPTL distribution parameters.
- MSE increases when parameter δ increased

Table 5. MLE and Bayesian at λ = 0.5.

	$\lambda = 0.5$			Point						Confidance						RE		
				SRS		RSS c = 1		RSS c = 3		SRS		RSS c = 1		RSS c = 3				
	δ	n		Bias	MSE	Bias	MSE	Bias	MSE	LCI	CP	LCI	CP	LCI	CP	RE1	RE2	RE3
MLE	0.5	8	λ	0.0855	0.2443	0.0281	0.0898	0.0241	0.0674	1.9103	94.00%	1.1708	95.20%	1.0055	95.40%	272%	363%	133%
			δ	0.3527	0.4438	0.1523	0.1346	0.0859	0.0985	2.2175	97.20%	1.3099	98.20%	1.1848	97.40%	330%	450%	137%
		12	λ	0.0420	0.1752	0.0241	0.0889	0.0201	0.0280	1.6342	93.80%	1.1904	97.00%	0.6514	97.60%	197%	627%	318%
			δ	0.3056	0.3524	0.0975	0.0885	0.0471	0.0506	2.4720	97.40%	1.1029	98.00%	0.8634	97.60%	398%	696%	175%
		17	λ	0.0251	0.1286	0.0324	0.0514	−0.0095	0.0187	1.4040	93.80%	0.8808	96.00%	0.5355	98.00%	250%	687%	275%
			δ	0.2812	0.3066	0.0671	0.0718	0.0529	0.0313	1.8718	97.40%	1.0178	98.00%	0.6623	97.40%	427%	980%	229%
	2	8	λ	0.1813	0.3690	0.0221	0.0468	0.0119	0.0116	2.2748	92.80%	0.8443	96.80%	0.4196	97.80%	789%	3188%	404%
			δ	0.1794	1.0304	0.0647	0.2270	−0.0054	0.0592	3.9204	96.40%	1.8523	93.20%	0.9547	96.80%	454%	1740%	383%
		12	λ	0.1628	0.2873	0.0242	0.0456	−0.0082	0.0016	2.0040	92.00%	0.9133	95.40%	0.1586	95.80%	630%	17,571%	2788%
			δ	0.0885	0.7759	−0.0063	0.2198	0.0243	0.0418	3.4388	91.40%	2.1409	94.40%	0.7969	95.40%	353%	1855%	525%
		17	λ	0.1030	0.1748	0.0214	0.0411	0.0072	0.0016	1.5898	93.00%	0.7911	98.60%	0.5394	99.40%	425%	10,996%	2586%
			δ	0.0675	0.5526	0.0061	0.0859	0.0100	0.0336	2.9049	91.00%	1.1496	96.80%	0.7184	96.60%	644%	1644%	255%
	3	8	λ	0.1337	0.3223	0.0029	0.0209	0.0045	0.0055	2.1651	94.80%	0.5676	99.00%	0.2901	99.00%	1540%	5874%	381%
			δ	0.0877	0.8553	0.0494	0.1465	−0.0041	0.0330	3.6126	92.20%	1.4892	95.00%	0.7131	97.20%	584%	2589%	443%
		12	λ	0.0793	0.1269	0.0142	0.0127	−0.0016	0.0007	1.3628	95.40%	0.4392	96.80%	0.1033	96.40%	997%	18,240%	1829%
			δ	0.0412	0.9558	−0.0008	0.1281	0.0167	0.0311	3.8329	92.80%	2.0797	95.20%	0.6886	98.20%	746%	3076%	412%
		17	λ	0.0564	0.0992	0.0024	0.0026	−0.0010	0.0005	1.2157	95.80%	0.2007	99.40%	0.0907	97.40%	3783%	18,510%	489%
			δ	0.0230	0.4447	0.0039	0.0472	0.0062	0.0157	2.6153	93.20%	0.8528	95.60%	0.4906	96.40%	941%	2838%	302%

Table 5. *Cont.*

	$\lambda = 0.5$			Point						Confidance						RE		
				SRS		RSS c = 1		RSS c = 3		SRS		RSS c = 1		RSS c = 3				
	δ	n		Bias	MSE	Bias	MSE	Bias	MSE	LCI	CP	LCI	CP	LCI	CP	RE1	RE2	RE3
Bayesian	0.5	8	λ	0.1083	0.0841	0.0517	0.0159	0.0517	0.0015	0.9130	93.86%	0.6465	98.57%	0.2523	89.00%	528%	5498%	1041%
			δ	0.1445	0.1479	0.0276	0.0114	0.0276	0.0001	1.0408	92.29%	0.6781	99.43%	0.2042	82.57%	1303%	224,969%	17272%
		12	λ	0.0832	0.0456	0.0297	0.0043	0.0297	0.0007	0.8189	95.71%	0.4811	100.00%	0.1731	90.29%	1054%	6853%	650%
			δ	0.1139	0.0902	0.0114	0.0032	0.0114	0.0000	0.9614	95.00%	0.5228	100.00%	0.1451	86.71%	2793%	334,035%	11959%
		17	λ	0.0655	0.0296	0.0186	0.0022	0.0186	0.0003	0.7501	98.29%	0.3906	100.00%	0.1314	91.80%	1352%	9137%	676%
			δ	0.0854	0.0577	0.0099	0.0016	0.0099	0.0000	0.8768	94.71%	0.4388	100.00%	0.1172	86.00%	3585%	356,028%	9930%
	2	8	λ	0.0733	0.0292	0.0116	0.0031	0.0116	0.0011	0.7491	98.14%	0.3264	99.29%	0.1657	90.57%	956%	2773%	290%
			δ	0.0728	0.1486	0.0046	0.0010	0.0046	0.0000	2.8695	99.57%	0.9168	100.00%	0.2086	99.71%	14,194%	632,266%	4454%
		12	λ	0.0458	0.0115	0.0027	0.0012	0.0027	0.0005	0.6043	99.57%	0.2217	100.00%	0.1119	89.71%	959%	2347%	245%
			δ	0.0452	0.0440	0.0028	0.0004	0.0028	0.0000	2.2306	100.00%	0.6511	100.00%	0.1449	100.00%	10,682%	276,372%	2587%
		17	λ	0.0292	0.0065	0.0034	0.0007	0.0034	0.0003	0.4974	99.71%	0.1648	99.71%	0.0788	89.00%	992%	2525%	255%
			δ	0.0329	0.0202	0.0012	0.0002	0.0012	0.0000	1.8829	100.00%	0.5275	100.00%	0.1164	100.00%	8281%	141,788%	1712%
	3	8	λ	0.0569	0.0185	0.0122	0.0026	0.0122	0.0010	0.6567	98.71%	0.2932	99.57%	0.1538	87.71%	704%	1800%	256%
			δ	0.0599	0.0772	0.0035	0.0006	0.0035	0.0000	3.2656	100.00%	0.9224	100.00%	0.2064	100.00%	12,553%	275,087%	2191%
		12	λ	0.0353	0.0103	0.0049	0.0013	0.0049	0.0004	0.5152	99.29%	0.1994	99.86%	0.1041	84.43%	816%	2291%	281%
			δ	0.0326	0.0190	0.0022	0.0002	0.0022	0.0000	2.4116	100.00%	0.6532	100.00%	0.1430	100.00%	8331%	79,911%	959%
		17	λ	0.0227	0.0056	0.0012	0.0006	0.0012	0.0003	0.4300	99.43%	0.1442	100.00%	0.0731	85.00%	885%	2252%	255%
			δ	0.0148	0.0098	0.0010	0.0002	0.0010	0.0000	1.9831	100.00%	0.5311	100.00%	0.1137	100.00%	5910%	42,117%	713%

Table 6. MLE and Bayesian at λ = 2.0.

	$\lambda = 2$			Point						Confidence						RE		
				SRS		RSS c = 1		RSS c = 3		SRS		RSS c = 1		RSS c = 3				
	δ	n		Bias	MSE	Bias	MSE	Bias	MSE	LCI	CP	LCI	CP	LCI	CP	RE1	RE2	RE3
MLE	0.5	8	λ	−0.3228	1.3168	−0.2769	0.5790	−0.1696	0.4183	4.3211	96.20%	2.7810	97.40%	2.4492	97.20%	227%	315%	138%
			δ	0.7846	1.6061	0.3501	0.4504	0.2261	0.2800	3.9052	97.20%	2.2466	95.20%	1.8773	93.80%	357%	574%	161%
		12	λ	−0.3533	1.1530	−0.1201	0.4148	−0.0659	0.1485	3.9788	96.80%	2.4830	96.20%	1.4901	94.00%	278%	776%	279%
			δ	0.6764	1.2864	0.1790	0.1909	0.0978	0.1891	3.5723	95.20%	1.5640	95.40%	1.6626	96.20%	674%	680%	101%
		17	λ	−0.3775	0.9961	−0.1037	0.2216	−0.0593	0.1354	3.6253	97.00%	1.8019	94.60%	1.4932	94.40%	449%	736%	164%
			δ	0.6290	1.0958	0.1070	0.0943	0.0812	0.1325	3.2834	96.00%	1.1294	94.40%	2.1871	97.80%	1162%	827%	71%
	2	8	λ	0.2120	2.3271	0.1444	0.9670	0.1247	0.5275	5.9261	94.86%	3.8158	92.71%	2.8068	95.71%	241%	441%	183%
			δ	1.1741	4.6654	0.3296	1.1130	0.2055	0.8799	7.1119	98.86%	3.9314	98.86%	3.5904	98.29%	419%	530%	126%
		12	λ	0.1358	1.8338	0.0884	0.6992	0.0429	0.2391	5.2854	93.43%	3.2618	93.43%	1.9107	94.14%	262%	767%	292%
			δ	0.9946	3.5393	0.3076	1.0125	0.0967	0.3363	6.2644	98.29%	4.1378	98.14%	2.2432	96.43%	350%	1052%	301%
		17	λ	0.1497	1.3820	0.0407	0.4263	0.0093	0.1806	4.5740	94.71%	2.5562	95.00%	1.6665	95.00%	324%	765%	236%
			δ	0.6744	2.4489	0.2204	0.6674	0.0913	0.3156	5.5394	98.43%	3.0859	97.00%	2.2826	95.86%	367%	776%	212%
	3	8	λ	0.3585	2.4085	0.2925	1.2106	0.1777	0.6230	5.9233	93.00%	4.1608	92.43%	3.0167	92.71%	199%	387%	194%
			δ	0.8685	4.3651	0.1212	1.7243	0.1372	1.3896	7.4541	99.43%	5.1292	99.00%	4.5929	97.86%	253%	314%	124%
		12	λ	0.2974	2.2241	0.0936	0.5135	0.0478	0.1616	5.7327	91.57%	2.7870	93.14%	1.5660	95.71%	433%	1376%	318%
			δ	0.8093	4.0765	0.2741	1.3776	0.0374	0.4212	7.7461	98.43%	4.4769	95.57%	2.5418	95.00%	296%	968%	327%
		17	λ	0.2908	1.6559	0.1129	0.3877	0.0281	0.1114	4.9173	93.14%	2.4022	92.29%	1.3047	95.71%	427%	1486%	348%
			δ	0.5567	3.2766	0.1160	1.0142	0.0475	0.3522	6.7566	98.86%	3.9243	96.00%	2.3204	94.43%	323%	930%	288%

Table 6. *Cont.*

	$\lambda = 2$			Point						Confidance						RE		
				SRS		RSS c = 1		RSS c = 3		SRS		RSS c = 1		RSS c = 3				
	δ	n		Bias	MSE	Bias	MSE	Bias	MSE	LCI	CP	LCI	CP	LCI	CP	RE1	RE2	RE3
Bayesian	0.5	8	λ	0.0245	0.0399	0.0178	0.0106	0.0178	0.0057	1.9912	100.00%	1.3898	100.00%	0.7058	99.71%	375%	703%	187%
			δ	0.1144	0.0697	0.0262	0.0049	0.0262	0.0002	0.9157	96.71%	0.5223	100.00%	0.1831	84.71%	1422%	44,461%	3126%
		12	λ	0.0137	0.0120	0.0072	0.0038	0.0072	0.0030	1.4957	100.00%	0.9943	100.00%	0.4938	100.00%	316%	398%	126%
			δ	0.0737	0.0319	0.0131	0.0019	0.0131	0.0001	0.7516	97.86%	0.3680	100.00%	0.1291	87.86%	1716%	43,486%	2534%
		17	λ	0.0060	0.0054	0.0011	0.0021	0.0011	0.0015	1.2491	100.00%	0.8081	100.00%	0.3874	100.00%	260%	359%	138%
			δ	0.0505	0.0166	0.0090	0.0010	0.0090	0.0000	0.6474	98.86%	0.2921	100.00%	0.1034	90.29%	1657%	39,855%	2405%
	2	8	λ	0.0525	0.0335	0.0216	0.0194	0.0216	0.0082	1.8112	100.00%	1.0216	99.86%	0.5501	95.29%	172%	410%	238%
			δ	0.1625	0.1404	0.0102	0.0019	0.0102	0.0000	2.6626	99.86%	0.8821	100.00%	0.2076	98.14%	7545%	439,806%	5829%
		12	λ	0.0341	0.0139	0.0059	0.0083	0.0059	0.0038	1.3486	100.00%	0.7071	100.00%	0.3718	96.14%	166%	363%	218%
			δ	0.0961	0.0615	0.0036	0.0007	0.0036	0.0000	2.0205	100.00%	0.6228	100.00%	0.1433	99.43%	8775%	280,496%	3196%
		17	λ	0.0235	0.0083	0.0064	0.0048	0.0064	0.0023	1.1121	100.00%	0.5422	100.00%	0.2719	94.43%	174%	357%	206%
			δ	0.0677	0.0329	0.0036	0.0004	0.0036	0.0000	1.6737	100.00%	0.5035	100.00%	0.1144	99.86%	8102%	188,280%	2324%
	3	8	λ	0.0525	0.0335	0.0216	0.0194	0.0216	0.0082	1.8112	100.00%	1.0216	99.86%	0.5501	95.29%	172%	410%	238%
			δ	0.1625	0.1404	0.0102	0.0019	0.0102	0.0000	2.6626	99.86%	0.8821	100.00%	0.2076	98.14%	7545%	439,806%	5829%
		12	λ	0.0341	0.0139	0.0059	0.0083	0.0059	0.0038	1.3486	100.00%	0.7071	100.00%	0.3718	96.14%	166%	363%	218%
			δ	0.0961	0.0615	0.0036	0.0007	0.0036	0.0000	2.0205	100.00%	0.6228	100.00%	0.1433	99.43%	8775%	280,496%	3196%
		17	λ	0.0235	0.0083	0.0064	0.0048	0.0064	0.0023	1.1121	100.00%	0.5422	100.00%	0.2719	94.43%	174%	357%	206%
			δ	0.0677	0.0329	0.0036	0.0004	0.0036	0.0000	1.6737	100.00%	0.5035	100.00%	0.1144	99.86%	8102%	188,280%	2324%

Table 7. MLE and Bayesian at λ = 3.0.

$\lambda = 3$				Point						Confidance						RE		
				SRS		RSS c = 1		RSS c = 3		SRS		RSS c = 1		RSS c = 3				
Methods	δ	**n**		**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**LCI**	**CP**	**LCI**	**CP**	**LCI**	**CP**	**RE1**	**RE2**	**RE3**
MLE	0.3	8	λ	−0.5190	1.6445	−0.0926	0.2282	−0.0331	0.0546	4.6001	98.43%	1.8386	95.14%	0.9072	96.00%	720%	3013%	418%
			δ	0.4907	0.9970	0.0389	0.0346	0.0100	0.0041	3.4111	92.29%	0.7140	98.00%	0.2494	98.86%	2878%	24,059%	836%
		12	λ	−0.3785	0.9832	−0.0357	0.0950	−0.0308	0.0479	3.5952	91.43%	1.2010	94.29%	1.0956	94.86%	1035%	2053%	198%
			δ	0.2685	0.4480	0.0165	0.0101	0.0063	0.0014	2.4050	92.14%	0.3898	97.43%	0.1470	95.43%	4415%	31,043%	703%
		17	λ	−0.1941	0.5198	−0.0280	0.0331	−0.0090	0.0048	2.7238	91.86%	0.7051	94.57%	0.2681	97.71%	1571%	10939%	696%
			δ	0.1332	0.2135	0.0056	0.0011	0.0004	0.0003	1.7356	94.43%	0.1297	95.71%	0.0685	95.43%	18,971%	69,994%	369%
	0.5	8	λ	−0.6023	2.7349	−0.3043	0.8875	−0.1534	0.5473	6.0419	98.14%	3.4976	95.43%	2.8390	94.14%	308%	500%	162%
			δ	0.9270	2.3289	0.2789	0.4770	0.1122	0.1048	4.7555	96.57%	2.4786	92.00%	1.1914	93.86%	488%	2221%	455%
		12	λ	−0.5364	1.8947	−0.1263	0.3544	−0.0401	0.0747	4.9727	97.43%	2.2821	94.71%	1.0606	96.29%	535%	2536%	474%
			δ	0.6496	1.4060	0.0806	0.0695	0.0155	0.0087	3.8912	94.43%	0.9846	97.00%	0.3603	98.14%	2023%	16,212%	801%
		17	λ	−0.4418	1.5273	−0.1640	0.3172	−0.0329	0.0672	4.5275	98.14%	2.1134	92.14%	1.0084	96.29%	482%	2274%	472%
			δ	0.4718	0.8682	0.0757	0.0602	0.0155	0.0073	3.1519	93.14%	1.0660	95.86%	0.6308	99.57%	1443%	11,957%	829%
	2	8	λ	0.0259	3.4728	−0.0331	1.7028	0.0438	1.1119	7.3096	94.29%	5.1173	95.29%	4.1330	96.14%	204%	312%	153%
			δ	1.4484	6.0130	0.7250	2.3908	0.4386	1.5867	7.7619	99.43%	5.3574	98.43%	4.6320	96.43%	252%	379%	151%
		12	λ	−0.0296	3.1821	−0.0065	1.0169	−0.0295	0.6357	6.9967	93.86%	3.9558	94.29%	3.1255	94.86%	313%	501%	160%
			δ	1.3666	5.5362	0.4086	1.1014	0.3101	0.9477	7.5135	98.71%	3.7919	98.43%	3.6200	96.43%	503%	584%	116%
		17	λ	−0.0406	2.3864	−0.1251	0.7429	−0.0254	0.2637	6.0578	95.00%	3.3453	96.14%	2.0118	95.43%	321%	905%	282%
			δ	1.1429	4.7424	0.3826	0.9196	0.1264	0.2775	7.2717	97.86%	3.8494	98.00%	2.0061	96.00%	516%	1709%	331%

Table 7. *Cont.*

$\lambda = 3$				Point						Confidance						RE		
				SRS		RSS c = 1		RSS c = 3		SRS		RSS c = 1		RSS c = 3				
Methods	δ	**n**		**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**LCI**	**CP**	**LCI**	**CP**	**LCI**	**CP**	**RE1**	**RE2**	**RE3**
Bayesian	0.3	8	λ	0.0081	0.0231	−0.0001	0.0062	−0.0001	0.0025	2.1588	100.00%	1.5521	100.00%	0.8862	100.00%	375%	936%	249%
			δ	0.0612	0.0201	0.0218	0.0030	0.0218	0.0003	0.5223	96.43%	0.3032	99.57%	0.1383	84.57%	675%	6813%	1009%
		12	λ	0.0031	0.0054	−0.0013	0.0019	−0.0013	0.0012	1.5645	100.00%	1.1041	100.00%	0.6220	100.00%	289%	462%	160%
			δ	0.0372	0.0096	0.0056	0.0010	0.0056	0.0001	0.4228	97.14%	0.2055	99.71%	0.0956	87.29%	924%	7430%	804%
		17	λ	0.0017	0.0024	−0.0020	0.0011	−0.0020	0.0007	1.2878	100.00%	0.9026	100.00%	0.5028	100.00%	220%	347%	158%
			δ	0.0294	0.0066	0.0053	0.0005	0.0053	0.0001	0.3623	98.43%	0.1577	100.00%	0.0746	91.71%	1379%	7877%	571%
	0.5	8	λ	−0.0011	0.0202	0.0054	0.0069	0.0054	0.0048	2.1389	100.00%	1.5040	100.00%	0.8219	100.00%	294%	420%	143%
			δ	0.1123	0.0569	0.0227	0.0048	0.0227	0.0003	0.8692	97.00%	0.4606	100.00%	0.1746	83.57%	1194%	22,408%	1877%
		12	λ	0.0015	0.0062	−0.0002	0.0023	−0.0002	0.0022	1.5528	100.00%	1.0743	100.00%	0.5750	100.00%	266%	287%	108%
			δ	0.0638	0.0259	0.0099	0.0020	0.0099	0.0001	0.6957	97.29%	0.3243	100.00%	0.1217	84.57%	1322%	23,434%	1772%
		17	λ	0.0034	0.0027	0.0019	0.0014	0.0019	0.0014	1.2806	100.00%	0.8677	100.00%	0.4551	100.00%	193%	187%	97%
			δ	0.0384	0.0141	0.0077	0.0012	0.0077	0.0001	0.5817	98.43%	0.2480	100.00%	0.0965	84.14%	1132%	20,160%	1782%
	2	8	λ	0.0414	0.0249	0.0285	0.0186	0.0285	0.0098	2.0417	100.00%	1.2539	100.00%	0.6903	99.57%	133%	253%	190%
			δ	0.1342	0.1354	0.0144	0.0027	0.0144	0.0000	2.5164	99.86%	0.8586	100.00%	0.2054	96.29%	5066%	314,626%	6211%
		12	λ	0.0224	0.0094	0.0123	0.0087	0.0123	0.0044	1.4773	100.00%	0.8766	100.00%	0.4703	98.86%	109%	215%	197%
			δ	0.0891	0.0633	0.0067	0.0013	0.0067	0.0000	1.9433	100.00%	0.6057	100.00%	0.1420	95.71%	4939%	227,059%	4597%
		17	λ	0.0109	0.0053	0.0028	0.0052	0.0028	0.0025	1.2093	100.00%	0.6824	100.00%	0.3538	99.14%	102%	212%	209%
			δ	0.0523	0.0341	0.0025	0.0008	0.0025	0.0000	1.5961	100.00%	0.4880	100.00%	0.1130	96.57%	4428%	147,107%	3322%

8. Applications

In this section, we used two real-world datasets to compare the KMPTL model to several other known competing models: power Topp–Leone (PTL), unit-Gompertz (UG), unit-Lindley (UL), Topp–Leone (TL), unit generalized log Burr XII (UGLBXII), Unit Exponential Pareto (UEPD), Kumaraswamy (Kw), Beta, and Marshall-Olkin Kumaraswamy (MOK). To estimate the parameters of the competing models, the MLE approach is employed. To choose the optimal model, different criteria have been used, such as the Akaike information criterion (AM1), correct Akaike information criterion (SM2), Bayesian information criterion (SM3), and Hannan–Quinn information criterion (SM4) values. In addition, different goodness of fit measures have been discussed, such as the Kolmogorov–Smirnov distance (KSD), Anderson–Darling (AD), and Cramer-von-Mises (CVM), for each distribution.

8.1. Economic Growth Data

The trade share variable's values in the famed "Determinants of Economic Growth Data" are taken into account in the first dataset, which is called the trade share dataset. Along with factors that may be associated with growth, the growth rates of up to 61 different countries are taken into consideration. The information is publicly accessible online as an addition to [82]. The trade share dataset consists of the following numbers: "0.1405 0.1566 0.1577 0.1604 0.1608 0.2215 0.2994 0.3131 0.3246 0.3247 0.3295 0.3300 0.3379 0.3397 0.3523 0.3589 0.3933 0.4176 0.4258 0.4356 0.4421 0.4444 0.4505 0.4558 0.4683 0.4733 0.4846 0.4889 0.5096 0.5177 0.5278 0.5347 0.5433 0.5442 0.5508 0.5527 0.5606 0.5607 0.5671 0.5753 0.5828 0.6030 0.6050 0.6136 0.6261 0.6395 0.6469 0.6512 0.6816 0.6994 0.7048 0.7292 0.7430 0.7455 0.7798 0.7984 0.8147 0.8230 0.8302 0.8342 0.9794". The data were initially examined in terms of outliers by drawing a box-plot, and TTT was obtained and compared to the hazard line, and we noticed that the data did not contain outliers and also TTT is an increasing vector, and this corresponds to the hazard line; see Figure 5.

When compared to all other models used to fit the existing economic growth data, the KMPTL model has the lowest distance KS in Table 8. The fit CDF with empirical CDF and estimated PDF with histogram are displayed for KMPTL and others models in Figure 6. In addition, QQ-plot and PP-plot are displayed for KMPTL distribution in Figure 7.

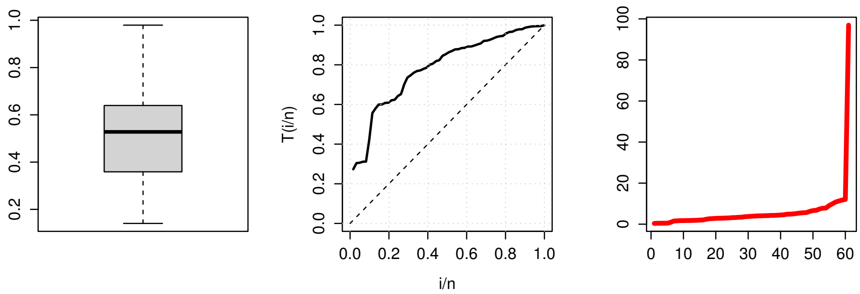

Figure 5. Box-plot, TTT plot, and hazard plot for KMPTL: economic growth data.

Table 8. MLE and different measures for KMPTL and comparative method: economic growth data.

		Estimates	SE	KSD	SM1	SM2	SM3	SM4	CVM	AD
KMPTL	λ	1.1672	0.7731	0.0568	−24.8680	−20.6462	−24.6611	−23.2135	0.0456	0.3864
	δ	2.7532	2.6813							
PTL	λ	0.4601	0.4046	0.0569	−24.8276	−20.6059	−24.6207	−23.1731	0.0493	0.4091
	δ	9.4177	3.8950							
Beta	a	2.7940	0.4880	0.0618	−23.9056	−19.6838	−23.6987	−22.2510	0.0491	0.3867
	b	2.6037	0.4519							

Table 8. *Cont.*

		Estimates	SE	KSD	SM1	SM2	SM3	SM4	CVM	AD
Kw	a	2.3289	0.3055	0.0689	−23.2431	−19.0213	−23.0362	−21.5886	0.0528	0.4009
	b	2.7624	0.5550							
MOK	α	0.2984	0.2974	0.0583	−22.6334	−16.3007	−22.2123	−20.1516	0.0492	0.4149
	β	3.0632	0.6398							
	θ	1.9447	0.9469							
UG	α	0.6157	0.2660	0.1098	−17.7512	−13.5295	−17.5443	−16.0967	0.1586	1.1548
	β	1.0927	0.2473							
UL	θ	0.7248	0.0687	0.2487	33.1918	35.3027	33.2596	34.0191	0.6153	3.7898
TL	θ	2.7394	0.3507	0.0860	−23.8362	−20.2725	−23.7684	−23.0090	0.0459	0.3974
UGLBXII	α	4.8911	7.0971	0.0569	−22.9393	−16.6067	−22.5183	−20.4575	0.0460	0.3947
	β	0.9787	0.1834							
	λ	1.7529	1.7128							
UEPD	α	0.8117	0.0647	0.1670	214.7548	221.0874	215.1758	217.2366	0.3447	2.2041
	β	1.3071	0.6696							
	λ	0.7641	41.1245							

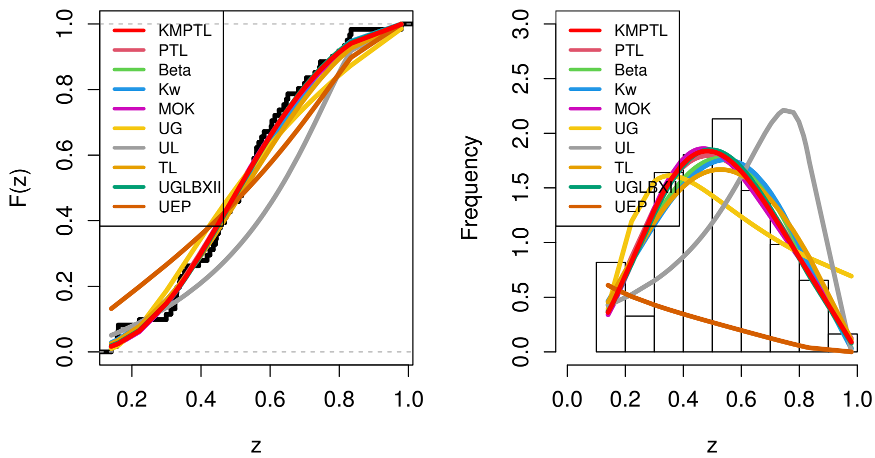

Figure 6. Estimated CDF and PDF for KMPTL and other models: economic growth data.

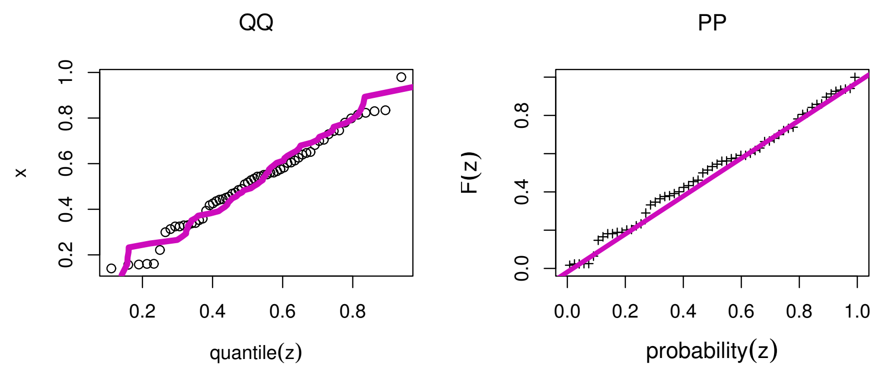

Figure 7. Estimated PP and QQ for KMPTL: economic growth data.

Figure 8 shows the profile ln-likelihood of the estimated parameters with economic growth data. They provide evidence of the MLE's existence and distinctiveness. Figure 9 shows the convergence line for MCMC results in a trace plot of the theses results for the KMPTL parameters with economic growth data. To confirm the normality plot of MCMC results, Figure 10 depicts the density plot of posterior MCMC results for KMPTL parameters for economic growth data.

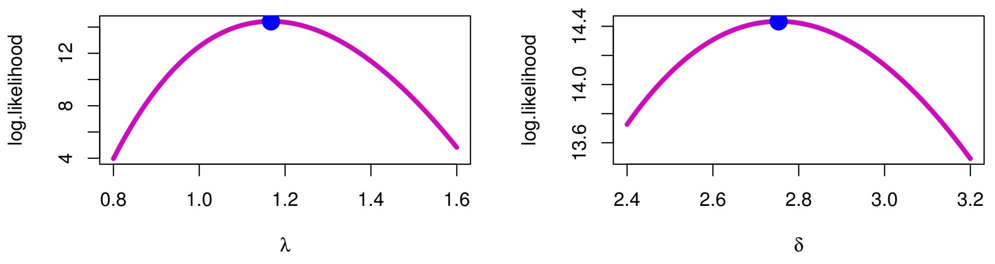

Figure 8. Examination of the MLE maxillary value of the KMPTL distribution: economic growth data.

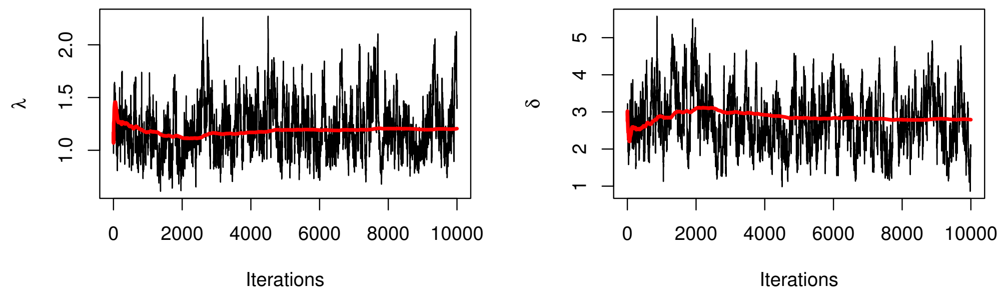

Figure 9. Examination of the Bayesian convergence measures of the KMPTL distribution: economic growth data.

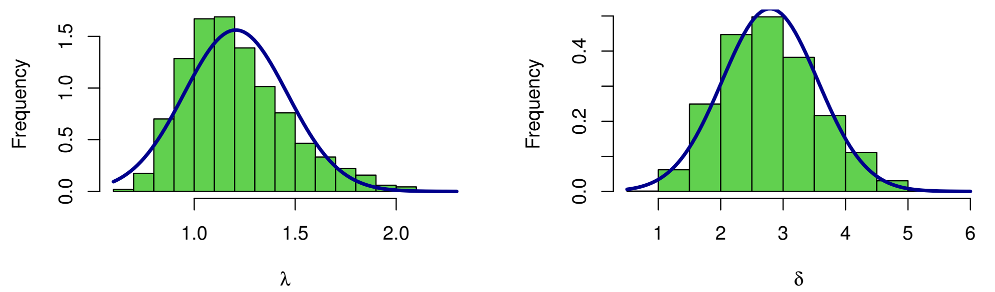

Figure 10. Examination of the Bayesian frequency measures of parameters of the KMPTL distribution: economic growth data.

8.2. Physics Data

The second dataset consisted of 30 measurements, which have been discussed by [83] and made by Quesenberry and Hales [84] to determine the tensile strength of polyester fibres. Theses data are as follows: 0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926. The data were initially examined in terms of outliers by drawing a box-plot, and TTT was obtained and compared to the hazard line, and we noticed that the data did not contain outliers and also that TTT is an increasing vector, and this corresponds to the hazard line; see Figure 11.

When compared to all other models used to fit the existing physical data, the KMPTL model has the lowest value of distance KS, as seen in Table 9. The fit CDF with empirical CDF and estimated PDF with histogram are displayed for KMPTL and other models in Figure 12. In addition, QQ-plot and PP-plot are displayed for KMPTL distribution in Figure 13.

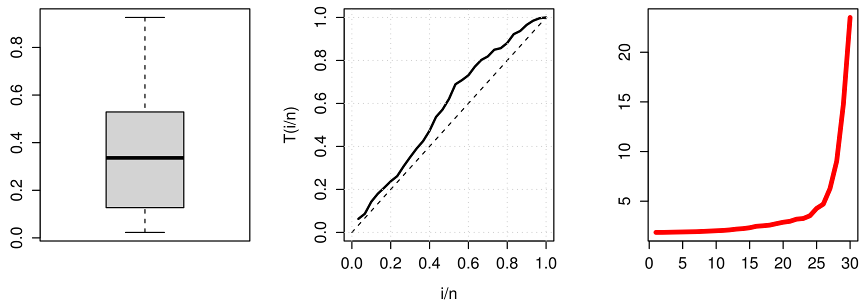

Figure 11. Box-plot, TTT plot, and hazard plot for KMPTL: polyester fibers data.

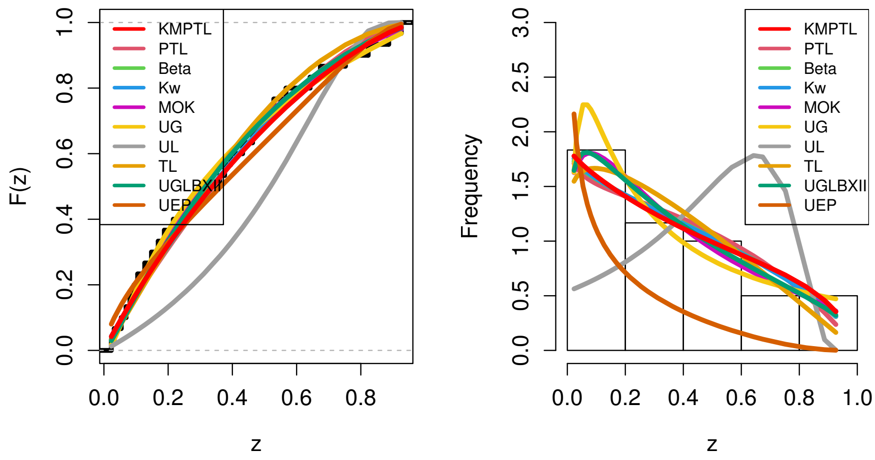

Figure 12. Estimated CDF and PDF for KMPTL and other models: polyester fibers data.

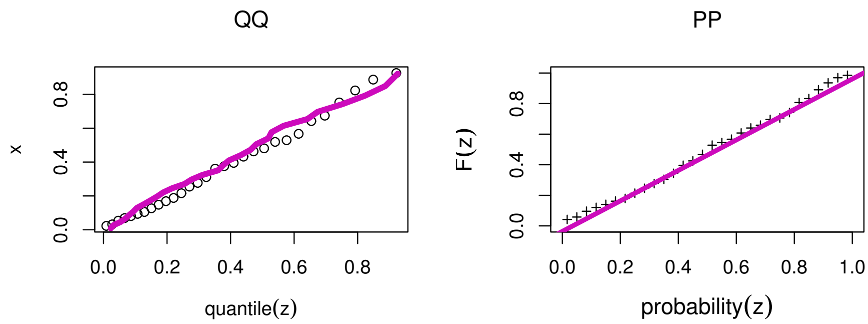

Figure 13. Estimated PP and QQ for KMPTL: polyester fibers data.

Table 9. MLE and different measures for KMPTL and comparative method: polyester fibers data.

		Estimates	SE	KSD	SM1	SM2	SM3	SM4	CVM	AD
KMPTL	λ	6.6639	1.5843	0.0624	−2.8736	−0.0712	−2.4291	−1.9771	0.0169	0.1450
	δ	0.1478	0.2745							
PTL	λ	1.8949	2.3814	0.0718	−2.0977	0.7047	−1.6533	−1.2012	0.0233	0.1955
	δ	0.4895	0.7559							
Beta	a	0.9666	0.2238	0.0669	−2.6101	0.1923	−2.1657	−1.7136	0.0184	0.1559
	b	1.6205	0.4107							

Table 9. *Cont.*

		Estimates	**SE**	**KSD**	**SM1**	**SM2**	**SM3**	**SM4**	**CVM**	**AD**
Kw	a	0.9622	0.2016	0.0649	−2.6221	0.1803	−2.1776	−1.7256	0.0183	0.1550
	b	1.6077	0.4135							
MOK	α	0.4363	0.4707	0.0630	−1.2087	2.9949	−0.2856	0.1361	0.0171	0.1456
	β	1.1869	0.3460							
	θ	1.2584	0.6442							
UG	α	1.0380	0.7702	0.0734	−2.8098	−0.0695	−2.4053	−1.9001	0.0184	0.1461
	β	0.4212	0.1917							
UL	θ	1.0505	0.1455	0.2721	20.1704	21.5716	20.3133	20.6187	0.1656	1.0870
TL	θ	1.1090	0.2025	0.0665	−3.8078	−2.4066	−3.6649	−3.3595	0.0189	0.1600
UGLBXII	α	1167.4551	810.1270	0.0658	−1.4306	2.7730	−0.5076	−0.0859	0.0173	0.1457
	β	0.6846	0.1003							
	λ	261.5642	1342.7096							
UEPD	α	0.6630	0.0882	0.1052	74.7790	78.9826	75.7021	76.1238	0.0637	0.4734
	β	0.9585	106.1478							
	λ	0.9751	71.5895							

Table 10 shows the Bayesian estimation for all data, and Figure 14 shows the profile ln-likelihood of ML parameters for polyester fibers data. They both provide evidence of the MLE's existence and distinctiveness. Figure 15 shows the convergence line of MCMC results with a trace plot of theses results for the KMPTL parameters with polyester fibers data. To confirm the normality plot for MCMC estimation, the density plot of posterior MCMC results for KMPTL parameters for polyester fibres data is shown in Figure 16.

Table 10. Bayesian estimation.

		Mean	**SE**	**Lower**	**Upper**
Economic Growth	λ	1.2055	0.2554	0.7643	1.7374
	δ	2.7903	0.7658	1.3756	4.2750
polyester fibers	λ	6.5853	2.0764	2.7525	10.6469
	δ	0.1703	0.0726	0.0524	0.3178

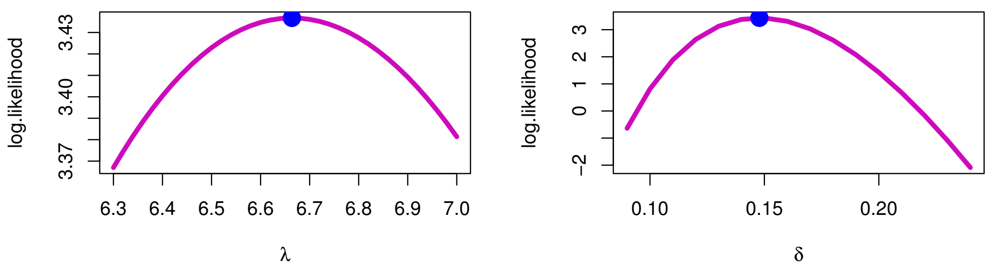

Figure 14. Examination of the MLE maxillary value of the KMPTL distribution: polyester fibers data.

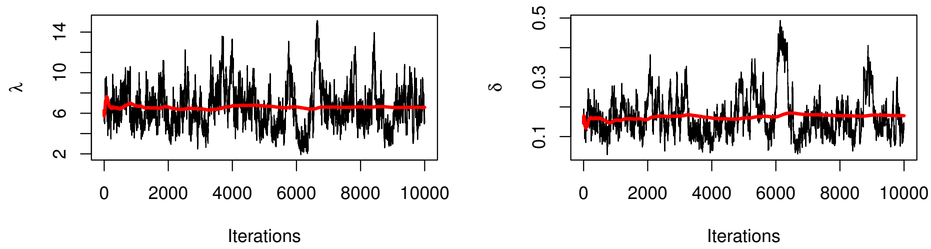

Figure 15. Examination of the Bayesian convergence measures of the KMPTL distribution: polyester fibers data.

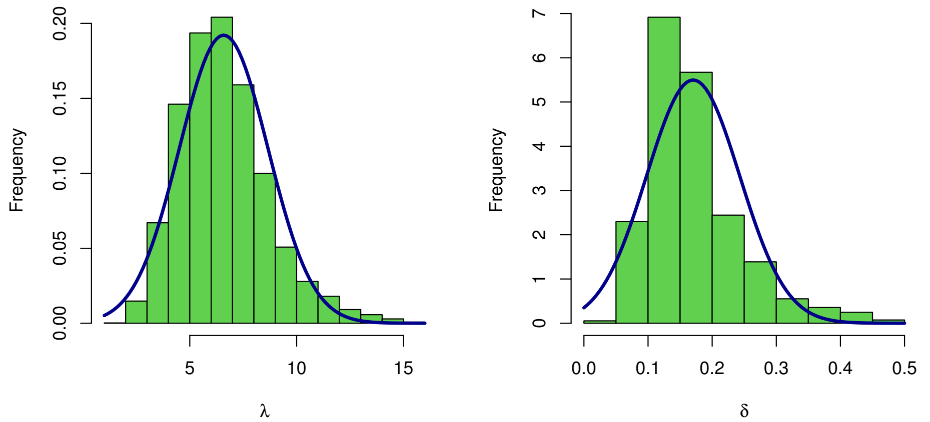

Figure 16. Examination of the Bayesian frequency measures of parameters of the KMPTL distribution: polyester fibers data.

9. Conclusions and Summary

In this article, we proposed a new two-parameter distribution called the KMPTL distribution. Several statistical and mathematical aspects of the KMPTL distribution, such as the qf, moments, generating function, and incomplete moments, are determined by calculating. Some measures of entropy are investigated. The CRRE is computed. To estimate the parameters of the KMPTL distribution, both ML and Bayesian estimation methods are used under SRS and RSS. The simulation investigation was conducted to estimate the model parameters of the KMPTL distribution utilizing SRS and RSS in order to demonstrate that RSS is better than SRS. Using two real-world datasets, we proved that the KMPTL distribution has greater flexibility than the PTL distribution and the other nine competing statistical distributions.

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