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First-Order Differential Subordinations and Their Applications

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Abstract: In this paper, we consider some relations related to the representations of starlike and convex functions, and obtain some sufficient conditions for starlike and convex functions by using the theory of differential subordination. Actually, we generalize a result by Suffridge for analytic functions with missing coefficients and then we apply that generalization for obtaining the different methods to the implications of starlike or convex functions. Our results generalize and improve the previous results in the literature.

Keywords: analytic functions; differential subordination; fixed initial coefficient; starlike functions; convex functions

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1. Introduction

We let $\mathcal{H}$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and define

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

where $a \in \mathbb{C}$ and n is a positive integer number. Furthermore, we introduce the subclass $\mathcal{A}_n$ of $\mathcal{H}$ as follows:

$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots\}.$$

In particular, we set $\mathcal{A}_1 \equiv \mathcal{A}$. As usual, let the subclass $\mathcal{S}$ of $\mathcal{A}$ be the class of all univalent functions in the open unit disk $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order $\gamma (0 \le \gamma < 1)$, written $f \in \mathcal{S}^*(\gamma)$, if it satisfies

$$\mathfrak{Re} \frac{z f'(z)}{f(z)} > \gamma \qquad (z \in \mathbb{U}).$$

Specifically, we put $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. Every element in $\mathcal{S}^*$ is called a starlike function. Furthermore, a function $f \in \mathcal{A}$ is said to be convex of order $\gamma (0 \le \gamma < 1)$, written $f \in \mathcal{K}(\gamma)$, if it satisfies

$$\mathfrak{Re} \left\{ \frac{z f''(z)}{f'(z)} + 1 \right\} > \gamma \qquad (z \in \mathbb{U}).$$

In particular, we put $\mathcal{K}(0) \equiv \mathcal{K}$. Every element in $\mathcal{K}$ is called a convex function. Now for analytic functions in $\mathbb{U}$ with the fixed initial coefficient, we define the class $\mathcal{H}_\beta[a, n]$ as follows:

$$\mathcal{H}_\beta[a, n] = \{f \in \mathcal{H} : f(z) = a + \beta z^n + a_{n+1} z^{n+1} + \dots\},$$

where n is a positive integer number, $a \in \mathbb{C}$ and $\beta \in \mathbb{C}$ are fixed numbers. Moreover we assume

$$\mathcal{A}_{n,b} = \{f \in \mathcal{H} : f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \dots\},$$

where n is a positive integer number and $b \in \mathbb{C}$ is a fixed number. In addition, we set $\mathcal{A}_{1,b} \equiv \mathcal{A}_b$. Assume f and g be in $\mathcal{H}$. We say that the function f is subordinate to g, denoted by $f \prec g$, if there exists an analytic function ω in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| \le |z| < 1$, such that $f(z) = g(\omega(z))$. Moreover, if g is an univalent function in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = 0$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

By considering the function $\frac{1+Az}{1+Bz}$, we can generalize the class of starlike functions as follows:

Let $f \in \mathcal{A}$ and $-1 \le B < A \le 1$. Then it is said that $f \in \mathcal{S}^*[A,B]$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}). \tag{1}$$

Through (1), it can be easily observed that $\frac{1+Az}{1+Bz}$ with $-1 < B < A < 1$ maps the open unit disc $\mathbb{U}$ onto the open disc with the center $C = (1-AB)/(1-B^2)$ and the radius $R = (A-B)/(1-B^2)$. Thus, for all $f \in \mathcal{S}^*[A,B]$, the relation $\frac{1-A}{1-B} < \Re\mathfrak{e}\frac{zf'(z)}{f(z)} < \frac{1+A}{1+B}$ holds and hence $\mathcal{S}^*[A,B] \subset \mathcal{S}^*(\frac{1-A}{1-B})$. Moreover for the different values of A and B, other types of the class $\mathcal{A}$ such as the class $\mathcal{S}^*[1,-1]$, which is equivalent to the class $\mathcal{S}^*$, also for $0 \le \alpha < 1$, the class $\mathcal{S}^*[1-2\alpha,-1]$ which is equivalent to $S^*(\alpha)$ and $\mathcal{S}^*[\alpha,0] = \left\{f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < \alpha\right\}$ are obtained.

The theory of differential subordination has a key role in the study of geometric function theory. In 1935 Goluzin [1] considered the subordination $zp'(z) \prec h(z)$ and proved that if h is convex then $p(z) \prec q(z) = \int_0^z h(t)t^{-1}dt$. Furthermore, in 1970 Suffridge [2] showed that Goluzin's result is true if h is starlike. Moreover, Miller and Mocanu by writing many research papers in this direction extended the concept of differential subordination (see for example [3] and references therein). Further, many authors have recently using different combinations of the representations of starlike and convex functions have obtained the simple conditions for starlikeness and convexity of analytic functions. In [4], Silverman gained the results for analytic functions including the terms as the quotient of the analytic representations of convex and starlike functions. For instance, Silverman proved that $G_b \subset \mathcal{S}^*\left(\frac{2}{1+\sqrt{1+8b}}\right)$, where $0 < b \le 1$ and

$$G_b = \left\{f \in \mathcal{A} : \left|\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1\right| < b, z \in \mathbb{U}\right\}.$$

Next, Obradovic and Tuneski [5], in view of $\frac{1}{1+b} \ge \frac{2}{1+\sqrt{1+8b}}$, improved the work of Silverman. Indeed, they established $G_b \subset \mathcal{S}^*(\frac{1}{1+b}) \subset \mathcal{S}^*\left(\frac{2}{1+\sqrt{1+8b}}\right)$ for $0 < b \le 1$. Nunokawa et al. [6], by applying the Silverman's quotient function [4], proved that f could be strongly starlike, strongly convex or starlike in $\mathbb{U}$. In [7–10], the authors have studied some conditions for the analytic functions to belong to the class $\mathcal{S}^*[A,B]$. In [11], some results related to the above discussion, with respect to n-symmetric points of functions, have been given. Inspired by [9,10], in this paper, we will extend and improve some results obtained in [9,10] and then we will determine some conditions which by means of them, a function belongs to the class $\mathcal{S}^*[A,B]$.

The contents of this article are regulated as follows: In Section 3, initially, we will prove a theorem that is the extension of a little change to the Suffridge theorem [3]. Next, we bring some applications of this theorem as the main results, making the functions be in the class $\mathcal{S}^*[A,B]$. These results extend and improve some results in [10]. In Section 4, we intend to bring some sufficient conditions for starlikeness of analytic functions. We also produce the functions belonging to the class $\mathcal{S}^*[A,B]$ by considering other conditions, and so we include some corollaries from the result acquired. Furthermore, these results extend and improve some results in [9]. Moreover, Suffridge's result is used in recent investigations like [12–15]. Note that some results related to this article for analytic functions with fixed initial coefficients are also mentioned.

In the continuation of the argument, in order to prove the main results, we require to remind a definition and two basic lemmas:

Definition 1. *(see [3]) Let Q denote the set of functions q that are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$, where*

$$E(q) := \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$.

Lemma 1. *(see [3]) Let $q \in Q$ with $q(0) = a$, and let $p(z) = a + a_n z^n + \dots$ be analytic in $\mathbb{U}$ with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(q)$, and an $m \geq n \geq 1$ for which $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$,*

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, *and*

(iii) $\mathfrak{Re}\left\{1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq m\mathfrak{Re}\left\{1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\}$.

Lemma 2. *(see [16]) Let $q \in Q$ with $q(0) = a$ and $p \in \mathcal{H}_c[a,n]$ with $p(z) \not\equiv a$. If there exist a point $z_0 \in \mathbb{U}$ such that $p(z_0) \in q(\partial\mathbb{U})$ and $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$, then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$$

and

$$\mathfrak{Re}\left\{1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq m\mathfrak{Re}\left\{1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\},$$

where $q^{-1}(p(z_0)) = \zeta_0 = e^{i\theta_0}$ and

$$m \geq n + \frac{|q'(0)| - |c||z_0|^n}{|q'(0)| + |c||z_0|^n}.$$

2. Main Results

First, we mention a lemma which is slightly different from the original one, ([3], Th 3.4 h).

Lemma 3. *Let q be univalent in $\mathbb{U}$ and functions θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ and $\phi(z) \neq 0$ for $z \in q(\mathbb{U})$. Moreover, let*

(i) $Q(z) = zq'(z)\phi[q(z)]$ *be starlike and*

(ii) $\mathfrak{Re}\left(\frac{\theta'[q(z)]}{\phi[q(z)]}\right) > 0$.

If $p \in \mathcal{H}_\beta[a,n]$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + nzq'(z)\phi[q(z)], \tag{2}$$

then $p \prec q$ and $p(z) = q(z^n)$ is the best dominant.

Proof. Let us define $h(z) = \theta[q(z)] + nzq'(z)\phi[q(z)]$. It easy to verify that the conditions (i) and (ii) imply that h is close-to-convex and hence univalent in $\mathbb{U}$. Now using the same argument as the proof of ([3], Th 3.4h), we get our result and we omit the details of the proof. □

By considering $\theta(z) \equiv 0$ in Lemma 3, we extend a little change of the Suffridge theorem [3] as follows:

Corollary 1. *Let q be univalent in $\mathbb{U}$ and $q(0) = a$. Moreover, let ϕ be analytic in a domain D containing $q(\mathbb{U})$ and let $p \in \mathcal{H}[a,n]$. If $zq'(z)\phi[q(z)]$ is starlike in $\mathbb{U}$, then*

$$zp'(z)\phi[p(z)] \prec nzq'(z)\phi[q(z)] \Longrightarrow p(z) \prec q(z). \tag{3}$$

and q is the best (a,n)-dominant.

Using the same argument of Lemma 3 and applying Lemma 1, we obtain the following theorem, and we omit its proof:

Theorem 1. *Let q be univalent in $\mathbb{U}$ and $q(0) = a$. Furthermore, let ϕ be analytic in a domain D containing $q(\mathbb{U})$ and let $p \in \mathcal{H}_\beta[a,n]$ with $0 \le \beta \le |q'(0)|$. If $zq'(z)\phi[q(z)]$ is starlike in $\mathbb{U}$, then*

$$zp'(z)\phi[p(z)] \prec \left(n + \frac{|q'(0)| - \beta}{|q'(0) + \beta|}\right) zq'(z)\phi[q(z)] \Longrightarrow p(z) \prec q(z).$$

By putting $\phi \equiv 1$ in Lemma 3 and Theorem 1, we reach to the following corollaries:

Corollary 2. *Let q be convex univalent in $\mathbb{U}$ with $q(0) = a$. If $p \in \mathcal{H}[a,n]$ and*

$$zp'(z) \prec nzq'(z) = h(z),$$

then $p \prec q$.

Corollary 3. *Let q be convex univalent in $\mathbb{U}$ with $q(0) = a$. If $p \in \mathcal{H}_\beta[a,n]$ with $0 \le \beta \le |q'(0)|$ and*

$$zp'(z) \prec \left(n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}\right) zq'(z) = h(z),$$

then $p \prec q$.

Corollary 4. *Let $p \in \mathcal{H}[1,n]$. Suppose that A and B are real numbers with $-1 \le B < A \le 1$. If*

$$\frac{zp'(z)}{p^2(z)} \prec n\frac{(A-B)z}{(1+Az)^2}, \tag{4}$$

then $p \prec \frac{1+Az}{1+Bz}$.

Proof. Let us define $g(z) = \frac{1}{p(z)}$ and $q(z) = \frac{1+Bz}{1+Az}$. A simple computation and (4) yield

$$zg'(z) \prec n\frac{(B-A)z}{(1+Az)^2} = nzq'(z).$$

On the other hand, q is convex in $\mathbb{U}$ with $q(0) = 1$, consequently by applying Corollary 2, we deduce that $\frac{1}{p} \prec \frac{1+Bz}{1+Az}$ and hence $p \prec \frac{1+Az}{1+Bz}$. □

By applying the same argument as Corollary 4, and using Corollary 3 we have:

Corollary 5. *Let $p \in \mathcal{H}_\beta[1,n]$, and A and B be real numbers with $-1 \leq B < A \leq 1$ and $B - A \leq \beta \leq 0$. If*

$$\frac{zp'(z)}{p^2(z)} \prec \left(n + \frac{A-B+\beta}{A-B-\beta}\right)\frac{(A-B)z}{(1+Az)^2},$$

then $p \prec \frac{1+Az}{1+Bz}$.

Setting $p(z) = \frac{zf'(z)}{f(z)}$ in Corollary 4, we obtain:

Corollary 6. *Let $f \in \mathcal{A}_n$. Moreover, let A and B be real numbers with $-1 \leq B < A \leq 1$. If*

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + n\frac{(A-B)z}{(1+Az)^2},$$

then $\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$.

Setting $p(z) = \frac{zf'(z)}{f(z)}$ in Corollary 5, we obtain:

Corollary 7. *Let $f \in \mathcal{A}_{n,b}$, and A and B be real numbers with $-1 \leq B < A \leq 1$ and $\frac{B-A}{n} \leq b \leq 0$. If*

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \left(n + \frac{A-B+nb}{A-B-nb}\right)\frac{(A-B)z}{(1+Az)^2},$$

then $\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$.

Putting $A = 1$ in Corollary 6, we get to the following corollary:

Corollary 8. *Let $f \in \mathcal{A}_n$. Moreover, let B be a real number with $-1 \leq B < 1$. If*

$$1 + \frac{zf''(z)}{f'(z)} \neq t\frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U})$$

for all $t \geq 1 + \frac{n(1-B)}{4}$, we have

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1+B}\right| < \frac{1}{1+B} \qquad (z \in \mathbb{U}).$$

Putting $A = 1$ in Corollary 7, we come to the following corollary:

Corollary 9. *Let $f \in \mathcal{A}_{n,b}$. Moreover, let B be a real number with $-1 \leq B < 1$ and $\frac{B-1}{n} \leq b \leq 0$. If*

$$1 + \frac{zf''(z)}{f'(z)} \neq t\frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U})$$

for all $t \geq 1 + \left(n + \frac{1-B+nb}{1-B-nb}\right)\frac{(1-B)}{4}$, then

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1+B}\right| < \frac{1}{1+B} \qquad (z \in \mathbb{U}).$$

Remark 1. *Corollary 4, Corollary 6 and Corollary 8, respectively, extend and improve Lemma 2.2, Theorem 2.3 and Corollary 2.4 in [10].*

3. Further Results about Analytic Functions to Settle in the Class $\mathcal{S}^*[A,B]$

It is well known that for $f \in \mathcal{A}$, the condition $|zf''(z)/f'(z)| < 2$ is sufficient for starlikeness of f. In this section, we will extend this result and will also try to bring other sufficient conditions for starlikeness.

Theorem 2. *Let A and B be real numbers with $-1 \leq B < A \leq 1$. Suppose that $p \in \mathcal{H}[1,n]$ and $p(z) \neq 0$ in $\mathbb{U}$. If*
$$p(z) + \frac{zp'(z)}{p(z)} - 1 \prec \frac{(A-B)(n+1+Az)z}{(1+Bz)(1+Az)},$$
then
$$p(z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Let us define $\theta(z) = z - 1$, $\phi(z) = \frac{1}{z}$ and $q(z) = \frac{1+Az}{1+Bz}$. For proving this theorem, it is sufficient to show that the conditions of Lemma 3 hold. However, we note that the condition (i) is equivalent to
$$\mathfrak{Re}\frac{zQ'(z)}{Q(z)} > 0,$$
where $Q(z) = \frac{(A-B)z}{(1+Az)(1+Bz)}$. Since
$$\mathfrak{Re}\frac{zQ'(z)}{Q(z)} = \mathfrak{Re}\left(1 - \frac{Bz}{1+Bz} - \frac{Az}{1+Az}\right) > \frac{1-|A||B|}{(1+|A|)(1+|B|)} \geq 0,$$
we attain the assertion of condition (i). On the other hand, from
$$\mathfrak{Re}\left(\frac{\theta'[q(z)]}{\phi[q(z)]}\right) = \mathfrak{Re}\left(\frac{1+Bz}{1+Az}\right) > 0,$$
we observe (ii). Moreover, the condition (2) is correct and consequently the proof is completed. □

By putting $A = 1, B = -1$ and $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2 we obtain:

Corollary 10. *Let $\Omega = \{w = -1 + ti : |t| \geq \sqrt{n(n+2)}\}$ and $f \in \mathcal{A}_n$ with $f \cdot f' \neq 0$ in $\mathbb{U}\backslash\{0\}$. If*
$$\frac{zf''(z)}{f'(z)} \notin \Omega,$$
then f is starlike.

Proof. Let us define $h(z) = \frac{2z(n+1+z)}{1-z^2}$. By some calculations, one can observe that $h(e^{it}) = -1 + \frac{(n+1)+\cos t}{\sin t} i$. However, the function $g(t) := \frac{n+1+\cos t}{\sin t}$ takes the the minimum value at the point $t_0 = \cos^{-1}(\frac{-1}{n+1})$ and so $|g(t)| \geq |g(t_0)| = \sqrt{n(n+2)}$. Hence, h maps unit disk onto the complement of Ω and the proof is completed. □

By using Corollary 10, we have:

Corollary 11. *Let $f \in \mathcal{A}_n$ with $f \cdot f' \neq 0$ in $\mathbb{U}\backslash\{0\}$. If*

(i) $\left|\frac{zf''(z)}{f'(z)}\right| < n+1$, *or*

(ii) $\left|\mathfrak{Im}(\frac{zf''(z)}{f'(z)})\right| < \sqrt{n(n+2)}$,

then f is starlike and the result is sharp for the function $f(z) = \frac{1+z^n}{1-z^n}$.

By putting $A = 0$, $B = -1$ and $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2, we obtain:

Corollary 12. *Let $f \in \mathcal{A}_n$ with $f \cdot f' \neq 0$ in $\mathbb{U} \backslash \{0\}$. If*

$$\mathfrak{Re}\left(\frac{zf''(z)}{f'(z)}\right) > -\frac{n+1}{2},$$

then f is starlike of order $1/2$ and the result is sharp for the function $f(z) = \frac{1}{1-z^n}$.

We remark that Corollary 12 is the generalization of Marx-Strohh*ä*cker Theorem (see [17]). By putting $A = 0$, $0 < b = -B < 1$ and $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2 we gain:

Corollary 13. *Let $0 < b < 1$, $f \in \mathcal{A}_n$ with $f \cdot f' \neq 0$ in $\mathbb{U} \backslash \{0\}$. If*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{(n+1)b^2}{1-b^2}\right| < \frac{(n+1)b}{1-b^2},$$

then

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-b^2}\right| < \frac{b}{1-b^2},$$

and the result is sharp for the function $f(z) = \frac{1}{1-bz^n}$.

Theorem 3. *Let $f \in \mathcal{A}_n$ with $f \cdot f' \neq 0$ in $\mathbb{U} \backslash \{0\}$. Moreover, let A and B be real numbers with $-1 \leq B < A \leq 0$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < N(A,B,n)\left|\frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathbb{U}), \tag{5}$$

where

$$N(A,B,n) = \frac{(A-B)(n-A+1)}{(1-A)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}.$$

In particular, if

$$\frac{A(n+3) + A^2(n-1) - A^3 - 1}{(n+2) + A(n-2)} \leq B < A \leq 0, \tag{6}$$

then f is convex in $\mathbb{U}$.

Proof. Let us define $p(z) = \frac{f(z)}{zf'(z)}$ and $q(z) = \frac{1+Bz}{1+Az}$. It can be readily observed that $p \in \mathcal{H}[1,n]$ and q is convex univalent in $\mathbb{U}$. We claim that $p \prec q$, otherwise there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \backslash E(q)$, and an $m \geq n \geq 1$ such that

$$p(z_0) = q(\zeta_0) \quad \textit{and} \quad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0). \tag{7}$$

With some calculations and so utilizing (5), one can obtain

$$|zp'(z) + p(z) - 1| < N(A,B,n) \qquad (z \in \mathbb{U}). \tag{8}$$

By letting $\zeta_0 = e^{it}$, where $0 \le t \le 2\pi$ and using (8), we have

$$\begin{aligned}|p(z_0) + z_0 p'(z_0) - 1|^2 &= |q(\zeta_0) + m\zeta_0 q'(\zeta_0) - 1|^2 \\ &= \frac{1}{|1 + Ae^{it}|^2}\left|e^{it}(B - A)\left(1 + m\frac{1}{1 + Ae^{it}}\right)\right|^2 \\ &= (A - B)^2 \frac{(m + 1)^2 + A^2 + 2A(m + 1)\cos t}{(1 + 2A\cos t + A^2)^2}.\end{aligned}$$

Letting $x = \cos t$ and defining

$$g(x) = \frac{(m + 1)^2 + A^2 + 2A(m + 1)x}{(1 + 2Ax + A^2)^2}, \qquad (-1 \le x \le 1)$$

we have

$$g'(x) = -2A\frac{(m + 1)(2(m + 1) - 1) - A^2(m - 1) + 2A(m + 1)x}{(1 + 2Ax + A^2)^3} \ge 0. \tag{9}$$

In view of (9) we deduce that g is an increasing function and takes its minimum at the point -1. Hence

$$g(x) \ge g(-1) = \frac{(m + 1 - A)^2}{(1 - A)^4}$$

for all $-1 \le x \le 1$. Therefore

$$|p(z_0) + z_0 p'(z_0) - 1| \ge (A - B)\frac{(m + 1 - A)}{(1 - A)^2} \ge (A - B)\frac{(n + 1 - A)}{(1 - A)^2},$$

which contradicts (8), and so this give the result. Since $\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$, we have

$$\left|\frac{zf'(z)}{f(z)}\right| < \frac{1 + A}{1 + B} \qquad (z \in \mathbb{U}). \tag{10}$$

Now combining (5), (6) and (10), we have

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{(A - B)(n - A + 1)}{(1 - A)^2}\frac{1 + A}{1 + B} \le 1,$$

and so f is convex. $\square$

With the same approach as the previous theorem and by applying Lemma 1, we attain the following theorem which we omit the proof of.

Theorem 4. *Let $f \in \mathcal{A}_{n,b}$ with $f \cdot f' \ne 0$ in $\mathbb{U}\backslash\{0\}$. Moreover, let A and B be real numbers with $-1 \le B < A \le 0$ and $0 < b \le \frac{A-B}{n}$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < N(A, B, n, b)\left|\frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathbb{U}),$$

where

$$N(A, B, n, b) = \frac{(A - B)[(A - B + bn)(n - A + 1) + A - B - bn]}{(1 - A)^2(A - B + nb)},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}.$$

In particular, if $b = \frac{A-B}{n}$ *and*

$$\frac{A(n+3) + A^2(n-1) - A^3 - 1}{(n+2) + A(n-2)} \le B < A \le 0,$$

then f *is convex in* $\mathbb{U}$.

By putting $A = 0$ and letting $N(0, B, n) = (n+1)\frac{1-\alpha}{\alpha}$, we obtain $B = \frac{\alpha-1}{\alpha}$, where $N(A, B, n)$ is given in Theorem 4. Now let $\alpha > 0$. Since $-1 \le B < A \le 0$, we have $\frac{1}{2} \le \alpha < 1$. Hence we gain:

Corollary 14. *Let* $f \in \mathcal{A}_n$ *with* $f \cdot f' \neq 0$ *in* $\mathbb{U}\backslash\{0\}$ *and* $\frac{1}{2} \le \alpha < 1$. *If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < (n+1)\frac{1-\alpha}{\alpha}\left|\frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathbb{U}),$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1-\alpha)z},$$

and $f \in \mathcal{S}^*(\alpha)$. *In particular, if* $\frac{n+2}{n+3} \le \alpha$, *then* $\left|\frac{zf''(z)}{f'(z)}\right| < 1$, *and* f *is convex in* $\mathbb{U}$.

Remark 2. *Theorem* 4 *and Corollary* 14 *extend and improve Theorem* 1 *and Corollary* 1 *in* [9], *respectively.*

By setting $A = 0$, $b = \frac{-B}{n}$ and letting $N(0, B, n, \frac{B}{n}) = (n+1)\frac{1-\alpha}{\alpha}$, we obtain $B = \frac{\alpha-1}{\alpha}$, where $N(A, B, n, b)$ is given in Theorem 4. Now let $\alpha > 0$. Since $-1 \le B < A \le 0$, we have $\frac{1}{2} \le \alpha < 1$. So we obtain:

Corollary 15. *Let* $f \in \mathcal{A}_{n, \frac{1-\alpha}{n\alpha}}$ *with* $f \cdot f' \neq 0$ *in* $\mathbb{U}\backslash\{0\}$ *and* $\frac{1}{2} \le \alpha < 1$. *If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < (n+1)\frac{1-\alpha}{\alpha}\left|\frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathbb{U}),$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1-\alpha)z},$$

and $f \in \mathcal{S}^*(\alpha)$. *In particular, if* $\frac{n+2}{n+3} \le \alpha$, *then* $\left|\frac{zf''(z)}{f'(z)}\right| < 1$, *and* f *is convex in* $\mathbb{U}$.

Corollary 16. *Let* $f \in \mathcal{A}$ *be an odd function with* $f \cdot f' \neq 0$ *in* $\mathbb{U}\backslash\{0\}$. *If* $\frac{1}{2} \le \alpha < 1$ *and*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3(1-\alpha)}{\alpha}\left|\frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathbb{U}),$$

then $f \in \mathcal{S}^*(\alpha)$. *In particular, if* $\frac{4}{5} \le \alpha < 1$, *then* $\left|\frac{zf''(z)}{f'(z)}\right| < 1$ *and* f *is convex in* $\mathbb{U}$.

Proof. Since f is odd function, we have $f(z) = z + \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$. Putting $n = 2$ in Corollary 15, the desired result is obtained. $\square$

Example 1. *Let* $f(z) = \frac{1}{\lambda}\sin\lambda z$ *with* $|\lambda| < \frac{\pi}{2}$. *We know that* f *is an odd analytic function. On the other hand, one can see that*

$$\left|\frac{f''(z)f(z)}{(f'(z))^2}\right| = |\tan^2\lambda z| \le \tan^2|\lambda| \qquad (z \in \mathbb{U}).$$

Therefore making use of Corollary 16, if $\frac{1}{2} \le \alpha < 1$ and $|\lambda| \le \arctan\left(\sqrt{\frac{3(1-\alpha)}{\alpha}}\right)$, then $f \in S^*(\alpha)$. In addition, if $\frac{4}{5} \le \alpha < 1$ and $|\lambda| \le \arctan\left(\sqrt{\frac{3(1-\alpha)}{\alpha}}\right)$, then f is convex.

Finally we prove the following result:

Theorem 5. *Let $\alpha > \frac{1}{n+2}$ and $f \in \mathcal{A}_n$ with $f \cdot f' \neq 0$ in $\mathbb{U}\backslash\{0\}$. If*

$$\left|1 - \alpha \frac{zf''(z)}{f'(z)}\right| \le \frac{\alpha}{2}(n+2)\left|1 + \frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathbb{U}), \tag{11}$$

then f is a starlike function.

Proof. Let us define $p(z) = \frac{zf'(z)}{f(z)}$ and $q(z) = \frac{1+z}{1-z}$. We will show that $p \prec q$. Suppose that p is not subordinate to q. Then from Lemma 1 there exist two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, where $m \ge n$. Thus, $p(z_0) = it$ with $t \in \mathbb{R}$ and $\zeta_0 = q^{-1}(p(z_0)) = \frac{it-1}{it+1}$. Furthermore with some calculations we find that

$$1 - \alpha \frac{zf''(z)}{f'(z)} = 1 - \alpha\left(p(z) + \frac{zp'(z)}{p(z)} - 1\right),$$

and so

$$\begin{aligned}
\left|\frac{1 - \alpha \frac{z_0 f''(z_0)}{f'(z_0)}}{1 + \frac{z_0 f'(z_0)}{f(z_0)}}\right|^2 &= \left|\frac{1 - \alpha(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} - 1)}{p(z_0) + 1}\right|^2 \\
&= \left|\frac{1 - \alpha\left(it + m(\frac{1+t^2}{2t})i - 1\right)}{it + 1}\right|^2 \\
&\ge \frac{(1+\alpha)^2 + \alpha^2(t + n\frac{1+t^2}{2t})^2}{1+t^2} \\
&= \alpha^2 n + \frac{4(1+\alpha)^2 t^2 + 4\alpha^2 t^4 + n^2\alpha^2(1+t^2)^2}{4t^2(1+t^2)}.
\end{aligned}$$

If we denote

$$h(x) = \alpha^2 n + \frac{4(1+\alpha)^2 x + 4\alpha^2 x^2 + n^2\alpha^2(1+x)^2}{4x(1+x)},$$

where $x = t^2$, then it is easy to check that $h'(x) \le 0$, then h is a decreasing function, Hence $h(x) > \frac{\alpha^2}{4}(n+2)^2$ and

$$\left|\frac{1 - \alpha \frac{z_0 f''(z_0)}{f'(z_0)}}{1 + \frac{z_0 f'(z_0)}{f(z_0)}}\right| > \frac{\alpha}{2}(n+2).$$

However, this contradicts (11). Hence the proof is completed. □

4. Conclusions

In this paper, we have proved a result similar to Suffridge's theorem and given some applications related to this result. Moreover, we have investigated some sufficient conditions for starlikeness of analytic functions and functions that are in the class $\mathcal{S}^*[A, B]$.

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References

1. Goluzin, G.M. On the majorization principle in function theory. *Dokl. Akad. Nauk SSSR* **1935**, *42*, 647–650.
2. Suffridge, T.J. Some remarks on convex maps of the unit disc. *Duke Math. J.* **1970**, *37*, 775–777. [CrossRef]
3. Miller, S.S.; Mocanu, P.T. *Differential Subordinations: Theory and Applications*; Marcel Dekker Inc.: New York, NY, USA, 2000.
4. Silverman, H. Convex and starlike criteria. *Int. J. Math. Math. Sci.* **1990**, *22*, 75–79. [CrossRef]
5. Obradowič, M.; Tuneski, N. On the starlike criteria defined by Silverman. *Zesz. Nauk. Politech. Rzesz. Mat.* **2001**, *24*, 59–64.
6. Nunokawa, M.; Owa, M.; Saitoh, S.; Ahuja, H. On the quotient of the analytic representations of convex and starlike functions. *Sūrikaisekikenkyūsho Kōkyūroku* **1999**, *112*, 63–69.
7. Avdiji, S.; Tuneski, N. Sufficient conditions for starlikeness using subordination method. *Adv. Math.* **2020**, *9*, 10707–10716. [CrossRef]
8. Kumar, S.S.; Kumar, V.; Ravichandran, V.; Cho, N.E. Sufficient conditions for starlike functions associated with the lemniscate of Bernoulli. *J. Inequal. Appl.* **2013**, *2013*, 1–13. [CrossRef]
9. Obradowič, M.; Owa, S. On certain properties for some classes of starlike functions. *J. Math. Anal. Appl.* **1999**, *145* , 357–364. [CrossRef]
10. Tuneski, N. On the quotient of the representations of convexity and starlikeness. *Math. Nachr.* **2003**, *248/249*, 200–203. [CrossRef]
11. Tuneski, N.; Darus, M. On functions that are Janowski starlike with respect to n-symmetric points. *Hacettepe J. Math. Statist.* **2012**, *41*, 271–275.
12. Liu, J.-L. On certain p-valent close-to-convex functions of order β and type α. *Rev. Real Acad. Ciencias Exactas Fís. Naturales. Ser. A Mat.* **2019**, *113*, 49–57. [CrossRef]
13. Darus, M.; Yavuz, E.; Owa, S. Some results on certain analytic functions. *Anal. Univ. Oradea Fasc. Mat.* **2020**, *27*, 31–35.
14. Oros, G.I.; Oros, G.; Owa, S. Subordination properties of certain operators concerning fractional integral and Libera integral operator. *Fractal Fract.* **2023**, *7*, 42. [CrossRef]
15. Lecko, A. A new inner approach for differential subordinations. *Proc. Royal Soc. Edin. Sec. A Math.* **2023**, 1–14 . [CrossRef]
16. Ali, R.M.; Nagpal, S.; Ravichandran, V. Second-order differential subordination for analytic functions with fixed initial coefficient. *Bull. Malays. Math. Sci. Soc.* **2011**, *34*, 611–629.
17. Strohhäcker, E. Beitrage zür Theorie der schlichten Funktionen. *Math. Z.* **1933**, *37*, 356–380. [CrossRef]